Can. J. Math., Vol. XXIV, No. 5, 1972, pp. 915–925

THE HARDY SPACE H^1 ON MANIFOLDS AND SUBMANIFOLDS

ROBERT S. STRICHARTZ

1. Introduction. It is well-known that the space $L^1(\mathbf{R}^n)$ of integrable functions on Euclidean space fails to be preserved by singular integral operators. As a result the rather large L^p theory of partial differential equations also fails for $p = 1$. Since L^1 is such a natural space, many substitute spaces have been considered. One of the most interesting of these is the space we will denote by $H^1(\mathbf{R}^n)$ of integrable functions whose Riesz transforms are integrable. Recall the Riesz transforms $R_1, \ldots, R_n$ are defined via the Fourier transform by

$$(R_j f)^\wedge(\xi) = \frac{\xi_j}{|\xi|}\hat{f}(\xi).$$

These are the n-dimensional analogues of the Hilbert transform (if $n = 1$ then R_1 is the Hilbert transform).

Now Stein [**6**] has shown that $H^1(\mathbf{R}^n)$ is preserved by all sufficiently smooth singular integral operators. In this paper we use that result to extend the basic L^p results used in the theory of elliptic boundary value problems to the class H^1. We will show that $H^1(\mathbf{R}^n)$ is locally preserved by pseudo-differential operators of order zero. This enables us to give an invariant definition of $H^1(M)$ where M is any compact C^∞ manifold without boundary: $H^1(M)$ is the space of all $f \in L^1(M)$ such that $Tf \in L^1(M)$ for every pseudo-differential operator T of order zero. We may also define Sobolev spaces $H_\alpha^1(M)$ of distributions having α derivatives in $H^1(M)$.

Next we characterize the restrictions of functions in $H_\alpha^1(M)$ to open submanifolds and lower dimensional submanifolds. After simple reductions of these problems to the Euclidean case our main results are as follows:

(1) A function $f(x, t) \in L^1(\mathbf{R}^{n-1} \times (0, \infty))$ is the restriction of an $H^1(\mathbf{R}^n)$ function if and only if the odd reflection

$$F(x, t) = \begin{cases} f(x, t) & \text{if } t > 0 \\ -f(x, -t) & \text{if } t < 0 \end{cases}$$

is in $H^1(\mathbf{R}^n)$.

(2) A function in $H_\alpha^1(\mathbf{R}^n)$ with $\alpha \geqq 1$ has a well-defined restriction to any hyperplane. Furthermore if $\alpha > 1$ the exact class of such restrictions is the Besov space $\Lambda(\alpha - 1; 1, 1)$. The case $\alpha = 1$ remains open.

Received September 14, 1971 and in revised form, October 22, 1971. This research was supported by NSF Grant GP-22820.

We will use freely without reference material found in Stein [**6**]. Most of our arguments are elementary, but they are based on two deep theorems of Stein and Fefferman (Theorems A and B below).

It is a great pleasure to acknowledge the assistance of my wife in proving Lemma 2.

2. The definition of $H^1(M)$**.** We consider $H^1(\mathbf{R}^n)$ as a Banach space with norm

$$||f||_{H^1} = ||f||_1 + \sum_{j=1}^{n} ||R_j f||_1.$$

We use the following result contained in Stein [**6**, p. 232]:

THEOREM A. *Let* $m(\xi) \in L^\infty(\mathbf{R}^n)$ *be* C^{n+1} *away from the origin and satisfy*

$$(1) \qquad ||\xi|^{|\alpha|}(\partial/\partial\xi)^\alpha m(\xi)| \leqq M \text{ for } |\alpha| \leqq n+1.$$

Then the multiplier transformation $Tf\hat{}(\xi) = m(\xi)\hat{f}(\xi)$ *is bounded on* $H^1(\mathbf{R}^n)$ *with norm dominated by a multiple of* M.

It is a simple matter to obtain from this an analogous local result for pseudo-differential operators of order zero.

THEOREM 1. *Let* $p(x, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ *have compact support in the* x-*variable and satisfy*

$$(2) \quad |(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta p(x, \xi)| \leqq M_{\alpha,\beta}(1 + |\xi|)^{-|\alpha|} \text{ for } |\alpha| \leqq n+1,\ |\beta| \leqq n.$$

Then the operator

$$Tf(x) = \int e^{ix\cdot\xi} p(x, \xi)\hat{f}(\xi)d\xi$$

is bounded from $H^1(\mathbf{R}^n)$ *to* $L^1(\mathbf{R}^n)$ *with a norm depending only on the* $M_{\alpha,\beta}$ *and the support of* $p(x, \xi)$.

