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The Homology of Abelian Covers of Knotted Graphs

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Abstract. Let $\tilde{M}$ be a regular branched cover of a homology 3-sphere M with deck group $G \cong \mathbb{Z}_2^d$ and branch set a trivalent graph Γ; such a cover is determined by a coloring of the edges of Γ with elements of G. For each index-2 subgroup H of G, $M_H = \tilde{M}/H$ is a double branched cover of M. Sakuma has proved that $H_1(\tilde{M})$ is isomorphic, modulo 2-torsion, to $\bigoplus_H H_1(M_H)$, and has shown that $H_1(\tilde{M})$ is determined up to isomorphism by $\bigoplus_H H_1(M_H)$ in certain cases; specifically, when $d = 2$ and the coloring is such that the branch set of each cover $M_H \to M$ is connected, and when $d = 3$ and Γ is the complete graph K_4. We prove this for a larger class of coverings: when $d = 2$, for any coloring of a connected graph; when $d = 3$ or 4, for an infinite class of colored graphs; and when $d = 5$, for a single coloring of the Petersen graph.

1 Introduction

In this paper we are concerned with invariants of graphs embedded in 3-space, which are known as *knotted* or *spatial* graphs. Although knotted graphs have been studied for some time, they have received more attention in the last ten years or so because of their potential applications to stereochemistry; a reader interested in these applications may consult Simon [6] or Kinoshita [2].

For our purposes, a graph is a 1-dimensional polyhedron Γ. A vertex of Γ is a point at which Γ is not a 1-manifold, and an edge is the closure of a component of the complement of the set of vertices. A component of Γ that contains a vertex is naturally a graph in the combinatorial sense (possibly with loops or multiple edges). A component without vertices is a single edge homeomorphic to S^1, which we call a circular edge (as opposed to a loop, which is homeomorphic to S^1, but contains a vertex).

Remark None of our theorems apply to graphs with circular edges, but they are needed for some lemmas.

All the graphs we consider are trivalent; this does not exclude circular edges. If Γ is a trivalent graph the number V of vertices and the Euler characteristic $\chi(\Gamma)$ are related by $V = -2\chi(\Gamma)$, and the number of non-circular edges is $-3\chi(\Gamma)$. By a cycle in a trivalent graph we mean a (possibly empty) subgraph homeomorphic to a disjoint union of circles; these are in one-to-one correspondence with the elements of $H_1(\Gamma; \mathbb{Z}_2)$. A cycle with one component is called a circuit. If Γ' is a subgraph of Γ, we use $\Gamma \setminus \Gamma'$ to denote the closure of the set-theoretic complement $\Gamma - \Gamma'$. We call a graph simple if it has no loops or multiple or circular edges.

Let d be an integer greater than 1, and let G be a (multiplicative) group isomorphic to $\mathbb{Z}_2^d$. Let M be a homology 3-sphere and let $\pi\colon \tilde{M} \to M$ be a regular branched cover with deck

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group G and branch set a graph $\Gamma \subset M$. A point in the inverse image of a vertex of Γ of valence n has a neighborhood that is a cone on a cover of S^2 branched over n points, and by the Riemann-Hurwitz formula any regular branched cover of S^2 by itself has 2 or 3 branch points. Thus $\tilde{M}$ is a manifold iff Γ is trivalent; we assume that this is the case. For each edge e of Γ, the stabilizer G_e of a lift of e to $\tilde{M}$ is a subgroup of G of order 2 (and is independent of the lift since G is abelian). We color e with the non-trivial element of G_e. The colors g_1, g_2 and g_3 of the edges at a vertex v are the non-trivial elements of the stabilizer of a lift of v (a group isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$), so they satisfy the relation $g_1g_2g_3 = 1$. Conversely, any coloring of the edges of Γ by non-trivial elements of G satisfying this relation at each vertex defines a homomorphism $H_1(M - \Gamma; \mathbb{Z}_2) \to G$ sending each meridian of an edge to the corresponding color. The corresponding branched covering is connected iff the colors of the edges generate G; we shall always assume this is so, and call Γ a G-colored graph. We regard two G-colorings of Γ as identical if they differ only by automorphisms of G and Γ. We sometimes write $G(d)$ for G to indicate the value of d under consideration. When we refer to a basis of G, or to independent elements of G, we are considering G as a $\mathbb{Z}_2$ vector space.

Remark Any coloring of the edges by elements of G satisfying the above relations defines a homomorphism from $H^1(\Gamma; \mathbb{Z}_2)$ to G, and vice-versa, so we can choose such a coloring with the colors generating G iff the first Betti number $b_1(\Gamma)$ of Γ is at least d. However, this may fail to be a G-coloring as just defined since some of the colors may be the identity. If G has a bridge e, the color of e must be 1 since e represents zero in $H^1(\Gamma; \mathbb{Z}_2)$. If Γ does not have a bridge, the existence of a G-coloring is not guaranteed; when $d = 2$, a G-coloring is just a Tait coloring, and the question of which bridgeless trivalent graphs have a Tait coloring has a long history.

Let $\mathcal{C}^\star = \mathcal{C}^\star(G)$ be the set of all subgroups of G of index 2, and let $\mathcal{C} = \mathcal{C}(G) = \mathcal{C}^\star \cup \{G\}$. If Γ is a G-colored graph, for $H \in \mathcal{C}$ we let Γ_H be the union of the edges of Γ whose colors are not in H; this is a cycle in Γ. If Γ is embedded in a homology 3-sphere M with branched cover $\pi\colon \tilde{M} \to M$, let $M_H = \tilde{M}/H$ for $H \in \mathcal{C}$. If $H \in \mathcal{C}^\star$, there is a 2-fold branched covering $\rho_H\colon M_H \to M$ whose branch set is the link Γ_H. There is also a branched covering $\pi_H\colon \tilde{M} \to M_H$ with group H, whose branch set Δ_H is the inverse image of $\Gamma \setminus \Gamma_H$. When $H = G$, $M_G = M$ and we let $\pi_G = \pi$ and $\rho_G = \mathrm{id}$. Sakuma showed that $H_1(\tilde{M})$ and $\bigoplus_{H \in \mathcal{C}^\star} H_1(M_H)$ are isomorphic modulo 2-torsion [5, Theorem 14.1], and determined the 2-torsion of $H_1(\tilde{M})$ when $d = 2$ and each Γ_H is connected, and when $d = 3$ and $\Gamma = K_4$ [5, Theorem 14.2]. Our first theorem generalizes part (1) of [5, Theorem 14.2], because $\bigoplus_{H \in \mathcal{C}^\star} H_1(M_H)$ has odd order when all the Γ_H are connected, so the exact sequence of the theorem is split.

Theorem 8.1 *If $d = 2$ and Γ is connected, then there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{H \in \mathcal{C}^\star} H_1(M_H) \xrightarrow{\beta} H_1(\tilde{M}) \longrightarrow \mathbb{Z}_2^{b_1(\Gamma)-2} \longrightarrow 0,$$

and $\beta\big(\bigoplus_{H \in \mathcal{C}^\star} H_1(M_H)\big) = 2H_1(\tilde{M})$.

There are infinitely many $G(2)$-colorings of connected graphs for which the Γ_H are not all connected; see Example 1.2. In this case, the above sequence does not split; nevertheless,

$H_1(\tilde{M})$ is determined up to isomorphism by $\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)$. This is a consequence of the case $p = 2$ and $e = 1$ of the following proposition, whose proof is a simple application of the structure theorem for finitely generated abelian groups, and is omitted.

Proposition 1.1 *Let A and B be finitely generated abelian groups, p a prime, and e a positive integer. If $p^eA \cong p^eB$ and $A/p^eA \cong B/p^eB$, then $A \cong B$.* ■

Example 1.2 Let Γ be an n-rung Möbius ladder. Recall that this graph consists of a $2n$-circuit (the rim) together with its diameters (the rungs). (It is usual to require $n \geq 3$, but the cases $n = 1$ or 2 make sense; when $n = 1$ we have the theta-curve, and when $n = 2$ we have K_4.) When $n \geq 2$, Γ is simple, and we take the vertices to be $v_0, \ldots, v_{2n-1}$ and the edges to be $\sigma_i = \{v_i, v_{i+1}\}$ and $\tau_i = \{v_i, v_{i+n}\}$, the subscripts being taken modulo $2n$. The σ_i form the rim, and the τ_i are the rungs. Let Γ' be a non-empty cycle in Γ that contains k rungs. If $k = 0$, Γ' is the rim; otherwise, Γ' is connected if k is odd, and has $\frac{k}{2}$ components if k is even.

Now take $d = 2$, and let the non-trivial elements of G be g_1, g_2 and g_3. Give all the rungs the color g_1, and give the edges of the rim the colors g_2 and g_3 alternately. If $H = \langle g_1\rangle$ then Γ_H is the rim, while if $H = \langle g_2\rangle$ or $\langle g_3\rangle$ then Γ_H contains all n rungs. Thus every Γ_H is connected iff n is odd or $n = 2$.

We say that a G-coloring of a graph Γ is unsplittable if, for any $g \in G$, deleting the edges of Γ with color g leaves a connected graph. If Γ has an unsplittable coloring, then either Γ is the theta-curve (in which case $d = 2$), or Γ is connected and simple. First, taking $g = 1$ shows that Γ is connected, and in particular has no circular edges. Since Γ has no bridges, it has no loops. If Γ is not the theta-curve and has a pair of multiple edges, these are adjacent to two distinct edges with the same color. Deleting these edges disconnects Γ, contrary to the definition.

A circuit C in a G-colored graph Γ will be called special if there is some $H \in \mathcal{C}^\star$ such that $\Gamma_H = C$ and $\Gamma \setminus C$ is connected. Note that if this is so then Γ is unsplittable iff the result of deleting from $\Gamma \setminus C$ all edges with color h is a forest whenever $1 \neq h \in H$.

Theorem 8.2 *Let Γ be a trivalent graph with an unsplittable $G(3)$-coloring with a special m-circuit. Then $3 \leq m \leq b_1(\Gamma)$, there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{H\in\mathcal{C}^\star} H_1(M_H) \xrightarrow{\beta} H_1(\tilde{M}) \longrightarrow \mathbb{Z}_4^{m-3} \oplus \mathbb{Z}_2^{2(b_1(\Gamma)-m)} \longrightarrow 0,$$

and $\beta\big(\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)\big) = 4H_1(\tilde{M})$.

This implies part (2) of [5, Theorem 14.2], since K_4 has a unique $G(3)$-coloring, which is unsplittable and has a special 3-circuit. Once again, when Theorem 8.2 applies, Proposition 1.1 shows that $H_1(\tilde{M})$ is determined up to isomorphism by $\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)$. We now show that Theorem 8.2 applies to infinitely many colored graphs.

Proposition 1.3 *Let m and b be integers with $3 \leq m \leq b$. Then there is a graph Γ with $b_1(\Gamma) = b$ and an unsplittable $G(3)$-coloring of Γ which has a special m-circuit.*

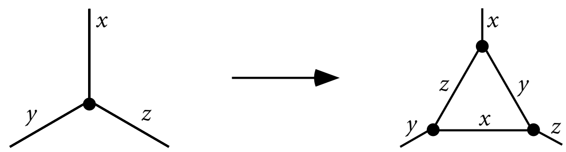

Figure 1: $x, y, z \in G, xyz = 1$

Proof First we show that for $m \geq 3$ there is a graph Γ with $b_1(\Gamma) = m$ and an unsplittable $G(3)$-coloring of Γ which has a special m-circuit. Let T be a tree with m vertices of valence 1 (its leaves) and $m-2$ vertices of valence 3 (its forks); such trees exist for any $m \geq 2$. Form Γ by adding an m-circuit C through the leaves of T. Pick $H_0 \in \mathcal{C}^\star$ and $g_0 \in G - H_0$. It is easy to color the edges of T with non-trivial elements of H_0 so that the required relation holds at each fork. Further pick an edge e_0 and a vertex v_0 of C. Give e_0 the color g_0. There is then a unique way to color the other edges of C so that the required relation holds at every vertex except perhaps v_0. If we take the product over all vertices v of the product of the edge-colors at v, the result is 1, since each edge-color appears twice. It follows that the required relation holds at v_0 as well. Since $m \geq 3$, T has at least one fork, and so all non-trivial elements of H_0 are used to color T, and the edge-colors of Γ generate G. Also, all the colors of C are in $G - H_0$. It follows first that they are non-trivial, so we do have a G-coloring, and second that $\Gamma_{H_0} = C$, so that C is a special m-circuit. Since deleting edges from a tree always leaves a forest, this coloring is unsplittable.

Now, if Γ is any unsplittable $G(3)$-colored graph with a special m-circuit, performing the operation of Figure 1 at any vertex not on that circuit yields a graph Γ' which is unsplittable, has a special m-circuit, and has $b_1(\Gamma') = b_1(\Gamma) + 1$; the general case follows. ∎

We give some specific examples of such colorings.

Example 1.4 Let Γ be an n-rung Möbius ladder ($n \geq 2$). It is possible to determine all unsplittable $G(3)$-colorings of Γ with a special circuit; we shall describe them but omit the verification that there are no others. First, an $(n+1)$-circuit consisting of one rung together with half the rim has complementary graph a tree. By the first part of the above proof, there is an unsplittable coloring for which this circuit is special. Next, suppose that $n \geq 3$ and let $\{x_1, x_2, x_3\}$ be a basis of G. Color the rim edge σ_0 with x_1, the rung τ_0 with x_2, the rung τ_1 with $x_2 x_3^{n-1}$, and all other rungs with x_3. There is a unique way to complete the coloring, and there is a special 4-circuit corresponding to the subgroup $\langle x_1 x_2, x_3 \rangle$; unsplittability is easily checked. Finally, there is an exceptional coloring when $n = 4$: color the rung τ_0 with $x_1 x_2 x_3$, τ_i with x_i for $1 \leq i \leq 3$, and the rim edge σ_0 with x_2. This determines an unsplittable coloring with a special 4-circuit corresponding to $\langle x_1, x_2 \rangle$.

Example 1.5 In [7], the generalized Petersen graph $P(n, k)$ was defined for $1 \leq k \leq n-1$ and $n \neq 2k$ as follows. It has $2n$ vertices $u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}$, and edges of three kinds, namely $\sigma_i = \{u_i, u_{i+1}\}$, $\tau_i = \{v_i, v_{i+k}\}$ and $\rho_i = \{u_i, v_i\}$, where the subscripts are

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taken modulo n. The edges σ_i form an n-circuit (the outer rim); if k is coprime to n (as we shall assume), so do the edges τ_i (the inner rim). The edges ρ_i are called rungs. Pick $H_0 \in \mathcal{C}^\star$ and $g_0 \in G - H_0$. Color the edges of the inner rim with non-trivial elements of H_0 so that adjacent edges receive distinct colors and all three elements appear. This forces colors on the rungs. If one edge of the outer rim is given the color g_0, there is a unique way to complete the G-coloring. Then Γ_{H_0} is the outer rim, whose complementary graph is connected; it is easy to see that this coloring is unsplittable. This example does not arise from the construction of Proposition 1.3.

If $n = 2m + 1$ and $k = 2$ there is also an unsplittable coloring with a special $(n + 1)$-circuit; the complementary graph to the circuit $u_1u_2 \cdots u_{2m}v_{2m}v_1u_1$ is a tree, so there is an unsplittable coloring for which this circuit is special.

We have one other theorem in the case $d = 3$.

Theorem 8.3 *Let Γ be an n-rung Möbius ladder ($n \geq 2$) with a $G(3)$-coloring, and let g_0 be the product of the colors on the rungs. Suppose that $g_0 \neq 1$, and let k be the number of rungs with color g_0. If $k = 0$, there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{H \in \mathcal{C}^\star} H_1(M_H) \xrightarrow{\beta} H_1(\tilde{M}) \longrightarrow \mathbb{Z}_4^{n-2} \longrightarrow 0,$$

while if $k > 0$ there is a short exact sequence

$$0 \longrightarrow \bigoplus_{H \in \mathcal{C}^\star} H_1(M_H) \xrightarrow{\beta} H_1(\tilde{M}) \longrightarrow \mathbb{Z}_4^{n-k-1} \oplus \mathbb{Z}_2^{2(k-1)} \longrightarrow 0.$$

In either case, $\beta\big(\bigoplus_{H \in \mathcal{C}^\star} H_1(M_H)\big) = 4H_1(\tilde{M})$.

There is considerable overlap between Theorems 8.2 and 8.3; all the colorings of Example 1.4 apart from the exceptional coloring for $n = 4$ satisfy the hypothesis of Theorem 8.3. However, it is easy to see that there are infinitely many colorings satisfying that hypothesis that do not have a special circuit.

Next we consider some $G(4)$-colorings of Möbius ladders.

Example 1.6 Let $d = 4$, and let $\{x_1, x_2, x_3, x_4\}$ be a basis of G. Let Γ be an n–rung Möbius ladder with $n \geq 3$. Give the colors x_1, x_2 and $x_1x_2x_3^n$ to one rung each, and give all other rungs the color x_3. If we give any rim edge the color x_4, there is then a unique way to color the remaining edges with elements of G so that the required relation holds at each vertex, and this does give a G-coloring. Here every Γ_H is connected; this can be seen by listing all the Γ_H, but it is easier to make use of the following lemma.

Lemma 1.7 *Let $e_1, \ldots, e_n$ be distinct edges of a G-colored graph Γ with colors $g_1, \ldots, g_n$. For $H \in \mathcal{C}^\star$, the number of these edges contained in Γ_H is even iff $g_1 \cdots g_n \in H$.*

Proof Let δ_H be the homomorphism from G to $\mathbb{Z}_2$ with kernel H, and let k of the edges $e_1, \ldots, e_n$ be contained in Γ_H. Since e_i is contained in Γ_H iff $\delta_H(g_i) = 1$, $\delta_H(g_1 \cdots g_n) = k \bmod 2$, and the result follows. ■

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For the colorings of Example 1.6, the product of the colors on the rungs is x_3. Let $H \in \mathcal{C}^\star$. If $x_3 \notin H$ then Γ_H contains an odd number of rungs by the lemma, while if $x_3 \in H$ then Γ_H contains at most three rungs; in either case, Γ_H is connected.

Theorem 8.7 *Let $d = 4$ and let Γ be an n-rung Möbius ladder with $n \geq 3$. Give Γ the $G(4)$-coloring of Example 1.6. Then*

$$H_1(\tilde{M}) \cong \begin{cases} \bigoplus_{H \in \mathcal{C}^\star} H_1(M_H) \oplus \mathbb{Z}_2, & \text{if } n = 3; \\ \bigoplus_{H \in \mathcal{C}^\star} H_1(M_H) \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2^{4n-14}, & \text{if } n \geq 4. \end{cases}$$

Our final theorem deals with a particular coloring of the Petersen graph.

Example 1.8 We use the notation of Example 1.5, and let Γ be the Petersen graph $P(5,2)$. Let $d = 5$, and let G have a basis $\{x_0, \dots, x_4\}$. Color the edge σ_i with x_i, the edge τ_i with $x_{i-1}x_{i+2}$, and the edge ρ_i with $x_{i-1}x_i$, all subscripts being taken modulo 5. We leave it to the reader to check that this is indeed a G-coloring. This graph has six disconnected cycles, all of which contain an odd number of the edges τ_i. Since the product of the colors on the τ_i is 1, it follows from Lemma 1.7 that every Γ_H is connected.

Theorem 8.8 *Let $d = 5$, and let Γ be the Petersen graph with the $G(5)$-coloring of Example 1.8. Then*

$$H_1(\tilde{M}) \cong \bigoplus_{H \in \mathcal{C}^\star} H_1(M_H) \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_4^4 \oplus \mathbb{Z}_2^2.$$

The rest of this section sets out some notation. In the next section we give the plan of the proof and explain the organization of the rest of the paper.

We deal often with direct sums $\bigoplus_{H \in \mathcal{C}'} \Lambda_H$, where the Λ_H are abelian groups indexed by a subset $\mathcal{C}'$ of $\mathcal{C}$. It is convenient to regard an element of $\bigoplus_{H \in \mathcal{C}'} \Lambda_H$ as a formal linear combination $\sum_{H \in \mathcal{C}'} \lambda_H H$ with $\lambda_H \in \Lambda_H$. When all the Λ_H are equal, we use the notation $\Lambda^{\mathcal{C}'}$ for $\bigoplus_{H \in \mathcal{C}'} \Lambda$. As in the proof of Lemma 1.7, for $H \in \mathcal{C}$, we let δ_H be the homomorphism $G \to \mathbb{Z}_2$ with kernel H; we also let ε_H the homomorphism with kernel H from G to the group $\{\pm 1\}$ of units of $\mathbb{Z}$ (a character of G).

If X is a polyhedron, $C(X; \Lambda)$ will denote the simplicial chain complex of some fixed but anonymous triangulation of X, with coefficients in the abelian group Λ. When the coefficient group is omitted, it is understood to be $\mathbb{Z}$, except in Section 7, where it is understood to be $\mathbb{Z}_2$. We assume that the simplices of the triangulation have been oriented, and by a simplex of X we shall mean a simplex of the triangulation with the chosen orientation; thus the simplices of X form a basis for $C(X)$. We let $S(X)$ be the set of all simplices of X, and $S_i(X)$ the subset of i-simplices. If $f \colon X \to Y$ is a simplicial map, the induced maps on chain complexes and homology will also be denoted by f without further decoration. If f is a regular branched covering and the triangulation of X is obtained by lifting that of Y, we have the transfer map $C(Y) \to C(X)$; recall that this sends a simplex σ to $\sum_{k \in K} k\tilde{\sigma}$, where K is the deck group and $\tilde{\sigma}$ is one lift of σ. This map and the induced map on homology will both be denoted by $f^!$. We let $b_i(X)$ be the i-th Betti number of X.

