J. Austral. Math. Soc. Ser. B 38(1997), 307–315

SUBHARMONIC ORBITS IN AN ANHARMONIC OSCILLATOR

J. R. CHRISTIE, K. GOPALSAMY and M. P. PANIZZA[1]

(Received 17 November 1994; revised 22 June 1995)

Abstract

In a recent paper, Christie and Gopalsamy [2] used Melnikov's method to establish a sufficient condition for the existence of chaotic behaviour, in the sense of Smale, in a particular time-periodically perturbed planar autonomous system of ordinary differential equations. They then concluded with an application to the dynamics of a one-dimensional anharmonic oscillator. In this paper, the same system is considered and a condition for the existence of subharmonic orbits in the perturbed system is deduced, using the subharmonic Melnikov theory. Finally, an application is given to the dynamical behaviour of the one-dimensional anharmonic oscillator system.

1. Introduction

This paper has been written as a sequel to a recent paper by Christie and Gopalsamy [2]. In that paper, the authors established a sufficient condition for the existence of chaotic behaviour, in the sense of Smale, in a particular time-periodically perturbed planar autonomous system of ordinary differential equations, by using Melnikov's method [12]. The paper concluded with an application to the dynamics of a one-dimensional anharmonic oscillator. In this paper, the subharmonic Melnikov theory [8], [14] is used to obtain a criterion for the existence of subharmonic periodic orbits of the same perturbed system as in the paper of Christie and Gopalsamy. For examples of applications of the subharmonic Melnikov theory, see Christie *et al.* [3], Greenspan and Holmes [5, 6], Grimshaw and Tian [7], Huilgol *et al.* [9] and Zhao *et al.* [15].

To apply the subharmonic Melnikov theory, we need to assume that the corresponding unperturbed system has a continuous family of periodic orbits, and we need to parametrise this family, which is difficult in many problems, to calculate the subharmonic Melnikov function. We want to determine if any of these periodic orbits persist under perturbation. In [2], Christie and Gopalsamy have established the existence

[1]School of Information Science and Technology, The Flinders University of South Australia GPO Box 2100, Adelaide, SA, 5001 Australia.

© Australian Mathematical Society, 1997, Serial-fee code 0334-2700/96

https://doi.org/10.1017/S0334270000000692 Published online by Cambridge University Press

of a double homoclinic orbit (a lemniscate) and three continuous families of periodic orbits for the unperturbed system. They derived a parametrisation for the continuous family of periodic orbits which filled the interior of the homoclinic loop in the first quadrant. This parametrisation is used in this paper to calculate the subharmonic Melnikov function for the perturbed system.

The plan of the paper is as follows. In Section 2, we calculate the subharmonic Melnikov function and we use the subharmonic Melnikov theory to deduce a condition for the existence of subharmonic periodic orbits of the perturbed system. Finally, in Section 3, we apply the results of Section 2 to the one-dimensional anharmonic oscillator problem.

2. Subharmonic orbits

In this section, we use the subharmonic Melnikov theory ([8], [14]) to investigate the existence of subharmonic periodic orbits of the system

$$\begin{aligned} \frac{dx}{dt} &= a^2x - 2y(x^2+y^2) + \epsilon\,(b\sin\omega t + cx), \\ \frac{dy}{dt} &= -a^2y + 2x(x^2+y^2), \end{aligned} \tag{2.1$_\epsilon$}$$

where $0 \le \epsilon \ll 1$ is a perturbation parameter, $a > 0$, $\omega > 0$ and b and c are real numbers. We know from Christie and Gopalsamy [2] that the unperturbed system $(2.1)_{\epsilon=0}$ has a continuous family of periodic solutions given (in the first quadrant) by

$$\left.\begin{aligned} x_k(t) &= \frac{a\sqrt{k+1}}{2}\,\frac{\operatorname{dn}(a^2t,k) - k\operatorname{sn}(a^2t,k)\operatorname{cn}(a^2t,k)}{k\operatorname{sn}^2(a^2t,k)+1} \\ y_k(t) &= \frac{a\sqrt{k+1}}{2}\,\frac{\operatorname{dn}(a^2t,k) + k\operatorname{sn}(a^2t,k)\operatorname{cn}(a^2t,k)}{k\operatorname{sn}^2(a^2t,k)+1} \end{aligned}\right\}, \qquad t \in \mathbb{R}, \tag{2.2}$$

