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SOME SUBGROUPS OF THE THOMPSON GROUP

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Abstract

We determine all conjugacy classes of maximal local subgroups of Thompson's sporadic simple group, and all maximal non-local subgroups except those with socle isomorphic to one of five particular small simple groups.

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1. Introduction

In this paper we classify all the maximal p-local subgroups of Thompson's simple group Th of order $90,745,943,887,872,000 = 2^{15}.3^{10}.5^3.7^2.13.19.31$, and also partially classify the non-local subgroups. The existence of this group was originally conjectured by J. G. Thompson as a subgroup of the then unconstructed Monster group, in which the $3C$-centralizer is $3 \times Th$, and was first constructed by P. E. Smith and J. G. Thompson (see [2]) as a group of real 248×248 matrices.

Our main result is the following theorem.

THEOREM. *Any maximal subgroup of Th is either*
(A) *a conjugate of one of the maximal subgroups given in Table* 1.
or (B) *a conjugate of a particular group* $L_2(19):2$ *if* $L_2(19)$ *is a subgroup of Th*

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or (C) *the normalizer of a simple group* S *with trivial centralizer, where* $S \cong A_6$, $L_2(7)$, $L_3(3)$ *or* $U_3(3)$.

$2_+^{1+8} \cdot A_9$	$5^2:GL_2(5)$
$2^5 \cdot L_5(2)$	$7^2:(3 \times 2S_4)$
$(3 \times G_2(3)):2$	$31:15$
$(3^3 \times 3_+^{1+2}) \cdot 3_+^{1+2}:2S_4$	${}^3D_4(2):3$
$3^2.[3^7].2S_4$	$U_3(8):6$
$(3 \times 3^4:2 \cdot A_6):2$	M_{10}
$5_+^{1+2}:4S_4$	S_5

TABLE 1

REMARK. Some further restrictions on possible subgroups of type (C) are given in the final section of the paper.

Note. Our notation for groups, conjugacy classes, characters, etc. follows the ATLAS [1].

Note added in proof. S. Linton has now shown that $L_2(19)$ is a subgroup of Th, and that the cases $S \cong A_6$, $L_2(7)$ and $U_3(3)$ do not arise in part (C) of the Theorem.

2. The 2-local subgroups

There is just one class of involutions in the Thompson group, with centralizer $2_+^{1+8} \cdot A_9$. In this group, the action of the A_9 on the 2^8 is not the deleted permutation representation, but may be obtained from the latter by applying the triality automorphism of $O_8^+(2)$. All the non-central involutions in 2^{1+8} are conjugate under the action of A_9, so we obtain one class of four-groups with centralizer $2^2 \cdot [2^9] \cdot L_3(2)$. Now the involutions of cycle shape $(2^2 1^5)$ in A_9 do not lift to involutions in Th. An involution of cycle shape $(2^4 1)$ has centralizer $2^3 S_4$ in A_9, and its fixed space in the 2^8 has order 2^4. Hence the stabilizer of a corresponding four-group in $2^{1+8}A_9$ has order at most $2^{11} \cdot 3$. On the other hand, the structure constant $\xi(2A, 2A, 2A) = 1/2^{14}.3.7 + 1/2^{10}.3$, so we have proved the following lemma.

Lemma 2.1. *There are just two classes of four-groups in* Th, *one with centralizer of order* $2^{14}.3.7$, *and the other with centralizer of order* $2^{10}.3$.

Using the geometry of $O_8^+(2)$, we can find the orbits of A_9 on the totally isotropic subspaces of the 2^8. (The easiest way to do this is to work in the deleted permutation representation, and then apply the triality automorphism, so that 1-spaces become 4-spaces and so on.) The orbits are as follows:

Orbit size(s)	
135	on the 135 points (1-spaces)
315 + 1260	on the 1575 lines (2-spaces)
135 + 1800	on the 2025 3-spaces
9 + 126	on the 135 4-spaces of the first kind
135	on the 135 4-spaces of the second kind

We define any isotropic space to be *nice* if it is contained in one of the 4-spaces in the orbit of size 9. The 9 nice 4-spaces are disjoint, and each contains 15 1-spaces, 35 2-spaces and 15 3-spaces, so by counting we see that any nice space is contained in a unique nice 4-space. Furthermore, a subspace is nice if and only if all its 2-dimensional subspaces are nice.

The normalizer of a four-group of the first type is $N(2A^2)_1 \cong 2^2 \cdot [2^9] \cdot (S_3 \times L_3(2))$, in which the $L_3(2)$ acts on the $[2^9]$ as one copy of the natural representation together with two copies of its dual. Furthermore, the natural module is a submodule, so gives rise to a normal subgroup 2^5 in $N(2A^2)_1$. This 2^5-group corresponds to a nice 4-space in the 2^8, so its normalizer contains both $2^5.2^4.A_8$ and $2^5.2^6.(L_3(2) \times S_3)$, and therefore has the shape $2^5.L_5(2)$. All four-groups in this 2^5-group are of the first type, and correspond to the nice 2-spaces in 2^{1+8}. Hence the isotropic 2-spaces in the 1260-orbit are conjugate to the second type of four-group. The normalizer of this latter four-group is $N(2A^2)_2 \cong (2^2 \times 2^{1+4}) \cdot (S_4 \times S_3) < 2^{1+8}A_9$, in which the $S_4 \times S_3$ acts on 6 + 3 letters in the A_9-image. Indeed, the four-group centralizer is $(2^2 \times 2^{1+4}) \cdot S_4$, in which the S_4 fixes 3 letters, since only $3A$-elements centralize isotropic 2-spaces. Hence all involutions in the S_4 have cycle type $(2^2 1^5)$ in the A_9, so do not lift to involutions in Th. Hence any elementary Abelian 2-group not in the nice 2^5 is in a unique group 2^{1+8}, and so its normalizer is in $2^{1+8}A_9$. This concludes the proof of

Theorem 2.2. *Any 2-local subgroup of* Th *is contained in either* $N(2A) \cong 2^{1+8} \cdot A_9$ *or* $N(2A^5) \cong 2^5 \cdot L_5(2)$.