2 Outline of the Proof

Consider a regular branched covering $\pi\colon \tilde{M} \to M$ of a homology 3-sphere M, with deck group G and branch set a G-colored graph Γ. Triangulate M so that Γ is triangulated by a subcomplex, and lift this triangulation to triangulations of the M_H and $\tilde{M}$. We have various transfer maps $\rho_H^!\colon C(M) \to C(M_H)$ and $\pi_H^!\colon C(M_H) \to C(\tilde{M})$. We define chain maps

$$\alpha\colon C(M)^{\mathcal{C}^\star} \longrightarrow \bigoplus_{H\in\mathcal{C}} C(M_H)$$

$$\text{by}\quad \alpha\Big(\sum_{H\in\mathcal{C}^\star} c_H H\Big) = \sum_{H\in\mathcal{C}^\star} \big(\rho_H^!(c_H)H - c_H G\big) \quad \text{for } c_H \in C(M), H \in \mathcal{C}^\star,$$

and

$$\beta\colon \bigoplus_{H\in\mathcal{C}} C(M_H) \longrightarrow C(\tilde{M})$$

$$\text{by}\quad \beta\Big(\sum_{H\in\mathcal{C}} d_H H\Big) = \sum_{H\in\mathcal{C}} \pi_H^!(d_H) \quad \text{for } d_H \in C(M_H), H \in \mathcal{C}.$$

We also let $\gamma\colon C(\tilde{M}) \to C(M;\mathbb{Z}_{2^{d-1}})$ be the composite of $\pi\colon C(\tilde{M}) \to C(M)$ and reduction of the coefficients modulo 2^{d-1}.

Consider the sequence

$$0 \longrightarrow C(M)^{\mathcal{C}^\star} \xrightarrow{\alpha} \bigoplus_{H\in\mathcal{C}} C(M_H) \xrightarrow{\beta} C(\tilde{M}) \xrightarrow{\gamma} C(M;\mathbb{Z}_{2^{d-1}}) \longrightarrow 0. \tag{2.1}$$

This is not exact, but we do have the following result.

Lemma 2.2 *The chain map α is injective, $\beta\alpha = 0$, $\gamma\beta = 0$, and γ is surjective.*

Proof Define $\alpha'\colon \bigoplus_{H\in\mathcal{C}} C(M_H) \to C(M)^{\mathcal{C}^\star}$ by

$$\alpha'\Big(\sum_{H\in\mathcal{C}} d_H H\Big) = \sum_{H\in\mathcal{C}^\star} \rho_H(d_H)H.$$

Then

$$\alpha'\alpha\Big(\sum_{H\in\mathcal{C}^\star} c_H H\Big) = \sum_{H\in\mathcal{C}^\star} \rho_H\rho_H^!(c_H)H = 2\sum_{H\in\mathcal{C}^\star} c_H H,$$

so α is injective. Next,

$$\beta\alpha\Big(\sum_{H\in\mathcal{C}^\star} c_H H\Big) = \sum_{H\in\mathcal{C}^\star} \big(\pi_H^!\rho_H^!(c_H) - \pi^!(c_H)\big) = 0,$$

so $\beta\alpha = 0$. Further,

$$\pi\beta\Big(\sum_{H\in\mathcal{C}} d_H H\Big) = \sum_{H\in\mathcal{C}} \rho_H\pi_H\pi_H^!(d_H) = \sum_{H\in\mathcal{C}} |H|\rho_H(d_H),$$

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so $\gamma\beta = 0$. Finally, $\pi: C(\tilde{M}) \to C(M)$ is clearly onto, and hence so is γ. ∎

The sequence (2.1) thus decomposes into four short exact sequences:

$$0 \longrightarrow C(M)^{\mathcal{C}^\star} \xrightarrow{\alpha} \operatorname{Ker}\beta \longrightarrow \operatorname{Ker}\beta/\operatorname{Im}\alpha \longrightarrow 0; \tag{2.3}$$

$$0 \longrightarrow \operatorname{Ker}\beta \overset{\iota}{\hookrightarrow} \bigoplus_{H\in\mathcal{C}} C(M_H) \xrightarrow{\beta} \operatorname{Im}\beta \longrightarrow 0; \tag{2.4}$$

$$0 \longrightarrow \operatorname{Im}\beta \hookrightarrow \operatorname{Ker}\gamma \longrightarrow \operatorname{Ker}\gamma/\operatorname{Im}\beta \longrightarrow 0; \quad \text{and} \tag{2.5}$$

$$0 \longrightarrow \operatorname{Ker}\gamma \hookrightarrow C(\tilde{M}) \xrightarrow{\gamma} C(M;\mathbb{Z}_{2^{d-1}}) \longrightarrow 0. \tag{2.6}$$

The last of these relates the homology groups of $\tilde{M}$ and the complex $\operatorname{Ker}\gamma$; the first homology is all we need.

Lemma 2.7 *We have* $H_1(\operatorname{Ker}\gamma) \cong H_1(\tilde{M})$.

Proof Since M is an integral homology sphere, it is also a $\mathbb{Z}_{2^{d-1}}$ homology sphere, so part of the long exact sequence of (2.6) is $0 \to H_1(\operatorname{Ker}\gamma) \to H_1(\tilde{M}) \to 0$. ∎

To extract information from the exact sequences (2.3)–(2.5), we need to study the complexes $\operatorname{Ker}\beta/\operatorname{Im}\alpha$ and $\operatorname{Ker}\gamma/\operatorname{Im}\beta$. This leads us to consider certain chain complexes associated to a G-colored graph Γ. These chain complexes are defined and studied in Section 4, after some preliminary results on the graded ring of G in Section 3. In Section 5, we determine the complex $\operatorname{Ker}\beta/\operatorname{Im}\alpha$, and in Section 6, we determine the quotients of a filtration of $\operatorname{Ker}\gamma/\operatorname{Im}\beta$. In Section 7 we prove some results on the $\mathbb{Z}_2$ homology of 2- and 4-fold branched covers, and in Section 8 we prove our theorems.

3 The Graded Ring of G

As always, G is a group isomorphic to $\mathbb{Z}_2^d$, but in this section we do not assume that $d \geq 2$. For $H \in \mathcal{C}$, the character ε_H extends to a ring homomorphism $\varepsilon_H: \mathbb{Z}[G] \to \mathbb{Z}$ on the group ring of G. The fundamental ideal $I = I[G]$ of G is the kernel of ε_G; we also let $J = J(G)$ be the ideal of those $\lambda \in \mathbb{Z}[G]$ for which $\varepsilon_G(\lambda) \equiv 0 \pmod 2$. Note that $J = I \oplus 2\mathbb{Z}$. We consider the associated graded rings $A = A(G) = \mathrm{G}_I(\mathbb{Z}[G])$ and $B = B(G) = \mathrm{G}_J(\mathbb{Z}[G])$. (See [8, p. 248].) Consider first the ring B. The group of homogeneous elements of degree k is $B_k = J^k/J^{k+1}$, and B is an algebra over $B_0 = \mathbb{Z}[G]/J$, which we identify with $\mathbb{Z}_2$. We denote the image in B_k of $\lambda \in J^k$ by $[\lambda]_k$; the product is given by $[\lambda]_k[\mu]_l = [\lambda\mu]_{k+l}$. Turning to A, we have $(1-g)^2 = 2(1-g)$ for $g \in G$, so $2I \leq I^2$. Let $k \geq 1$. It follows that $2I^k \leq I^{k+1}$, and hence $J^k = I^k \oplus 2^k\mathbb{Z}$. Therefore we may identify A_k with its image in B_k, and B_k is the direct sum of A_k and a copy of $\mathbb{Z}_2$ generated by $[2^k]_k$. Note also that $A_kB_l = A_{k+l}$. Of course $A_0 \cong \mathbb{Z}$; below, when we refer to A_k, it is to be understood that $k \geq 1$.

We shall determine the structure of the algebra B, and hence that of A. (The structure of $\mathrm{G}_I(\mathbb{Z}[G])$ when G is free abelian was determined by Massey in [4].) We define a function $\omega: G \to A_1$ by $\omega(g) = [1-g]_1$.

Lemma 3.1 *The function ω is an isomorphism, and for any $\lambda = \sum_{g\in G} \lambda_g g \in I$ we have $\omega\big(\prod_{g\in G} g^{\lambda_g}\big) = [\lambda]_1$.*

Proof We compute

$$(1-g) + (1-h) - (1-gh) = 1 - g - h + gh = (1-g)(1-h) \in I^2,$$

so $[1-g]_1 + [1-h]_1 = [1-gh]_1$, and ω is a homomorphism. The function $[\lambda]_1 \mapsto \prod_{g\in G} g^{\lambda_g}$ is a well-defined homomorphism $A_1 \to G$ sending $\omega(g)$ to g. For $\lambda \in I$ we have $\lambda = -\sum_{g\in G} \lambda_g(1-g)$, so $[\lambda]_1 = \sum_{g\in G} \lambda_g \omega(g) = \omega\big(\prod_{g\in G} g^{\lambda_g}\big)$, and the result follows. ■

For $0 \le l \le d$, we let $\mathcal{I}_l$ be the set of l-tuples $\vec{\imath} = (i_1, i_2, \dots, i_l)$ of integers with $1 \le i_1 < i_2 < \cdots < i_l \le d$.

Lemma 3.2 *Let $\{x_1, \dots, x_d\}$ be a basis of G, and (for $k \ge 0$) let $\mathcal{B}_k$ be the set consisting of the elements $(1 - x_{i_1}) \cdots (1 - x_{i_l})$ for $k \le l \le d$ and $\vec{\imath} \in \mathcal{I}_l$, together with the elements $2^{k-l}(1 - x_{i_1}) \cdots (1 - x_{i_l})$ for $0 \le l < k$, $l \le d$ and $\vec{\imath} \in \mathcal{I}_l$. (When $l = 0$, the empty product $(1 - x_{i_1}) \cdots (1 - x_{i_l})$ is taken to be 1.) Then $\mathcal{B}_k$ is a basis of J^k (as a $\mathbb{Z}$-module). Further, an element λ of $\mathbb{Z}[G]$ is in J^k iff $\varepsilon_H(\lambda) \equiv 0 \pmod{2^k}$ for all $H \in \mathcal{C}$.*

Proof Every element g of G can be written uniquely in the form $g = x_{i_1} \cdots x_{i_l}$ for $0 \le l \le d$ and $\vec{\imath} \in I_l$; call l the length of g. Then g is the unique element of maximal length appearing in $(1 - x_{i_1}) \cdots (1 - x_{i_l})$, and it follows that the $(1 - x_{i_1}) \cdots (1 - x_{i_l})$ are linearly independent; therefore so are the elements of $\mathcal{B}_k$. Let V_k be the additive subgroup of $\mathbb{Z}[G]$ spanned by $\mathcal{B}_k$, and let W_k be the subgroup of those $\lambda \in \mathbb{Z}[G]$ such that $\varepsilon_H(\lambda) \equiv 0 \pmod{2^k}$ for all $H \in \mathcal{C}$. Clearly $V_k \le J^k$. Since $\varepsilon_H(\lambda) \equiv \varepsilon_G(\lambda) \pmod 2$, we have $J = W_1$, and it follows that $J^k \le W_k$ for all $k \ge 0$.

It remains to prove that $W_k \le V_k$. Let $\lambda = \sum_{g\in G} \lambda_g g$ be a non-zero element of $\mathbb{Z}[G]$. Let l be the maximum length of those g with $\lambda_g \ne 0$, and let n be the number of those g of length l with $\lambda_g \ne 0$. Call the pair (l, n) the weight of λ, and order weights lexicographically. Suppose that $W_k \not\le V_k$, and let λ be an element of $W_k - V_k$ of minimum weight (l, n). Let $h = x_{j_1} \cdots x_{j_l}$ ($\vec{\jmath} \in \mathcal{I}_l$) have $\lambda_h \ne 0$. If $l \ge k$, then $\lambda - (-1)^l \lambda_h (1 - x_{j_1}) \cdots (1 - x_{j_l})$ is an element of $W_k - V_k$ of smaller weight than λ, a contradiction. Suppose that $l < k$. Let G' be the subgroup of G generated by $x_{j_1}, \dots, x_{j_l}$, and G'' the subgroup generated by the other x_i, so $G = G' \oplus G''$. Since $\lambda \in W_k$,

$$\sum_{H' \in \mathcal{C}(G')} \varepsilon_{H'}(h) \varepsilon_{H' \oplus G''}(\lambda) \equiv 0 \pmod{2^k}.$$

Now

$$\sum_{H' \in \mathcal{C}(G')} \varepsilon_{H'}(h) \varepsilon_{H' \oplus G''}(\lambda) = \sum_{g \in G} \Big(\sum_{H' \in \mathcal{C}(G')} \varepsilon_{H'}(h) \varepsilon_{H' \oplus G''}(g) \Big) \lambda_g.$$

Let $g = g'g''$, with $g' \in G'$ and $g'' \in G''$. Then $\varepsilon_{H'}(h)\varepsilon_{H' \oplus G''}(g) = \varepsilon_{H'}(hg')$, and $\sum_{H' \in \mathcal{C}(G')} \varepsilon_{H'}(hg')$ is 0 if $g' \ne h$, and 2^l if $g' = h$. If $g' = h$ and $g'' \ne 1$, then $\lambda_g = 0$

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by our choice of h. It follows that $2^l\lambda_h \equiv 0 \pmod{2^k}$, or $\lambda_h \equiv 0 \pmod{2^{k-l}}$. Now $\lambda - (-1)^l\big(\lambda_h/2^{k-l}\big)2^{k-l}(1-x_{j_1})\cdots(1-x_{j_l})$ is an element of $W_k - V_k$ of smaller weight than λ, and this contradiction proves that $W_k \le V_k$. ■

As an immediate consequence of this lemma, we have bases for A_k and B_k.

Lemma 3.3 *Let $\{x_1,\dots,x_d\}$ be a basis of G. The elements*

$$[2^{k-l}(1-x_{i_1})\cdots(1-x_{i_l})]_k = [2]_1^{k-l}\omega(x_{i_1})\cdots\omega(x_{i_l})$$

for $0 \le l \le \min\{k,d\}$ and $\vec\imath \in \mathfrak{I}_l$ form a basis of B_k (as a $\mathbb{Z}_2$ vector space), and those for $1 \le l \le \min\{k,d\}$ form a basis for A_k. ■

Note that this implies that multiplication by $[2]_1$ defines injections $B_k \to B_{k+1}$ for $k \ge 0$ and $A_k \to A_{k+1}$ for $k \ge 1$, and that these are onto for $k \ge d$.

Lemma 3.4 *The graded algebra B is the quotient of the symmetric algebra of B_1 by the relations $a^2 = [2]_1 a$ for $a \in A_1$.*

Proof The given relations do hold in B: by Lemma 3.1, any element of A_1 equals $\omega(g)$ for some $g \in G$, and $\omega(g)^2 = [(1-g)^2]_2 = [2(1-g)]_2 = [2]_1\omega(g)$. Therefore, if $\hat B$ is the quotient of the symmetric algebra of B_1 by these relations, there is an epimorphism $\hat B \to B$. But if $\{x_1,\dots,x_d\}$ is a basis of G, then $\hat B_k$ is generated by the elements $[2]_1^{k-l}\omega(x_{i_1})\cdots\omega(x_{i_l})$ for $0 \le l \le \min\{k,d\}$ and $\vec\imath \in \mathfrak{I}_l$, and these map to independent elements in B_k by Lemma 3.3. ■

The $\mathbb{Z}_2$ vector space $\mathbb{Z}_2^{\mathcal{C}^\star}$ is a commutative algebra under componentwise multiplication. Its identity element $\sum_{H\in\mathcal{C}^\star} H$ will be denoted by $1^{\mathcal{C}^\star}$. We may define a linear map $\Omega\colon B_1 \to \mathbb{Z}_2^{\mathcal{C}^\star}$ by $\Omega\big(\omega(g)\big) = \sum_{H\in\mathcal{C}^\star}\delta_H(g)H$ for $g \in G$, and $\Omega([2]_1) = 1^{\mathcal{C}^\star}$. Since $x^2 = x$ for all $x \in \mathbb{Z}_2^{\mathcal{C}^\star}$, it follows from Lemma 3.4 that Ω extends (uniquely) to an algebra homomorphism $\Omega\colon B \to \mathbb{Z}_2^{\mathcal{C}^\star}$.

Lemma 3.5 *The map Ω restricts to an injection on A_k for $1 \le k$, and on B_k for $0 \le k \le d-1$. Further, Ω maps each of A_d and B_{d-1} onto $\mathbb{Z}_2^{\mathcal{C}^\star}$.*

Proof We show first that Ω maps A_d isomorphically onto $\mathbb{Z}_2^{\mathcal{C}^\star}$. For any $g_1,\dots,g_d \in G$, we have $\Omega\big(\omega(g_1)\cdots\omega(g_d)\big) = \sum_{H\in\mathcal{C}^\star}\delta_H(g_1)\cdots\delta_H(g_d)H$. Given $H_0 \in \mathcal{C}^\star$ we may find a basis $\{x_1,\dots,x_d\}$ of G with $x_i \notin H_0$ for $1 \le i \le d$. Then $\delta_{H_0}(x_1)\cdots\delta_{H_0}(x_d) = 1$ and $\delta_H(x_1)\cdots\delta_H(x_d) = 0$ for any $H \ne H_0$, so $\Omega\big(\omega(x_1)\cdots\omega(x_d)\big) = H_0$. Thus Ω maps A_d onto $\mathbb{Z}_2^{\mathcal{C}^\star}$. Since $\dim A_d = 2^d - 1 = \dim\mathbb{Z}_2^{\mathcal{C}^\star}$, Ω is also injective on A_d.

Next, let $\{x_1,\dots,x_d\}$ be any basis of G, and consider the basis elements $b_{\vec\imath} = [2]_1^{d-l}\omega(x_{i_1})\cdots\omega(x_{i_l})$ ($0 \le l \le d$, $\vec\imath \in \mathfrak{I}_l$) of B_d. Let s be the sum of all $\Omega(b_{\vec\imath})$. For each $H \in \mathcal{C}^\star$, the coefficient of H in $\Omega(b_{\vec\imath})$ is 1 if $x_{i_1},\dots,x_{i_l} \notin H$, and 0 otherwise. Therefore the coefficient of H in s is the number of subsets of $\{x_1,\dots,x_d\} \cap (G-H)$, taken modulo 2. Since $\{x_1,\dots,x_d\} \cap (G-H)$ is non-empty, this number is even, so $s = 0$. It follows that Ω maps the subspace of B_d spanned by the $b_{\vec\imath}$ for $l < d$ isomorphically onto $\mathbb{Z}_2^{\mathcal{C}^\star}$. Since

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multiplication by $[2]_1$ maps B_{d-1} isomorphically onto this space and $\Omega([2]_1 b) = \Omega(b)$ for all $b \in B$, Ω also maps B_{d-1} isomorphically onto $\mathbb{Z}_2^{\mathcal{C}^\star}$. Since multiplication by $[2]_1$ maps B_k injectively into B_{k+1}, it follows that Ω is injective on B_k for $0 \le k \le d-1$, and therefore on A_k for $1 \le k \le d-1$. Finally, multiplication by $[2]_1$ maps A_k isomorphically onto A_{k+1} for $k \ge d$, and hence Ω is injective on A_k for $k \ge d$. ■

There is an inner product on $\mathbb{Z}_2^{\mathcal{C}^\star}$ given by

$$\Big(\sum_{H\in\mathcal{C}^\star} a_H H\Big)\cdot\Big(\sum_{H\in\mathcal{C}^\star} b_H H\Big) = \sum_{H\in\mathcal{C}^\star} a_H b_H.$$

Note that for any x and y in $\mathbb{Z}_2^{\mathcal{C}^\star}$, $x \cdot y = 1^{\mathcal{C}^\star} \cdot (xy)$.

Lemma 3.6 *For $1 \le k \le d-1$, we have $\Omega(A_k) = \Omega(B_{d-k-1})^\perp$, where $^\perp$ denotes the orthogonal complement with respect to the above inner product.*

Proof Since $\dim A_k = \sum_{i=1}^{k} \binom{d}{i}$ and $\dim B_{d-k-1} = \sum_{i=0}^{d-k-1} \binom{d}{i} = \sum_{i=k+1}^{d} \binom{d}{i}$, we have $\dim A_k + \dim B_{d-k-1} = 2^d - 1 = \dim \mathbb{Z}_2^{\mathcal{C}^\star}$. Therefore it suffices to prove that $\Omega(a)\cdot\Omega(b) = 0$ for $a \in A_k$ and $b \in B_{d-k-1}$. Since $\Omega(a)\cdot\Omega(b) = 1^{\mathcal{C}^\star}\cdot\Omega(ab)$ and $ab \in A_{d-1}$, it is enough to show that $1^{\mathcal{C}^\star}\cdot\Omega(A_{d-1}) = 0$. For $g_1,\dots,g_{d-1} \in G$, $1^{\mathcal{C}^\star}\cdot\Omega\big(\omega(g_1)\cdots\omega(g_{d-1})\big)$ is the number of $H \in \mathcal{C}^\star$ that contain none of $g_1,\dots,g_{d-1}$, taken modulo 2. Since $g_1,\dots,g_{d-1}$ do not generate G, this number is even, and we are done. ■

Now let G' be a subgroup of G, and set $G'' = G/G'$. We have an epimorphism $\mathbb{Z}[G] \to \mathbb{Z}[G'']$ inducing epimorphisms $I[G]^k \to I[G'']^k$ and $A_k(G) \to A_k(G'')$ for all $k \ge 1$. We denote the kernels of these maps by $\mathbb{Z}[G,G']$, $I^k[G,G']$, and $A_k(G,G')$. For $k = 1$, $A_1(G,G')$ is just the image of G' under the isomorphism $\omega\colon G \to A_1(G)$.