where $\operatorname{sn}(\cdot,k)$, $\operatorname{cn}(\cdot,k)$ and $\operatorname{dn}(\cdot,k)$ denote Jacobi elliptic functions [1] with elliptic modulus $k = \sqrt{a^4 - 8\sigma}/a^2 \in (0,1)$, in which the solution curves of $(2.1)_{\epsilon=0}$ are level sets of the Hamiltonian given by

$$H(x,y) = a^2xy - \frac{1}{2}(x^2+y^2)^2 = \sigma, \tag{2.3}$$

where σ is a constant such that $\sigma \le a^4/8$.

We concentrate on the dynamics of the one-parameter family of periodic orbits under perturbation. We want to know if any of these periodic orbits will persist under

https://doi.org/10.1017/S0334270000000692 Published online by Cambridge University Press

perturbation. The system $(2.1)_\epsilon$ is of the form

$$\left.\begin{aligned}\frac{dx}{dt} &= f_1(x, y) + \epsilon\, g_1(x, y, t)\\ \frac{dy}{dt} &= f_2(x, y) + \epsilon\, g_2(x, y, t)\end{aligned}\right\}, \qquad (x, y, t) \in \mathbb{R}^3,$$

and we write

$$f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} \quad \text{and} \quad g(x, y, t) = \begin{pmatrix} g_1(x, y, t) \\ g_2(x, y, t) \end{pmatrix}.$$

Since the unperturbed system $(2.1)_{\epsilon=0}$ is Hamiltonian, then for relatively prime positive integers m and n, the subharmonic Melnikov function is defined ([8], [14]) by

$$M^{m/n}(t_0) = \int_{-mT/2}^{mT/2} f(q_k(t)) \wedge g(q_k(t), t + t_0)\, dt, \qquad t_0 \in \mathbb{R}, \tag{2.4}$$

where $q_k(t) = (x_k(t), y_k(t))$, and the resonance condition is $T(k) = mT/n$, in which $T(k)$ is the period of q_k and $T = 2\pi/\omega$ is the period of the perturbation. If the subharmonic Melnikov function has a simple zero, and the condition $\frac{dT(k)}{d\sigma(k)} \neq 0$ is satisfied, then for $0 < \epsilon \leq \epsilon(n)$, $(2.1)_\epsilon$ has a subharmonic orbit of period mT [8, Theorem 4.6.2].

From (2.2), the resonance condition is

$$T(k) = 2K(k)/a^2 = mT/n = 2\pi m/(\omega n), \tag{2.5}$$

where $K(k)$ is the complete elliptic integral of the first kind [1]. In what follows, we choose $n = 1$ for simplicity and abbreviate $K(k)$ by K as necessary. From $(2.1)_\epsilon$, (2.2) and (2.4), we have for $t_0 \in \mathbb{R}$,

$$\begin{aligned} M^{m/1}(t_0) = &- b \int_{-\frac{mT}{2}}^{\frac{mT}{2}} \frac{dy_k}{dt}(t) \sin[\omega(t + t_0)]\, dt \\ &- c \int_{-\frac{mT}{2}}^{\frac{mT}{2}} [-a^2 x_k(t) y_k(t) + 2 x_k{}^2(t)(x_k{}^2(t) + y_k{}^2(t))]\, dt. \end{aligned} \tag{2.6}$$

https://doi.org/10.1017/S0334270000000692 Published online by Cambridge University Press

Using integration by parts on the first integral results in

$$\int_{-mT/2}^{mT/2} \frac{dy_k}{dt}(t)\sin[\omega(t+t_0)]\,dt \tag{2.7}$$
$$= y_k(t)\sin[\omega(t+t_0)]\Big|_{-mT/2}^{mT/2} - \omega\int_{-mT/2}^{mT/2} y_k(t)\cos[\omega(t+t_0)]\,dt$$
$$= -\omega\int_{-mT/2}^{mT/2} y_k(t)\cos[\omega(t+t_0)]\,dt$$
$$= -\omega\cos\omega t_0\int_{-mT/2}^{mT/2} y_k(t)\cos\omega t\,dt + \omega\sin\omega t_0\int_{-mT/2}^{mT/2} y_k(t)\sin\omega t\,dt.$$