3. The p-local subgroups, $p \geqslant 5$

For each prime $p \geqslant 5$ dividing the order of the Thompson group, there is a unique class of subgroups of order p, and their normalizers are as follows:

$$N(5A) \cong 5_+^{1+2}{:}4S_4,$$
$$N(7A) \cong (7:3 \times L_2(7)){:}2 < {}^3D_4(2){:}3,$$
$$N(13A) \cong (13:6 \times 3) \cdot 2 < (3 \times G_2(3)){:}2,$$
$$N(19A) \cong 19:18 < U_3(8){:}6,$$
$$N(31AB) \cong 31:15.$$

The groups $N(5A)$ and $N(31AB)$ will turn out to be maximal subgroups of Th.

There is a unique class of groups of each of the orders 5^2 and 7^2, and so their normalizers are

$$N(5A^2) \cong 5^2 : GL_2(5),$$
$$N(7A^2) \cong 7^2 : (3 \times 2S_4)$$

both of which are maximal subgroups of Th.

4. The 3-local subgroups

There are three classes of elements of order 3 in Th, with normalizers

$$N(3A) \cong (3 \times G_2(3)){:}2,$$
$$N(3B) \cong (3^3 \times 3_+^{1+2}) \cdot 3_+^{1+2}{:}2S_4,$$
$$N(3C) \cong (3 \times 3^4 : 2A_6){:}2.$$

For a proof that $N(3B)$ has the above structure, and for further details, see below.

Now by looking at the character value on involutions, we see that the 248-character of Th restricts to $3 \times G_2(3)$ as $1 \otimes (1 + 91) + (\omega + \bar{\omega}) \otimes 78$, where the characters of the group of order 3 are denoted by their values on a generator, and those of $G_2(3)$ by their degrees. Hence we have the following class fusion

$G_2(3)$-class	$3A/B$	$3C$	$3D$	$3E$
Th-class	$3B$	$3B$	$3C$	$3A$
diagonal elements	$3A$	$3A$	$3B$	$3C$
3^2-type	$3A_3B_1$	$3A_3B_1$	$3A_1B_2C_1$	$3A_2C_2$
3^2-centralizer	$[3^6]{:}2A_4$	$[3^7]$	$3^5{:}2$	$3^5{:}2$

Now let Y be an elementary Abelian 3-group generated by $3A$-elements. If every pair of $3A$-elements in Y generates a group of type $3A_3B_1$, then Y contains a unique $3B$-pure subgroup of index 3, and so $N(Y)$ is contained in the normalizer

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of a $3B$-pure group. Otherwise, Y contains a group of type $3A_1B_2C_1$ or $3A_2C_2$, each of which has centralizer of shape $3^5:2$. Hence $C(Y)$ contains a unique Sylow 3-subgroup, which in each case is an elementary Abelian group of order 3^5, whose normalizer we will find later. In fact we will see that these two 3^5-groups are conjugate in Th.

Next we consider the case of an elementary Abelian group Y generated by $3C$-elements, and containing no $3A$-elements. We first need to study the $3C$-normalizer in some detail. We have $C(3C) \cong 3 \times 3^4:SL_2(9)$, in which the group $SL_2(9) \cong 2A_6$ acts naturally on $3^4 \cong (\mathbf{F}_9)^2$, where $\mathbf{F}_9$ is the field $\{0, \pm 1, \pm i, \pm 1 \pm i\}$ of order 9.

The 243 linear representations of the normal 3^5-subgroup E are fused by $2A_6$ to give representations of degrees 1 and 80, which we denote by $1a$, $1b$, $1c$, $80a$, $80b$, $80c$. Furthermore the outer elements of $2S_6$ fuse $1b$ to $1c$ and $80b$ to $80c$. Hence from the character value on the $3C$-element we see that the 248-character of Th restricts to $3^5:2S_6$ as $1b^4c^4 + 80abc$, that is, four copies of each of the non-trivial linear characters, plus one copy of each of the 80-dimensional characters. This shows that $E \cong 3^5$ has type $3C_1B_{40}C_{80}$, and $N(E)$ is not transitive on the $3C$-elements in E, since the order of the Thompson group is not divisible by 3^{11}.

Now consider the subgroup $3 \times SL_2(9) \cong 3 \times 2A_6$. Since we have already seen 3^2-groups of types $3A_1B_2C_1$ and $3A_2C_2$ in the involution centralizer, it follows that both of these are represented in $3 \times 2A_6$. Furthermore, since the Sylow 3-subgroup of $SL_2(9)$ fixes a 1-space in the natural representation over $\mathbf{F}_9$, we see that there is a group $F \cong 3^5$ containing both these 3^2-groups.