Proof. We will show in fact that

$$\int \sup_y \left| \int e^{ix\cdot\xi} p(y, \xi)\hat{f}(\xi)d\xi \right| dx \leqq c||f||_H.$$

Indeed by Sobolev's inequality

$$\sup_y \left| \int e^{ix\cdot\xi} p(y, \xi)\hat{f}(\xi)d\xi \right| \leqq c \sum_{|\beta| \leqq n} \int \left| \int e^{ix\cdot\xi}(\partial/\partial y)^\beta p(y, \xi)\hat{f}(\xi)d\xi \right| dy.$$

On the other hand by Theorem A we have

$$\int \left| \int e^{ix\cdot\xi}(\partial/\partial y)^\beta p(y, \xi)\hat{f}(\xi)d\xi \right| dx$$

$$\leqq c \sup_\xi \sup_{|\alpha| \leqq n+1} |\xi|^{|\alpha|}(\partial/\partial\xi)^\alpha(\partial/\partial y)^\beta p(y, \xi)| \, ||f||_{H^1}.$$

Integrating with respect to y we obtain

$$\int \sup_y \left| \int e^{ix\cdot\xi} p(y,\xi)\hat{f}(\xi)d\xi \right| dx$$

$$\leqq c \int \sum_{|\beta|\leqq n} \sup_{|\alpha|\leqq n+1} \sup_{\xi} |\xi|^{|\alpha|} (\partial/\partial\xi)^{\alpha}(\partial/\partial y)^{\beta} p(y,\xi)|dy| |f||_{H^1}.$$

The integrand vanishes for y outside the support of $p(y, \xi)$ and is bounded because of (2), so the integral is finite.

At this point it is convenient to handle the problem of localizing H^1 functions. Note that multiplying by a function in $C^\infty{}_{\mathrm{com}}(\mathbf{R}^n)$ will not preserve the class H^1 because all functions in H^1 must have total integral zero. The next lemma says in effect that this is the only difficulty.

LEMMA 1. *Let $f(x) \in L^1(\mathbf{R}^n)$ have compact support and total integral zero. If the Riesz transforms $R_j f(x)$ are integrable on a neighbourhood of the support of f, then $f \in H^1(\mathbf{R}^n)$.*

Proof. We must show that $R_j f(x)$ is integrable over the set of x whose distance to the support of f exceeds ϵ. Using the formula

$$R_j f(x) = c\text{P.V.} \int \frac{x_j - y_j}{|x-y|^{n+1}} f(y) dy$$

we see that $R_j f$ is bounded on this set. For large values of x we use the fact that $\int f(y)dy = 0$ to write

$$R_j f(x) = c \int \left(\frac{x_j - y_j}{|x-y|^{n+1}} - \frac{x_j}{|x|^{n+1}} \right) f(y) dy$$

and apply the mean value theorem to estimate

$$|R_j f(x)| \leqq c|x|^{-n-1} \int |y| \, |f(y)| dy$$

which is integrable.

We are now in a position to define $H^1(M)$ for a compact C^∞ manifold without boundary M. We fix a smooth measure dx on M equivalent to Lebesgue measure in every coordinate system. Let $\{\varphi_i\}$ be a C^∞ partition of unity subordinate to a covering by coordinate neighbourhoods and let ψ_i be a C^∞ function supported in a coordinate neighbourhood satisfying $\psi_i \varphi_i = \varphi_i$. We define $H^1(M)$ to be the subspace of $L^1(M)$ of functions f for which $\psi_i R_j(\varphi_i f)$ is integrable for all i, j. Here the Riesz transform R_j is taken with respect to the local coordinate system. The norm on $H^1(M)$ is

$$||f||_1 + \sum_i \sum_j ||\psi_i R_j(\varphi_i f)||_1.$$

https://doi.org/10.4153/CJM-1972-091-5 Published online by Cambridge University Press

This definition appears to depend on the choice of the partition of unity and the local coordinates, but we shall see that in fact it does not.