Lemma 3.7 *Let $G' \le G$ and $a \in A_k(G)$ ($1 \le k \le d$). Let $\Omega(a) = \sum_{H\in\mathcal{C}^\star(G)} a_H H$. Then $a \in A_k(G,G')$ iff $a_H = 0$ whenever $H \ge G'$.*

Proof Let $G'' = G/G'$. There is a linear map $\mathbb{Z}_2^{\mathcal{C}^\star(G)} \to \mathbb{Z}_2^{\mathcal{C}^\star(G'')}$ sending $H \in \mathcal{C}^\star(G)$ to H/G' if $H \ge G'$, and to zero otherwise; its kernel consists of all $\sum_{H\in\mathcal{C}^\star(G)} a_H H$ such that $a_H = 0$ whenever $H \ge G'$. We also have the algebra homomorphism $\Omega''\colon B(G'') \to \mathbb{Z}_2^{\mathcal{C}^\star(G'')}$. Restricting to $A_k(G)$, we have a commutative diagram

$$\begin{array}{ccc} A_k(G) & \xrightarrow{\ \Omega\ } & \mathbb{Z}_2^{\mathcal{C}^\star(G)} \\ \downarrow & & \downarrow \\ A_k(G'') & \xrightarrow{\ \Omega''\ } & \mathbb{Z}_2^{\mathcal{C}^\star(G'')}. \end{array}$$

By Lemma 3.5, Ω'' is injective, and the result follows. ■

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4 Homology Groups of Colored Graphs

Let Γ be a $G(d)$-colored graph ($d \geq 2$), and fix a triangulation of Γ. In this section we study chain complexes $C(\Gamma \mid k)$ (for $k = 1, 2, \dots, d$) of $\mathbb{Z}_2$ vector spaces associated to this triangulation. (Why we should want to do this will emerge in later sections.) Let σ be a simplex of Γ. If σ is a vertex of Γ, we let G_σ be the subgroup of G generated by the colors of the edges of Γ incident to σ, which is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. If σ is any other simplex of Γ, then σ is contained in a unique edge of Γ, and we let g_σ be the color of this edge, and G_σ the subgroup of G generated by g_σ. We also set $A_k^\sigma = A_k(G, G_\sigma)$. We let $C(\Gamma \mid k)$ be the subcomplex of $C(\Gamma; A_k)$ generated by all chains of the form $a\sigma$ where σ is a simplex of Γ and $a \in A_k^\sigma$. (This is a subcomplex because if τ is a face of σ then $A_k^\sigma \leq A_k^\tau$.) We let $b_i(\Gamma \mid k)$ be the dimension of the i-th homology group $H_i(\Gamma \mid k)$ of $C(\Gamma \mid k)$. Of course, the homology groups are zero except in dimensions 0 and 1, and $H_1(\Gamma \mid k)$ is equal to the space $Z_1(\Gamma \mid k)$ of 1-cycles. We let $\chi(\Gamma \mid k) = b_0(\Gamma \mid k) - b_1(\Gamma \mid k)$ be the $\mathbb{Z}_2$ Euler characteristic of $C(\Gamma \mid k)$. It is clear that the homology of $C(\Gamma \mid k)$ is unchanged by subdivision, and therefore independent of the triangulation.

Lemma 4.1 *We have $\chi(\Gamma \mid k) = -\binom{d-2}{k-1}\chi(\Gamma)$. (In the case $k = d$ we are using the convention that $\binom{n}{r} = 0$ for $r > n$.)*

Proof By Lemma 3.3, the dimension of A_k^σ is $a = \sum_{i=1}^{k}\left(\binom{d}{i} - \binom{d-2}{i}\right)$ if σ is a vertex of Γ, and $b = \sum_{i=1}^{k}\left(\binom{d}{i} - \binom{d-1}{i}\right)$ otherwise. Therefore $\chi(\Gamma \mid k) = b\chi(\Gamma) + (a - b)V$, where V is the number of vertices of Γ. Since Γ is trivalent, $V = -2\chi(\Gamma)$, so $\chi(\Gamma \mid k) = -(2a - 3b)\chi(\Gamma)$, and it is easy to compute that $2a - 3b = \binom{d-2}{k-1}$. ■

Lemma 4.2 *We have $b_0(\Gamma \mid 1) = b_1(\Gamma)$ and $b_1(\Gamma \mid 1) = b_0(\Gamma)$.*

Proof Let $a = \sum_{\sigma \in S_1(\Gamma)} a_\sigma \sigma$ ($a_\sigma \in A_1^\sigma$) be a 1-chain of $C(\Gamma \mid 1)$. For each 1-simplex σ of Γ, $A_1^\sigma \cong G_\sigma \cong \mathbb{Z}_2$, with non-trivial element $\omega(g_\sigma)$. If g_1, g_2, and g_3 are the colors of three edges meeting at a vertex, $\omega(g_1) + \omega(g_2) + \omega(g_3) = \omega(g_1 g_2 g_3) = 0$. It follows that a is a cycle iff, for each component Γ' of Γ, the a_σ for 1-simplices σ of Γ' are either all zero or all non-zero. This proves that $H_1(\Gamma \mid 1) \cong \mathbb{Z}_2^{b_0(\Gamma)}$, or $b_1(\Gamma \mid 1) = b_0(\Gamma)$, and it then follows from Lemma 4.1 that $b_0(\Gamma \mid 1) = b_1(\Gamma)$. ■

Lemma 4.3 *For $1 \leq k \leq d$, there is an injection $\iota_k \colon B_{k-1} \to Z_1(\Gamma \mid k)$ defined by $\iota_k(b) = \sum_{\sigma \in S_1(\Gamma)} \omega(g_\sigma) b\sigma$.*

Proof It is clear that the given formula defines a linear map from B_{k-1} to $C_1(\Gamma \mid k)$. Let $b \in B_{k-1}$, and let τ be a 0-simplex of Γ. If τ is not a vertex of Γ, it is clear that the coefficient of τ in $\partial\iota_k(b)$ is zero. If τ is a vertex and the adjacent edge-colors are g_1, g_2 and g_3, this coefficient is $\sum_{i=1}^{3} \omega(g_i) b = \omega(g_1 g_2 g_3) b = 0$. Thus $\iota_k(b)$ is a cycle. It remains to show that ι_k is injective; suppose that $\iota_k(b) = 0$. Then $\omega(g_\sigma) b = 0$ for every 1-simplex σ of Γ. Since the g_σ generate G, this implies that $ab = 0$ for every $a \in A_1$, and therefore for every $a \in A_d$. Now $\Omega(a) \cdot \Omega(b) = 1^{\mathcal{C}^*} \cdot \Omega(ab) = 0$. Since Ω maps A_d onto $\mathbb{Z}_2^{\mathcal{C}^*}$, it follows that $\Omega(b) = 0$; since Ω is injective on B_{k-1}, we have $b = 0$. ■

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We call Γ k–taut if ι_k is an isomorphism; by Lemma 3.3, this occurs iff $b_1(\Gamma \mid k) = \sum_{i=0}^{k-1} \binom{d}{i}$. By Lemma 4.2, Γ is 1-taut iff it is connected. To give examples of k-taut graphs for $k > 1$, we use a different description of the chain complex $C(\Gamma \mid k)$. For $1 \leq k \leq d$, the injection $\Omega\colon A_k \to \mathbb{Z}_2^{\mathcal{C}^\star}$ induces an injection $\Omega\colon C(\Gamma; A_k) \to C(\Gamma; \mathbb{Z}_2^{\mathcal{C}^\star})$, which is onto for $k = d$. We identify $C(\Gamma; \mathbb{Z}_2^{\mathcal{C}^\star})$ with $C(\Gamma; \mathbb{Z}_2)^{\mathcal{C}^\star}$. Since $\Omega([2]_1 a) = \Omega(a)$ and $[2]_1 A_k \leq A_{k+1}$, we have a chain of subcomplexes

$$\Omega C(\Gamma; A_1) \leq \Omega C(\Gamma; A_2) \leq \cdots \leq \Omega C(\Gamma; A_d) = C(\Gamma; \mathbb{Z}_2)^{\mathcal{C}^\star}.$$

A chain of $C(\Gamma; \mathbb{Z}_2)^{\mathcal{C}^\star}$ is of the form $\sum_{\sigma \in S(\Gamma), H \in \mathcal{C}^\star} a_{\sigma,H} \sigma H$ with coefficients $a_{\sigma,H}$ in $\mathbb{Z}_2$. It belongs to $\Omega C(\Gamma; A_k)$ iff, for each simplex σ, $\sum_{H \in \mathcal{C}^\star} a_{\sigma,H} H \in \Omega(A_k)$. We let $C'(\Gamma \mid k)$ be the subcomplex $\Omega C(\Gamma \mid k)$ of $\Omega C(\Gamma; A_k)$. For $1 \leq k \leq d-1$ and $a \in A_k$, we have $a \in A_k^\sigma$ iff $[2]_1 a \in A_{k+1}^\sigma$; it follows that $C'(\Gamma \mid k) = \Omega C(\Gamma; A_k) \cap C'(\Gamma \mid k+1)$. By Lemma 3.7, a chain $\sum_{\sigma \in S(\Gamma), H \in \mathcal{C}^\star} a_{\sigma,H} \sigma H \in C(\Gamma; \mathbb{Z}_2)^{\mathcal{C}^\star}$ belongs to $C'(\Gamma \mid d)$ iff $a_{\sigma,H} = 0$ whenever $H \geq G_\sigma$. Now $H \geq G_\sigma$ iff σ is not a simplex of Γ_H, so we may identify $C'(\Gamma \mid d)$ with $\bigoplus_{H \in \mathcal{C}^\star} C(\Gamma_H; \mathbb{Z}_2)$. It follows that a 1-chain $\sum_{\sigma,H} a_{\sigma,H} \sigma H$ of $C'(\Gamma \mid k)$ is a cycle iff, for each $H \in \mathcal{C}^\star$, $a_{\sigma,H}$ is constant on each component of Γ_H. We let $W(\Gamma \mid k)$ be the subspace of $Z_1'(\Gamma \mid k)$ consisting of all 1-chains of $C'(\Gamma \mid k)$ such that, for each $H \in \mathcal{C}^\star$, $a_{\sigma,H}$ is constant on all of Γ_H.

Lemma 4.4 *For $1 \leq k \leq d$, $W(\Gamma \mid k) = \Omega \iota_k (B_{k-1})$.*

Proof We first prove the case $k = d$. Ω maps B_{d-1} isomorphically onto $\mathbb{Z}_2^{\mathcal{C}^\star}$, and there is an isomorphism $\mathbb{Z}_2^{\mathcal{C}^\star} \to W(\Gamma \mid d)$ sending $\sum_{H \in \mathcal{C}^\star} b_H H$ to $\sum_{H \in \mathcal{C}^\star} \sum_{\sigma \in S_1(\Gamma_H)} b_H \sigma H$. We show that the composite is equal to $\Omega \iota_d$. If $b \in B_{d-1}$ and $\Omega(b) = \sum_{H \in \mathcal{C}^\star} b_H H$ then

$$\Omega \iota_d(b) = \sum_{\sigma \in S_1(\Gamma)} \Omega\big(\omega(g_\sigma) b\big) \sigma = \sum_{\sigma \in S_1(\Gamma), H \in \mathcal{C}^\star} \delta_H(g_\sigma) b_H \sigma H = \sum_{H \in \mathcal{C}^\star} \sum_{\sigma \in S_1(\Gamma_H)} b_H \sigma H,$$

and this case is proved.

Now let $k < d$. Since $W(\Gamma \mid k) = W(\Gamma \mid d) \cap C'(\Gamma \mid k)$, it is enough to prove that $\Omega \iota_k(B_{k-1}) = \Omega \iota_d(B_{d-1}) \cap C'(\Gamma \mid k)$. Suppose that $b_k \in B_{k-1}$ and $b_d \in B_{d-1}$ are such that $\Omega(b_k) = \Omega(b_d)$. Then

$$\Omega \iota_k(b_k) = \sum_{\sigma \in S_1(\Gamma)} \Omega\big(\omega(g_\sigma)\big) \Omega(b_k) \sigma = \sum_{\sigma \in S_1(\Gamma)} \Omega\big(\omega(g_\sigma)\big) \Omega(b_d) \sigma = \Omega \iota_d(b_d).$$

Since, for any $b \in B_{k-1}$, $[2]_1^{d-k} b \in B_{d-1}$ and $\Omega([2]_1^{d-k} b) = \Omega(b)$, it follows that $\Omega \iota_k(B_{k-1})$ is contained in $\Omega \iota_d(B_{d-1}) \cap C'(\Gamma \mid k)$. Conversely, for $b \in B_{d-1}$, we have $\Omega \iota_d(b) \in C'(\Gamma \mid k)$ iff, for each $\sigma \in S_1(\Gamma)$, $\Omega\big(\omega(g_\sigma) b\big) \in \Omega(A_k) = \Omega(B_{d-k-1})^\perp$ (using Lemma 3.6). For $b' \in B_{d-k-1}$, $\Omega\big(\omega(g_\sigma) b\big) \cdot \Omega(b') = \Omega(b) \cdot \Omega\big(\omega(g_\sigma) b'\big)$. Since the g_σ generate G, the elements $\omega(g_\sigma) b'$ generate $A_1 B_{d-k-1} = A_{d-k}$. Therefore $\Omega \iota_d(b) \in C'(\Gamma \mid k)$ iff $\Omega(b) \in \Omega(A_{d-k})^\perp = \Omega(B_{k-1})$, and the proof is complete. ∎

Thus Γ is k-taut iff $W(\Gamma \mid k) = Z_1'(\Gamma \mid k)$. Since $W(\Gamma \mid k) = W(\Gamma \mid k+1) \cap C'(\Gamma \mid k)$ for $k < d$, we have:

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Lemma 4.5 *For $1 \leq k < d$, if Γ is $(k+1)$-taut then it is k-taut.* ∎

Since $C'(\Gamma \mid d) = \bigoplus_{H \in \mathcal{C}^\star} C(\Gamma_H; \mathbb{Z}_2)$, we have:

Lemma 4.6 *A $G(d)$-colored graph Γ is d-taut iff Γ_H is connected for all $H \in \mathcal{C}^\star$.* ∎

Thus the $G(d)$-colored graphs of Theorems 8.7 and 8.8 are d-taut. Those of Theorems 8.1, 8.2 and 8.3 are not, in general, but they are $(d-1)$-taut. For Theorem 8.1 this is clear; for the remaining cases we need the following description of $C'(\Gamma \mid d-1)$.

Lemma 4.7 *Let $a = \sum_{\sigma \in S(\Gamma), H \in \mathcal{C}^\star} a_{\sigma,H} \sigma H$ be an element of $C'(\Gamma \mid d)$. Then $a \in C'(\Gamma \mid d-1)$ iff $\sum_{H \in \mathcal{C}^\star} a_{\sigma,H} = 0$ for all $\sigma \in S(\Gamma)$.*

Proof We know that $a \in C'(\Gamma \mid d-1)$ iff $\sum_{H \in \mathcal{C}^\star} a_{\sigma,H} H \in \Omega(A_{d-1})$ for all $\sigma \in S(\Gamma)$. By Lemma 3.6, $\Omega(A_{d-1}) = \Omega(B_0)^\perp$. Now B_0 is generated by $[1]_0$ and

$$\Omega([1]_0) \cdot \sum_{H \in \mathcal{C}^\star} a_{\sigma,H} H = \sum_{H \in \mathcal{C}^\star} a_{\sigma,H}.$$

The result follows. ∎

If a $G(d)$-colored graph Γ is $(d-1)$-taut, we shall say simply that Γ is taut.

Lemma 4.8 *If $d = 3$ and Γ has an unsplittable G-coloring with a special circuit, then Γ is taut.*

Proof Since Γ is simple, we may use the natural triangulation in which the 0-simplices are the vertices and the 1-simplices are the edges. Let $H_0 \in \mathcal{C}^\star$ be such that $\Gamma_0 = \Gamma_{H_0}$ is a special circuit, and let the non-trivial elements of H_0 be h_1, h_2 and h_3. The remaining elements of $\mathcal{C}^\star$ fall into three pairs depending on their intersections with H_0; we let H_i and H_i' be those for which $H_i \cap H_0 = \langle h_i \rangle = H_i' \cap H_0$. We also let $\Gamma_i = \Gamma_{H_i}$ and $\Gamma_i' = \Gamma_{H_i'}$. Let $\sum_{\sigma,H} a(\sigma, H)\sigma H$ be any 1-cycle of $C'(\Gamma \mid 2)$, the sum being over edges σ and $H \in \mathcal{C}^\star$. Since Γ_0 is connected, $a(\sigma, H_0)$ is constant on Γ_0. For notational simplicity, we show only that $a(\sigma, H_1)$ is constant on Γ_1.

Let S be the set of all edges colored h_3. If $\sigma \in S$, we have $a(\sigma, H_0) = a(\sigma, H_3) = a(\sigma, H_3') = 0$ and

$$a(\sigma, H_1) + a(\sigma, H_1') + a(\sigma, H_2) + a(\sigma, H_2') = 0. \tag{4.9}$$

Define an equivalence relation $\sim$ on S by setting $\sigma_1 \sim \sigma_2$ if

$$a(\sigma_1, H_1) + a(\sigma_1, H_1') = a(\sigma_2, H_1) + a(\sigma_2, H_1').$$

Suppose that σ_1 and σ_2 are in S and each have a vertex in common with an edge τ of $\Gamma \setminus \Gamma_0$. If the color of τ is h_2 then σ_1 and σ_2 lie in the same component of Γ_1, and in the same component of Γ_1'. Therefore $a(\sigma_1, H_1) = a(\sigma_2, H_1)$ and $a(\sigma_1, H_1') = a(\sigma_2, H_1')$, so $\sigma_1 \sim \sigma_2$. Now, by (4.9), $\sigma_1 \sim \sigma_2$ iff

$$a(\sigma_1, H_2) + a(\sigma_1, H_2') = a(\sigma_2, H_2) + a(\sigma_2, H_2'),$$

and it follows similarly that $\sigma_1 \sim \sigma_2$ if τ has color h_1. Since $\Gamma \setminus \Gamma_0$ is connected, it follows that $\sigma_1 \sim \sigma_2$ for any σ_1 and σ_2 in S.

Now define an equivalence relation $\approx$ on S by setting $\sigma_1 \approx \sigma_2$ if $a(\sigma_1, H_1) = a(\sigma_2, H_1)$. If σ_1 and σ_2 belong to the same component of Γ_1 then $\sigma_1 \approx \sigma_2$. Since $\sigma_1 \sim \sigma_2$, we have $\sigma_1 \approx \sigma_2$ iff $a(\sigma_1, H_1') = a(\sigma_2, H_1')$, and so $\sigma_1 \approx \sigma_2$ if σ_1 and σ_2 belong to the same component of Γ_1'. Now $\Gamma_1 \cup \Gamma_1'$ is the result of deleting all edges colored h_1 from Γ, which is connected since Γ is unsplittable. It follows that $\sigma_1 \approx \sigma_2$ for all σ_1 and σ_2 in S; *i.e.*, that $a(\sigma, H_1)$ is constant on S. Now any component of Γ_1 contains an edge of S, so $a(\sigma, H_1)$ is constant on Γ_1. ■

Lemma 4.10 *Let $d = 3$, and let Γ be a Möbius ladder with a G-coloring in which the product of the colors on the rungs is non-trivial. Then Γ is taut.*

Proof We make $\mathcal{C}$ into an (additive) abelian group by setting $H + K = \operatorname{Ker}(\delta_H + \delta_K)$. For $H \in \mathcal{C}$, we let $w_H = \sum_{\sigma \in S_1(\Gamma_H)} \sigma = \sum_{\sigma \in S_1(\Gamma)} \delta_H(g_\sigma)\sigma \in Z_1(\Gamma; \mathbb{Z}_2)$. Then $w_{H+K} = w_H + w_K$, and the w_H form a subgroup of $Z_1(\Gamma; \mathbb{Z}_2)$ isomorphic to $\mathcal{C}$. A 1-cycle of $C'(\Gamma \mid d)$ may be written in the form $z = \sum_{H \in \mathcal{C}^\star} z_H H$, with $z_H \in Z_1(\Gamma_H; \mathbb{Z}_2)$. Then (by Lemma 4.7) z is in $C'(\Gamma \mid d-1)$ iff $\sum_{H \in \mathcal{C}^\star} z_H = 0$ (the sum being taken in $Z_1(\Gamma; \mathbb{Z}_2)$), while z is in $W(\Gamma \mid d)$ iff each z_H is a multiple of w_H. Therefore Γ is taut iff, given $z_H \in Z_1(\Gamma_H)$ for $H \in \mathcal{C}^\star$ with $\sum_{H \in \mathcal{C}^\star} z_H = 0$, each z_H is a multiple of w_H; since Γ_G is empty, we may replace $\mathcal{C}^\star$ by $\mathcal{C}$ in this statement.

Now let g_0 be the product of the colors on the rungs. If the color of a rim edge σ_i is g, then the color of the edge σ_{i+n} (where n is the number of rungs) is gg_0, so $g \neq g_0$. Since G is generated by the colors on the rungs and a single edge of the rim, at least two distinct elements appear as rung colors. Thus the edges colored g_0 form a proper subset of the rungs, and deleting them leaves a Möbius ladder Γ'. The G-coloring of Γ induces a G'-coloring of Γ', where $G' = G/\langle g_0 \rangle \cong \mathbb{Z}_2^2$. Since Γ' is connected, it is taut. Let $\mathcal{C}'$ be the set of $H \in \mathcal{C}$ such that $g_0 \in H$. There is a bijection $\mathcal{C}' \to \mathcal{C}(G')$ given by $H \mapsto H' = H/\langle g_0 \rangle$, and $\Gamma'_{H'} = \Gamma_H$. By Lemma 1.7, Γ_H contains an even number of rungs iff $H \in \mathcal{C}'$. Suppose now that $z_H \in Z_1(\Gamma_H; \mathbb{Z}_2)$ for $H \in \mathcal{C}$ and $\sum_{H \in \mathcal{C}} z_H = 0$, and set $z = \sum_{H \in \mathcal{C}'} z_H = \sum_{H \in \mathcal{C} - \mathcal{C}'} z_H$. If $H \notin \mathcal{C}'$, Γ_H is connected, so z_H is automatically a multiple of w_H. It follows that z is equal to w_K for some $K \in \mathcal{C}$. For $H \in \mathcal{C}'$, each component of Γ_H contains zero or two rungs, and so the sum of the coefficients of the rungs in z_H is zero. Therefore the same is true of w_K, which implies that $K \in \mathcal{C}'$. Now $(z_K + w_K) + \sum_{H \in \mathcal{C}' - \{K\}} z_H = 0$, and it follows from the tautness of Γ' that z_H is a multiple of w_H for $H \in \mathcal{C}'$ as well. Therefore Γ is taut. ■

Much of the approach outlined in Section 2 goes through for any taut G-colored graph, but not for non-taut graphs. This raises the question of how extensive the class of taut graphs is. For Möbius ladders, one can determine all the taut colorings. If $d = 3$ then any taut coloring satisfies the hypothesis of Theorem 8.3, apart from the exceptional coloring of the 4-rung ladder in Example 1.4. (Actually, there is a coloring of the 3-rung ladder for which the product of the colors on the rungs is 1, but an automorphism of the graph takes it to one for which the product is non-trivial.) For $d = 4$, apart from the colorings of Example 1.6, all taut colorings are obtained as follows. Suppose that $n \geq 4$, and let $\{x_1, x_2, x_3, x_4\}$ be a basis of G. Give the colors x_1, x_2, x_3, and $x_1x_2x_3x_4^{n-1}$ to one rung each, and give all other rungs the color x_4. It is possible to complete the coloring, and the result is taut (but not 4-taut). For $d \geq 5$, there is no taut coloring of any Möbius ladder.