Now the Sylow 3-group in $(\mathbf{F}_9)^2:SL_2(9)$ is a group 3^{2+4} in which each non-central element has order 3 and centralizer 3^4. There are 10 such groups 3^4, of which one is the vector space $(\mathbf{F}_9)^2$ and another contains elements of the complementary $SL_2(9)$. The remaining 8 groups are permuted transitively by the Sylow 3-normalizer $3^{2+4}:[2^4]$ in $3^4:2S_6$. Hence these also give rise to 3^5-groups of type $3B_{40}C_{81}$, conjugate to E but seen from the point of view of one of the $3C$-elements in the 80-orbit. This determines the conjugacy classes of all elements of order 3 in $N(3C)$, and in particular shows that the 3^5-group F defined above has type $3A_{54}B_{40}C_{27}$, and that any two commuting $3C$-elements generate a 3^2-group of type $3A_2C_2$ or $3B_1C_3$.

Now any elementary Abelian group generated by $3C$-elements is either in a conjugate of E, in which case it contains a unique $3B$-pure subgroup of index 3, or else its centralizer has a unique Sylow 3-group, which is conjugate to F. Now F contains $3A_2C_2$-subgroups, so is conjugate to both the 3^5-groups discussed earlier. Hence, in order to complete the reduction to the $3B$-pure case, it suffices to find the normalizer of F. But now its intersection with $(\mathbf{F}_9)^2$ is a $3B_4$-group, and this is determined as the intersection of all the $3B$-pure 3^3-groups in F.

We have now reduced to the $3B$-pure case, so we must study the structure of the centralizer of a typical $3B$-element t in some detail. Here it is necessary to use the notation introduced in [3] to describe the subgroup $3^{1+12}\cdot 2Suz$ of the Monster. Briefly, 3^{1+12} is written as the central product of 6 copies of 3^{1+2}, and $2Suz$ is written as 6×6 matrices acting on this decomposition. The matrix elements are quaternions reduced modulo 3, and the vector coordinates (that is, the elements of 3^{1+2} modulo the centre) are quaternions reduced modulo $\theta = i + j + k$ (on the left). First we obtain generators for most of the group $C(t)$ by centralizing the $3C$-element $(i, i, 0, 0, 0, 0)\cdot\langle \omega, \bar{\omega}, +, \longleftrightarrow\rangle$ in the Monster. We obtain $N(3B)\cong(3^3\times 3^{1+2})\cdot 3^{1+2}{:}2S_4$, where the bracketed normal subgroup $3^3\times 3^{1+2}$, which we denote by T, is contained in the corresponding group 3^{1+12} in the Monster, and may be generated by the elements $(0,0,0,1,1,1)$, $(0,0,0,i,i,i)$, $(1,-1,0,0,0,0)$, $(0,0,1,0,0,0)$ and $(0,0,i,0,0,0)$. We can extend by an outer automorphism group $3\times 2A_4$ generated by $\langle 1,1,i,i,i,i\rangle$ and $(-i,i,0,0,0,0)\cdot\langle\omega,\omega,\omega,\omega,\omega,\omega\rangle$ together with the central 3-element $(i,-i,0,0,0,0)\cdot d$, where d is the matrix

$$\begin{pmatrix} 1 & \theta & 0 & 0 & 0 & 0\\ \theta & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 1 & 1 & 1\\ 0 & 0 & -1 & -1 & 1 & 1\\ 0 & 0 & -1 & 1 & -1 & 1\\ 0 & 0 & -1 & 1 & 1 & -1 \end{pmatrix}.$$

[Warning: d is an element of the Monster, but not of Th.]

First we study the normal subgroup T in some detail. This group T contains four conjugate elementary Abelian 3^5-groups, whose union is the whole group. Now elements of Th-classes $3A$, $3B$, $3C$ are of M-classes $3A$, $3B$, $3B$ respectively. Hence the group generated by $(1,-1,0,0,0,0)$ and $(0,0,1,0,0,0)$ contains two $3A$-elements and two elements of class $3B$ or $3C$, so it is of type $3A_2C_2$. This implies that these 3^5-groups are conjugate to F, and we can use this to determine the classes of the elements in T. We have the following orbits under $2A_4$:

$(1,-1,0,0,0,0)$, $(1,-1,0,1,1,1)$	9 elements of class $3A$
$(0,0,1,0,0,0)$, $(0,0,\pm 1,1,1,1)$, $*(1,-1,1,i,i,i)$	9 elements of class $3A$
$*(0,0,-1,i,i,i)$, $(1,-1,\pm 1,0,0,0)$, $(1,-1,\pm 1,1,1,1)$, $(-1,1,\pm 1,1,1,1)$	9 elements of class $3C$
$*(0,0,1,i,i,i)$, $*(1,-1,-1,i,i,i)$	9 elements of class $3B$
$(0,0,0,1,1,1)$	4 elements of class $3B$

This gives the conjugacy classes of all the elements in T, since multiplying by t does not change the class. Note that the signs in the cases marked * are rather subtle. In order to prove that they are as given, we need to use some later results. However, we do not use these subtleties, so no details are given here.

