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ON A BANACH SPACE WITHOUT A WEAK MID-POINT LOCALLY UNIFORMLY ROTUND NORM

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In this paper we show that (i) l^∞ does not admit an equivalent weak mid-point locally uniformly rotund norm and (ii) l^∞/c_0 does not admit an equivalent rotund norm.

1. INTRODUCTION

In [**3**], Lindenstrauss showed that l^∞ does not admit an equivalent weakly locally uniformly rotund norm. In this paper, we refine his argument to show that l^∞ does not even admit an equivalent weak mid-point locally uniformly rotund norm. In addition, our argument also shows that l^∞/c_0 does not admit an equivalent rotund norm, a result previously proven by Bourgain in [**2**]. We say that a norm $\|\cdot\|$ on a Banach space X is *rotund* if each point of the unit sphere $S(X)$ is an extreme point of the closed unit ball $B(X)$. Further, we say that a norm $\|\cdot\|$ is *weak mid-point locally uniformly rotund* (or weak MLUR for short) if for each $x \in X \setminus \{0\}$ and each sequence $\{h_n : n \in \mathbb{N}\}$ in X, $h_n \to 0$ weakly whenever $\lim_{n\to\infty} \|x \pm h_n\| = \|x\|$. In [**4**], it is shown that a norm $\|\cdot\|$ is weak MLUR if and only if each point of $S(X)$ (when considered as a subset of the second dual ball) is an extreme point of the second dual ball.

THEOREM.

(i) *l^∞ does not admit an equivalent weak MLUR norm;*
(ii) *l^∞/c_0 does not admit an equivalent rotund norm.*

PROOF: (i) Let $\|\cdot\|_\infty$ denote the usual sup norm on l^∞ and let $|||\cdot|||$ denote any equivalent norm on l^∞. By the support of $x \in l^\infty$ we mean the set $\sigma(x) \equiv \{k \in \mathbb{N} : x(k) \neq 0\}$. Let $F_0 \equiv \{x \in l^\infty : \|x\|_\infty = 1$ and $\mathbb{N} \setminus \sigma(x)$ is infinite$\}$. Let $m_0 \equiv \inf\{|||x||| : x \in F_0\}$ and $M_0 \equiv \sup\{|||x||| : x \in F_0\}$. As $|||\cdot|||$ is an equivalent norm on l^∞, $0 < m_0 \leqslant M_0 < \infty$. We proceed by induction.

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STEP 1. Choose $x_1 \in F_0$ so that $(3M_0 + m_0)/4 \leqslant |||x_1|||$ and choose distinct integers i_1 and $j_1 \in \mathbb{N} \setminus \sigma(x_1)$. Then define $F_1 \equiv \{x \in F_0 : x$ agree with x_1 on $\sigma(x_1) \cup \{i_1, j_1\}\}$ and set $m_1 \equiv \inf\{|||x||| : x \in F_1\}$ and $M_1 \equiv \sup\{|||x||| : x \in F_1\}$.

Now, after the first n steps of the induction, we shall have constructed elements $\{x_1, x_2, \ldots, x_n\} \subseteq F_0 \subseteq l^\infty$, non-empty subsets $F_n \subseteq F_{n-1} \subseteq \ldots F_1 \subseteq F_0$, positive real numbers $m_0 \leqslant m_1 \leqslant \ldots \leqslant m_{n-1} \leqslant m_n \leqslant M_n \leqslant M_{n-1} \leqslant \ldots \leqslant M_1 \leqslant M_0$ and distinct positive integers $\{i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_n\}$ such that, for each k, $(1 \leqslant k \leqslant n)$;

(a) $x_k \in F_{k-1}$, $(3M_{k-1} + m_{k-1})/4 \leqslant |||x_k|||$;
(b) $\sigma(x_k) \cap \{i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k\} = \emptyset$;
(c) $F_k \equiv \{x \in F_0 : x$ agrees with x_k on $\sigma(x_k) \cup \{i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_k\}\}$;
(d) $m_k \equiv \inf\{|||x||| : x \in F_k\}$ and $M_k \equiv \sup\{|||x||| : x \in F_k\}$.

STEP $n+1$. Choose $x_{n+1} \in F_n$ so that $(3M_n + m_n)/4 \leqslant |||x_{n+1}|||$ and choose distinct integers i_{n+1} and $j_{n+1} \in \mathbb{N} \setminus (\sigma(x_{n+1}) \cup \{i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_n\})$. Then define $F_{n+1} \equiv \{x \in F_0 : x$ agrees with x_{n+1} on $\sigma(x_{n+1}) \cup \{i_1, i_2, \ldots, i_{n+1}, j_1, j_2, \ldots, j_{n+1}\}\}$ and set $m_{n+1} \equiv \inf\{|||x||| : x \in F_{n+1}\}$ and $M_{n+1} \equiv \sup\{|||x||| : x \in F_{n+1}\}$. This completes the induction.