Also, the operation of Figure 1 takes taut graphs to taut graphs, and so generates infinitely many further examples for $d \leq 5$; I know of no taut graphs for $d \geq 6$. Even for taut graphs, one encounters some difficulties which will be discussed after Lemma 6.6, and which I have been unable to overcome for the added examples just mentioned.

5 The Chain Complex $\operatorname{Ker}\beta / \operatorname{Im}\alpha$

We now return to the consideration of a regular branched covering $\pi\colon \tilde{M} \to M$ of a homology 3-sphere, with deck group G and branch set a G-colored graph Γ, and of the chain maps α, β and γ defined in Section 2. For each simplex σ of M, choose a lift $\tilde{\sigma}$ of σ to $\tilde{M}$, and for $H \in \mathcal{C}$ let $\sigma_H = \pi_H(\tilde{\sigma})$. (In particular, $\sigma_G = \sigma$.) Let G_σ be the stabilizer of $\tilde{\sigma}$, and let $A_k^\sigma = A_k(G, G_\sigma)$. If σ is a simplex of Γ, these definitions agree with those of the previous section; otherwise, $G_\sigma = 1$ and $A_k^\sigma = 0$. Also let $\mathcal{C}_\sigma$ be the set of $H \in \mathcal{C}$ such that $H \geq G_\sigma$, and $\mathcal{C}_\sigma^\star = \mathcal{C}_\sigma - \{G\}$.

Lemma 5.1 *$\bigl(\bigoplus_{H\in\mathcal{C}} C(M_H)\bigr) / \operatorname{Im}\alpha$ is generated by the $\sigma_H H$ for $\sigma \in S(M)$ and $H \in \mathcal{C}$, and $\operatorname{Im}\beta$ is generated by the $\pi_H^!(\sigma_H)$.*

Proof If $H = G$ or $H \notin \mathcal{C}_\sigma$, then σ_H is the unique lift of σ to M_H, while if $H \in \mathcal{C}_\sigma^\star$ there is one other lift σ'_H of σ to M_H. In the last case, $\alpha(\sigma H) = (\sigma_H + \sigma'_H)H - \sigma_G G$. This gives the first statement, and the second follows since $\operatorname{Im}\alpha \leq \operatorname{Ker}\beta$. ■

For $\sigma \in S(M)$ and $g \in G$, the simplex $g\tilde{\sigma}$ of $\tilde{M}$ depends only on the image of g in G/G_σ. We fix once and for all a right inverse for the projection $G \to G/G_\sigma$, and thereby identify G/G_σ with a complement of G_σ in G. A basis for $C(\tilde{M})$ is given by all $g\tilde{\sigma}$ for $\sigma \in S(M)$ and $g \in G/G_\sigma$. Note that there is a bijection $\mathcal{C}_\sigma \to \mathcal{C}(G/G_\sigma)$, namely $H \mapsto H/G_\sigma$.

Lemma 5.2 *For each $\sigma \in S(M)$, the elements $\pi_H^!(\sigma_H) \in C(\tilde{M})$ for $H \in \mathcal{C}_\sigma$ are linearly independent. For $H \in \mathcal{C} - \mathcal{C}_\sigma$, $2\pi_H^!(\sigma_H) = \pi_G^!(\sigma_G)$.*

Proof For $H \in \mathcal{C}_\sigma$, we have

$$\pi_H^!(\sigma_H) = \sum_{h\in H} h\tilde{\sigma} = |G_\sigma| \sum_{h\in H/G_\sigma} h\tilde{\sigma} = |G_\sigma| \sum_{g\in G/G_\sigma} \tfrac{1}{2}\bigl(\varepsilon_H(g) + 1\bigr) g\tilde{\sigma}.$$

Let T be the matrix with rows indexed by $H \in \mathcal{C}_\sigma$, columns indexed by $g \in G/G_\sigma$, and entries $\varepsilon_H(g)$, and let J be the matrix with all entries 1. To prove the first statement, we must show that $\det(T + J) \neq 0$. Now T is just the character table of G/G_σ, and the orthogonality relations show that $\det T \neq 0$ (in fact, that $\det T = \pm n^{n/2}$ where $n = |G/G_\sigma|$). Expand

$\det(T+J)$ by multilinearity in the rows. Since the row of T corresponding to $G \in \mathcal{C}_\sigma$ consists entirely of ones, all but two of the terms are zero, and the remaining two are equal to $\det T$, so $\det(T+J) = 2\det T \neq 0$.

Now let $H \in \mathcal{C} - \mathcal{C}_\sigma$. Then $\rho_H^!(\sigma_G) = 2\sigma_H$, so $\pi_G^!(\sigma_G) = \pi_H^!\rho_H^!(\sigma_G) = 2\pi_H^!(\sigma_H)$. ■

Lemma 5.3 *The chain complex* $\operatorname{Ker}\beta/\operatorname{Im}\alpha$ *is isomorphic to* $C'(\Gamma \mid d-1)$.

Proof The complex $C'(\Gamma \mid d-1)$ was defined as a subcomplex of $C(\Gamma;\mathbb{Z}_2)^{\mathcal{C}^\star}$, which is in turn a subcomplex of $C(M;\mathbb{Z}_2)^{\mathcal{C}^\star}$. As a subcomplex of $C(M;\mathbb{Z}_2)^{\mathcal{C}^\star}$, $C'(\Gamma \mid d-1)$ consists of those chains $\sum_{\sigma\in S(M), H\in\mathcal{C}^\star} a_{\sigma,H}\sigma H$ such that, for each σ, $\sum_{H\in\mathcal{C}^\star} a_{\sigma,H} = 0$ and $a_{\sigma,H} = 0$ if $H \in \mathcal{C}_\sigma^\star$ (because these equations imply that $a_{\sigma,H} = 0$ whenever σ is not in Γ).

Let $\bar\rho_H \colon C(M_H) \to C(M;\mathbb{Z}_2)$ be the composite of $\rho_H \colon C(M_H) \to C(M)$ and reduction of the coefficients modulo 2. Define $\zeta\colon \bigoplus_{H\in\mathcal{C}} C(M_H) \to C(M;\mathbb{Z}_2)^{\mathcal{C}^\star}$ by

$$\zeta\Big(\sum_{H\in\mathcal{C}} d_H H\Big) = \sum_{H\in\mathcal{C}^\star} \bar\rho_H(d_H)H \quad \text{for } d_H \in C(M_H), H \in \mathcal{C}.$$

Then

$$\zeta\alpha\Big(\sum_{H\in\mathcal{C}^\star} c_H H\Big) = \sum_{H\in\mathcal{C}^\star} \bar\rho_H\rho_H^!(c_H)H = 0 \quad \text{for } c_H \in C(M), H \in \mathcal{C}^\star.$$

Thus ζ induces a map from $\big(\bigoplus_{H\in\mathcal{C}} C(M_H)\big)/\operatorname{Im}\alpha$ to $C(M;\mathbb{Z}_2)^{\mathcal{C}^\star}$; we shall show that $\operatorname{Ker}\beta/\operatorname{Im}\alpha$ is mapped isomorphically to $C'(\Gamma \mid d-1)$. By Lemmas 5.1 and 5.2, any element of $\operatorname{Ker}\beta/\operatorname{Im}\alpha$ has a representative of the form

$$c = \sum_{\sigma\in S(M)} \Big(a_\sigma \sigma G + \sum_{H\notin\mathcal{C}_\sigma} b_{\sigma,H}\sigma_H H\Big) \quad \text{for } a_\sigma, b_{\sigma,H} \in \mathbb{Z},$$

and such an element is in $\operatorname{Ker}\beta$ iff $2a_\sigma + \sum_{H\notin\mathcal{C}_\sigma} b_{\sigma,H} = 0$ for each σ. It follows immediately that the image of $\operatorname{Ker}\beta/\operatorname{Im}\alpha$ in $C(M;\mathbb{Z}_2)^{\mathcal{C}^\star}$ is $C'(\Gamma \mid d-1)$. Further, the chain c is in $\operatorname{Ker}\zeta$ iff each $b_{\sigma,H}$ is even, and then

$$\begin{aligned}\alpha\Big(\sum_{\sigma\in S(M), H\notin\mathcal{C}_\sigma} \tfrac12 b_{\sigma,H}\sigma H\Big) &= \sum_{\sigma\in S(M), H\notin\mathcal{C}_\sigma} \Big(b_{\sigma,H}\sigma_H H - \tfrac12 b_{\sigma,H}\sigma G\Big)\\ &= \sum_{\sigma\in S(M)} \Big(a_\sigma\sigma G + \sum_{H\notin\mathcal{C}_\sigma} b_{\sigma,H}\sigma_H H\Big) = c\end{aligned}$$

provided $c \in \operatorname{Ker}\beta$. This completes the proof. ■

Lemma 5.4 *There is a short exact sequence*

$$0 \longrightarrow \mathbb{Z}^{\mathcal{C}^\star} \longrightarrow H_0(\operatorname{Ker}\beta) \longrightarrow H_0(\Gamma \mid d-1) \longrightarrow 0,$$

and $H_1(\operatorname{Ker}\beta) \cong H_1(\Gamma \mid d-1)$.

Proof By Lemma 5.3, the sequence (2.3) becomes

$$0 \longrightarrow C(M)^{\mathcal{C}^\star} \xrightarrow{\alpha} \operatorname{Ker}\beta \longrightarrow C'(\Gamma \mid d-1) \longrightarrow 0.$$

In the long exact homology sequence, the map $H_1(\Gamma \mid d-1) \to H_0(M)^{\mathcal{C}^\star}$ is zero since $H_1(\Gamma \mid d-1)$ is torsion and $H_0(M) \cong \mathbb{Z}$. Therefore the long exact sequence gives exact sequences

$$0 \longrightarrow H_1(\operatorname{Ker}\beta) \longrightarrow H_1(\Gamma \mid d-1) \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow \mathbb{Z}^{\mathcal{C}^\star} \longrightarrow H_0(\operatorname{Ker}\beta) \longrightarrow H_0(\Gamma \mid d-1) \longrightarrow 0. \qquad \blacksquare$$

We now turn to the sequence (2.4). Note that the map induced on first homology by the map $\beta\colon \bigoplus_{H\in\mathcal{C}} C(M_H) \to \operatorname{Im}\beta$ from that sequence may be regarded as a map from $\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)$ to $H_1(\operatorname{Im}\beta)$ since $H_1(M_G) = H_1(M) = 0$.

Lemma 5.5 *If Γ is taut, the map β from $\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)$ to $H_1(\operatorname{Im}\beta)$ is injective.*

Proof By Lemma 5.4, part of the long exact sequence of (2.4) becomes

$$H_1(\Gamma \mid d-1) \xrightarrow{\iota} \bigoplus_{H\in\mathcal{C}^\star} H_1(M_H) \xrightarrow{\beta} H_1(\operatorname{Im}\beta).$$

We must show that the map ι in this sequence is trivial. Any element of $H_1(\Gamma \mid d-1) = Z_1'(\Gamma \mid d-1)$ has the form $z = \sum_{H\in\mathcal{C}^\star} z_H H$, where $z_H \in Z_1(\Gamma_H;\mathbb{Z}_2)$ and $\sum_{H\in\mathcal{C}^\star} z_H = 0$ in $Z_1(\Gamma;\mathbb{Z}_2)$. Let $\bar\rho_H\colon C(M_H) \to C(M;\mathbb{Z}_2)$ be as in the proof of Lemma 5.3. The inverse image of Γ_H in M_H is a link L_H, and we may take $w_H \in Z_1(L_H) \leq Z_1(M_H)$ with $\bar\rho_H(w_H) = z_H$. Then $\sum_{H\in\mathcal{C}^\star} \bar\rho_H(w_H) = 0$, so there is an element w_G of $Z_1(M)$ with $2w_G + \sum_{H\in\mathcal{C}^\star} \rho_H(w_H) = 0$. Let $w = \sum_{H\in\mathcal{C}} w_H H \in \bigoplus_{H\in\mathcal{C}} Z_1(M_H)$. For $H \in \mathcal{C}^\star$, $\rho_H^!\rho_H(w_H) = 2w_H$ since w_H is in $Z_1(L_H)$. Therefore

$$0 = \pi^!\Big(2w_G + \sum_{H\in\mathcal{C}^\star} \rho_H(w_H)\Big) = 2\pi^!(w_G) + \sum_{H\in\mathcal{C}^\star} \pi_H^!\rho_H^!\rho_H(w_H) = 2\sum_{H\in\mathcal{C}} \pi_H^!(w_H) = 2\beta(w),$$

so $w \in \operatorname{Ker}\beta$. It follows from the proof of Lemma 5.3 that the element of $H_1(\operatorname{Ker}\beta)$ represented by w corresponds to z under the isomorphism $H_1(\operatorname{Ker}\beta) \cong H_1(\Gamma \mid d-1)$ of Lemma 5.4, and so $\iota(z)$ is the element of $\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)$ represented by w.

Since Γ is taut, for each $H \in \mathcal{C}^\star$, z_H is a multiple of the mod 2 fundamental class of Γ_H, and we may take w_H to be a multiple of the fundamental class of L_H for some orientation of L_H. Since L_H bounds a lift of a Seifert surface for Γ_H, w_H represents zero in $H_1(M_H)$, and so $\iota(z) = 0$ as required. $\blacksquare$

Lemma 5.6 *We have $H_0(\operatorname{Im}\beta) \cong \mathbb{Z}$.*

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Proof The end of the long exact sequence of (2.4) shows that

$$\beta\colon \bigoplus_{H\in\mathcal{C}} H_0(M_H) \to H_0(\operatorname{Im}\beta)$$

is surjective. In fact the restriction of β to $\bigoplus_{H\in\mathcal{C}^\star} H_0(M_H)$ is surjective since $\pi_G^!$ factors through $\pi_H^!$ for any $H \in \mathcal{C}^\star$. Let $H \in \mathcal{C}^\star$. For $\sigma \in S_0(M)$, the 0-simplices σ_H all represent the same generator of $H_0(M_H) \cong \mathbb{Z}$. The image x_H of this generator in $H_0(\operatorname{Im}\beta)$ is represented by $\pi_H^!(\sigma_H)$ for any $\sigma \in S_0(M)$. Define an equivalence relation on $\mathcal{C}^\star$ by setting $H \sim K$ if $x_H = x_K$. Suppose that there is some $\sigma \in S_0(M)$ such that neither H nor K is in $\mathcal{C}_\sigma$. Then, by Lemma 5.2, $\pi_H^!(\sigma_H) = \frac{1}{2}\pi_G^!(\sigma_G) = \pi_K^!(\sigma_K)$, and so $H \sim K$. Now suppose H_1 and H_2 are any two elements of $\mathcal{C}^\star$. For $i = 1$ or 2, there is some color $g_i \notin H_i$ (since the colors generate G), and a 0-simplex σ_i of Γ with $g_i \in G_{\sigma_i}$. Thus $H_i \notin \mathcal{C}_{\sigma_i}$. We may find $K \in \mathcal{C}^\star$ containing neither g_1 nor g_2. Then $K \notin \mathcal{C}_{\sigma_i}$, so $H_i \sim K$ for $i = 1$ or 2. Therefore $H_1 \sim H_2$, and there is only one equivalence class. This shows that $H_0(\operatorname{Im}\beta)$ is cyclic.

On the other hand, the image of x_H under the map $H_0(\operatorname{Im}\beta) \to H_0(\tilde{M}) \cong \mathbb{Z}$ induced by inclusion is 2^{d-1} times a generator, and therefore $H_0(\operatorname{Im}\beta) \cong \mathbb{Z}$. ■

Lemma 5.7 *If Γ is taut, there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{H\in\mathcal{C}^\star} H_1(M_H) \xrightarrow{\beta} H_1(\operatorname{Im}\beta) \longrightarrow \mathbb{Z}_2^{b_1(\Gamma)-d} \longrightarrow 0.$$

Proof By the previous lemma, part of the long exact sequence of (2.4) is

$$\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H) \xrightarrow{\beta} H_1(\operatorname{Im}\beta) \longrightarrow H_0(\operatorname{Ker}\beta) \xrightarrow{\iota} \mathbb{Z}^{\mathcal{C}} \longrightarrow \mathbb{Z} \longrightarrow 0, \tag{5.8}$$

and the first map is injective by Lemma 5.5. It remains to prove that $\operatorname{Ker}\iota \cong \mathbb{Z}_2^{b_1(\Gamma)-d}$. We show that there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{\mathcal{C}^\star} & \xrightarrow{\phi} & H_0(\operatorname{Ker}\beta) & \xrightarrow{\psi} & \mathbb{Z}_2^{b_0(\Gamma|d-1)} & \longrightarrow & 0\\
 & & {\scriptstyle\mathrm{id}}\downarrow & & {\scriptstyle\iota}\downarrow & & {\scriptstyle\theta}\downarrow & & \\
0 & \longrightarrow & \mathbb{Z}^{\mathcal{C}^\star} & \xrightarrow{\iota\phi} & \mathbb{Z}^{\mathcal{C}} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z}_2^{|\mathcal{C}|-2} & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

in which the rows and the central column are exact. The central column is part of (5.8), and the top row is the short exact sequence of Lemma 5.4. The composite $\iota\phi\colon \mathbb{Z}^{\mathcal{C}^\star} \to \mathbb{Z}^{\mathcal{C}}$ is the map $H_0(M)^{\mathcal{C}^\star} \to \bigoplus_{H\in\mathcal{C}} H_0(M_H)$ induced by α, so it is given by $\iota\phi(\sum_{H\in\mathcal{C}^\star} a_H H) =$

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$\sum_{H\in\mathcal{C}^\star} 2a_H H - (\sum_{H\in\mathcal{C}^\star} a_H)G$. This is injective and has cokernel isomorphic to $\mathbb{Z}\oplus\mathbb{Z}_2^{|\mathcal{C}|-2}$, so we obtain the exact second row. The maps in the right-hand column may now be defined to make the diagram commute. Diagram-chasing shows that the right-hand column is exact, so that $\operatorname{Im}\theta \cong \mathbb{Z}_2^{|\mathcal{C}|-2}$ and $\operatorname{Ker}\theta \cong \mathbb{Z}_2^{b_0(\Gamma|d-1)-|\mathcal{C}|+2}$. More diagram-chasing shows that ψ maps $\operatorname{Ker}\iota$ isomorphically onto $\operatorname{Ker}\theta$. Since Γ is taut, $b_1(\Gamma \mid d-1) = \dim B_{d-2} = 2^d - d - 1$ and Γ is connected. By Lemma 4.1, $\chi(\Gamma \mid d-1) = -\chi(\Gamma) = b_1(\Gamma) - 1$. Hence

$$b_0(\Gamma \mid d-1) - |\mathcal{C}| + 2 = 2^d - d - 1 + b_1(\Gamma) - 1 - 2^d + 2 = b_1(\Gamma) - d,$$

and we are done. ■

6 The Chain Complex $\operatorname{Ker}\gamma/\operatorname{Im}\beta$

We may regard $C(\tilde{M})$ as a $\mathbb{Z}[G]$-module. As such, it is generated (though not freely) by the $\tilde{\sigma}$ for $\sigma \in S(M)$. For $1 \le k \le d-1$, we let $D(k)$ be the subcomplex of $C(\tilde{M})$ consisting of chains $c = \sum_{\sigma\in S(M)} \lambda_\sigma\tilde{\sigma}$ ($\lambda_\sigma \in \mathbb{Z}[G]$) satisfying, for all σ,

$$\varepsilon_G(\lambda_\sigma) \equiv 0 \pmod{2^{d-1}} \quad \text{and}$$
$$\varepsilon_H(\lambda_\sigma) \equiv 0 \pmod{2^k} \quad \text{for } H \in \mathcal{C}^\star_\sigma.$$

This is well-defined because, for $\lambda \in \mathbb{Z}[G]$ and $\sigma \in S(M)$, the chain $\lambda\tilde{\sigma}$ determines the image $\bar{\lambda}$ of λ in $\mathbb{Z}[G/G_\sigma]$, and hence determines $\varepsilon_H(\lambda) = \varepsilon_{H/G_\sigma}(\bar{\lambda})$ for $H \in \mathcal{C}_\sigma$. By Lemma 3.2, $I^kC(\tilde{M}) \le D(k)$. Recall that we have identified G/G_σ with a subgroup of G, and hence $\mathbb{Z}[G/G_\sigma]$ with a subring of $\mathbb{Z}[G]$.

Lemma 6.1 *We have* $\operatorname{Ker}\gamma = D(1)$ *and* $\operatorname{Im}\beta = D(d-1)$.