Now let us consider the action of the quotient group $3^{1+2}:2S_4$ on the group T. Certainly it fixes the centre, which is an elementary Abelian group of order 3^4, and also fixes the subgroup of index 3 therein consisting of all the $3B$-elements. Furthermore, the central element of $3^{1+2}:2S_4$ acts non-trivially on this 3^4, so we have a faithful representation of this group. Since the 2-space stabilizer $3^4:(2A_4 \times 2A_4)$ in $SL_4(3)$ does not contain a group $3^{1+2}:2A_4$, it follows that the 4-dimensional module for the latter group is uniserial $1 + 2 + 1$. In particular, $N(3B)$ is transitive on the central $3A$-elements in T, and on the central $3B$-elements in T outside the derived group $\langle t \rangle$.

Now there are also non-central $3A$-elements in T, and since there are only two classes of $3A_3B_1$-groups in Th, it follows that $N(3B)$ is also transitive on these $3A$-elements. It is clear then that $N(3B)$ is also transitive on the $3C$-elements and the non-central $3B$-elements in T.

For the sake of convenience we call the central $3B^2$-group in T a *type* 1 group, and the non-central one *type* 2 (both containing the original element t). It will be clear later that each is the unique $3B^2$-group with the appropriate centralizer order, and therefore the normalizer is in each case transitive on the non-trivial elements. We have $N(3B^2)_1 \cong 3^2.[3^7].2S_4$, which will later turn out to be a maximal subgroup of Th, and $N(3B^2)_2 \cong 3^2.[3^5].2S_4$, which we proceed to show is contained in $N(3B)$. Consider the $3B^2$-group which is central in T but does not contain t. There is a unique class of such groups, and the normalizer of one of these in $N(t)$ is $(3^3 \times 3^{1+2}).3.2S_4$. Now this $3B^2$-group is not of type 1, for if it were then there would be a $3B$-pure 3^3-group such that all the $3B^2$-subgroups were of type 1, and hence the normalizer would have to be $3^3.3^4.L_3(3)$, which is absurd since then Th would have a subgroup $3^3:13 \times 3$. Indeed, this all happens inside the group F described earlier, in which we can see there are just two classes of $3B^2$-groups, one being of type 1 and all the rest of type 2. Hence $N(3B^2)_2$ is contained in $N(3B)$, as claimed. Now any larger elementary Abelian $3B$-pure subgroup of T is again in the 3^5-group, which is conjugate to F, so has order 3^3 and contains a unique $3B^2$-group of type 1. Hence its normalizer is in $N(3B^2)_1$.

Having dealt with all subgroups of T, we must next find the conjugacy classes of elements of order 3 in $N(t)/\langle t \rangle$ outside $T/\langle t \rangle$. The quotient group $N(t)/T \cong 3^{1+2}:2S_4$ contains five classes of elements of order 3, four of which we have nice representatives for:

Name	Representative	Class in Suz	Centralizer in $3^{1+2}:2A_4$
$3a$	$(i,-i,0,0,0,0)\cdot d$	$3B$	$3^{1+2}:2A_4$
$3b$	?	$3B$	3^3
$3c$	$(-i,i,0,0,0,0)\cdot\langle\omega\rangle$	$3A$	$3^2 \times S_3$
$3d$	$\langle\omega\rangle\cdot d$	$3B$	$3^2 \times S_3$
$3e$	$(-i,i,0,0,0,0)\cdot\langle\bar{\omega}\rangle\cdot d$	$3B$	$3^2 \times S_3$

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Note: in this table, and below, we write $\langle\omega\rangle$ for the diagonal matrix $\langle\omega,\omega,\omega,\omega,\omega,\omega\rangle$. Each element has an *associated vector*, which is the vector in 3^{12} that we have to divide by in order to get an element in $6Suz$—so that for example the vector associated with the $3a$-element given above is $(i,-i,0,0,0,0)$.

For each of these five classes we must find the classes they lift to in the non-split extension of the 5-dimensional vector space $T/\langle t\rangle$ by the group $C(t)/T \cong 3^{1+2}:2A_4$. Now by general principles, an element x is conjugate to its multiples by vectors in the image of $x-1$. Hence we must find the orbits of the centralizer of x in $3^{1+2}:2S_4$ on the vectors modulo the image of $x-1$. Or to be more precise, we need the orbits on the coset of this generated by the vector associated with x.

We deal first with the case $3a$, and we take the element $(i,-i,0,0,0,0)\cdot d$, where d is the matrix displayed above. Then the image of $x-1$ is generated by $(0,0,0,1,1,1)$ and $(0,0,0,i,i,i)$, and the vector associated with x is of course $(i,-i,0,0,0,0)$. Then the group $3^{1+2}2A_4$ acts on the relevant coset of $T/\mathrm{Im}(x-1)$ with orbits of sizes $3+24$, so we have two conjugacy classes of elements in $C(3B)$. In order to indentify their classes in Th, we first identify their classes in M. The given vectors are not in the subspace of 3^{1+12} generated by the fixed space of d and its centralizer in 3^{1+12}. Hence by [3] the elements either have order 9 or are of M-class $3C$. But the latter case cannot happen, so by looking at the centralizer orders of the elements of order 9 we have the two classes:

Representative	Centralizer order	Type
$(i,-i,0,0,0,0)\cdot d$	$2^3\cdot 3^6$	$9A$
$(i,-i,1,0,0,0)\cdot d$	3^6	$9B$