Let us recall briefly the definition of pseudo-differential operators [**3**]. An operator $T: C^\infty(M) \to C^\infty(M)$ is called a pseudo-differential operator of order r if $\psi_i T(\varphi_i f)$ is given in local coordinates by

$$\int e^{ix\cdot\xi} p(x, \xi)(\varphi_i f)^\wedge(\xi) d\xi$$

where

$$p(x, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$$

satisfies

$$|(\partial/\partial\xi)^\alpha (\partial/\partial x)^\beta p(x, \xi)| \leqq M_{\alpha,\beta}(1 + |\xi|)^{r-|\alpha|} \tag{3}$$

for all α and β, and $(1 - \psi_i)T(\varphi_i f)$ is given by

$$\int_M K(x, y) f(y) dy$$

where $K \in C^\infty(M \times M)$.

THEOREM 1′. *If T is a pseudo-differential operator of order zero, then T is a bounded operator on $H^1(M)$.*

Proof. Since $f \to \psi_i R_j(\varphi_i f)$ is also a pseudo-differential operator of order zero, and these form an algebra under composition, it suffices to show $||Tf||_1 \leqq c||f||_{H^1}$. Now

$$Tf = \sum_i \psi_i T(\varphi_i f) + \sum_i (1 - \psi_i) T(\varphi_i f),$$

so it suffices to estimate each term separately. We have easily

$$||(1 - \psi_i) T(\varphi_i f)||_1 \leqq c||f||_1$$

so it remains to estimate $\psi_i T(\varphi_i f)$.

Now $\varphi_i f$ in local coordinates need not be in H^1, but this is easily remedied. We consider $\varphi_i f + g$ where g is chosen to make the total integral zero in local coordinates, and $g \in C^\infty{}_{\text{com}}$ with support on the set where $\psi_i = 1$. Easy estimates show $\psi_i R_j g \in L^1$, so by the hypotheses and Lemma 1 we have $\varphi_i f + g \in H^1(\mathbf{R}^n)$ in local coordinates (it is clear we may also choose g so that

$$||\varphi_i f + g||_{H^1} \leqq c\left(||f||_1 + \sum_{j=1}^n ||\psi_i R_j(\varphi_i f)||_1\right)\Bigg).$$

Applying Theorem 1 we obtain

$$||\psi_i T(\varphi_i f + g)||_1 \leqq c||\varphi_i f + g||_{H^1}.$$

But $||\psi_i Tg||_1$ can be easily estimated in terms of g and its derivatives, so we have

$$||\psi_i T(\varphi_i f)||_1 \leqq c\left(||f||_1 + \sum_{j=1}^n ||\psi_i R_j(\varphi_i f)||_1\right)$$

as desired.

COROLLARY. *A function f belongs to $H^1(M)$ if and only if $f \in L^1(M)$ and $Tf \in L^1(M)$ for every pseudo-differential operator T of order zero.*

Proof. If $f \in H^1(M)$ then $Tf \in L^1(M)$ by Theorem 1′. For the converse we have only to observe that $f \to \psi_i R_j(\varphi_i f)$ is a pseudo-differential operator of order zero.

We may now define the space $H_\alpha{}^1(M)$ for any real α to be the image of $H^1(M)$ under any invertible elliptic pseudo-differential operator of order $-\alpha$. Equivalently, $f \in H_\alpha{}^1(M)$ if and only if $Tf \in L^1(M)$ for every pseudo-differential operator T of order α.

3. Restrictions and extensions. Let us denote points in $\mathbf{R}^n$ by (x, t) with $x \in \mathbf{R}^{n-1}$ and $t \in R^1$. Let $g(x)$ be a real-valued function on R^{n-1} which is uniformly Lipschitz, $|g(x) - g(y)| \leqq A|x - y|$. Let $\Omega \subseteq \mathbf{R}^n$ be the set of points where $t > g(x)$. We define $H^1(\Omega)$ to be the set of restrictions to Ω of functions in $H^1(\mathbf{R}^n)$. We give $H^1(\Omega)$ the natural norm

$$||f||_{H^1} = \inf\{||F||_{H^1} : F \in H^1(\mathbf{R}^n) \quad \text{and} \quad F|\Omega = f\}.$$

Suitably interpreted, the above definitions make sense for $n = 1$ where $\Omega = \{t : t > g\}$ and g is a constant.

THEOREM 2. *Let $E : L^1(\Omega) \to L^1(\mathbf{R}^n)$ be the odd reflection*

$$Ef(x, t) = \begin{cases} f(x, t), & \text{if } t > g(x) \\ -f(x, 2g(x) - t), & \text{if } t < g(x) \end{cases}$$

(since the boundary $t = g(x)$ has measure zero we need not define Ef there). Then $f \in H^1(\Omega)$ if and only if $Ef \in H^1(\mathbf{R}^n)$, and $||Ef||_{H^1} \leqq c(A)||f||_{H^1}$.