Proof From the definition of γ, $\sum_{\sigma\in S(M)} \lambda_\sigma\tilde{\sigma} \in \operatorname{Ker}\gamma$ iff $\varepsilon_G(\lambda_\sigma) \equiv 0 \pmod{2^{d-1}}$ for each σ. Since $\varepsilon_H(\lambda) \equiv \varepsilon_G(\lambda) \pmod 2$ for all $H \in \mathcal{C}$ and $\lambda \in \mathbb{Z}[G]$, it follows that $\operatorname{Ker}\gamma = D(1)$. For $\sigma \in S(M)$, let $(\operatorname{Im}\beta)_\sigma = \operatorname{Im}\beta \cap \mathbb{Z}[G]\tilde{\sigma}$. To show that $\operatorname{Im}\beta = D(d-1)$, it is enough to show that $\lambda\tilde{\sigma} \in (\operatorname{Im}\beta)_\sigma$ iff $\varepsilon_H(\lambda) \equiv 0 \pmod{2^{d-1}}$ for all $H \in \mathcal{C}_\sigma$. We may assume that $\lambda = \sum_{g\in G/G_\sigma} \lambda_g g \in \mathbb{Z}[G/G_\sigma]$, and so $\varepsilon_H(\lambda) = \varepsilon_{H/G_\sigma}(\lambda)$ for $H \in \mathcal{C}_\sigma$. Consider the chain $\sum_{H\in\mathcal{C}_\sigma} \varepsilon_H(\lambda)\pi^!_H(\sigma_H) \in C(\tilde{M})$. We have

$$\begin{aligned}
\sum_{H\in\mathcal{C}_\sigma} \varepsilon_H(\lambda)\pi^!_H(\sigma_H) &= \sum_{H\in\mathcal{C}_\sigma} \varepsilon_{H/G_\sigma}(\lambda)|G_\sigma| \sum_{h\in H/G_\sigma} h\tilde{\sigma} \\
&= |G_\sigma| \sum_{g\in G/G_\sigma, H\in\mathcal{C}_\sigma} \varepsilon_{H/G_\sigma}(\lambda)\tfrac{1}{2}\big(\varepsilon_{H/G_\sigma}(g) + 1\big)g\tilde{\sigma} \\
&= \tfrac{1}{2}|G_\sigma| \sum_{g\in G/G_\sigma} \Big(\sum_{H\in\mathcal{C}_\sigma} \big(\varepsilon_{H/G_\sigma}(\lambda g) + \varepsilon_{H/G_\sigma}(\lambda)\big)\Big)g\tilde{\sigma} \\
&= \tfrac{1}{2}|G_\sigma| \sum_{g\in G/G_\sigma} |G/G_\sigma|(\lambda_g + \lambda_1)g\tilde{\sigma}.
\end{aligned}$$

That is,

$$\sum_{H\in\mathcal{C}_\sigma} \varepsilon_H(\lambda)\pi_H^!(\sigma_H) = 2^{d-1}\lambda\tilde{\sigma} + \tfrac{1}{2}|G/G_\sigma|\lambda_1\pi_G^!(\sigma_G). \tag{6.2}$$

Suppose first that $G_\sigma = 1$. Then $\mathcal{C}_\sigma = \mathcal{C}$, and by Lemmas 5.1 and 5.2, a basis for $(\operatorname{Im}\beta)_\sigma$ consists of the $\pi_H^!(\sigma_H)$ for $H \in \mathcal{C}$. In this case, (6.2) gives

$$\sum_{H\in\mathcal{C}} \varepsilon_H(\lambda)\pi_H^!(\sigma_H) = 2^{d-1}\bigl(\lambda\tilde{\sigma} + \lambda_1\pi_G^!(\sigma_G)\bigr),$$

and it follows that $\lambda\tilde{\sigma} \in (\operatorname{Im}\beta)_\sigma$ iff $\varepsilon_H(\lambda) \equiv 0 \pmod{2^{d-1}}$ for all $H \in \mathcal{C}$. Now suppose that $G_\sigma \neq 1$. Then a basis for $(\operatorname{Im}\beta)_\sigma$ consists of the $\pi_H^!(\sigma_H)$ for $H \in \mathcal{C}_\sigma^\star$ and $\pi_{H_0}^!(\sigma_{H_0})$ for any one $H_0 \in \mathcal{C} - \mathcal{C}_\sigma$, and (6.2) gives

$$2\varepsilon_G(\lambda)\pi_{H_0}^!(\sigma_{H_0}) + \sum_{H\in\mathcal{C}_\sigma^\star} \varepsilon_H(\lambda)\pi_H^!(\sigma_H) = 2^{d-1}\lambda\tilde{\sigma} + |G/G_\sigma|\lambda_1\pi_{H_0}^!(\sigma_{H_0}).$$

In this case, $\lambda\tilde{\sigma} \in (\operatorname{Im}\beta)_\sigma$ iff $\varepsilon_H(\lambda) \equiv 0 \pmod{2^{d-1}}$ for all $H \in \mathcal{C}_\sigma^\star$ and $2\varepsilon_G(\lambda) \equiv |G/G_\sigma|\lambda_1 \pmod{2^{d-1}}$. But $|G/G_\sigma|\lambda_1 = \sum_{H\in\mathcal{C}_\sigma} \varepsilon_H(\lambda)$, so this is true iff $\varepsilon_H(\lambda) \equiv 0 \pmod{2^{d-1}}$ for all $H \in \mathcal{C}_\sigma$, as required. ∎

Thus we have a filtration $\operatorname{Im}\beta = D(d-1) \leq \cdots \leq D(1) = \operatorname{Ker}\gamma$, and instead of dealing directly with the complex $\operatorname{Ker}\gamma/\operatorname{Im}\beta$, we consider the quotients of this filtration.

The following notation will be used in the proofs of the next lemma and Lemma 6.5. Let $\sigma \in S(M)$, and let $\partial\sigma = \sum_{\tau\in S(M)} i_{\sigma,\tau}\tau$. Thus $i_{\sigma,\tau} = \pm 1$ if τ is a face of σ, and $i_{\sigma,\tau} = 0$ otherwise. If τ is a face of σ, there is a unique element $g_{\sigma,\tau}$ of $G/G_\tau \leq G$ such that $g_{\sigma,\tau}\tilde{\tau}$ is a face of $\tilde{\sigma}$; we set $g_{\sigma,\tau} = 1$ otherwise. Then $\partial\tilde{\sigma} = \sum_{\tau\in S(M)} i_{\sigma,\tau}g_{\sigma,\tau}\tilde{\tau}$.

Recall that $C(\Gamma \mid k)$ was defined as a subcomplex of $C(\Gamma; A_k)$, which is in turn a subcomplex of $C(M; A_k)$. $C(\Gamma \mid k)$ is the subcomplex of $C(M; A_k)$ generated by all chains $a\sigma$ where σ is a simplex of M and $a \in A_k^\sigma$, because $A_k^\sigma = 0$ if σ is not a simplex of Γ.

Lemma 6.3 *For $1 \leq k \leq d-2$, we have $D(k)/D(k+1) \cong C(M; A_k)/C(\Gamma \mid k)$.*

Proof Since $C(M; \mathbb{Z}[G])$ is the free $\mathbb{Z}[G]$-module on the simplices of M, there is a unique $\mathbb{Z}[G]$-module homomorphism η from $C(M; \mathbb{Z}[G])$ to $C(\tilde{M})$ sending $\sigma \in S(M)$ to $\tilde{\sigma}$; of course, η is not a chain map. Nevertheless, its kernel is a subcomplex; it is generated by $\lambda\sigma$ for $\sigma \in S(M)$ and $\lambda \in \mathbb{Z}[G, G_\sigma]$. The subcomplex $C(M; I^k)$ is sent by η to $I^kC(\tilde{M}) \leq D(k)$; the kernel of $\eta \mid C(M; I^k)$ is the subcomplex $E(k)$ generated by $\lambda\sigma$ for $\sigma \in S(M)$ and $\lambda \in I^k[G, G_\sigma]$. For $1 \leq k \leq d-2$, we may identify $C(M; I^k)/C(M; I^{k+1})$ with $C(M; A_k)$, and $E(k)/E(k+1)$ with $C(\Gamma \mid k)$. Then we have an induced map $\bar{\eta}_k$ from $C(M; A_k)$ to $D(k)/D(k+1)$, whose kernel contains $C(\Gamma \mid k)$. For $\lambda \in I^k$ and $\sigma \in S(M)$, we have

$$(\eta\partial - \partial\eta)(\lambda\sigma) = \sum_{\tau\in S(M)} i_{\sigma,\tau}\lambda(1 - g_{\sigma,\tau})\tilde{\tau} \in I^{k+1}C(\tilde{M}) \leq D(k+1),$$

which shows that $\bar{\eta}_k$ is a chain map.

Suppose that $\lambda\tilde{\sigma} \in D(k)$. We may assume that $\lambda \in \mathbb{Z}[G/G_\sigma]$. For any $H \in \mathcal{C}$, there is some $H' \in \mathcal{C}_\sigma$ with $H \cap G/G_\sigma = H' \cap G/G_\sigma$, and so $\varepsilon_H(\lambda) = \varepsilon_{H'}(\lambda) \equiv 0 \pmod{2^k}$. By Lemma 3.2, $\lambda - \varepsilon_G(\lambda) \in I^k$; also $\eta\Big(\big(\lambda - \varepsilon_G(\lambda)\big)\sigma\Big) = \lambda\tilde{\sigma} - \varepsilon_G(\lambda)\tilde{\sigma}$. But $\varepsilon_G(\lambda) \equiv 0 \pmod{2^{d-1}}$, and so $\varepsilon_G(\lambda)\tilde{\sigma} \in D(k+1)$. Therefore $\bar{\eta}_k$ maps $C(M; A_k)$ onto $D(k)/D(k+1)$.

Next, suppose $\lambda \in I^k$ and $\sigma \in S(M)$ are such that $[\lambda]_k\sigma$ is in the kernel of $\bar{\eta}_k$; that is, $\lambda\tilde{\sigma} \in D(k+1)$. Take $\mu \in \mathbb{Z}[G/G_\sigma]$ so that $\mu\tilde{\sigma} = \lambda\tilde{\sigma}$. As before, for $H \in \mathcal{C}$, there is some $H' \in \mathcal{C}_\sigma$ with $H \cap G/G_\sigma = H' \cap G/G_\sigma$, and so $\varepsilon_H(\mu) = \varepsilon_{H'}(\mu) = \varepsilon_{H'}(\lambda) \equiv 0 \pmod{2^{k+1}}$. Also $\varepsilon_G(\mu) = \varepsilon_G(\lambda) = 0$, so it follows from Lemma 3.2 that $\mu \in I^{k+1}$. Since $\lambda\tilde{\sigma} = \mu\tilde{\sigma}$, $\lambda - \mu \in I^k[G, G_\sigma]$, so $[\lambda]_k = [\lambda - \mu]_k$ is in A_k^σ. It follows that the kernel of $\bar{\eta}_k$ is equal to $C(\Gamma \mid k)$, and so $\bar{\eta}_k$ induces the desired isomorphism of chain complexes from $C(M; A_k)/C(\Gamma \mid k)$ to $D(k)/D(k+1)$. ∎

Lemma 6.4 *For $1 \le k \le d-2$, we have $H_0\big(D(k)/D(k+1)\big) = 0$, $H_1\big(D(k)/D(k+1)\big) \cong \mathbb{Z}_2^{b_0(\Gamma|k) - \dim A_k}$, and $H_2\big(D(k)/D(k+1)\big) \cong H_1(\Gamma \mid k)$.*

Proof Lemma 6.3 gives a short exact sequence

$$0 \longrightarrow C(\Gamma \mid k) \longrightarrow C(M; A_k) \longrightarrow D(k)/D(k+1) \longrightarrow 0.$$

The long homology sequence gives exact sequences

$$0 \longrightarrow H_2\big(D(k)/D(k+1)\big) \longrightarrow H_1(\Gamma \mid k) \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow H_1\big(D(k)/D(k+1)\big) \longrightarrow H_0(\Gamma \mid k) \longrightarrow A_k \longrightarrow H_0\big(D(k)/D(k+1)\big) \longrightarrow 0.$$

The map $H_0(\Gamma \mid k) \to A_k$ in the second of these has image containing $\omega(g)b$ for any $b \in B_{k-1}$ and any $g \in G$ that appears as an edge color. Since the colors generate G, this map is onto, and the result follows. ∎

Lemma 6.5 *If $1 \le k \le d-2$ and Γ is k-taut, there is a short exact sequence*

$$0 \longrightarrow H_1\big(D(k+1)\big) \longrightarrow H_1\big(D(k)\big) \longrightarrow \mathbb{Z}_2^{\binom{d-2}{k-1}(b_1(\Gamma)-1) - \binom{d}{k} + 1} \longrightarrow 0.$$

Proof By Lemma 6.4, part of the long exact sequence of

$$0 \longrightarrow D(k+1) \longrightarrow D(k) \longrightarrow D(k)/D(k+1) \longrightarrow 0$$

is

$$\begin{aligned} H_2\big(D(k)\big) \xrightarrow{\phi_k} H_1(\Gamma \mid k) \longrightarrow H_1\big(D(k+1)\big) \longrightarrow H_1\big(D(k)\big) \xrightarrow{\psi_k} \mathbb{Z}_2^{b_0(\Gamma|k) - \dim A_k} \\ \longrightarrow H_0\big(D(k+1)\big) \longrightarrow H_0\big(D(k)\big) \longrightarrow 0, \end{aligned}$$

whether or not Γ is k-taut. Suppose that $H_0\big(D(k+1)\big) \cong \mathbb{Z}$. It follows that ψ_k is onto, and that $H_0\big(D(k)\big) \cong \mathbb{Z}$. Since $H_0\big(D(d-1)\big) \cong \mathbb{Z}$ by Lemmas 6.1 and 5.6, a downward induction on k shows that ψ_k is onto for $1 \le k \le d-2$. Now, since Γ is k-taut, $b_1(\Gamma \mid k) = \dim B_{k-1}$ and

$$\chi(\Gamma \mid k) = -\tbinom{d-2}{k-1}\chi(\Gamma) = \tbinom{d-2}{k-1}\big(b_1(\Gamma)-1\big),$$

and so

$$\begin{aligned} b_0(\Gamma \mid k) - \dim A_k &= \tbinom{d-2}{k-1}\big(b_1(\Gamma)-1\big) + \dim B_{k-1} - \dim A_k \\ &= \tbinom{d-2}{k-1}\big(b_1(\Gamma)-1\big) - \tbinom{d}{k} + 1. \end{aligned}$$

It only remains to prove that ϕ_k is onto.

In the rest of the proof, σ always denotes a 3-simplex of M, τ a 2-simplex, and υ a 1-simplex, so that, for example, $\sum_{\sigma,\tau}$ indicates a sum over $\sigma \in S_3(M)$ and $\tau \in S_2(M)$. We assume that the orientations of the 3-simplices of M are induced by an orientation of M, so that $c = \sum_\sigma \sigma$ represents a generator of $H_3(M)$. Consider the chain $\tilde{c} = \sum_\sigma \tilde{\sigma} \in C_3(\tilde{M})$. We have $\partial\tilde{c} \in \operatorname{Ker}\big(C_2(\tilde{M}) \to C_2(M)\big) = IC_2(\tilde{M})$. Let $\lambda \in J^{k-1}$. Then $\lambda I \le I^k$, so $\lambda\partial\tilde{c} \in I^k C_2(M) \le D_2(k)$, and the cycle $\lambda\partial\tilde{c}$ represents an element x of $H_2\big(D(k)\big)$. Now

$$\lambda\partial\tilde{c} = \lambda\sum_{\sigma,\tau} i_{\sigma,\tau}g_{\sigma,\tau}\tilde{\tau} = \eta\Big(\lambda\sum_{\sigma,\tau} i_{\sigma,\tau}g_{\sigma,\tau}\tau\Big),$$

where η is the map $C(M;\mathbb{Z}[G]) \to C(\tilde{M})$ from the proof of Lemma 6.3. It follows that $\phi_k(x)$ is the image in $C_1(M;A_k)$ of $c' = \partial(\lambda\sum_{\sigma,\tau} i_{\sigma,\tau}g_{\sigma,\tau}\tau)$. Now

$$c' = \lambda\sum_{\sigma,\tau,\upsilon} i_{\sigma,\tau}i_{\tau,\upsilon}g_{\sigma,\tau}\upsilon = \lambda\sum_{\upsilon}\mu_\upsilon\upsilon \quad \text{where} \quad \mu_\upsilon = \sum_{\sigma,\tau} i_{\sigma,\tau}i_{\tau,\upsilon}g_{\sigma,\tau} \in \mathbb{Z}[G].$$

Fix a 1-simplex υ of M. Let the 2-simplices of M having υ as a face be $\tau_1,\dots,\tau_n$, and the 3-simplices $\sigma_1,\dots,\sigma_n$. Let $\sigma_0 = \sigma_n$, and choose the numbering so that τ_j is a face of σ_{j-1} and σ_j for $1 \le j \le n$. Let $i_j = i_{\sigma_{j-1},\tau_j}i_{\tau_j,\upsilon} = \pm 1$. Then $i_{\sigma_j,\tau_j}i_{\tau_j,\upsilon} = -i_j$, so $\mu_\upsilon = \sum_{j=1}^n i_j(g_{\sigma_{j-1},\tau_j} - g_{\sigma_j,\tau_j}) \in I$. By Lemma 3.1, $[\mu_\upsilon]_1 = \omega(\prod_{j=1}^n g_{\sigma_{j-1},\tau_j}g_{\sigma_j,\tau_j})$. Considering a lift to $\tilde{M}$ of a meridian of υ, we see that $\prod_{j=1}^n g_{\sigma_{j-1},\tau_j}g_{\sigma_j,\tau_j}$ is the color g_υ if υ is a 1-simplex of Γ, and 1 otherwise. Therefore

$$\phi_k(x) = \sum_{\upsilon\in S_1(\Gamma)} [\lambda]_{k-1}\omega(g_\upsilon)\upsilon = \iota_k([\lambda]_{k-1}).$$

Since Γ is k-taut, this shows that ϕ_k is onto. ∎

Lemma 6.6 *If Γ is taut, there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{H\in\mathcal{C}^\star} H_1(M_H) \xrightarrow{\beta} H_1(\tilde{M}) \longrightarrow \Lambda \longrightarrow 0,$$

where Λ satisfies $2^{d-1}\Lambda = 0$ and $|\Lambda| = 2^m$ for $m = 2^{d-2}(b_1(\Gamma)-5)+d+1$.

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Proof Since $D(d-1) = \operatorname{Im}\beta$, $D(1) = \operatorname{Ker}\gamma$ and $H_1(\operatorname{Ker}\gamma) \cong H_1(\tilde{M})$, Lemmas 5.7 and 6.5 give an exact sequence as claimed with $2^{d-1}\Lambda = 0$ and $|\Lambda| = 2^m$ where m is the sum of $b_1(\Gamma) - d$ and $\binom{d-2}{k-1}\bigl(b_1(\Gamma)-1\bigr) - \binom{d}{k} + 1$ for $1 \le k \le d-2$. Since $\binom{d-2}{k-1}\bigl(b_1(\Gamma)-1\bigr) - \binom{d}{k} + 1$ is equal to $b_1(\Gamma) - d$ when $k = d-1$,

$$\begin{aligned} m &= \sum_{k=1}^{d-1} \Bigl(\tbinom{d-2}{k-1}\bigl(b_1(\Gamma)-1\bigr) - \tbinom{d}{k} + 1 \Bigr) \\ &= 2^{d-2}\bigl(b_1(\Gamma)-1\bigr) - 2^d + d + 1 = 2^{d-2}\bigl(b_1(\Gamma)-5\bigr) + d + 1. \end{aligned}$$ ∎

Even accepting the limitation to taut graphs, Lemma 6.6 is unsatisfactory in two respects. First, it gives incomplete information about the group Λ. (Theorem 8.2 and Proposition 1.3 show that, at least for $d = 3$, Λ may be any group satisfying the conditions of the lemma.) Second, it gives no information at all about the extension of $\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)$ by Λ. All the examples I know are consistent with the conjecture that $\beta\bigl(\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)\bigr) = 2^{d-1}H_1(\tilde{M})$ whenever Γ is taut, but I have been unable to prove this. The following lemma suffices in some cases.

Lemma 6.7 *Let Γ be taut and suppose that, for every $H \in \mathcal{C}^\star$ such that Γ_H is disconnected, the cover $\pi_H\colon \tilde{M} \to M_H$ can be factored through 2-fold covers $\tilde{M} = M_d \to \cdots \to M_2 \to M_1 = M_H$ so that each transfer map $H_1(M_i;\mathbb{Z}_2) \to H_1(M_{i+1};\mathbb{Z}_2)$ is trivial. Then $\beta\bigl(\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)\bigr) = 2^{d-1}H_1(\tilde{M})$.*

Proof Lemma 6.6 implies that $2^{d-1}H_1(\tilde{M}) \le \beta\bigl(\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)\bigr)$, so it is enough to show that $\pi_H^!\bigl(H_1(M_H)\bigr) \le 2^{d-1}H_1(\tilde{M})$ for all $H \in \mathcal{C}^\star$. If Γ_H is connected, $H_1(M_H)$ has odd order and there is nothing to prove. If Γ_H is disconnected and π_H is factored as above then the image of the transfer $H_1(M_i) \to H_1(M_{i+1})$ $(1 \le i < d)$ on integer homology is contained in $2H_1(M_{i+1})$, and the result follows. ∎

7 The Mod 2 Homology of 2- and 4-Fold Branched Covers

In this section, the coefficients for homology will always be $\mathbb{Z}_2$, and will be omitted from the notation. Let L be a link in a closed, connected, orientable 3-manifold N. There is a double cover of N with branch set L iff L represents zero in $H_1(N)$; suppose this is so. Let $\theta\colon H_1(N-L) \to \mathbb{Z}_2$ be a homomorphism sending each meridian of L to 1, and let $p\colon \tilde{N} \to N$ be the corresponding branched cover. We wish to allow the possibility that L is empty (*i.e.*, p is unbranched); in this case we insist that θ be onto, so that $\tilde{N}$ is connected. There is an intersection pairing $H_1(N-L) \times H_2(N,L) \to \mathbb{Z}_2$ inducing an isomorphism $H_2(N,L) \to \hom\bigl(H_1(N-L),\mathbb{Z}_2\bigr)$; we let $\theta' \in H_2(N,L)$ correspond to θ. There is also an intersection pairing $H_2(N) \times H_2(N,L) \to H_1(N,L)$, and we let $\theta''\colon H_2(N) \to H_1(N,L)$ be given by intersection with θ'.