Next consider the case $3c$, taking x to be the element $(-i,i,0,0,0,0)\cdot\langle\omega\rangle$. Then the image of $x-1$ is generated by $(0,0,1,0,0,0)$ and $(0,0,0,1,1,1)$, and the vector associated with x is $(-i,i,0,0,0,0)$. The quotient of T by the image of $x-1$ may therefore be generated by $(1,-1,0,0,0)$, $(0,0,0,i,i,i)$ and $(0,0,i,0,0,0)$. If we add a vector in the space generated by the first two vectors, then we get an element of order 9, see [3]. Hence all these elements are of class $9C$ and are conjugate. Then the $3c$-centralizer in $3^{1+2}2S_4$ has 3 orbits on the remaining vectors in the coset, and two of these orbits are interchanged by the outer automorphism. In all 3 cases the corresponding $[3^2]$-group has type $3B_4$ in the Monster, and centralizes an element of Th-class $3C$ in T, so from what we know about the $3C$-centralizer we can deduce that the whole centralizer is conjugate to E. Thus we have the 3 classes:

Representative	Centralizer	Type
$(-i, i, 0, 0, 0, 0)\langle\omega\rangle$	$9 \times 3^2{:}2$	$9C$
$(-i, i, i, 0, 0, 0)\langle\omega\rangle$	3^5	$3B_4$
$(-j, j, i, 0, 0, 0)\langle\omega\rangle$	3^5	$3B_1C_3$

Let us turn now to the case $3d$, taking x to be the element $\langle\omega\rangle d$. In this case the image of $x - 1$ is generated by $(0, 0, 0, 1, 1, 1)$ and $(0, 0, -1, i, i, i)$, and does not contain our original $3B$-element t. Also, the vector associated with x is $(0, 0, 0, 0, 0, 0)$. Note that this implies that elements are not necessarily conjugate to their multiples by t. We must determine the orbits of $C(x)$ on the vectors of the space generated by $(1, -1, 0, 0, 0, 0)$, $(0, 0, 1, i, i, i)$ and $(0, 0, i, 0, 0, 0)$, say. If we ever add in the last generator, then the element has order 9 modulo $\langle t\rangle$, so we can neglect this case. Now x itself gives rise to a 3^2-group of M-type $3A_1B_3$, so of *Th*-type $3A_1B_2C_1$. But the centralizer of such a 3^2-group is just $3^5{:}2$, so by counting we see that *all* 3^2-groups of type $3d$ are conjugate to it. We therefore have only one class of 3^2-group as follows:

Representative	Centralizer	Type
$\langle\omega\rangle \cdot d$	$3^5{:}2$	$3A_1B_2C_1$

Next we turn to the $3e$ case, taking x to be the element $(-i, i, 0, 0, 0, 0) \cdot \langle\overline{\omega}\rangle \cdot d$, so that the fixed space of x is generated by $(0, 0, 0, 1, 1, 1)$ and $(0, 0, 1, i, i, i)$, and the vector associated with x is $(-i, i, 0, 0, 0, 0)$. We may suppose that the added vector is in the space generated by $(1, -1, 0, 0, 0, 0)$, $(0, 0, -1, i, i, i)$, and $(0, 0, i, 0, 0, 0)$, say. If ever we add in the last generator, then the resulting element has order 9 modulo $\langle t\rangle$, so we can neglect it. If we add in only $(1, -1, 0, 0, 0, 0)$, then the resulting 3^2-group is still in the involution centralizer, so the structure of $2^{1+8} \cdot A_9$ implies that it is conjugate to the first 3^2-group $\langle t, x\rangle$. And finally, the element $(0, 0, -1, i, i, i)$ is in the fixed space of x, so gives rise to a 3^2-group with the same centralizer. We have the two cases:

Representative	Centralizer	Type
$(-i, i, 0, 0, 0, 0) \cdot \langle\overline{\omega}\rangle \cdot d$	$3^5{:}2$	$3B_4$
$(-i, i, 1, i, i, i) \cdot \langle\overline{\omega}\rangle \cdot d$	3^5	$3B_1C_3$

In here we identify the conjugacy classes by observing that E must be a subgroup of $C(3B)$, and can intersect T in at most a 3^3-subgroup. Hence it maps onto a 3^2-subgroup of $3^{1+2}{:}2S_4$, containing no $3a$-elements or $3d$-elements, so containing $3b$, $3c$ and $3e$-elements. In fact this 3^2-group has type $3b_1c_2e_1$.

Before considering the $3b$-case, which is the only one left, let us find the normalizers of everything not involving the $3b$-cosets. We can ignore the $3a$-elements, since they lift only to elements of order 9. If we have a $3c$, $3d$ or $3e$-element, then the centralizer is an elementary Abelian 3^5-group containing a $3C$-element, so its normalizer has already been found. Hence we can restrict to the case when we have only a $3b$-element outside T.

Here we are somewhat hampered by not having an explicit element to work with, so we have to use a rather clumsy theoretical argument to classify the conjugacy classes.