Now substituting (2.2) produces

$$\int_{-mT/2}^{mT/2} y_k(t)\cos\omega t\,dt \tag{2.8}$$
$$= \frac{a\sqrt{k+1}}{2}\int_{-mT/2}^{mT/2}\frac{\mathrm{dn}(a^2t,k)+k\,\mathrm{sn}(a^2t,k)\,\mathrm{cn}(a^2t,k)}{k\,\mathrm{sn}^2(a^2t,k)+1}\cos\omega t\,dt$$
$$= \frac{\sqrt{k+1}}{2a}\int_{-K}^{K}\frac{\mathrm{dn}(u,k)+k\,\mathrm{sn}(u,k)\,\mathrm{cn}(u,k)}{k\,\mathrm{sn}^2(u,k)+1}\cos\left(\frac{m\pi u}{K}\right)du$$
$$= \frac{K\sqrt{k+1}}{2a}A_m,$$

where A_m is the mth Fourier cosine coefficient of the function

$$F_k(u) = \frac{\mathrm{dn}(u,k)+k\,\mathrm{sn}(u,k)\,\mathrm{cn}(u,k)}{k\,\mathrm{sn}^2(u,k)+1}, \qquad u\in(-K,K).$$

Similarly,

$$\int_{-mT/2}^{mT/2} y_k(t)\sin\omega t\,dt \tag{2.9}$$
$$= \frac{\sqrt{k+1}}{2a}\int_{-K}^{K}\frac{\mathrm{dn}(u,k)+k\,\mathrm{sn}(u,k)\,\mathrm{cn}(u,k)}{k\,\mathrm{sn}^2(u,k)+1}\sin\left(\frac{m\pi u}{K}\right)du$$
$$= \frac{K\sqrt{k+1}}{2a}B_m,$$

where B_m is the mth Fourier sine coefficient of F_k. By using contour integration ([10]), one can show that the Fourier series expansion of F_k is ([11])

$$F_k(u) = \frac{\pi}{2K\sqrt{k+1}} \tag{2.10}$$
$$+ \frac{\pi}{K\sqrt{k+1}}\sum_{n=1}^{\infty}\left[\frac{\cosh(2nW)}{\cosh(2nW_0)}\cos\left(\frac{n\pi u}{K}\right)+\frac{\sinh(2nW)}{\cosh(2nW_0)}\sin\left(\frac{n\pi u}{K}\right)\right],$$

https://doi.org/10.1017/S0334270000000692 Published online by Cambridge University Press

where $W_0 = \pi K'/2K$ and $W = \pi(K' - u_0)/2K$ in which $K' = K(k')$, where $k' = \sqrt{1-k^2}$ is the complementary modulus [1], and u_0 satisfies $\operatorname{cn}(u_0, k') = \sqrt{k/(k+1)}$, $0 < u_0 < K'$. From (2.10), we obtain

$$A_m = \frac{\pi \cosh(2mW)}{K\sqrt{k+1}\cosh(2mW_0)} \quad \text{and} \quad B_m = \frac{\pi \sinh(2mW)}{K\sqrt{k+1}\cosh(2mW_0)}, \tag{2.11}$$

and thus (2.8) and (2.9) lead respectively to

$$\int_{-mT/2}^{mT/2} y_k(t)\cos\omega t\,dt = \frac{\pi\cosh(2mW)}{2a\cosh(2mW_0)} \tag{2.12}$$

and

$$\int_{-mT/2}^{mT/2} y_k(t)\sin\omega t\,dt = \frac{\pi\sinh(2mW)}{2a\cosh(2mW_0)}. \tag{2.13}$$

Hence, from (2.7),

$$\begin{aligned}&\int_{-mT/2}^{mT/2} \frac{dy_k}{dt}(t)\sin[\omega(t+t_0)]\,dt \\ &\quad = \frac{\pi\omega}{2a\cosh(2mW_0)}[\sinh(2mW)\sin\omega t_0 - \cosh(2mW)\cos\omega t_0].\end{aligned} \tag{2.14}$$