The transfer map with $\mathbb{Z}_2$ coefficients, $p^!\colon C(N) \to C(\tilde{N})$, kills $C(L)$, so there is an induced map $p^!\colon C(N,L) \to C(\tilde{N})$. More generally, if X is any subcomplex of N and

$\tilde{X} = p^{-1}(X)$, there is a map $p^!\colon C(N, L \cup X) \to C(\tilde{N}, \tilde{X})$. It was observed by Lee and Weintraub [3, Theorem 1] that the sequence

$$0 \longrightarrow C(N, L \cup X) \xrightarrow{p^!} C(\tilde{N}, \tilde{X}) \xrightarrow{p} C(N, X) \longrightarrow 0 \tag{7.1}$$

is exact. When N is a $\mathbb{Z}_2$ homology sphere, it follows (taking $X = \varnothing$) that $p^!\colon H_1(N, L) \to H_1(\tilde{N})$ is an isomorphism, which gives a different proof of Sublemma 15.4 of [5], that $\dim H_1(\tilde{N}) = b_0(L) - 1$. The following lemma generalizes this to other manifolds.

Lemma 7.2 *In the above situation, let $n = \dim H_1(N)$, let r be the rank of the map $H_1(L) \to H_1(N)$ induced by inclusion, and let s be the rank of θ''. Then $r \le s \le n$, $\dim H_1(\tilde{N}) = b_0(L) - 1 + 2n - r - s$, and the rank of the map $p^!\colon H_1(N) \to H_1(\tilde{N})$ equals $n - s$.*

Note a special case of this lemma: if $H_1(L) \to H_1(N)$ is onto then $\dim H_1(\tilde{N}) = b_0(L) - 1$ and $p^!\colon H_1(N) \to H_1(\tilde{N})$ is the zero map.

Proof Certainly $s \le \dim H_2(N) = n$. To see that $r \le s$, consider the composite of θ'' and the connecting homomorphism $H_1(N, L) \to H_0(L)$. This is the map $H_2(N) \to H_0(L)$ given by intersection with L, which is dual to $H_1(L) \to H_1(N)$; therefore it has rank r.

From (7.1) with $X = \varnothing$ we get an exact sequence

$$H_2(\tilde{N}) \xrightarrow{p} H_2(N) \xrightarrow{\partial} H_1(N, L) \to H_1(\tilde{N}) \longrightarrow H_1(N) \longrightarrow H_0(N, L) \longrightarrow 0.$$

We claim that the connecting homomorphism labelled ∂ in this sequence is equal to θ''. We may take a (possibly non-orientable) surface F in N with boundary L representing $\theta' \in H_2(N, L)$. Then $\tilde{N}$ may be constructed by gluing together two copies of N cut open along F. Let $x \in H_2(N)$, and represent x by a surface F' transverse to F. Then $p^{-1}(F')$ is the union of two copies of F' cut open along $F \cap F'$. Either one of these carries a 2-chain mapping to F' under p, and their common boundary is the image under $p^!$ of the element of $C_1(N, L)$ carried by $F \cap F'$. Therefore $\partial(x)$ is represented by $F \cap F'$, which represents $\theta''(x)$ by the definition of θ'', and the claim is proved. It follows that $\dim H_1(\tilde{N}) = \dim H_1(N, L) + n - s - \dim H_0(N, L)$, and that the map $p\colon H_2(\tilde{N}) \to H_2(N)$ has rank $n - s$. Now the exact sequence

$$H_1(L) \longrightarrow H_1(N) \longrightarrow H_1(N, L) \longrightarrow H_0(L) \longrightarrow \mathbb{Z}_2 \longrightarrow H_0(N, L) \longrightarrow 0$$

shows that $\dim H_1(N, L) = b_0(L) - 1 + n - r + \dim H_0(N, L)$, so $\dim H_1(\tilde{N})$ is as claimed. Also, the map $p^!\colon H_1(N) \to H_1(\tilde{N})$ is dual to $p\colon H_2(\tilde{N}) \to H_2(N)$, and so has rank $n - s$. ■

Now let Γ be a $G(2)$-colored graph embedded in a $\mathbb{Z}_2$ homology sphere M. Just as when M is an integral homology sphere, this determines a branched covering $\pi\colon \tilde{M} \to M$. We let the non-trivial elements of G be g_1, g_2 and g_3, and set $H_i = \langle g_i \rangle \in \mathcal{C}^\star$. Where H_i would appear as a subscript, we just use i; thus we have 2-fold covers $\rho_i\colon M_i \to M$ branched over Γ_i and $\pi_i\colon \tilde{M} \to M_i$ branched over Δ_i for $1 \le i \le 3$. If $\tilde{\Gamma} = \pi^{-1}(\Gamma)$,

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the map $\pi\colon H_1(\tilde{M}-\tilde{\Gamma}) \to H_1(M-\Gamma)$ kills each meridian of $\tilde{\Gamma}$, so it induces a map $\bar{\pi}\colon H_1(\tilde{M}) \to H_1(M-\Gamma)$.

We wish to determine $\dim H_1(\tilde{M})$. We deal first with the case where Γ is connected, since here we need some additional information.

Lemma 7.3 *When Γ is connected, $\dim H_1(\tilde{M}) = b_1(\Gamma) - 2$ and $\pi_i^!\colon H_1(M_i) \to H_1(\tilde{M})$ is the zero map for $1 \le i \le 3$. Further, the map $\bar{\pi}\colon H_1(\tilde{M}) \to H_1(M-\Gamma)$ is injective.*

Proof Let $1 \le i \le 3$. Since M is a $\mathbb{Z}_2$ homology sphere, $\rho_i^!\colon H_1(M,\Gamma_i) \to H_1(M_i)$ is an isomorphism. Since Γ is connected, every element of $H_1(M,\Gamma_i)$ is represented by a chain of $\Gamma \setminus \Gamma_i$; since Δ_i is the inverse image of $\Gamma \setminus \Gamma_i$ in M_i, the map $H_1(\Delta_i) \to H_1(M_i)$ induced by inclusion is onto. Since Δ_i is a link of $b_1(\Gamma) - 1$ components, the first two claims follow from the special case of Lemma 7.2 noted above. The image of $\bar{\pi}$ is the kernel of the homomorphism $H_1(M-\Gamma) \to G$ corresponding to π; since this kernel has the same dimension as $H_1(\tilde{M})$, it follows that $\bar{\pi}$ is injective. ■

Now let the components of Γ be Γ^k for $1 \le k \le b_0(\Gamma)$. We let $A = \{1,\dots,b_0(\Gamma)\}$ be the index set for these components. For $1 \le i \le 3$, we partition A into two sets A_i and A_i', with $k \in A_i$ iff Γ^k is a circular edge colored g_i. We also set $\Gamma_i^k = \Gamma^k \cap \Gamma_i$. (If Γ^k is a circular edge, then Γ_i^k is empty if Γ^k has color g_i, and equal to Γ^k otherwise.) If γ is a 1-cycle of $M - \Gamma^k$, we have $\sum_{i=1}^3 \operatorname{Lk}(\gamma,\Gamma_i^k) = 0$, where Lk denotes mod 2 linking number. Hence, for $k \ne l$,

$$\begin{aligned}\operatorname{Lk}(\Gamma_1^k,\Gamma_2^l) + \operatorname{Lk}(\Gamma_2^k,\Gamma_1^l) &= \big(\operatorname{Lk}(\Gamma_2^k,\Gamma_2^l) + \operatorname{Lk}(\Gamma_3^k,\Gamma_2^l)\big) + \big(\operatorname{Lk}(\Gamma_2^k,\Gamma_2^l) + \operatorname{Lk}(\Gamma_2^k,\Gamma_3^l)\big)\\ &= \operatorname{Lk}(\Gamma_2^k,\Gamma_3^l) + \operatorname{Lk}(\Gamma_3^k,\Gamma_2^l),\end{aligned}$$

and similarly $\operatorname{Lk}(\Gamma_2^k,\Gamma_3^l) + \operatorname{Lk}(\Gamma_3^k,\Gamma_2^l) = \operatorname{Lk}(\Gamma_3^k,\Gamma_1^l) + \operatorname{Lk}(\Gamma_1^k,\Gamma_3^l)$; we let $\lambda_{kl} \in \mathbb{Z}_2$ be this common value. Note that if $k \in A_i$ then $\lambda_{kl} = \operatorname{Lk}(\Gamma^k,\Gamma_i^l)$, and if also $l \in A_j$ then λ_{kl} equals $\operatorname{Lk}(\Gamma^k,\Gamma^l)$ if $i \ne j$ and 0 if $i = j$. We also set $\lambda_{kk} = \sum_{l\in A, l\ne k} \lambda_{kl}$, and let Λ be the symmetric matrix $[\lambda_{kl}]_{k,l\in A}$.

Lemma 7.4 *We have $\dim H_1(\tilde{M}) = b_0(\Gamma) + b_1(\Gamma) - 3 - \operatorname{rank}\Lambda$.*

Proof We shall prove this by applying Lemma 7.2 to the covering $\pi_1\colon \tilde{M} \to M_1$. First we establish some notation.

(a) If A' and A'' are subsets of A, we let $\Lambda(A',A'')$ be the submatrix $[\lambda_{kl}]_{k\in A', l\in A''}$ of Λ. Note that $\Lambda(A_1,A_1)$ is a diagonal matrix with diagonal entries $\lambda_{kk} = \operatorname{Lk}(\Gamma^k,\Gamma_1)$ for $k \in A_1$.

(b) We let F be a surface in M with $\partial F = \Gamma_1$. Then M_1 can be constructed by gluing together two copies of M cut open along F.

(c) We denote the connecting homomorphisms in the exact sequences of the pairs (M,Γ), (M,Γ_1) and $(M,\Gamma\setminus\Gamma_1)$ by $\partial_i\colon H_{i+1}(M,\Gamma) \to \tilde{H}_i(\Gamma)$, $\partial_i'\colon H_{i+1}(M,\Gamma_1) \to \tilde{H}_i(\Gamma_1)$ and $\partial_i''\colon H_{i+1}(M,\Gamma\setminus\Gamma_1) \to \tilde{H}_i(\Gamma\setminus\Gamma_1)$. (Here $\tilde{H}$ denotes reduced homology.) We need these maps only for $i = 0$ or 1, where they are isomorphisms.

(d) The case of (7.1) for the cover $M_1 \to M$ with $X = \varnothing$ is

$$0 \longrightarrow C(M, \Gamma_1) \xrightarrow{\rho_1^!} C(M_1) \xrightarrow{\rho_1} C(M) \longrightarrow 0.$$

The long exact sequence shows that

$$\alpha_i = \rho_1^!(\partial_i')^{-1} \colon \tilde{H}_i(\Gamma_1) \to H_{i+1}(M_1)$$

is an isomorphism for $i = 0$ and an epimorphism for $i = 1$.

(e) The case of (7.1) for $M_1 \to M$ with $X = \Gamma \setminus \Gamma_1$ is

$$0 \longrightarrow C(M, \Gamma) \xrightarrow{\rho_1^!} C(M_1, \Delta_1) \xrightarrow{\rho_1} C(M, \Gamma \setminus \Gamma_1) \longrightarrow 0.$$

Denote the connecting homomorphisms in the long exact sequence by

$$\partial_i''' \colon H_{i+1}(M, \Gamma \setminus \Gamma_1) \to H_i(M, \Gamma).$$

We get an exact sequence

$$H_1(\Gamma) \xrightarrow{\gamma_1} H_2(M_1, \Delta_1) \xrightarrow{\delta_1} H_1(\Gamma \setminus \Gamma_1) \xrightarrow{\beta} \tilde{H}_0(\Gamma)$$
$$\xrightarrow{\gamma_0} H_1(M_1, \Delta_1) \xrightarrow{\delta_0} \tilde{H}_0(\Gamma \setminus \Gamma_1) \longrightarrow 0,$$

where $\beta = \partial_0 \partial_1'''(\partial_1'')^{-1}$, $\gamma_i = \rho_1^! \partial_i^{-1}$ and $\delta_i = \partial_i'' \rho_1$.

The isomorphism $\alpha_0 \colon \tilde{H}_0(\Gamma_1) \to H_1(M_1)$ gives

(7.5) $$\dim H_1(M_1) = b_0(\Gamma_1) - 1.$$

Next we determine $b_0(\Delta_1)$ and the rank of the map $H_1(\Delta_1) \to H_1(M_1)$. Consider a non-circular edge e of Γ with color g_1; the number of such edges is $b_1(\Gamma) - b_0(\Gamma)$, and the inverse image $\rho_1^{-1}(e)$ is a single component of Δ_1. The image under α_0^{-1} of the homology class of $\rho_1^{-1}(e)$ is represented by ∂e, and the subspace of $\tilde{H}_0(\Gamma_1)$ spanned by such elements is $\hat{H}_0(\Gamma_1) = \bigoplus_{k \in A_1'} \tilde{H}_0(\Gamma_1^k)$, which has dimension $b_0(\Gamma_1) - |A_1'|$. The remaining components of $\Gamma \setminus \Gamma_1$ are the Γ^k for $k \in A_1$, and such a Γ^k is covered by a single component of Δ_1 if $\lambda_{kk} = 1$, and by two components if $\lambda_{kk} = 0$. The number of $k \in A_1$ with $\lambda_{kk} = 1$ is the rank of the diagonal matrix $\Lambda(A_1, A_1)$, and so

(7.6) $$b_0(\Delta_1) = b_1(\Gamma) - b_0(\Gamma) + 2|A_1| - \operatorname{rank} \Lambda(A_1, A_1).$$

For $k \in A_1$ with $\lambda_{kk} = 1$, the component of Δ_1 covering Γ^k is null-homologous. Let B be the set of $k \in A_1$ with $\lambda_{kk} = 0$, and $k \in B$. The two components of Δ_1 covering Γ^k represent the same element of $H_1(M_1)$, which we call x_k^1. We may assume that the surface F is disjoint from Γ^k, and take a surface F' with boundary Γ^k which is transverse to F. Then $\rho_1^{-1}(F')$ is the union of two copies of F' cut open along $F \cap F'$, either one of which shows that x_k^1 is the image under $\rho_1^!$ of the element of $H_1(M, \Gamma_1)$ represented by $F \cap F'$. Thus $\alpha_0^{-1}(x_k^1)$ is

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represented by $\partial(F \cap F')$. Now $H_0(\Gamma_1)/\hat{H}_0(\Gamma_1)$ has a basis with one element x_l^0 for each $l \in A_1'$, and the image of $\alpha_0^{-1}(x_k^1)$ in this quotient is $\sum_{l\in A_1'} \lambda_{kl} x_l^0$. Therefore the rank of $H_1(\Delta_1) \to H_1(M_1)$ is $\dim \hat{H}_0(\Gamma_1) + \operatorname{rank} \Lambda(B, A_1')$. But $\operatorname{rank} \Lambda(B, A_1') = \operatorname{rank} \Lambda(A_1, A) - \operatorname{rank} \Lambda(A_1, A_1)$, so

$$(7.7)\quad \operatorname{rank}\bigl(H_1(\Delta_1) \longrightarrow H_1(M_1)\bigr) = b_0(\Gamma_1) - |A_1'| + \operatorname{rank} \Lambda(A_1, A) - \operatorname{rank} \Lambda(A_1, A_1).$$

The 2-fold covering $\pi_1 : \tilde{M} \to M_1$ corresponds to a homomorphism $\theta : H_1(M_1 - \Delta_1) \to \mathbb{Z}_2$, to which are associated $\theta' \in H_2(M_1, \Delta_1)$ and $\theta'' : H_2(M_1) \to H_1(M_1, \Delta_1)$; we must determine the rank of θ''. We first identify θ'. For $x \in H_1(M - \Gamma)$, $\operatorname{Lk}(x, \Gamma_i)$ is well-defined for $1 \le i \le 3$, and $\sum_{i=1}^3 \operatorname{Lk}(x, \Gamma_i) = 0$. Define a homomorphism $\phi : H_1(M - \Gamma) \to G$ by $\phi(x) = \prod_{i=1}^3 g_i^{\operatorname{Lk}(x,\Gamma_i)}$. Then ϕ sends the meridian of an edge of Γ to the color of that edge, so it is the homomorphism corresponding to the cover $\tilde{M} \to M$. Let $\hat{\Gamma} = \rho_1^{-1}(\Gamma)$, and let $\iota : H_1(M_1 - \hat{\Gamma}) \to H_1(M_1 - \Delta_1)$ be the surjection induced by inclusion. For $y \in H_1(M_1 - \hat{\Gamma})$, we have $\rho_1(y) \in H_1(M - \Gamma)$ and $\phi\rho_1(y) = g_1^{\theta\iota(y)}$. It follows that $\operatorname{Lk}\bigl(\rho_1(y), \Gamma_2\bigr) = \operatorname{Lk}\bigl(\rho_1(y), \Gamma_3\bigr), \operatorname{Lk}\bigl(\rho_1(y), \Gamma_1\bigr) = 0$, and $\theta\iota(y) = \operatorname{Lk}\bigl(\rho_1(y), \Gamma_2\bigr)$. There are intersection pairings $H_1(M_1 - \Delta_1) \times H_2(M_1, \Delta_1) \to \mathbb{Z}_2$ and $H_1(M - \Gamma) \times H_2(M, \Gamma) \to \mathbb{Z}_2$ and a linking pairing $H_1(M - \Gamma) \times H_1(\Gamma) \to \mathbb{Z}_2$, and they are related by

$$\iota(y) \cdot \gamma_1(z) = \iota(y) \cdot \rho_1^! \partial_1^{-1}(z) = \rho_1(y) \cdot \partial_1^{-1}(z) = \operatorname{Lk}\bigl(\rho_1(y), z\bigr)$$

for $y \in H_1(M_1 - \hat{\Gamma})$ and $z \in H_1(\Gamma)$. For $k \in A$, let $z_k \in H_1(\Gamma)$ be represented by Γ_2^k. Then $\sum_{k\in A} z_k$ is represented by Γ_2, and so $\iota(y) \cdot \gamma_1(\sum_{k\in A} z_k) = \theta\iota(y)$ for $y \in H_1(M_1 - \hat{\Gamma})$. Therefore $\theta' = \gamma_1(\sum_{k\in A} z_k)$.

We have an epimorphism $\alpha_1 : H_1(\Gamma_1) \to H_2(M_1)$. For $k \in A_1'$, let $y_k^1 \in H_1(\Gamma_1)$ be represented by Γ_1^k, and let $\hat{H}_1(\Gamma_1)$ be the subspace of $H_1(\Gamma_1)$ generated by these elements. Also let $\hat{\alpha}_1$ be the restriction of α_1 to $\hat{H}_1(\Gamma_1)$. We shall show that $\operatorname{Ker}(\theta''\alpha_1) \le \hat{H}_1(\Gamma_1)$, from which it will follow that

$$\operatorname{rank} \theta'' = \operatorname{rank}(\theta''\alpha_1) = \operatorname{rank}(\theta''\hat{\alpha}_1) + \dim H_1(\Gamma_1) - \dim \hat{H}_1(\Gamma_1),$$

or

$$(7.8)\qquad \operatorname{rank} \theta'' = \operatorname{rank}(\theta''\hat{\alpha}_1) + b_0(\Gamma_1) - |A_1'|.$$

Consider the composite $\delta_0\theta''\alpha_1 : H_1(\Gamma_1) \to \tilde{H}_0(\Gamma \setminus \Gamma_1)$. This may be described geometrically as follows. If $x \in H_1(\Gamma_1)$ is represented by a circuit C, take surfaces F' and F'' with $\partial F' = C$ and $\partial F'' = \Gamma_2$ that meet transversely except along the common part of their boundaries, $C \cap \Gamma_2$. Then the closure of $(F' \cap F'') - (C \cap \Gamma_2)$ represents an element y of $H_1(M, \Gamma)$. Now $\theta''\alpha_1(x) \in H_1(M_1, \Delta_1)$ is represented by $\rho_1^{-1}(F' \cap F'')$, and is therefore the sum of $\rho_1^!(y)$ and the element represented by $\rho_1^{-1}(C \cap \Gamma_2)$. Since $\rho_1\rho_1^!(y) = 0$, $\rho_1\theta''\alpha_1(x) \in H_1(M, \Gamma \setminus \Gamma_1)$ is represented by $C \cap \Gamma_2$. Hence $\delta_0\theta''\alpha_1(x)$ is represented by $\partial(C \cap \Gamma_2)$, which is just the sum of the vertices of Γ lying on C. It follows that $\delta_0\theta''\alpha_1(x) = 0$ iff $x \in \hat{H}_1(\Gamma_1)$, so $\operatorname{Ker}(\theta''\alpha_1) \le \hat{H}_1(\Gamma_1)$, as claimed.