Now we have already shown, in the discussion of the structure of $N(3B)$, that a $3b$-element centralizes a 3^3-subgroup of T, and we know what this is by looking inside F. Indeed, it has type $3B_4C_9$, and so may be taken to be generated by $(0,0,0,1,1,1)$ and $(1,-1,1,0,0,0)$. Now if we multiply by any element of the central 3^4 of T then the resulting element still centralizes a $3C$-element, so has order 3 and is in one of the 3^5-groups already considered. Finally we wish to prove that if we multiply by any other element of T then we get an element of order 9. Now $T/\mathrm{Im}(x-1) \cong 3^3$ so there are 27 cosets to consider, of which we have dealt with 9. But now $C(9A) \cong 3^3 \cdot 3^{1+2}:Q_8$ regarded as an element of type $3a$ (that is, if it cubes to t then the normal 3^3 is the intersection with T). Hence there exist elements of order 9 of type $3b$, for otherwise we could multiply a $3b$-type element of order 3 by a commuting $9A$-element to obtain a $3b$-type element of order 9. Hence all the remaining elements have order 9, as we have already seen that multiplying by a *central* element of T does not affect the order of the element modulo $\langle t\rangle$. (This is true for elements of order 3, so it is also true for elements of order 9, since $T/\langle t\rangle$ is Abelian.) Now these 9-elements do not cube to t, so there are just three classes of elements of order 3 modulo $\langle t\rangle$, each with centralizer 3^5, the corresponding 3^2-groups being one of type $3B_4$ and two of type $3B_1C_3$. Finally we notice that in this case also the centralizer of such an outer element of order 3 is just the group $E \cong 3^5$, whose normalizer we have already found.

This concludes the proof of

THEOREM 4.1. *Any* 3-*local subgroup of Th is contained in one of the following maximal* 3-*local subgroups*

$$
\begin{aligned}
N(3A) &\cong (3 \times G_2(3)):2,\\
N(3B) &\cong \left(3^3 \times 3_+^{1+2}\right) \cdot 3_+^{1+2}:2S_4,\\
N(3B^2) &\cong 3^2.[3^7].2S_4,\\
N(3C) &\cong \left(3 \times 3^4:2A_6\right):2.
\end{aligned}
$$

5. Non-local subgroups

Using the classification of finite simple groups, we can find all non-Abelian simple groups whose order divides that of the Thompson group, $|Th| = 2^{15}.3^{10}.5^3.7^2.13.19.31$. We divide these into two cases:

(1) Known or possible subgroups:

A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(13)$, $L_2(19)$?, $L_3(3)$, $U_3(3)$, $U_3(8)$, $G_2(3)$, and ${}^3D_4(2)$.

(2) Non-subgroups:

A_7, A_8, A_9, A_{10}, $L_2(25)$, $L_2(27)$, $L_2(31)$, $L_2(49)$, $L_2(64)$, $L_2(125)$, $U_3(4)$, $U_3(5)$, $L_3(4)$, $L_3(5)$, $L_3(9)$, $U_4(2)$, $U_4(3)$, $L_4(3)$, $L_5(2)$, $L_6(2)$, $S_6(2)$, $O_8^+(2)$, $S_4(8)$, $S_6(3)$, $O_7(3)$, $Sz(8)$, ${}^2F_4(2)'$, $G_2(4)$ and J_2.

We prove that none of the groups listed in (2) is in Th. First note that it suffices to prove it for A_7, $L_3(5)$, $L_2(25)$, $L_2(27)$, $L_3(4)$, $Sz(8)$, $L_2(31)$, ${}^2F_4(2)'$, $L_2(49)$, $L_2(64)$, $L_2(125)$, $L_3(9)$, $U_3(4)$, J_2, and $U_4(2)$.

Now the 3-elements in any A_5 are of class $3B$ (see Proposition 5.1 below), and $C(3B)$ does not contain A_4, so there is no A_7. Similarly, the 3-elements in $31:3$ are of class $3C$, so there is no $L_3(5)$. For $L_2(25)$, note that the 4-elements in S_5 are of class $4B$ (see below), but there is no class of 12-elements in Th which powers to both $3B$-elements and $4B$-elements.

We eliminate the groups $L_2(31)$, $L_2(49)$, $L_2(64)$, $L_2(125)$, $L_3(9)$ and ${}^2F_4(2)'$ since they contain elements of orders 16, 25, 63, 63, 91 and 16 respectively. Similarly, $U_3(4)$ and J_2 contain $5 \times A_5$ and $L_2(27)$ contains $3^3:13$, while $L_3(4)$ and $U_4(2)$ contain subgroups of the shape $2^4:A_5$. In each case we know from the local analysis that Th does not contain such a group. Finally, it is easy to show that there is no restriction of the character of degree 248 to $Sz(8)$.

Conversely, J. G. Thompson has shown that Th contains subgroups of the shapes $U_3(8):6$ and ${}^3D_4(2):3$, by looking inside the Monster, and S. P. Norton has shown similarly that Th contains $M_{10} \cong A_6 \cdot 2$ (see Proposition 5.8 below). Then the $3A$-centralizer contains $G_2(3)$, which contains all the remaining groups on the list except for $L_2(19)$. I do not yet know whether $L_2(19)$ is a subgroup of Th.

In what follows, we make considerable use of structure constants. If X, Y, and Z are three conjugacy classes in G, then $\xi_G(X, Y, Z)$ denotes the value of the expression

$$\frac{|G|}{|C(x)|\,|C(y)|\,|C(z)|} \sum \frac{\chi(x) \cdot \chi(y) \cdot \chi(z)}{\chi(1)}$$

where $x \in X$, $y \in Y$, $z \in Z$, and the sum is over all irreducible characters χ of G. It is a well-known fact that

$$\xi_G(X, Y, Z) = \sum \frac{1}{|C(x, y, z)|}$$

where the sum is over all conjugacy classes of triples (x, y, z) of such elements with $xyz = 1$.