To prove the theorem we make use of Fefferman's characterization of the dual space of $H^1(\mathbf{R}^n)$ [**2**]. We denote by $\mathrm{BMO}(\mathbf{R}^n)$ the space of functions of bounded mean oscillation on $\mathbf{R}^n$ defined as follows: an element of $\mathrm{BMO}(\mathbf{R}^n)$ is an equivalence class of locally integrable functions, modulo constant functions, satisfying

$$\frac{1}{m(Q)} \int_Q |f(x) - a_Q| dx \leqq M \tag{4}$$

for some constant a_Q and every cube $Q \subseteq \mathbf{R}^n$. The norm is the least such M. It is clear that the optimal value for a_Q is the mean value of f on the cube Q.

THEOREM B [**2**]. *The dual space of $H^1(\mathbf{R}^n)$ is* $\mathrm{BMO}(\mathbf{R}^n)$ *under the usual pairing and with an equivalent norm.*

The restriction problem for BMO is easier to deal with than for H^1. Let us first observe that (4) holds for a larger class of sets. In fact we have

$$\frac{1}{m(D)} \int_D |f(x) - a_D| dx \leqq bM \tag{5}$$

for some constant a_D for every set D with the property that there exists a cube $Q \supseteq D$ such that $m(Q) \leqq bm(D)$. Indeed we need only set $a_D = a_Q$ and observe

$$\frac{1}{m(D)}\int_D |f(x) - a_Q|dx \leqq \frac{1}{m(D)}\int_Q |f(x) - a_Q|dx \leqq \frac{m(Q)}{m(D)} M.$$

We call such sets D b-quasicubes. We may then define $\mathrm{BMO}_b(\Omega)$ to be the set of equivalence classes of locally integrable functions on Ω for which (5) holds for every b-quasicube contained in Ω.

LEMMA 2. *There exists a constant depending only on n and A such that the following three conditions are equivalent* (*with norm equivalence*):

(i) $f \in \mathrm{BMO}_b(\Omega)$ for some $b \geqq c$;

(ii) there exists $F \in \mathrm{BMO}(\mathbf{R}^n)$ such that $F|\Omega = f$;

(iii) $E'f \in \mathrm{BMO}(\mathbf{R}^n)$, where E' is the even reflection

$$E'f(x, t) = \begin{cases} f(x, t), & \text{if } t > g(x) \\ f(x, 2g(x) - t), & \text{if } t < g(x). \end{cases}$$

Proof. Clearly (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) so it suffices to show (i) $\Rightarrow$ (iii). The gist of the argument is that because g satisfies a Lipschitz condition the reflection of a cube is a b-quasicube.

Let Q be any cube lying outside Ω, and let $D = \{(x, t) : (x, 2g(x) - t) \in Q\}$. We have $m(D) = m(Q)$, $D \subseteq \Omega$ and

$$\frac{1}{m(Q)}\int_Q |E'f(x, t) - a|dxdt = \frac{1}{m(D)}\int_D |f(x, t) - a|dxdt$$

so to establish (4) for Q it suffices to show that D is a b-quasicube. Now suppose Q is the cube $\{(x, t) : |x_j - y_j| \leqq r \text{ and } |t - s| \leqq r\}$. Then $m(Q) = (2r)^n$. Let $(x, 2g(x) - t) \in D$. Then $|x_j - y_j| \leqq r$ and $|2g(x) - t - (2g(y) - s)| \leqq r + 2|g(x) - g(y)| \leqq r + 2A\sqrt{n}r$ so D is contained in the cube with diameter $2r(1 + 2\sqrt{n}A)$ and centre $(y, 2g(y) - s)$. Thus D is a b-quasicube if $b \geqq (1 + 2\sqrt{n}A)^n$.