We now evaluate the second integral in (2.6). Substituting (2.2) in this integral and simplifying produces

$$\begin{aligned}&\int_{-mT/2}^{mT/2} [-a^2 x_k(t) y_k(t) + 2x_k^2(t)(x_k^2(t) + y_k^2(t))]\,dt \\
&= \frac{a^4 k(k+1)}{4}\int_{-mT/2}^{mT/2}\left[\frac{1+(k-2)\operatorname{sn}^2(a^2t,k)+(k^2-2k)\operatorname{sn}^4(a^2t,k)}{(1+k\operatorname{sn}^2(a^2t,k))^3}\right. \\
&\quad + \frac{k^3\operatorname{sn}^6(a^2t,k)+(-2k-2)\operatorname{sn}(a^2t,k)\operatorname{cn}(a^2t,k)\operatorname{dn}(a^2t,k)}{(1+k\operatorname{sn}^2(a^2t,k))^3} \\
&\quad \left. + \frac{(2k^2+2k)\operatorname{sn}^3(a^2t,k)\operatorname{cn}(a^2t,k)\operatorname{dn}(a^2t,k)}{(1+k\operatorname{sn}^2(a^2t,k))^3}\right] dt \\
&= \frac{a^2 k(k+1)}{4}\int_{-K}^{K}\left[\frac{1+(k-2)\operatorname{sn}^2(u,k)+(k^2-2k)\operatorname{sn}^4(u,k)}{(1+k\operatorname{sn}^2(u,k))^3}\right. \\
&\quad + \frac{k^3\operatorname{sn}^6(u,k)+(-2k-2)\operatorname{sn}(u,k)\operatorname{cn}(u,k)\operatorname{dn}(u,k)}{(1+k\operatorname{sn}^2(u,k))^3} \\
&\quad \left. + \frac{(2k^2+2k)\operatorname{sn}^3(u,k)\operatorname{cn}(u,k)\operatorname{dn}(u,k)}{(1+k\operatorname{sn}^2(u,k))^3}\right] du \\
&= \frac{a^2 k(k+1)}{4}\int_{-K}^{K}\frac{1+(k-2)\operatorname{sn}^2(u,k)+(k^2-2k)\operatorname{sn}^4(u,k)+k^3\operatorname{sn}^6(u,k)}{(1+k\operatorname{sn}^2(u,k))^3}\,du,\end{aligned} \tag{2.15}$$

https://doi.org/10.1017/S0334270000000692 Published online by Cambridge University Press

since the functions

$$u \to \frac{\operatorname{sn}(u,k)\operatorname{cn}(u,k)\operatorname{dn}(u,k)}{(1+k\operatorname{sn}^2(u,k))^3} \qquad \text{and} \qquad u \to \frac{\operatorname{sn}^3(u,k)\operatorname{cn}(u,k)\operatorname{dn}(u,k)}{(1+k\operatorname{sn}^2(u,k))^3}$$

are both odd. The integrand in (2.15) is an even function of u, so

$$\begin{aligned}\int_{-mT/2}^{mT/2} &[-a^2 x_k(t) y_k(t) + 2x_k^2(t)(x_k^2(t)+y_k^2(t))]\,dt \qquad (2.16)\\ &= \frac{a^2k(k+1)}{2}\int_0^K \frac{1+(k-2)\operatorname{sn}^2(u,k)+(k^2-2k)\operatorname{sn}^4(u,k)+k^3\operatorname{sn}^6(u,k)}{(1+k\operatorname{sn}^2(u,k))^3}\,du\\ &= \frac{a^2k(k+1)}{2}\left[\int_0^K \frac{du}{(1+k\operatorname{sn}^2(u,k))^3} + (k-2)\int_0^K \frac{\operatorname{sn}^2(u,k)\,du}{(1+k\operatorname{sn}^2(u,k))^3}\right.\\ &\qquad\left. +k(k-2)\int_0^K \frac{\operatorname{sn}^4(u,k)\,du}{(1+k\operatorname{sn}^2(u,k))^3} + k^3\int_0^K \frac{\operatorname{sn}^6(u,k)\,du}{(1+k\operatorname{sn}^2(u,k))^3}\right].\end{aligned}$$