PROPOSITION 5.1. *There is a unique class of A_5 in Th, and it has normalizer S_5.*

PROOF. The only non-zero structure constant of type $(2, 3, 5)$ is $\xi(2A, 3B, 5A) = 1$. But there is no A_5 in any of the element centralizers.

PROPOSITION 5.2. (a) *There is a unique class of $L_2(8)$, and its normalizer is $S_3 \times L_2(8):3$, which is contained in $N(3A) \cong (3 \times G_2(3)):2$.*
(b) *There is a unique class of ${}^3D_4(2)$, and its normalizer is ${}^3D_4(2):3$.*

PROOF. The elements of order 3 in $L_2(8)$ are cubes, so are of class $3B$. Now the 248-character restricts to ${}^3D_4(2)$ as the direct sum of the irreducible representations of degrees 52 and 196, so in particular the elements of ${}^3D_4(2)$-class $3B$ are of Th-class $3B$. Then the structure constant $\xi_{Th}(2A, 3B, 7A) = 7/6$ is entirely accounted for by the contributions from the known classes of $L_2(8)$ and ${}^3D_4(2)$, since $\xi_{L_2(8)}(2, 3, 7) = 3$ and in ${}^3D_4(2)$ we have $\xi(2B, 3B, 7D) = 3$.

REMARK. It is possible to give an alternative proof of the uniqueness of ${}^3D_4(2)$ as a subgroup of Th, by constructing the group out of its 7-local subgroups.

PROPOSITION 5.3. *There is a unique class of $U_3(8)$ in Th, and its normalizer is $U_3(8):6$.*

PROOF. Any group $U_3(8)$ may be constructed by taking a group $3 \times L_2(8)$, and extending the 2^3-normalizer from $3 \times 2^3:7$ to $2^{3+6}:(7 \times 3)$. Now there is a unique class of $3 \times L_2(8)$ in Th, which is contained in $3 \times G_2(3)$, and the entire 2^3-normalizer in Th has the shape $2^3 \cdot [2^8] \cdot (S_3 \times L_2(7))$. But in the latter group the elements of order 21 act on the $[2^8]$-factor as the direct sum of irreducible representations of degrees 6 and 2. Hence there is a unique group $2^{3+6}:21$ containing a given $2^3:21$, and the result follows.

PROPOSITION 5.4. *There is a unique class of $L_2(13)$ in Th, and its normalizer is $(3 \times L_2(13)):2$, which is contained in $N(3A) \cong (3 \times G_2(3)):2$.*

PROOF. Since the total $(2, 3, 7)$-structure constant in $L_2(13)$ is 6, and the centralizer of any $L_2(13)$ in Th has order at most 3, it follows that any $L_2(13)$ contributes at least 1 to the $(2, 3, 7)$-structure constant in Th. But $\xi(2A, 3A, 7A) = 3/14$, and we have already accounted for all of $\xi(2A, 3B, 7A) = 7/6$, so the class fusion must be $(2A, 3C, 6A, 7A, 13A)$. Now the only $L_2(13)$ with non-trivial centralizer is the one with normalizer $(3 \times L_2(13)):2$, contained in $(3 \times G_2(3)):2$, and this $L_2(13)$ extends to $L_2(13):2$. Furthermore, $\xi(2A, 3C, 7A) = 4$, so any $L_2(13)$ with trivial centralizer also extends to $L_2(13):2$. If we restrict the 248-dimensional representation to each of the groups $13:12$, $L_2(13):2$ and $(3 \times G_2(3)):2$ in turn, we find that each of these groups fixes a unique 1-space pointwise. Thus any $L_2(13)$ is contained in $G_2(3)$, and the result follows.

Proposition 5.5. *There is a unique class of $G_2(3)$ in Th, and its normalizer is* $(3 \times G_2(3)):2$.

Proof. Any group $G_2(3)$ may be constructed from $L_2(13)$ by extending D_{14} to $7:6$. But both the $L_2(13)$-normalizer $(3 \times L_2(13)):2$ and the D_{14}-normalizer $7:6 \times S_3$ are contained in $(3 \times G_2(3)):2$, and the result follows.

Proposition 5.6. *If $L_2(19)$ is a subgroup of Th, then there is exactly one class, and its normalizer is $L_2(19):2$.*

Proof. The 248-character restricts to $U_3(8)$ as $1a + 57ab + 133a$, so the 9-elements in $19:9$ are $9C$-elements. Thus any $L_2(19)$ has type $(2A, 3B, 5A, 9C, 10A, 19A)$. Now $\xi_{Th}(2A, 3B, 9C) = 6$, of which an amount $1/6$ is attributable to $L_2(8)$. But the total $(2, 3, 9)$-structure constant in $L_2(19)$ is 6, and the result follows from the fact that the elements of order 19 in Th are self-centralizing.

We conclude with a few remarks about subgroups isomorphic to A_6, $L_2(7)$, $L_3(3)$ and $U_3(3)$.

Proposition 5.7. *Any A_6 in Th is of type $(2A, 3B, 3B, 4B, 5A)$. Hence the degree* 248 *character of Th restricts to any A_6 as* $5a^3b^3 + 8a^4b^4 + 9a^6 + 10a^{10}$.