Finally we must verify (4) for cubes Q meeting the boundary of Ω. In this case we show by similar reasoning that the set D which is the union of $Q \cap \Omega$ with the reflection of $Q \cap -\Omega$ is a b-quasicube for sufficiently large b. We then have

$$\frac{1}{m(Q)}\int_Q |E'f(x, t) - a_D|dxdt \leqq \frac{2}{m(Q)}\int_D |f(x, t) - a_D|dxdt \leqq 2bM.$$

Proof of Theorem 2. Let $f \in H^1(\Omega)$ and let $F \in H^1(\mathbf{R}^n)$ with $F|\Omega = f$ and $||F||_{H^1} \leqq 2||f||_{H^1}$. First we claim that F may be approximated in $H^1(\mathbf{R}^n)$ norm by bounded functions of compact support. For if not there would exist a non-constant function $h \in \mathrm{BMO}(\mathbf{R}^n)$ such that $\int h(x)F(x)dx = 0$ for every $F \in H^1(\mathbf{R}^n)$ which is bounded with compact support. But Lemma 1 together

https://doi.org/10.4153/CJM-1972-091-5 Published online by Cambridge University Press

with simple estimates shows that any bounded function with compact support and total integral zero is in $H^1(\mathbf{R}^n)$. Thus we may choose $F(x)$ to show that h is constant, a contradiction.

Let $||F_k - F||_{H^1} \leqq k^{-1}$ with F_k bounded with compact support. Let $f_k = F_k|\Omega$. Then $Ef_k \to Ef$ in L^1, so it suffices to show Ef_k is Cauchy in $H^1(\mathbf{R}^n)$ norm, and $||Ef_k||_{H^1} \leqq c||F_k||_{H^1}$.

Now observe that Ef_k is bounded with compact support and total integral zero, so $Ef_k \in H^1(\mathbf{R}^n)$. Thus

$$||Ef_k||_{H^1} \leqq \sup \left\{ \int hEf_k : ||h||_{\mathrm{BMO}} \leqq 1 \right\}.$$

Now we write $h = h_1 + h_2$ where $h_1 = E'(h|\Omega)$. In view of Lemma 2 we have $||h_1||_{\mathrm{BMO}} \leqq c||h||_{\mathrm{BMO}}$ hence also $||h_2||_{\mathrm{BMO}} \leqq (c+1)||h||_{\mathrm{BMO}}$. But since h_1 is even and Ef_k is odd we have $\int h_1(x,t)Ef_k(x,t)dxdt = 0$. Thus

$$\int hEf_k = \int h_2 Ef_k = \int h_2 F_k$$

since h_2 vanishes off Ω. Thus $||Ef_k||_{H^1} \leqq c||F_k||_{H^1}$ and similarly

$$||E(f_k - f_j)||_{H^1} \leqq c||F_k - F_j||_{H^1}.$$

Remark. If $n = 1$ and $\Omega = \{t > 0\}$, the condition that $Ef \in H^1(\mathbf{R}^1)$ is equivalent to $f \in L^1(0, \infty)$ and

$$\text{P.V.} \int_0^\infty \frac{t}{x^2 - t^2} f(t)dt \in L^1(0, \infty).$$

For $n > 1$ there does not appear to be any simpler way to formulate the condition.

It is now a routine matter to generalize Theorem 2. If G is any open subset of M we may define $H^1(G)$ to be the space of restrictions to G of functions in $H^1(M)$.

COROLLARY 1. *If the boundary of G is a compact Lipschitz manifold then there exists a bounded linear extension map $E : H^1(G) \to H^1(M)$ so that $f \in H^1(G)$ if and only if $Ef \in H^1(M)$.*

We omit the proof and refer the reader to [**6**, Chapter 6] for similar arguments.

It is also possible to characterize restrictions to G of functions in $H_\alpha^1(M)$. For simplicity we consider only the case $\alpha = k$, a positive integer. Here we may define $H_k^1(G)$ to be the space of $f \in H^1(G)$ for which $Df \in H^1(G)$ for every differential operator D of order $\leqq k$.

COROLLARY 2. *There exists a bounded linear extension operator*

$$E_k : H_k^1(G) \to H_k^1(M).$$

Proof. By a partition of unity argument we may reduce the problem to finding $E_k : H_k^1(\Omega) \to H_k^1(\mathbf{R}^n)$, where $H_k^1(\Omega)$ is the space of $f \in H^1(\Omega)$ for which $D^\alpha f \in H^1(\Omega)$ for all α with $|\alpha| \leqq k$. We first extend f and $D^\alpha f$ to elements of $H^1(\mathbf{R}^n)$ by odd reflection, and then apply the construction of Calderón [**1**] for obtaining an extension operator from $L_k^p(\Omega) \to L_k^p(\mathbf{R}^n)$. We may apply Calderón's proof almost verbatim, using Theorem A for the preservation of $H^1(\mathbf{R}^n)$ by singular integrals.