Evaluating the integrals in (2.16) by using [1] produces the results

$$\int_0^{K(k)} \frac{du}{(1+k\operatorname{sn}^2(u,k))^3} = \frac{1}{4(k+1)^2}\left[(2k^2+k-1)K(k)+3E(k)+\frac{\pi(2k^2+3k+2)}{2(k+1)}\right], \qquad (2.17)$$

$$\int_0^{K(k)} \frac{\operatorname{sn}^2(u,k)\,du}{(1+k\operatorname{sn}^2(u,k))^3} = \frac{1}{4k(k+1)^2}\left[(k+1)K(k)-E(k)+\frac{\pi k}{2(k+1)}\right], \qquad (2.18)$$

$$\int_0^{K(k)} \frac{\operatorname{sn}^4(u,k)\,du}{(1+k\operatorname{sn}^2(u,k))^3} = \frac{1}{4k^2(k+1)^2}\left[(k+1)K(k)-E(k)-\frac{\pi k}{2(k+1)}\right], \qquad (2.19)$$

$$\int_0^{K(k)} \frac{\operatorname{sn}^6(u,k)\,du}{(1+k\operatorname{sn}^2(u,k))^3} = \frac{1}{4k^3(k+1)^2}\left[(2k^2+k-1)K(k)+3E(k)-\frac{\pi(2k^2+3k+2)}{2(k+1)}\right], \qquad (2.20)$$

where $E(k)$ is the complete elliptic integral of the second kind [1]. Substituting (2.17)–(2.20) in (2.16) and simplifying, finally produces

$$\int_{-mT/2}^{mT/2} [-a^2 x_k(t) y_k(t) + 2x_k^2(t)(x_k^2(t)+y_k^2(t))]\,dt = \frac{a^2}{2}(k^2-1)K(k)+\frac{a^2}{2}E(k). \qquad (2.21)$$

Overall, from (2.6), (2.14) and (2.21), the subharmonic Melnikov function for $n=1$ and $t_0 \in \mathbb{R}$ is

$$\begin{aligned}M^{m/1}(t_0) &= \frac{\pi b\omega}{2a\cosh(2mW_0)}[\cosh(2mW)\cos\omega t_0 - \sinh(2mW)\sin\omega t_0] \qquad (2.22)\\ &\quad + \frac{a^2c}{2}(1-k^2)K(k) - \frac{a^2c}{2}E(k)\\ &= \frac{\pi b\omega\sqrt{\cosh(4mW)}}{2a\cosh(2mW_0)}\sin(\omega t_0+\beta) + \frac{a^2c}{2}(1-k^2)K(k) - \frac{a^2c}{2}E(k),\end{aligned}$$

https://doi.org/10.1017/S0334270000000692 Published online by Cambridge University Press

where $\beta = -\arctan(\coth(2mW))$ and $K(k) = \pi m a^2/\omega$ is the resonance condition. It has been verified that

$$\lim_{m\to\infty} M^{m/1}(t_0) = M(t_0), \qquad t_0 \in \mathbb{R},$$

as required by Guckenheimer and Holmes [8, Theorem 4.6.4], where M is the standard Melnikov function obtained in Christie and Gopalsamy [2].

From (2.22), the subharmonic Melnikov function has infinitely many zeros provided

$$\sin(\omega t_0 + \beta) = \frac{a^3 c \cosh(2mW_0)}{\pi b\, \omega \sqrt{\cosh(4mW)}}[(k^2 - 1)K(k) + E(k)]. \tag{2.23}$$

The zeros of the subharmonic Melnikov function will be simple if $dM^{m/1}/dt_0 \neq 0$ at the zeros of $M^{m/1}$. From (2.22), we have

$$\frac{dM^{m/1}}{dt_0}(t_0) = \frac{\pi b\, \omega^2 \sqrt{\cosh(4mW)}}{2a \cosh(2mW_0)} \cos(\omega t_0 + \beta), \qquad t_0 \in \mathbb{R}. \tag{2.24}$$