We let the elements of the natural basis for $H_0(\Gamma)$ be y_k^0 for $k \in A$, and define $\hat{\beta} : \hat{H}_1(\Gamma_1) \to \tilde{H}_0(\Gamma)$ by $\hat{\beta}(y_k^1) = \sum_{l\in A} \lambda_{kl} y_l^0$ for $k \in A_1'$. We claim that $\gamma_0\hat{\beta} = \theta''\hat{\alpha}_1$:

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$\hat{H}_1(\Gamma_1) \to H_1(M_1, \Delta_1)$. Let $k \in A_1'$ and $l \in A$. We have $\hat{\alpha}_1(y_k^1) \in H_2(M_1)$ and $\gamma_1(z_l) \in H_2(M_1, \Delta_1)$, with intersection $\hat{\alpha}_1(y_k^1) \cdot \gamma_1(z_l) \in H_1(M_1, \Delta_1)$. Suppose $k \neq l$. Then $(\partial_1')^{-1}(y_k^1) \in H_2(M, \Gamma_1)$ and $\partial_1^{-1}(z_l) \in H_2(M, \Gamma)$ may be represented by transverse surfaces F' and F'' with boundaries Γ_1^k and Γ_2^l, respectively, and $\hat{\alpha}_1(y_k^1) \cdot \gamma_1(z_l)$ is the image under $\rho_1^! \colon H_1(M, \Gamma) \to H_1(M_1, \Delta_1)$ of the class represented by $F' \cap F''$. Since the image of this class under $\partial_0 \colon H_1(M, \Gamma) \to \tilde{H}_0(\Gamma)$ is $\mathrm{Lk}(\Gamma_1^k, \Gamma_2^l)(y_k^0 + y_l^0)$, we have

$$\hat{\alpha}_1(y_k^1) \cdot \gamma_1(z_l) = \gamma_0\big(\mathrm{Lk}(\Gamma_1^k, \Gamma_2^l)(y_k^0 + y_l^0)\big) \quad \text{for } k \in A_1', l \in A, k \neq l.$$

Now $\sum_{k \in A_1'} (\partial_1')^{-1}(y_k^1)$ is represented by F, whose inverse image in M_1 is null homologous, so $\sum_{k \in A_1'} \hat{\alpha}_1(y_k^1) = 0$. Therefore, for $k \in A_1'$,

$$\hat{\alpha}_1(y_k^1) \cdot \gamma_1(z_k) = \sum_{l \in A_1', l \neq k} \hat{\alpha}_1(y_l^1) \cdot \gamma_1(z_k) = \sum_{l \in A, l \neq k} \gamma_0\big(\mathrm{Lk}(\Gamma_1^l, \Gamma_2^k)(y_k^0 + y_l^0)\big),$$

where in the last term we may sum over A since Γ_1^l is empty for $l \notin A_1'$. Hence, again for $k \in A_1'$,

$$\begin{aligned} \theta''\hat{\alpha}_1(y_k^1) = \hat{\alpha}_1(y_k^1) \cdot \theta' &= \sum_{l \in A} \hat{\alpha}_1(y_k^1) \cdot \gamma_1(z_l) \\ &= \sum_{l \in A, l \neq k} \gamma_0\big(\lambda_{kl}(y_k^0 + y_l^0)\big) = \sum_{l \in A} \gamma_0(\lambda_{kl} y_l^0) = \gamma_0 \hat{\beta}(y_k^1), \end{aligned}$$

and so indeed $\gamma_0 \hat{\beta} = \theta'' \hat{\alpha}_1$. Thus we have a commutative diagram

$$\begin{array}{ccccc} & & \hat{H}_1(\Gamma_1) & \xrightarrow{\hat{\alpha}_1} & H_2(M_1) \\ & & {\scriptstyle\hat{\beta}}\downarrow & & {\scriptstyle\theta''}\downarrow \\ H_1(\Gamma \setminus \Gamma_1) & \xrightarrow{\beta} & \tilde{H}_0(\Gamma) & \xrightarrow{\gamma_0} & H_1(M_1, \Delta_1) \end{array}$$

in which the bottom row is exact. Therefore

$$\mathrm{rank}(\theta''\hat{\alpha}_1) = \mathrm{rank}(\gamma_0\hat{\beta}) = \dim(\mathrm{Im}\,\beta + \mathrm{Im}\,\hat{\beta}) - \mathrm{rank}\,\beta.$$

For $k \in A_1$, Γ^k represents an element y_k^1 of $H_1(\Gamma \setminus \Gamma_1)$, and these form a basis. We claim that $\beta(y_k^1) = \sum_{l \in A} \lambda_{kl} y_l^0$ for $k \in A_1$, from which it will follow that

$$\mathrm{rank}(\theta''\hat{\alpha}_1) = \mathrm{rank}\,\Lambda - \mathrm{rank}\,\Lambda(A_1, A). \tag{7.9}$$

For $k \in A_1$, $(\partial_1'')^{-1}(y_k^1)$ is represented by a surface F' with boundary Γ^k, which we may take to be transverse to F. Then $\rho_1^{-1}(F')$ is the union of two copies of F' cut open along $F \cap F'$. The boundary of either one is the union of $\rho_1^{-1}(F \cap F')$ and part of $\rho_1^{-1}(\Gamma^k) \subseteq \Delta_1$, so it represents the same element of $C_1(M_1, \Delta_1)$ as $\rho_1^{-1}(F \cap F')$. It follows that $\partial_1'''(\partial_1'')^{-1}(y_k^1)$ is represented by $F \cap F'$, and hence that

$$\beta(y_k^1) = \partial_0\partial_1'''(\partial_1'')^{-1}(y_k^1) = \sum_{l \in A_1'} \mathrm{Lk}(\Gamma^k, \Gamma_1^l)(y_l^0 + y_k^0) = \sum_{l \in A} \lambda_{kl} y_l^0,$$

as claimed.

The proof of the lemma is completed by applying Lemma 7.2 to the covering $\tilde{M} \to M_1$ and using the equations (7.5)–(7.9). ■

In applying Lemma 7.4, we compute the matrix Λ using the following result, which is implicit in the proof of Lemma 1 of Flapan [1].

Lemma 7.10 *Let K be a knot in a $\mathbb{Z}_2$ homology 3-sphere N, and let A and B be disjoint arcs in N meeting K in their endpoints. Let $\tilde{N}$ be the 2-fold cover of N branched over K, and let $\tilde{A}$ and $\tilde{B}$ be the inverse images of A and B in the $\mathbb{Z}_2$ homology sphere $\tilde{N}$. Then $\mathrm{Lk}(\tilde{A}, \tilde{B}) = 1$ iff the endpoints of A separate those of B on K.* ■

8 Proofs of Theorems

Recall that in the statement of each theorem, Γ is a $G(d)$-colored graph embedded in a homology 3-sphere M, with corresponding branched cover $\tilde{M}$.

Theorem 8.1 *If $d = 2$ and Γ is connected, then there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{H \in \mathcal{C}^\star} H_1(M_H) \xrightarrow{\beta} H_1(\tilde{M}) \longrightarrow \mathbb{Z}_2^{b_1(\Gamma)-2} \longrightarrow 0,$$

and $\beta\big(\bigoplus_{H \in \mathcal{C}^\star} H_1(M_H)\big) = 2H_1(\tilde{M})$.

Proof Lemma 6.6 gives the exact sequence, while Lemma 7.3 shows that the mod 2 transfer $\pi_H^!\colon H_1(M_H;\mathbb{Z}_2) \to H_1(\tilde{M};\mathbb{Z}_2)$ is zero for $H \in \mathcal{C}^\star$, which implies the second assertion by Lemma 6.7. ■

Theorem 8.2 *Let Γ be a trivalent graph with an unsplittable $G(3)$-coloring with a special m-circuit. Then $3 \le m \le b_1(\Gamma)$, there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{H \in \mathcal{C}^\star} H_1(M_H) \xrightarrow{\beta} H_1(\tilde{M}) \longrightarrow \mathbb{Z}_4^{m-3} \oplus \mathbb{Z}_2^{2(b_1(\Gamma)-m)} \longrightarrow 0,$$

and $\beta\big(\bigoplus_{H \in \mathcal{C}^\star} H_1(M_H)\big) = 4H_1(\tilde{M})$.

Proof Let $H_0 \in \mathcal{C}^\star$ be such that $\Gamma_0 = \Gamma_{H_0}$ is a special m-circuit, and let $M_0 = M_{H_0}$ and $\Delta_0 = \Delta_{H_0}$. Since the coloring is unsplittable, Γ is simple, so any circuit has length at least 3. Further, Γ is connected, so $\chi(\Gamma \setminus \Gamma_0) = 1 - b_1(\Gamma) + m$; since $\Gamma \setminus \Gamma_0$ is connected, this gives $m \le b_1(\Gamma)$. By Lemma 4.8, Γ is taut, so Lemma 6.6 gives an exact sequence

$$0 \longrightarrow \bigoplus_{H \in \mathcal{C}^\star} H_1(M_H) \xrightarrow{\beta} H_1(\tilde{M}) \longrightarrow \mathbb{Z}_4^{a} \oplus \mathbb{Z}_2^{b} \longrightarrow 0$$

for some a and b with $2a + b = 2b_1(\Gamma) - 6$.

Suppose $1 \neq h \in H \in \mathcal{C}^\star$. There is a cover $\tilde{M}/\langle h\rangle \to M$ with group $H/\langle h\rangle \cong \mathbb{Z}_2^2$; its branch set is obtained from Γ by deleting all edges with color h. Since Γ is unsplittable, we may apply Lemma 7.3 to this cover to show that the transfer $H_1(M_H;\mathbb{Z}_2) \to H_1(\tilde{M}/\langle h\rangle;\mathbb{Z}_2)$ is zero. By Lemma 6.7, to show that $\beta\big(\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)\big) = 4H_1(\tilde{M})$ it is then enough to show that, for each $H \in \mathcal{C}^\star$, there is some non-trivial h in H such that $H_1(\tilde{M}/\langle h\rangle;\mathbb{Z}_2) \to H_1(\tilde{M};\mathbb{Z}_2)$ is zero. Now consider the cover $\tilde{M} \to M_0$, with group $H_0 \cong \mathbb{Z}_2^2$ and branch set Δ_0. Since Γ_0 is a circuit, M_0 is a $\mathbb{Z}_2$ homology sphere. Since $\Gamma \setminus \Gamma_0$ is connected, so is Δ_0, and Lemma 7.3 applies to this cover, showing that $H_1(\tilde{M}/\langle h\rangle;\mathbb{Z}_2) \to H_1(\tilde{M};\mathbb{Z}_2)$ is zero whenever $1 \neq h \in H_0$. Since $H \cap H_0$ contains a non-trivial element for all $H \in \mathcal{C}^\star$, the proof that $\beta\big(\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)\big) = 4H_1(\tilde{M})$ is complete. It follows that $H_1(\tilde{M};\mathbb{Z}_2) \cong H_1(\tilde{M})/2H_1(\tilde{M}) \cong \mathbb{Z}_2^{a+b}$. On the other hand, Lemma 7.3 applied to $\tilde{M} \to M_0$ also shows that $\dim H_1(\tilde{M};\mathbb{Z}_2) = b_1(\Delta_0) - 2$. Since $b_1(\Delta_0) = 2b_1(\Gamma) - m - 1$, we have $a + b = 2b_1(\Gamma) - m - 3$. It follows that $a = m - 3$ and $b = 2(b_1(\Gamma) - m)$, and we are done. ∎

Theorem 8.3 *Let Γ be an n-rung Möbius ladder ($n \geq 2$) with a $G(3)$-coloring, and let g_0 be the product of the colors on the rungs. Suppose that $g_0 \neq 1$, and let k be the number of rungs with color g_0. If $k = 0$, there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{H\in\mathcal{C}^\star} H_1(M_H) \xrightarrow{\beta} H_1(\tilde{M}) \longrightarrow \mathbb{Z}_4^{n-2} \longrightarrow 0,$$

while if $k > 0$ there is a short exact sequence

$$0 \longrightarrow \bigoplus_{H\in\mathcal{C}^\star} H_1(M_H) \xrightarrow{\beta} H_1(\tilde{M}) \longrightarrow \mathbb{Z}_4^{n-k-1} \oplus \mathbb{Z}_2^{2(k-1)} \longrightarrow 0.$$

In either case, $\beta\big(\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)\big) = 4H_1(\tilde{M})$.

Proof By Lemma 4.10, Γ is taut, so Lemma 6.6 gives an exact sequence

$$0 \longrightarrow \bigoplus_{H\in\mathcal{C}^\star} H_1(M_H) \xrightarrow{\beta} H_1(\tilde{M}) \longrightarrow \mathbb{Z}_4^a \oplus \mathbb{Z}_2^b \longrightarrow 0$$

for some a and b with $2a + b = 2n - 4$. Consider the cover $\pi'\colon \tilde{M}/\langle g_0\rangle \to M$ with group $G' = G/\langle g_0\rangle \cong \mathbb{Z}_2^2$. Its branch set is the $(n-k)$-rung Möbius ladder Γ' obtained by deleting the rungs colored g_0, so Lemma 7.3 shows that $H_1(M_H;\mathbb{Z}_2) \to H_1(\tilde{M}/\langle g_0\rangle;\mathbb{Z}_2)$ is zero whenever $g_0 \in H \in \mathcal{C}^\star$. By Lemma 1.7, Γ_H is connected if $g_0 \notin H$, so Lemma 6.7 will imply that $\beta\big(\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)\big) = 4H_1(\tilde{M})$ provided that $H_1(\tilde{M}/\langle g_0\rangle;\mathbb{Z}_2) \to H_1(\tilde{M};\mathbb{Z}_2)$ is also zero. Lemma 7.3 also gives $\dim H_1(\tilde{M}/\langle g_0\rangle;\mathbb{Z}_2) = n - k - 1$. The 2-fold cover $\tilde{M} \to \tilde{M}/\langle g_0\rangle$ has as branch set a link L, which is the inverse image of the rungs of Γ labelled g_0. Let r be the rank of $H_1(L;\mathbb{Z}_2) \to H_1(\tilde{M}/\langle g_0\rangle;\mathbb{Z}_2)$. If $k = 0$ then L is empty and $r = 0$. Suppose $k > 0$, and consider a rung e labelled g_0. The endpoints of e lie on two edges of Γ' with the same color in the G'-labelling determining π'. Hence, if $D \subset M$ is a 2-disk containing e in its interior and meeting Γ' only in the endpoints of e, then $(\pi')^{-1}(D)$ consists of two annuli. This shows first that $(\pi')^{-1}(e)$ has two components, so $b_0(L) = 2k$. It also shows that under the map $\bar{\pi}'\colon H_1(\tilde{M}/\langle g_0\rangle;\mathbb{Z}_2) \to H_1(M - \Gamma';\mathbb{Z}_2)$, each component

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of $(\pi')^{-1}(e)$ is sent to the element of $H_1(M - \Gamma'; \mathbb{Z}_2)$ represented by ∂D. This element is non-trivial and independent of the choice of e. By Lemma 7.3, $\bar{\pi}'$ is injective, and it follows that $r = 1$. Using the Kronecker delta, we may say that in all cases $b_0(L) = 2k$ and $r = 1 - \delta_{k0}$. It now follows from Lemma 7.2 applied to the cover $\tilde{M} \to \tilde{M}/\langle g_0 \rangle$ that

$$\dim H_1(\tilde{M}; \mathbb{Z}_2) - \operatorname{rank}\bigl(H_1(\tilde{M}/\langle g_0\rangle; \mathbb{Z}_2) \longrightarrow H_1(\tilde{M}; \mathbb{Z}_2)\bigr) = n + k - 3 + \delta_{k0}. \tag{8.4}$$

Now choose $H \in \mathcal{C}^\star$ with $g_0 \notin H$. Then M_H is a $\mathbb{Z}_2$ homology sphere, and we may compute $\dim H_1(\tilde{M}; \mathbb{Z}_2)$ by applying Lemma 7.4 to the cover $\pi_H \colon \tilde{M} \to M_H$. We must compute the matrix Λ of that lemma. Suppose that Γ_H contains the m rungs τ_{i_j} for $0 \le j < m$, where $0 \le i_0 < \cdots < i_{m-1} < n$, and let the color of τ_{i_j} be $h_j \in G - H$. Then $\Gamma \setminus \Gamma_H$ has m components $C_0, \ldots, C_{m-1}$, and the components of Δ_H are $\rho_H^{-1}(C_0), \ldots, \rho_H^{-1}(C_{m-1})$. We may choose the numbering of the C_j so that τ_{i_j} has one vertex on C_j and the other on C_{j+1}. (The subscripts on the C_j are to be taken modulo m.) By Lemma 7.10, all the off-diagonal elements of Λ except $\lambda_{j,j\pm 1}$ are zero. Also, if the edges of C_j and C_{j+1} that meet τ_j have colors h'_j and h''_j, then $\lambda_{j,j+1} = 1$ iff $h'_j \neq h''_j$. However, $h'_j h''_j = g_0 h_j$, so $\lambda_{j,j+1} = 1$ iff $h_j \neq g_0$. Since exactly k of the h_j are equal to g_0, it follows that $\operatorname{rank} \Lambda = m - k - \delta_{k0}$.

The trivalent graph Δ_H has $2(n-m)$ vertices, so $\chi(\Delta_H) = m - n$, and since $b_0(\Delta_H) = m$ we have $b_1(\Delta_H) = n$. Now Lemma 7.4 gives $\dim H_1(\tilde{M}; \mathbb{Z}_2) = n + k - 3 + \delta_{k0}$. Comparing this to (8.4), we see that $H_1(\tilde{M}/\langle g_0\rangle; \mathbb{Z}_2) \to H_1(\tilde{M}; \mathbb{Z}_2)$ is the zero map, and hence $\beta\bigl(\bigoplus_{H \in \mathcal{C}^\star} H_1(M_H)\bigr) = 4H_1(\tilde{M})$. It then follows that $a + b = \dim H_1(\tilde{M}; \mathbb{Z}_2) = n + k - 3 + \delta_{k0}$, giving $a = n - k - 1 - \delta_{k0}$ and $b = 2(k - 1 + \delta_{k0})$, completing the proof. ∎

Suppose that Γ is taut. By Lemmas 5.7 and 6.5, we may identify $\bigoplus_{H \in \mathcal{C}^\star} H_1(M_H)$ and $H_1(\operatorname{Im}\beta)$ with their images in $H_1(\tilde{M})$; thus $\bigoplus_{H \in \mathcal{C}^\star} H_1(M_H) \le H_1(\operatorname{Im}\beta) \le H_1(\tilde{M})$. For any $x \in H_1(\tilde{M})$, $2^{d-1}x \in \bigoplus_{H \in \mathcal{C}^\star} H_1(M_H)$. In the proofs of the remaining theorems, we need to show that we may choose x so that $2^{d-2}x \notin \bigoplus_{H \in \mathcal{C}^\star} H_1(M_H)$. Now $2^{d-2}x$ is in $H_1(\operatorname{Im}\beta)$, and so it is in $\bigoplus_{H \in \mathcal{C}^\star} H_1(M_H)$ iff it is in the kernel of the map $H_1(\operatorname{Im}\beta) \to \mathbb{Z}_2^{b_1(\Gamma)-d}$ from Lemma 5.7. From the proof of that lemma, the kernel of this map is equal to the kernel of the composite of the maps $H_1(\operatorname{Im}\beta) \to H_0(\operatorname{Ker}\beta)$ from the long exact sequence of (2.4), and $H_0(\operatorname{Ker}\beta) \to H_0(\Gamma \mid d-1)$ from Lemma 5.4.

Lemma 8.5 *Suppose that Γ is taut, and let $e_1, \ldots, e_n$ be edges of Γ with colors $g_1, \ldots, g_n$ such that $g_1 \cdots g_n = 1$. For $1 \le i \le n$, pick a vertex v_i of e_i. Then there is an element x of $H_1(\tilde{M})$ such that the image of $2^{d-2}x$ in $H_0(\Gamma \mid d-1)$ is represented by $\sum_{i=1}^{n} \sum_{H \in \mathcal{C}^\star} \delta_H(g_i) v_i H \in C'_0(\Gamma \mid d-1)$.*

Proof Consider an element x of $H_1(\tilde{M})$ represented by a cycle of the form $z = \sum_{\sigma \in S_1(M)} (1 - h_\sigma)\tilde{\sigma}$ for some $h_\sigma \in G$. Let S' be the set of those σ for which $h_\sigma \neq 1$, and for each $\sigma \in S'$, define an element c_σ of $\sum_{H \in \mathcal{C}} C_1(M_H)$ by

$$c_\sigma = -2^{d-2}\sigma G + \sum_{H \in \mathcal{C}} \tfrac{1}{2}\bigl(1 - \varepsilon_H(h_\sigma)\bigr)\sigma_H H.$$

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Then

$$\begin{aligned}\beta(c_\sigma) &= -2^{d-2}\sum_{g\in G} g\tilde\sigma + \sum_{H\in\mathcal{C}} \tfrac12\bigl(1-\varepsilon_H(h_\sigma)\bigr)\sum_{h\in H} h\tilde\sigma\\ &= \sum_{g\in G}\Bigl(-2^{d-2} + \sum_{H\in\mathcal{C}} \tfrac12\bigl(1-\varepsilon_H(h_\sigma)\bigr)\tfrac12\bigl(1+\varepsilon_H(g)\bigr)\Bigr)g\tilde\sigma\\ &= \sum_{g\in G}\sum_{H\in\mathcal{C}} \tfrac14\bigl(\varepsilon_H(g)-\varepsilon_H(h_\sigma)-\varepsilon_H(gh_\sigma)\bigr)g\tilde\sigma\\ &= 2^{d-2}(1-h_\sigma)\tilde\sigma.\end{aligned}$$

Therefore $\beta(\sum_{\sigma\in S'} c_\sigma) = 2^{d-2}z$, and so the image of $2^{d-2}x$ in $H_0(\operatorname{Ker}\beta)$ is represented by $\sum_{\sigma\in S'}\partial c_\sigma$. From the proofs of Lemmas 5.4 and 5.3, the image of $2^{d-2}x$ in $H_0(\Gamma \mid d-1)$ is represented by

$$z' = \sum_{\sigma\in S', H\in\mathcal{C}^\star} \delta_H(h_\sigma)(\partial\sigma)H = \sum_{\sigma\in S_1(M), H\in\mathcal{C}^\star} \delta_H(h_\sigma)(\partial\sigma)H$$

(since $\frac12\bigl(1-\varepsilon_H(g)\bigr) \bmod 2 = \delta_H(g)$).

