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Direct products of monogenic semigroups

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This thesis is concerned with characterizing finite (external) direct products of finite cyclic semigroups and finite (external) direct products of finite monogenic inverse semigroups, and includes an analysis of the lattice of congruences on direct products of finite cyclic semigroups.

Let $S = S_1 \times \ldots \times S_n$ be a finite (external) direct product of finite cyclic semigroups. In Chapter 1, we characterize S , showing that it is a unipotent, archimedean semigroup, and (hence) contains a unique maximal subgroup; if at least two of the direct factors are non-trivial and at least one is not a group, then S is not a cyclic semigroup and contains a proper maximal cyclic subsemigroup of maximal index and maximal period with respect to the other cyclic subsemigroups of S ; and if each direct factor has index greater than 1 , then S contains a unique minimal set of generators. We determine the index and period of a cyclic subsemigroup of S generated by any given element of S , as well as the numbers of certain elements in S which generate cyclic subsemigroups of given indices and periods. We prove that a finite cyclic semigroup which is not a group cannot be decomposed into a direct product of semigroups.

We then prove that S has a unique decomposition into cyclic semigroups which are not groups. There are several known results concerning the unique factorization into direct factors for various kinds of algebras and relational structures - for example, every finite structure with a partial binary operation $(\cdot)$ and an element 1 such that $x \cdot 1 = 1 \cdot x = x$ for every element x has the unique factorization property (see Jónsson [*10*]); any algebra with a one-element subalgebra, congruences which commute, and a

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lattice of congruences of finite length, has the unique factorization property (the Birkhoff-Ore Theorem; see Birkhoff [1]). But a direct product of finite cyclic semigroups which are not groups does not contain an identity element and, in general, congruences do not commute. So these results do not apply to such semigroups.

Chang, Jónsson and Tarski [2] and Jonsson [10] have investigated when relational structures have the strict refinement and/or unique factorization properties, but their results do not give much information about a direct product of finite cyclic semigroups which are not groups or which do not satisfy their special conditions. We do not use their results in proving, nor do their results imply, that if $S = S_1 \times \ldots \times S_n$ is isomorphic to $T = T_1 \times \ldots \times T_l$, where S and T are (external) direct products of non-trivial finite cyclic semigroups, with $S_1, \ldots, S_n$ not groups, then $l = n$ and, with suitable rearrangement, S_i is isomorphic to T_i, $1 \leq i \leq n$. An analogous result is also true for a direct product of finite cyclic monoids which are not groups, a monoid being a semigroup with an identity adjoined, but, unlike the proof for semigroups, we show that this follows easily from the existence of a unique minimal set of generators. The result for semigroups does not appear to be a corollary to the result for monoids, and we use a quite different argument. We then ascertain when an arbitrary finite commutative semigroup is a direct product of two or more cyclic semigroups which are not groups, and how a decomposition of such a semigroup into its direct factors can be determined. I have had two articles published - [11] and [12] - in which the results of this chapter appear.

Let $U = U_1 \times \ldots \times U_n$ be a finite (external) direct product of finite monogenic inverse semigroups, a monogenic inverse semigroup I being one generated by a single element $u \in I$, say, and its inverse u^{-1}, written $I = \langle u, u^{-1} \rangle$. Chapter 2 is concerned with characterizing these emigroups. Djadčenko and Šaĭn [6] showed that any element of a finite monogenic inverse semigroup $\langle u, u^{-1} \rangle$, say, is either in the maximal subgroup of $\langle u \rangle$, or has a unique representation in the form $u^{-q}u^{s}u^{-t}$, where q, s and t are integers, with $0 \leq q \leq s$, $0 \leq t \leq s$

and $s > 0$, and $s < r$, where r is the index of the semigroup $\langle u \rangle$. We firstly collect together some properties of $\langle u, u^{-1} \rangle$. The index and period clearly uniquely determines $\langle u, u^{-1} \rangle$. (This is a consequence of a result of Eberhart and Selden [7], although not stated by them; it was missed by Gluskin [9], but follows from the results of Eršova [8].) So, for any two finite monogenic inverse semigroups, say $\langle u, u^{-1} \rangle$ and $\langle v, v^{-1} \rangle$, if $\langle u \rangle$ and $\langle v \rangle$ are isomorphic, then $\langle u, u^{-1} \rangle$ and $\langle v, v^{-1} \rangle$ are also isomorphic. We prove that the converse is also true, and that $\{u, u^{-1}\}$ is a unique minimal set of semigroup generators for $\langle u, u^{-1} \rangle$. Chapter 2 then examines cyclic subsemigroups of a finite monogenic inverse semigroup $\langle u, u^{-1} \rangle$, and we prove that if u has index r and period m , then any element of $\langle u, u^{-1} \rangle$ has index less than or equal to r and period dividing m , and if $r > 1$ and an element has index r , then it has period m .