Hence, $M^{m/1}$ has simple zeros if

$$\sin^2(\omega t_0 + \beta) < 1$$

and using (2.23), this reduces to

$$a^6 c^2 \cosh^2(2mW_0)[(1 - k^2)K(k) - E(k)]^2 < \pi^2 b^2 \omega^2 \cosh(4mW). \tag{2.25}$$

Note from (2.25) that the subharmonic Melnikov function has simple zeros for $c = 0$, and if $c \neq 0$, then c has to be small enough to satisfy (2.25). Now in order to use Guckenheimer and Holmes [8, Theorem 4.6.2] to establish the existence of a subharmonic periodic orbit of $(2.1)_\epsilon$ of period mT for sufficiently small ϵ, we need to show that $\frac{dT(k)}{d\sigma(k)} \neq 0$, where $\sigma(k) = H(q_k(t))$. We use (2.5) and $k = \sqrt{a^4 - 8\sigma}/a^2$ to obtain

$$\frac{dT(k)}{d\sigma(k)} = \frac{dT(k)}{dk}\frac{dk}{d\sigma(k)} = -\frac{8}{a^4\sqrt{a^4 - 8\sigma}}\frac{dK}{dk} < 0,$$

since $\frac{dK}{dk} > 0$. Hence, we conclude that if (2.25) is satisfied where $K(k) = \pi m a^2/\omega$, then for sufficiently small ϵ, the system $(2.1)_\epsilon$ has a subharmonic orbit of period mT.

3. An application to a one-dimensional oscillator

In [2], Christie and Gopalsamy considered the dynamics of an anharmonic oscillator parametrically driven at twice the resonant frequency ω_0. Such a system has recently been applied in optics in relation to the squeezing of light by DiFilippo *et al.* [4], and

https://doi.org/10.1017/S0334270000000692 Published online by Cambridge University Press

Wielinga and Milburn [13] have considered an equivalent model in the context of quantum-mechanical tunnelling. Christie and Gopalsamy [2] studied the potential for a one-dimensional oscillator with a small ($|\alpha|z^2 \ll 1$) quartic anharmonic correction whose frequency is modulated at $2\omega_0$ by a weak ($\epsilon_1 \ll 1$) parametric drive :

$$U(z,t) = \frac{1}{2}\mathscr{M}\omega_0^2 z^2\left(1 + \epsilon_1 \sin(2\omega_0 t) + \frac{1}{2}\alpha z^2\right), \tag{3.1}$$

where $\mathscr{M}$ denotes the mass of a particle undergoing the oscillatory motion. Neglecting the higher-order harmonics, oscillations with frequency ω_0 together with a dynamic phase were considered as follows :

$$\begin{aligned} z(t) &= r(t)\cos(\omega_0 t - \theta(t)) \\ &= C(t)\cos\omega_0 t + S(t)\sin\omega_0 t, \end{aligned} \tag{3.2}$$

where $C(t) = r(t)\cos(\theta(t))$ and $S(t) = r(t)\sin(\theta(t))$.

By making use of an approximation, Christie and Gopalsamy [2] derived the equations of motion

$$\begin{aligned} \frac{dC}{dt} &= \frac{\epsilon_1\omega_0}{4}C + \frac{3\alpha\omega_0}{8}S(C^2+S^2), \\ \frac{dS}{dt} &= -\frac{\epsilon_1\omega_0}{4}S - \frac{3\alpha\omega_0}{8}C(C^2+S^2). \end{aligned} \tag{3.3}$$

If we assume that $\epsilon_1 > 0$ and $\alpha < 0$, the system (3.3) is identical to $(2.1)_{\epsilon=0}$ with $a^2 = \epsilon_1\omega_0/4$ and $\alpha = -16/3\omega_0$.