Next we consider the problem of restrictions to hypersurfaces. Let us define $H_\alpha^1(\mathbf{R}^n)$ to be the image of $H^1(\mathbf{R}^n)$ under the Bessel potential G_α defined by $(G_\alpha f)^\wedge(\xi) = (1 + |\xi|^2)^{-\alpha/2}\hat{f}(\xi)$. Since G_α is an invertible elliptic pseudo-differential operator of order $-\alpha$ this is consistent with our previous definition of $H_\alpha^1(M)$, and it is not hard to see that $H_\alpha^1(M)$ is modelled on $H_\alpha^1(\mathbf{R}^n)$ in the same way that $H^1(M)$ is modelled on $H^1(\mathbf{R}^n)$.

There are many ways of defining the Besov spaces $\Lambda(\alpha; p, q)(\mathbf{R}^n)$. The following definition is due to Peetre [**4**]: let σ, τ be C^∞ functions on $[0, \infty)$ with $\sigma \equiv 1$ on $(1, 2)$ and supported on $(\frac{1}{2}, 4)$, and $\tau \equiv 1$ on $[0, 4)$ and supported on $[0, 8)$. Then $\Lambda(\alpha; p, q)(\mathbf{R}^n)$ is the space of tempered distributions on $\mathbf{R}^n$ for which the following norm is finite:

$$||f : \Lambda(\alpha; p, q)|| = ||\mathscr{F}^{-1}(\tau(|\xi|)\hat{f}(\xi))||_p + \left(\int_0^1 ||\mathscr{F}^{-1}(\sigma(t|\xi|)|\xi|^\alpha \hat{f}(\xi))||_p{}^q \frac{dt}{t}\right)^{1/q}$$

(if $q = \infty$, replace the integral by sup). It is clear from other equivalent definitions (see [**6**]) that $\Lambda(\alpha; p, q)$ is locally invariant under diffeomorphisms and multiplication by $C^\infty{}_{\text{com}}$ functions, so we may define $\Lambda(\alpha; p, q)(M)$ for a compact C^∞ manifold without boundary M in the usual manner.

THEOREM 3. *If $\alpha \geqq 1$, the restriction map $Rf(x, t) = f(x, 0)$ is well defined from $H_\alpha^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^{n-1})$. Furthermore, if $\alpha > 1$ then*

$$R : H_\alpha^1(\mathbf{R}^n) \to \Lambda(\alpha - 1; 1, 1)(\mathbf{R}^{n-1})$$

and there exists an extension map $\mathscr{E} : \Lambda(\alpha - 1; 1, 1)(\mathbf{R}^{n-1}) \to H_\alpha^1(\mathbf{R}^n)$ such that $R\mathscr{E}f = f$.

Proof. If $f \in H_\alpha^1(\mathbf{R}^n)$ for $\alpha \geqq 1$ then $\partial f/\partial t \in L^1(\mathbf{R}^n)$ by Theorem A and

$$\int |f(x, 0)|dx \leqq \int_0^\infty \int_{\mathbf{R}^{n-1}} |(\partial f/\partial t)(x, t)|dxdt,$$

so $R : H_\alpha^1(\mathbf{R}^n) \to L^1(\mathbf{R}^{n-1})$. Now the boundedness of

$$R : H_\alpha^1(\mathbf{R}^n) \to \Lambda(\alpha - 1; 1, 1)(\mathbf{R}^{n-1}) \quad \text{for} \quad \alpha > 1$$

is essentially proved in Stein [**7**]. In fact, Stein proves the boundedness of $R : L_\alpha^p(\mathbf{R}^n) \to \Lambda(\alpha - 1/p; p, p)(\mathbf{R}^{n-1})$ if $\alpha > 1/p$ and $1 < p < \infty$ where $L_\alpha^p(\mathbf{R}^n)$ is the image of $L^p(\mathbf{R}^n)$ under G_α. But the same proof works for $p = 1$ (the restrictions $0 < \alpha < 1$ and $1/p < \alpha < 1$ on p. 579 of [**7**] may be replaced by $0 < \alpha < 2$ and $1/p < \alpha < 1 + 1/p$).