We then characterize $U = U_1 \times \ldots \times U_n$, and prove that it is a non-commutative non-monogenic inverse semigroup containing a proper maximal cyclic subsemigroup of maximal index and maximal period with respect to the other cyclic subsemigroups of U , and find the number of idempotents it contains. We show that U is a group if and only if each direct factor is a group, and prove that a finite monogenic inverse semigroup which is not a group is directly indecomposable. We show that if an inverse semigroup is cyclic, then it is a group, and find a generating set for U with $4^n - 2^{n+1}$ elements.

As for cyclic semigroups, the known results on unique factorization into direct factors of certain algebras and relational structures, do not apply to direct products of finite monogenic inverse semigroups which are not groups. We prove that if $U = U_1 \times \ldots \times U_n$ is isomorphic to $V = V_1 \times \ldots \times V_l$, where U and V are (external) direct products of non-trivial finite monogenic inverse semigroups, with $U_1, \ldots, U_n$ not groups, then $l = n$ and, with suitable rearrangement, U_i is isomorphic to V_i , $1 \le i \le n$. We then determine when a finite semigroup is a

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direct product of two or more monogenic inverse semigroups which are not groups, and show how the direct factors, if they exist, can be found. I have had some of the results of this chapter published in [12].

Let $S = S_1 \times \ldots \times S_n$ again denote a direct product of finite cyclic semigroups, with maximal subgroup K, say. Let ρ be a congruence on S. In Chapter 3, we prove: S/ρ is a commutative archimedean semigroup with a unique idempotent and maximal subgroup $K' = \{a\rho \mid a \in K\}$; S/ρ is (completely) simple if and only if S/ρ is a group, that is, if and only if S/ρ has an identity element; S/ρ has a zero element if and only if every element of S/ρ has period 1 . We note two well-known results: if S/ρ is a group, then $S/\rho \cong K/\sigma$ for some congruence σ on K ; and for any congruence σ on K, there exists a congruence ρ on S such that $S/\rho \cong K/\sigma$ (the latter following from a result of Clifford and Miller [3]). We then prove that if S/ρ is cyclic of index t, greater than 3, then, for some i, $1 \le i \le n$, there exists a congruence ξ_i on S_i such that $S/\rho \cong S_i/\xi_i$; however, when $t \le 3$ and t is less than or equal to the maximal index of the S_i's, $1 \le i \le n$, then, for any congruence δ on K such that K/δ is cyclic of order q, there exists a congruence ρ on S such that S/ρ is cyclic of index t and period q.

Although any cyclic semigroup has a distributive lattice of congruences (see Clifford and Preston [5], §9.3, Exercise 6), we prove, in Chapter 4, that the lattice L of congruences on $S = S_1 \times \ldots \times S_n$, where at least one direct factor is not a group and at least two are non-trivial, is not lower semimodular, and hence neither modular nor distributive. But L is upper semimodular - this follows from our proof of a stronger result: any finite ideal extension of a group by a nil semigroup has an upper semimodular lattice of congruences. (S is an ideal extension of a group by a nil semigroup.)

We begin Chapter 5 by noting that finite cyclic semigroups can have, in some cases, directly decomposable lattices of congruences (see Clifford and Preston [5], §9.3, Exercise 6). We prove that $S = S_1 \times \ldots \times S_n$, where at least two direct factors are not groups, has a directly

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indecomposable lattice of congruences. We deduce this result by proving a somewhat stronger result: the lattice of congruences on a semigroup W, with $W \backslash W^2$ of finite order, greater than 1, and such that $W^2 \subseteq (W \backslash W^2)^2$, is directly indecomposable. We conclude by proving that the lattice of equivalences of a finite set with 3 or more elements is directly indecomposable.

For definitions of terms used above, see [4], [5], or the relevant references cited.

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