Then, an external potential of the form $\epsilon_2 z \sin\omega_0 t \sin\omega t$ was applied, where $0 \le \epsilon_2 \ll 1$ is a perturbation parameter and $\omega > 0$ the frequency of the perturbation. The perturbed system is governed by the equations

$$\begin{aligned} \frac{dC}{dt} &= \frac{\epsilon_1\omega_0}{4}C + \frac{3\alpha\omega_0}{8}S(C^2+S^2) + \frac{\epsilon_2}{2\mathscr{M}\omega_0}\sin\omega t, \\ \frac{dS}{dt} &= -\frac{\epsilon_1\omega_0}{4}S - \frac{3\alpha\omega_0}{8}C(C^2+S^2). \end{aligned} \tag{3.4}$$

The system (3.4) is identical to the system $(2.1)_\epsilon$ with $a^2 = \epsilon_1\omega_0/4$, $\alpha = -16/3\omega_0$, $b = 1$, $c = 0$ and $\epsilon = \epsilon_2/2\mathscr{M}\omega_0$. From (2.25), if $K(k) = \pi m a^2/\omega$, the subharmonic Melnikov function for the system has simple zeros, so a subharmonic orbit of period $2\pi m/\omega$ exists for sufficiently small ϵ.

Acknowledgements

The first author is the holder of a Research Science Cadetship from the Defence Science and Technology Organisation (DSTO), Australia. The third author receives

https://doi.org/10.1017/S0334270000000692 Published online by Cambridge University Press

partial support through a grant from the Board of Research of Flinders University. The authors wish to thank Professor Jibin Li of the Kunming Institute of Technology, People's Republic of China, for explaining how to calculate the Fourier series expansion of Jacobi elliptic functions and combinations of Jacobi elliptic functions, during his visit to Flinders University in the summer of 1993-4.

References

[1] P. F. Byrd and M. D. Friedman, *Handbook of elliptic integrals for engineers and scientists* (Springer, New York, 1971).

[2] J. R. Christie and K. Gopalsamy, "Chaos in an anharmonic oscillator", *J. Austral. Math. Soc. Ser. B* **37** (1995) 186–207.

[3] J. R. Christie, K. Gopalsamy and J. Li, "Chaos in sociobiology", *Bull. Austral. Math. Soc.* **51** (1995) 439–451.

[4] F. DiFilippo, V. Natarajan, K. R. Boyce and D. E. Pritchard, "Classical amplitude squeezing for precision measurements", *Phys. Rev. Lett.* **68** (1992) 2859–2862.

[5] B. D. Greenspan and P. J. Holmes, "Homoclinic orbits, subharmonics and global bifurcations in forced oscillations", in *Nonlinear dynamics and turbulence* (eds. G. Barenblatt, G. Iooss and D. D. Joseph), (Pitman, London, 1983) 172–214.

[6] B. D. Greenspan and P. J. Holmes, "Repeated resonance and homoclinic bifurcation in a periodically forced family of oscillators", *SIAM J. Math. Anal.* **15** (1984) 69–97.

[7] R. Grimshaw and X. Tian, "Periodic and chaotic behaviour in a reduction of the perturbed Korteweg-de Vries equation", *Proc. R. Soc. Lond. A* **445** (1994) 1–21.

[8] J. Guckenheimer and P. J. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields* (Springer, New York, 1983).

[9] R. R. Huilgol, J. R. Christie and M. P. Panizza, "The motion of a mass hanging from an overhead crane", *Chaos, Solitons & Fractals* **5** (1995) 1619–1631.

[10] R. G. Langebartel, "Fourier expansions of rational fractions of elliptic integrals and Jacobian elliptic functions", *SIAM J. Math. Anal.* **11** (1980) 506–513.

[11] J. Li, *Chaos and Melnikov's method* (Chongqing University, Chongqing, 1989).

[12] V. K. Melnikov, "On the stability of the center for time-periodic perturbations", *Trans. Moscow Math. Soc.* **12** (1963) 1–57.

[13] B. Wielinga and G. J. Milburn, "Quantum tunneling in a Kerr medium with parametric pumping", *Phys. Rev. A* **48** (1993) 2494–2496.

[14] S. Wiggins, *Introduction to applied nonlinear dynamical systems and chaos* (Springer, New York, 1990).

[15] X. Zhao, K. Kwek, J. Li and K. Huang, "Chaotic and resonant streamlines in the ABC flow", *SIAM J. Appl. Math.* **53** (1993) 71–77.

https://doi.org/10.1017/S0334270000000692 Published online by Cambridge University Press