We define the extension map by means of the Fourier transform:

$$(6)\qquad (\mathscr{E}f)^\wedge(\xi,\eta) = c_1\tau(|\xi|)\hat{f}(\xi)\sigma(|\eta|) + c_2\frac{(1-\tau(|\xi|))|\xi|^{\beta-1}\hat{f}(\xi)}{(|\xi|^2+\eta^2)^{\beta/2}}$$

where $\beta > \alpha + 2 + n$ and c_1, c_2 are chosen so that

$$(7)\qquad \int_{-\infty}^{\infty} (\mathscr{E}f)^\wedge(\xi,\eta)d\eta = \hat{f}(\xi).$$

Now (7) means $R\mathscr{E}f = f$. To show $\mathscr{E} : \Lambda(\alpha - 1; 1, 1)(\mathbf{R}^{n-1}) \to H_\alpha{}^1(\mathbf{R}^n)$ we handle each term of (6) separately. The first term is trivial because it has compact support away from the origin. For the second term we use the identity

$$(1-\tau(|\xi|))\hat{f}(\xi) = \int_0^1 (1-\tau(|\xi|))\sigma^2(s|\xi|)\hat{f}(\xi)\frac{ds}{s}.$$

The problem is then to show that the following three expressions are Fourier transforms of L^1 functions:

$$(8)\qquad \int_0^1 \frac{|\xi|^{\beta-1}\hat{f}(\xi)\sigma^2(s|\xi|)}{(|\xi|^2+\eta^2)^{(\beta-\alpha)/2}}\frac{ds}{s}$$

$$(9)\qquad \int_0^1 \frac{|\xi|^{\beta-1}\xi_j\hat{f}(\xi)\sigma^2(s|\xi|)}{(|\xi|^2+\eta^2)^{(\beta-\alpha+1)/2}}\frac{ds}{s}$$

$$(10)\qquad \int_0^1 \frac{|\xi|^{\beta-1}\eta\hat{f}(\xi)\sigma^2(s|\xi|)}{(|\xi|^2+\eta^2)^{(\beta-\alpha+1)/2}}\frac{ds}{s}.$$

Let us consider for example (10) (the others are similar). We have

$$\left\|\mathscr{F}_{x,t}{}^{-1}\left(\int_0^1 \frac{|\xi|^{\beta-1}\eta\hat{f}(\xi)\sigma^2(s|\xi|)}{(|\xi|^2+\eta^2)^{(\beta-\alpha+1)/2}}\frac{ds}{s}\right)\right\|_1$$
$$\leqq \int_{-\infty}^{\infty}\int_0^1 \left\|\mathscr{F}_x{}^{-1}\left[\int_{-\infty}^{\infty}\frac{|\xi|^{\beta-\alpha}\eta\sigma(s|\xi|)}{(|\xi|^2+\eta^2)^{(\beta-\alpha+1)/2}}e^{it\eta}d\eta\right]\right\|_1$$
$$\times \|\mathscr{F}_x{}^{-1}(|\xi|^{\alpha-1}\sigma(s|\xi|)\hat{f}(\xi)\|_1\cdot\frac{ds}{s}dt.$$

Since

$$\int_0^1 \|\mathscr{F}_x{}^{-1}(|\xi|^{\alpha-1}\sigma(s|\xi|)\hat{f}(\xi))\|_1\frac{ds}{s} \leqq \|f : \Lambda(\alpha-1;1,1)\|$$

it suffices to show that

$$(11)\qquad \int_{-\infty}^{\infty}\left\|\mathscr{F}_x{}^{-1}\left(\int_{-\infty}^{\infty}\frac{|\xi|^{\beta-\alpha}\eta\sigma(s|\xi|)}{(|\xi|^2+\eta^2)^{(\beta-\alpha+1)/2}}e^{it\eta}d\eta\right)\right\|_1 dt$$

is bounded independent of s. But the change of variable $t \to st$, $\eta \to \eta/s$ and the dilation $\xi \to \xi/s$ (dilation on the Fourier transform side does not change

https://doi.org/10.4153/CJM-1972-091-5 Published online by Cambridge University Press

the L^1 norm) shows that (11) is independent of s, so we may set $s = 1$. Next we make the change of variable $\eta \to |\xi|\eta$ to obtain

$$\int_{-\infty}^{\infty} \left\| \mathscr{F}_x^{-1} \left[\int_{-\infty}^{\infty} |\xi|\sigma(|\xi|) \frac{\eta e^{it|\xi|\eta}}{(1+\eta^2)^{(\beta-\alpha+1)/2}} d\eta \right] \right\|_1 dt. \tag{12}$$

To estimate (12) we will use the well-known version of Sobolev's inequality,

$$||\mathscr{F}_x^{-1}(g(\xi))||_1 \leqq c \sum_{|\gamma| \leqq n+1} ||(\partial/\partial\xi)^\gamma g(\xi)||_1. \tag{13}$$