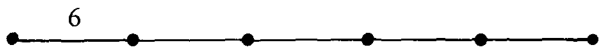
Proof. Firstly, it contains $3B$-elements since it contains A_5. Secondly, it contains $4B$-elements since it has trivial centralizer and $\xi(2A, 4A, 5A) = 1/4$.

Remark. This proof also shows that the S_5 contains $4B$-elements.

Proposition 5.8. *There exists a subgroup A_6 with normalizer M_{10}. This group A_6 together with S_5 generate Th.*

Proof. In the Monster there is a group $(A_6 \times A_6 \times A_6) \cdot (2 \times S_4)$. Centralizing a $3C$-element permuting the three factors of the minimal normal subgroup of this, we have a group M_{10}. Now using the "Y"-generators for M (see [1], page

232), this $3C$-element rotates the three arms of the Y, giving a subgroup of Th as a quotient of the inner half of the infinite Coxeter group with diagram



By covering up each of the nodes in turn we obtain the subgroups A_6, S_5, $(6 \times A_4):2, \ldots$ of Th. Now the first two of these groups intersect in A_5, and generate the third, which does not normalize the A_6 and so extends it to Th.

PROPOSITION 5.9. *If there is a subgroup S_6 in Th, then there is a unique class.*

PROOF. Any group S_6 can be constructed by taking S_5 and adjoining an involution commuting with a subgroup S_4. But the A_4-normalizer is $(A_4 \times 2A_4):2$, so there is a unique way of making this extension.

REMARK. S. P. Norton has shown that S_6 is not a subgroup of Th, as follows. Let a, b, c, d, e, f be generators of Th *wr* 2 corresponding to the nodes of the above Coxeter graph in order. Then the putative S_6 in Th would be generated by $\{(ab)^3, ac, ad, ae, af\}$, and in particular would contain the element $(ab)^3ac$. But we can calculate the order of this element in the Monster, since it is contained in a known subgroup $O_7(3)$. It turns out to have order 9, which is a contradiction.

PROPOSITION 5.10. (a) *There is a unique class of $L_2(7)$ containing $3A$-elements, and its normalizer is $(L_2(7) \times 7:3):2$, which is contained in ${}^3D_4(2):3$.*
(b) *There is no $L_2(7)$ containing $3B$-elements.*

PROOF. (a) The structure constant $\xi(2A, 3A, 7A) = 3/14$ is completely accounted for by the contributions $1/21$ from $(L_2(7) \times 7:3):2$ inside ${}^3D_4(2):3$ and $1/6$ from $S_4 \times 2^3 \cdot L_3(2)$ inside $2^5 \cdot L_5(2)$.
(b) The structure constant $\xi(2A, 3B, 7A) = 7/6$ has already been fully accounted for by $L_2(8)$ and ${}^3D_4(2):3$ (see Proposition 5.2).

REMARK. $\xi(2A, 3C, 7A) = 4$, of which an amount 1 has already been accounted for by $L_2(13)$.

PROPOSITION 5.11. *There is a unique class of $U_3(3)$ whose non-3-central 3-elements are of class $3A$, and its normalizer is $3 \times U_3(3):2$, which is contained in $(3 \times G_2(3)):2$.*

PROOF. Any group $U_3(3)$ can be constructed by taking a group $L_2(7)$ and extending a subgroup S_3 to $S_3 \times 3$. But by Proposition 5.10 any $L_2(7)$ containing $3A$-elements has normalizer contained in ${}^3D_4(2):3$. Furthermore the S_3 has normalizer $S_3 \times L_2(8):3$, which is contained in the same group ${}^3D_4(2):3$. Since there is a unique class of $U_3(3)$ in ${}^3D_4(2)$, the result follows.

PROPOSITION 5.12. *Any other $U_3(3)$ in Th has type* $(2A, 3B, 3C, 4A, 4A, 6C, 7A, 8A, 12C)$.

PROOF. In any $U_3(3)$ the 3-central 3-elements have centralizer $3^{1+2}:4$, so are of class $3B$. The remaining 3-elements are contained in $L_2(7)$, so are of class $3C$, since we have excluded the $3A$-case. The only difficulty now is to identify the second class of elements of order 4, but only $4A$ gives integral trace on restricting the 248-character.

In the case of $L_3(3)$, we have very little information. The 3-central 3-elements have centralizer $3^{1+2}:2$, so again are of class $3B$. The remaining 3-elements are of class $3B$ or $3C$, since they normalize elements of order 13. Furthermore, they are contained in S_4, so if they are $3B$-elements then the 4-elements are of class $4B$, since $\xi(2A, 3B, 4A) = 0$.

REMARK. It may well be possible to complete the enumeration of the maximal subgroups of *Th* by computer. The first, and perhaps biggest, problem is to reconstruct *Th* as a group of 248 by 248 matrices, preferably over $\mathbf{F}_2$ for efficiency of calculation. Then the enumeration of $L_2(7)$ and $U_3(3)$ is almost algorithmic, by taking $7:3$ and extending the 3-normalizer. Similarly, the cases $L_3(3)$ and $L_2(19)$ could probably be dealt with by taking $13:3$ or $19:3$ and again extending the 3-normalizer. The case of A_6 seems rather harder, but can perhaps be approached via the 2-local subgroups.

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