For each $n \in \mathbb{N}$, define $h_n \in l^\infty$ by

$$h_n(k) \equiv \begin{cases} 1 & \text{if } k \in \{i_n, i_{n+1}, \ldots\} \\ 0 & \text{otherwise.} \end{cases}$$

Also define $x_\infty \in l^\infty$ by

$$x_\infty(k) \equiv \begin{cases} x_n(k) & \text{if } k \in \sigma(x_n) \text{ for some } n \\ 0 & \text{otherwise.} \end{cases}$$

It is readily verfied that x_∞ is well-defined and that $x_\infty \in \bigcap\{F_n : n \in \mathbb{N}\}$. It is also clear that $x_\infty \pm h_{n+1} \in F_n$ for each $n \in \mathbb{N}$. Next, choose $f \in (l^\infty)^*$ so that $f(h_1) = \|f\|_\infty = 1$ and $f(y) = 0$ for each $y \in c_0$. Clearly, for such an element f, we have that $f(h_n) = 1$ for all $n \in \mathbb{N}$. We complete the proof of part (i) by showing that $\lim\limits_{n\to\infty} |||x_\infty \pm h_n||| = |||x_\infty|||$. To see this, observe that $2x_n - F_n \subseteq F_n$ for each n. This, of course, implies that $|||2x_n - y||| \leqslant M_n$ for each $y \in F_n$, and this in turn implies that $3M_{n-1}/2 + m_{n-1}/2 \leqslant |||2x_n||| \leqslant M_n + |||y|||$ for each $y \in F_n$. Now, by taking the infimum over $y \in F_n$, we get that $3M_{n-1}/2 + m_{n-1}/2 \leqslant M_n + m_n \leqslant M_{n-1} + m_n$ and so $(M_{n-1} + m_{n-1})/2 \leqslant m_n \leqslant M_n \leqslant M_{n-1}$.

Therefore, $0 \leqslant \big|\, |||x_\infty \pm h_{n+1}||| - |||x_\infty|||\, \big| \leqslant M_n - m_n \leqslant (M_{n-1} - m_{n-1})/2$, since $x_\infty \pm h_{n+1}$ and $x_\infty \in F_n$. Hence, by induction, $0 \leqslant \big|\, |||x_\infty \pm h_{n+1}||| - |||x_\infty|||\, \big| \leqslant (M_0 - m_0)/2^n$; which shows that $\lim\limits_{n\to\infty} |||x_\infty \pm h_n||| = |||x_\infty|||$.

(ii) Let $|||\cdot|||$ be any equivalent norm on l^∞/c_0 and let $\pi\colon l^\infty \to l^\infty/c_0$ denote the usual quotient mapping. We apply the construction from part (i) to the equivalent norm (on l^∞) $\|x\| \equiv \|x\|_\infty + |||\pi(x)|||$. Indeed, from part (i) we have the existence of an element $x_\infty \in l^\infty$ and a sequence $\{h_n\colon n \in \mathbb{N}\} \subseteq l^\infty$ such that $\lim_{n\to\infty} \|x_\infty \pm h_n\| = \|x_\infty\|$, $\|x_\infty \pm h_n\|_\infty = \|x_\infty\|_\infty = 1$ for all n and $\pi(h_n) = \pi(h_1) \neq 0$ for each n. Therefore, $|||\pi(x_\infty)||| = |||\pi(x_\infty) \pm \pi(h_1)|||$; which shows that the $|||\cdot|||$ norm on l^∞/c_0 is not rotund. ∎

COROLLARY. *l^∞ cannot be equivalently renormed so that its unit sphere (considered as a subset of the second dual ball) is an extremal subset of its second dual ball.*

PROOF: Suppose to the contrary that such a norm exists. Call it $\|\cdot\|_1$ say. Let $\|\cdot\|_2$ be any equivalent rotund norm on l^∞ and define $|||\cdot||| : l^\infty \to R$ by $|||x||| \equiv \|x\|_1 + \|x\|_2$. It is easy to check that each point of the unit sphere of the $|||\cdot|||$ norm is an extreme point of its second dual ball and so the $|||\cdot|||$ norm is weak MLUR. But this contradicts the above Theorem. Therefore, no such norm exists. ∎

REMARK. If $|||\cdot|||$ is an equivalent Kadec norm then its unit sphere is an extremal subset of its second dual ball. Hence, l^∞ does not admit an equivalent Kadec norm.

NOTE ADDED IN PROOF: It has recently come to the attention of the authors that the paper [**1**] contains a proof of the fact that l^∞ does not admit an equivalent weak mid-point locally uniformly rotund norm.

REFERENCES

[**1**] G.A. Aleksandrov and I.P. Dimitrov, 'On the equivalent weakly midpoint locally uniformly rotund renorming of the space l^∞', (in Russian), in *Mathematics and Mathematical Education* (Sunny Beach, 1985), **pp.** 189–191.

[**2**] J. Bourgain, 'l^∞/c_0 has no equivalent strictly convex norm', *Proc. Amer. Math. Soc.* **78** (1980), 225–226.

[**3**] J. Lindenstrauss, 'Weakly compact sets – their topological properties and the Banach spaces they generate', *Ann. of Math. Studies* **69** (1972), 235–273.

[**4**] M.A. Smith, 'A curious generalisation of local uniform rotundity', *Comment Math. Univ. Carol* **25** (1984), 659–665.

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