Now

$$\int_{-\infty}^{\infty} \frac{\eta^k e^{i\tau\eta}}{(1+\eta^2)^{(\beta-\alpha+1)/2}} d\eta$$

is the kth derivative of the one-dimensional Bessel potential of order $\beta - \alpha + 1$. Thus we have

$$\left| \int_{-\infty}^{\infty} \frac{\eta^k e^{i\tau\eta}}{(1+\eta^2)^{(\beta-\alpha+1)/2}} d\eta \right| \leqq ce^{-A\tau}$$

provided $\beta - \alpha + 1 - k > 1$ (see [**1**]). Thus

$$\left| \int_{-\infty}^{\infty} (\partial/\partial\xi)^\gamma \left(|\xi|\sigma(|\xi|) \frac{\eta e^{it|\xi|\eta}}{(1+\eta^2)^{(\beta-\alpha+1)/2}} \right) d\eta \right| \leqq c\sigma(|\xi|)(1+|t|)^{|\gamma|} e^{-At|\xi|}$$

for $|\gamma| \leqq n + 1$, so by (13)

$$\left\| \mathscr{F}_x^{-1} \left(\int_{-\infty}^{\infty} |\xi|\sigma(|\xi|) \frac{\eta e^{it|\xi|\eta}}{(1+\eta^2)^{(\beta-\alpha+2)/2}} \right) d\eta \right\|_1 \leqq c(1+|t|)^{|\gamma|} e^{-At/2},$$

which shows (12) is bounded.

COROLLARY. *Let $N \subseteq M$ be a compact C^∞ submanifold of codimension one. Let $\partial/\partial t$ denote any transversal derivative to N. Let $T_k : C^\infty(M) \to C^\infty(N)^{k+1}$ be defined by $T_k f = (f|N, (\partial/\partial t)f|N, \ldots (\partial/\partial t)^k f|N)$. Then*

$$T_k : H_\alpha^1(M) \to \prod_{j=0}^{k} \Lambda(\alpha - 1 - j; 1, 1) \text{ for all } \alpha > k + 1,$$

and there exists an extension map

$$\mathscr{E}_k : \prod_{j=0}^{k} \Lambda(\alpha - 1 - j; 1, 1) \to H_\alpha^1(M)$$

such that $T_k \mathscr{E}_k$ is the identity.

We omit the routine proof of the above corollary, which is similar to one in Seeley [**5**].

Using the results of this section we can generalize to H^1 most of the results on elliptic boundary value problems valid for L^p, $1 < p < \infty$. We state one typical example from Seeley [**5**]:

THEOREM 4. *Let $M_0 \subseteq M$ be an open C^∞ submanifold with compact C^∞ boundary Γ, and let A be a uniformly elliptic differential operator on M of order m.*

Let u be any C^∞ function on M_0 which satisfies $Au = 0$ and which is the restriction to M_0 of a distribution on M. Then $u \in H_\alpha{}^1(M_0)$ if and only if the Cauchy data of u belongs to $\Pi_{j=0}^{m-1} \Lambda(\alpha - 1 - j; 1, 1)(\Gamma)$, for any non-negative integer α.

Remark. Comparing this with the results of [**8**], we see that for a solution of $Au = 0$ on M_0, $u \in H^1(M_0)$ if and only if $u \in L^1(M_0)$.

References

1. A. P. Calderón, *Lebesgue spaces of differentiable functions and distributions*, Symp. on Pure Math. *5* (1961), 33–49.

2. C. Fefferman, *Characterizations of bounded mean oscillation*, Bull. Amer. Math. Soc. *77* (1971), 587–588.

3. L. Nirenberg, *Pseudo-differential operators*, Proc. Symp. Pure Math. *16* (1970), 149–167.

4. J. Peetre, *Sur les espaces de Besov*, C.R. Acad. Sci. Paris Sér. A-B *264* (1967), A281–A283.

5. R. T. Seeley, *Singular integrals and boundary value problems*, Amer. J. Math. *88* (1966), 781–809.

6. E. M. Stein, *Singular integrals and differentiability properties of functions* (Princeton University Press, 1970).

7. ——— *The characterization of functions arising as potentials. II*, Bull. Amer. Math. Soc. *68* (1962), 577–582.

8. R. Strichartz, *Boundary values of solutions of elliptic equations satisfying H^p conditions* (to appear in Trans. Amer. Math. Soc.).

Cornell University,
Ithaca, New York