We now construct a specific 1-cycle. Take a disc D in M meeting Γ transversely in n points $p_1,\dots,p_n$, where p_i lies on the edge e_i. Take disjoint arcs $A_1,\dots A_n$ on D, where A_i joins p_i to a point q_i of ∂D and q_i is adjacent to q_{i+1} on ∂D. (Here and in the rest of the proof, subscripts are to be taken modulo n.) We may assume that D and each A_i are triangulated by subcomplexes of M (and hence the p_i and q_i are 0-simplices of M). Let $c_i \in C_1(M)$ be a 1-chain carried by A_i with $\partial c_i = q_i - p_i$. Also let $d_i \in C_1(M)$ be carried by one of the arcs into which the q_i divide ∂D, with $\partial d_i = q_{i+1} - q_i$. Let $\tilde c_i$ and $\tilde d_i$ be the images of c_i and d_i under the $\mathbb{Z}$-module homomorphism $C(M) \to C(\tilde M)$ taking σ to $\tilde\sigma$ ($\sigma \in S(M)$). Now $\pi^{-1}(D)$ is the union of 2^d copies of D cut open along the A_i; let $\tilde D$ be one copy. If σ is either p_i or a 1-simplex of ∂D, there is just one lift of σ lying in $\partial\tilde D$; we take this to be $\tilde\sigma$. If σ is either q_i or a 1-simplex of A_i, there are two lifts of σ lying in $\partial\tilde D$, and g_i takes one to the other. We may choose $\tilde\sigma$ to be one of these lifts in such a way that $\partial\tilde d_i = g_{i+1}\tilde q_{i+1} - \tilde q_i$ and $\partial\tilde c_i = \tilde q_i - \tilde p_i$. With these choices, $z_1 = \sum_{i=1}^n(\tilde c_i - g_i\tilde c_i + \tilde d_i)$ is a 1-cycle of $\tilde M$ carried by $\partial\tilde D$. Set $g_i' = \prod_{j=1}^i g_j$; g_i' depends only on $i \bmod n$ since $g_1\cdots g_n = 1$, and so $z_2 = \sum_{i=1}^n g_i'\tilde d_i$ is another 1-cycle of $\tilde M$. (It is carried by a single lift of ∂D.) Let $x \in H_1(\tilde M)$ be represented by $z = z_1 - z_2 = \sum_{i=1}^n\bigl((1-g_i)\tilde c_i + (1-g_i')\tilde d_i\bigr)$. By the previous paragraph, the image of $2^{d-2}x$ in $H_0(\Gamma \mid d-1)$ is represented by

$$\begin{aligned}z' &= \sum_{i=1}^n\sum_{H\in\mathcal{C}^\star}\bigl(\delta_H(g_i)(q_i+p_i) + \delta_H(g_i')(q_{i+1}+q_i)\bigr)H\\ &= \sum_{i=1}^n\sum_{H\in\mathcal{C}^\star}\Bigl(\delta_H(g_i)p_i + \bigl(\delta_H(g_i)+\delta_H(g_i')+\delta_H(g_{i-1}')\bigr)q_i\Bigr)H\\ &= \sum_{i=1}^n\sum_{H\in\mathcal{C}^\star}\delta_H(g_i)p_iH \quad \text{(because } g_ig_i'g_{i-1}' = 1).\end{aligned}$$

Now $\sum_{i=1}^{n}\sum_{H\in\mathcal{C}^\star}\delta_H(g_i)p_iH$ is homologous to $\sum_{i=1}^{n}\sum_{H\in\mathcal{C}^\star}\delta_H(g_i)\nu_iH$, and the proof is complete. ■

The next lemma will be used in the proof of Theorem 8.7 to show that the 0-chain of the previous one is not a boundary.

Lemma 8.6 *Let Γ be an m-rung Möbius ladder, and let $0 \le i_1 < i_2 < \cdots < i_k < m$, where either m is odd and k is even, or $m = k = 2$. Let the two circuits of Γ that contain all the rungs be Γ_1 and Γ_2, and for $\alpha = 1$ or 2, let $c_\alpha \in C_1(\Gamma_\alpha;\mathbb{Z}_2)$ be such that $\partial c_\alpha = \sum_{j=1}^{k}\nu_{i_j}$. Let $a \in \mathbb{Z}_2$ be the sum of the coefficients of the rungs τ_{i_j} in c_α for $1 \le j \le k$ and $\alpha = 1$ or 2. Then $a = 1$ iff $m = 2$.*

Proof Since each Γ_α contains all the rungs τ_{i_j}, a is independent of the choice of c_1 and c_2. Suppose first that $m = k = 2$, and so $i_1 = 0$ and $i_2 = 1$. If Γ_1 is taken to be the circuit containing σ_0, we may take $c_1 = \sigma_0$ and $c_2 = \tau_0 + \sigma_1$, and so $a = 1$.

Now suppose that m is odd. We show that for $1 \le j \le \frac{1}{2}k$, there are chains $c_{\alpha j} \in C_1(\Gamma_\alpha;\mathbb{Z}_2)$ with $\partial c_{\alpha j} = \nu_{i_{2j-1}} + \nu_{i_{2j}}$ such that

$$c_{1j} + c_{2j} = \tau_{i_{2j-1}} + \tau_{i_{2j}} + \sum_{i=i_{2j-1}}^{i_{2j}-1}(\sigma_i + \sigma_{i+n}),$$

the sum being taken in $C_1(\Gamma;\mathbb{Z}_2)$. Then we may set $c_\alpha = \sum_{j=1}^{k/2} c_{\alpha j}$ and conclude that $a = 0$. Given j, let Γ_α be that one of Γ_1 and Γ_2 that contains $\sigma_{i_{2j-1}}$, and Γ_β the other. If $i_{2j} - i_{2j-1}$ is odd, we may set

$$c_{\alpha j} = \sigma_{i_{2j-1}} + \tau_{i_{2j-1}+1} + \sigma_{i_{2j-1}+1+n} + \tau_{i_{2j-1}+2} + \cdots + \tau_{i_{2j}-1} + \sigma_{i_{2j}-1} \quad \text{and}$$
$$c_{\beta j} = \tau_{i_{2j-1}} + \sigma_{i_{2j-1}+n} + \tau_{i_{2j-1}+1} + \sigma_{i_{2j-1}+1}\cdots + \sigma_{i_{2j}-1+n} + \tau_{i_{2j}},$$

while if $i_{2j} - i_{2j-1}$ is even, we may set

$$c_{\alpha j} = \sigma_{i_{2j-1}} + \tau_{i_{2j-1}+1} + \sigma_{i_{2j-1}+1+n} + \tau_{i_{2j-1}+2} + \cdots + \sigma_{i_{2j}-1+n} + \tau_{i_{2j}} \quad \text{and}$$
$$c_{\beta j} = \tau_{i_{2j-1}} + \sigma_{i_{2j-1}+n} + \tau_{i_{2j-1}+1} + \sigma_{i_{2j-1}+1}\cdots + \tau_{i_{2j}-1} + \sigma_{i_{2j}-1}.$$

■

Theorem 8.7 *Let $d = 4$ and let Γ be an n-rung Möbius ladder with $n \ge 3$. Give Γ the $G(4)$-coloring of Example 1.6. Then*

$$H_1(\tilde{M}) \cong \begin{cases} \bigoplus_{H\in\mathcal{C}^\star} H_1(M_H) \oplus \mathbb{Z}_2, & \text{if } n = 3; \\ \bigoplus_{H\in\mathcal{C}^\star} H_1(M_H) \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2^{4n-14}, & \text{if } n \ge 4. \end{cases}$$

Proof This coloring is 4-taut, so Lemma 6.6 applies and $\bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)$ has odd order. Therefore

$$H_1(\tilde{M}) \cong \bigoplus_{H\in\mathcal{C}^\star} H_1(M_H) \oplus \mathbb{Z}_8^a \oplus \mathbb{Z}_4^b \oplus \mathbb{Z}_2^c$$

for some a, b and c with $3a + 2b + c = 4n - 11$, and $a + b + c = \dim H_1(\tilde{M}; \mathbb{Z}_2)$. If $n = 3$, we must have $a = b = 0$ and $c = 1$, which proves this case of the theorem. From now on we assume that $n \geq 4$.

Let $H_0 = \langle x_1, x_2, x_3\rangle \in \mathcal{C}^\star$, and set $\Gamma_0 = \Gamma_{H_0}$, $M_0 = M_{H_0}$, and $\Delta_0 = \Delta_{H_0}$; note that Γ_0 is the rim. The chain of subgroups $1 \leq \langle x_3\rangle \leq H_0 \leq G$ determines a chain of coverings $\tilde{M} \to M_1 \to M_0 \to M$, of which the middle one has group $H_0/\langle x_3\rangle \cong \mathbb{Z}_2^2$ and the others are 2-fold. The branch set Δ_0 of $\tilde{M} \to M_0$ is a link of n components, any two of which have linking number 1 by Lemma 7.10. The branch set of $M_1 \to M_0$ is a 3-component sublink L_0 of Δ_0, lying over the three rungs whose color is not x_3, and it follows from Lemma 7.4 that $H_1(M_1; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Also, each component of $\Delta_0 - L_0$ is covered by four simple closed curves in M_1, so $\tilde{M} \to M_1$ is branched over a link Δ_1 of $4n - 12$ components. Each element of $H_1(M_1; \mathbb{Z}_2)$ represented by a component of Δ_1 has non-trivial image under the map $H_1(M_1; \mathbb{Z}_2) \to H_1(M_0 - L_0)$ defined just before Lemma 7.3, and is therefore non-trivial. Hence $H_1(\Delta_1; \mathbb{Z}_2) \to H_1(M_1; \mathbb{Z}_2)$ is onto, and the special case of Lemma 7.2 shows that $\dim H_1(\tilde{M}; \mathbb{Z}_2) = 4n - 13$. It follows that $2a + b = 2$. If we show that $a > 0$, it will follow that $a = 1$, $b = 0$ and $c = 4n - 14$, completing the proof.

Let $g_i \in G$ be the color of the rung τ_i. Since $n \geq 4$, there is at least one rung with color x_3, which we may take to be τ_0. Then $g_1 \cdots g_{n-1} = 1$, and applying Lemma 8.5 to the rungs $\tau_1, \ldots, \tau_{n-1}$ we see that it is enough to show that

$$z = \sum_{i=1}^{n-1} \sum_{H\in\mathcal{C}^\star} \delta_H(g_i) v_i H \in C_0'(\Gamma \mid 3)$$

represents a non-zero element of $H_0(\Gamma \mid 3)$. Recall that $C'(\Gamma \mid 3)$ is a subcomplex of $C'(\Gamma \mid 4) = \bigoplus_{H\in\mathcal{C}^\star} C(\Gamma_H; \mathbb{Z}_2)$. Since, for each $H \in \mathcal{C}^\star$, Γ_H is a circuit and there are an even number of i $(1 \leq i \leq n-1)$ with $\delta_H(g_i) = 1$, z is a boundary in $C'(\Gamma \mid 4)$. Let $c \in C'(\Gamma \mid 4)$, with $c = \sum_{\sigma\in S_1(\Gamma), H\in\mathcal{C}^\star} c(\sigma, H)\sigma H$, and set $\phi(c) = \sum_{i=1}^{n-1} \sum_{H\in\mathcal{C}^\star} c(\tau_i, H) \in \mathbb{Z}_2$. If c is a cycle, then $\phi(c) = 0$, so if c_1 and c_2 both have boundary z, then $\phi(c_1) = \phi(c_2)$. On the other hand, if c lies in $C'(\Gamma \mid 3)$, then $\phi(c) = 0$. Thus if we can find $c \in C'(\Gamma \mid 4)$ with $\partial c = z$ and $\phi(c) = 1$, it will follow that z represents a non-zero element of $H_0(\Gamma \mid 3)$.

For $H \in \mathcal{C}^\star$, let $z_H = \sum_{i=1}^{n-1} \delta_H(g_i) v_i \in C_0(\Gamma_H; \mathbb{Z}_2)$, so $z = \sum_{H\in\mathcal{C}^\star} z_H H$. A chain $c \in C'(\Gamma \mid 4)$ with $\partial c = z$ has the form $c = \sum_{H\in\mathcal{C}^\star} c_H H$ with $c_H \in C_1(\Gamma_H; \mathbb{Z}_2)$ and $\partial c_H = z_H$ for $H \in \mathcal{C}^\star$. Now $z_{H_0} = 0$ and we may take $c_{H_0} = 0$. The remaining elements of $\mathcal{C}^\star(G)$ are in 2–1 correspondence with the elements of $\mathcal{C}^\star(H_0)$. For $H \in \mathcal{C}^\star(H_0)$ let H_1 and H_2 be the two elements of $\mathcal{C}^\star(G)$ with $H_1 \cap H_0 = H = H_2 \cap H_0$. Then Γ_{H_1} and Γ_{H_2} contain the same rungs; let m_H be the number of these rungs, and k_H the number of them distinct from τ_0. The union of Γ_{H_1} and Γ_{H_2} is an m_H-rung Möbius ladder. If $c_{H_1} \in C_1(\Gamma_{H_1}; \mathbb{Z}_2)$ and $c_{H_2} \in C_1(\Gamma_{H_2}; \mathbb{Z}_2)$ both have boundary $z_{H_1} = z_{H_2}$, we may compute the sum a_H of the coefficients of the τ_i for $1 \leq i \leq n-1$ in c_{H_1} and c_{H_2} using Lemma 8.6 (provided that m_H and k_H satisfy the hypotheses of that lemma, as we shall see they do), and then $c = \sum_{H\in\mathcal{C}^\star(H_0)} (c_{H_1} H_1 + c_{H_2} H_2)$ is an element of $C_1'(\Gamma \mid 4)$ with $\partial c = z$ and

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	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}
H_{1i} :	1	0	0	0	0
H_{2i} :	0	1	1	1	1
H_{3i} :	1	1	0	0	0
H_{4i} :	0	0	1	1	1
H_{5i} :	1	0	1	0	0
H_{6i} :	0	1	0	1	1

Table 1

$\phi(c) = \sum_{H\in\mathcal{C}^\star(H_0)} a_H$. Now, for any n and $H = \langle x_1, x_3\rangle$, $\langle x_2, x_3\rangle$ or $\langle x_1x_2, x_3\rangle$ we have $m_H = k_H = 2$, and so $a_H = 1$. For n even we have $m_H = n-1$ and $k_H = n-2$ for $H = \langle x_1, x_2x_3\rangle$, $\langle x_2, x_1x_3\rangle$ or $\langle x_1x_2, x_1x_3\rangle$, and $m_H = n-3$ and $k_H = n-4$ for $H = \langle x_1, x_2\rangle$; while for n odd we have $m_H = n-2$ and $k_H = n-3$ for $H = \langle x_1, x_2\rangle$, $\langle x_1, x_2x_3\rangle$ or $\langle x_2, x_1x_3\rangle$, and $m_H = n$ and $k_H = n-1$ for $H = \langle x_1x_3, x_2x_3\rangle$; in all these cases, $a_H = 0$. This gives $\phi(c) = 1$, completing the proof. ■

Theorem 8.8 *Let $d = 5$, and let Γ be the Petersen graph with the $G(5)$-coloring of Example 1.8. Then*

$$H_1(\tilde{M}) \cong \bigoplus_{H\in\mathcal{C}^\star} H_1(M_H) \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_4^4 \oplus \mathbb{Z}_2^2.$$

Proof Let $S = \bigoplus_{H\in\mathcal{C}^\star} H_1(M_H)$. This coloring is 5-taut, and therefore taut. By Lemmas 5.7 and 6.5, we may identify S and $H_1\big(D(k)\big)$ $(1 \le k \le 4)$ with their images in $H_1(\tilde{M})$, so we have a filtration

$$S \le H_1(\operatorname{Im}\beta) = H_1\big(D(4)\big) \le H_1\big(D(3)\big) \le H_1\big(D(2)\big) \le H_1\big(D(1)\big) = H_1(\tilde{M}).$$

Moreover, $H_1\big(D(4)\big)/S \cong \mathbb{Z}_2$, and there are exact sequences

$$0 \longrightarrow H_1\big(D(4)\big)/S \longrightarrow H_1\big(D(3)\big)/S \longrightarrow \mathbb{Z}_2^6 \longrightarrow 0, \tag{8.9}$$

$$0 \longrightarrow H_1\big(D(3)\big)/S \longrightarrow H_1\big(D(2)\big)/S \longrightarrow \mathbb{Z}_2^6 \longrightarrow 0, \quad \text{and} \tag{8.10}$$

$$0 \longrightarrow H_1\big(D(2)\big)/S \longrightarrow H_1(\tilde{M})/S \longrightarrow \mathbb{Z}_2 \longrightarrow 0. \tag{8.11}$$

We show first that $H_1(\tilde{M})/S$ has an element of order 16. Applying Lemma 8.5 to the edges $\tau_0, \ldots, \tau_4$, we see that it is enough to show that

$$z = \sum_{i=0}^{4} \sum_{H\in\mathcal{C}^\star} \delta_H(x_{i-1}x_{i+2})\nu_i H \in C_0'(\Gamma \mid 4)$$

represents a non-zero element of $H_0(\Gamma \mid 4)$. Now $C'(\Gamma \mid 4)$ is a subcomplex of $C'(\Gamma \mid 5) = \bigoplus_{H\in\mathcal{C}^\star} C(\Gamma_H;\mathbb{Z}_2)$. For each $H \in \mathcal{C}^\star$, $\delta_H(x_{i-1}x_{i+2})$ is non-zero for an even number of i $(0 \le i \le 4)$, and so z is a boundary in $C'(\Gamma \mid 5)$. Let $c \in C'(\Gamma \mid 5)$, with

	τ_i	τ_{i+1}	τ_{i+2}	τ_{i+3}	τ_{i+4}	ρ_i	ρ_{i+1}	ρ_{i+2}	ρ_{i+3}	ρ_{i+4}
H_{1i}, H_{2i} :	0	1	0	1	0	1	1	0	0	0
H_{3i}, H_{4i} :	0	1	1	1	1	1	0	1	0	0
H_{5i}, H_{6i} :	1	1	0	0	0	1	1	1	1	0

Table 2

$c = \sum_{\sigma \in S_1(\Gamma), H \in \mathcal{C}^\star} c(\sigma, H)\sigma H$, and set $\phi(c) = \sum_{i=0}^{4} \sum_{H \in \mathcal{C}^\star} c(\tau_i, H) \in \mathbb{Z}_2$. If c is a cycle, or if $c \in C'(\Gamma \mid 4)$, then $\phi(c) = 0$. Thus if we can find $c \in C'(\Gamma \mid 5)$ with $\partial c = z$ and $\phi(c) = 1$, it will follow that z represents a non-zero element of $H_0(\Gamma \mid 4)$.

For $H \in \mathcal{C}^\star$, let $z_H = \sum_{i=0}^{4} \delta_H(x_{i-1}x_{i+2})v_i \in C_0(\Gamma_H; \mathbb{Z}_2)$. If $c_H \in C_1(\Gamma_H; \mathbb{Z}_2)$ has $\partial c_H = z_H$, then $c = \sum_{H \in \mathcal{C}^\star} c_H H \in C'(\Gamma \mid 5)$ has $\partial c = z$. Let $H_0 \in \mathcal{C}^\star$ have $\delta_{H_0}(x_i) = 1$ for all i. Then Γ_{H_0} is the outer rim and $z_{H_0} = 0$, so we may take $c_{H_0} = 0$. The remaining elements of $\mathcal{C}^\star$ may be numbered as H_{ji}, $1 \le j \le 6$ and $0 \le i \le 4$. In Table 1, we list for $H = H_{ji}$ the values of δ_H on the basis $x_0, \ldots, x_4$; it will be apparent from the table that we have listed every $H \ne H_0$. The x_i are the colors on the σ_i; in Table 2 we list the values of the δ_H on the colors of the other edges. We can read off the 0-chains z_H from these tables; we list these below, together with $c_H \in C_1(\Gamma; \mathbb{Z}_2)$ with $\partial c_H = z_H$; reference to the tables will show that in fact $c_H \in C_1(\Gamma_H; \mathbb{Z}_2)$.

$$H = H_{1i}: \quad z_H = v_{i+1} + v_{i+3}, \quad c_H = \tau_{i+1};$$

$$H = H_{2i}: \quad z_H = v_{i+1} + v_{i+3}, \quad c_H = \tau_{i+1};$$

$$H = H_{3i}: \quad z_H = v_{i+1} + v_{i+2} + v_{i+3} + v_{i+4}, \quad c_H = \tau_{i+1} + \tau_{i+2};$$

$$H = H_{4i}: \quad z_H = v_{i+1} + v_{i+2} + v_{i+3} + v_{i+4}, \quad c_H = \tau_{i+1} + \tau_{i+2};$$

$$H = H_{5i}: \quad z_H = v_i + v_{i+1}, \quad c_H = \rho_i + \sigma_i + \rho_{i+1};$$

$$H = H_{6i}: \quad z_H = v_i + v_{i+1}, \quad c_H = \tau_i + \rho_{i+2} + \sigma_{i+1} + \rho_{i+1}.$$

Now $\phi(\sum_{H \in \mathcal{C}^\star} c_H H) = 1$, and the proof that $H_1(\tilde{M})/S$ has an element of order 16 is complete. It follows that $H_1\big(D(3)\big)/S$ has an element of order 4. Since $H_1\big(D(4)\big)/S \cong \mathbb{Z}_2$, the sequence (8.9) gives $H_1\big(D(3)\big)/S \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2^5$. Also, $H_1\big(D(2)\big)/S$ has an element of order 8, so the sequence (8.10) gives $H_1\big(D(2)\big)/S \cong \mathbb{Z}_8 \oplus \mathbb{Z}_4^a \oplus \mathbb{Z}_2^b$ for some a and b with $2a + b = 10$, and then (8.11) gives $H_1(\tilde{M})/S \cong \mathbb{Z}_{16} \oplus \mathbb{Z}_4^a \oplus \mathbb{Z}_2^b$. Since S has odd order, we have $H_1(\tilde{M}) \cong S \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_4^a \oplus \mathbb{Z}_2^b$.

Consider the tower of coverings $\tilde{M} \to M_2 \to M_1 \to M_0 \to M$ corresponding to the chain of subgroups $1 \le \langle x_0x_1, x_4x_0 \rangle \le \langle x_0x_1, x_2x_3, x_4x_0 \rangle \le H_0 \le G$. Here $\tilde{M} \to M_2$ has group $\langle x_0x_1, x_4x_0 \rangle \cong \mathbb{Z}_2^2$ and the others are 2-fold. Now M_0 is a $\mathbb{Z}_2$ homology sphere, and the branch set Δ_{H_0} of $\tilde{M} \to M_0$ depends on the mod 2 linking number of the inner and outer rims of Γ in M. If this number is 0 then Δ_{H_0} is a graph with vertices $w_0, \ldots, w_4, w_0', \ldots, w_4'$ and edges $\{w_i, w_{i+2}\}$ colored $x_{i-1}x_{i+2}$, $\{w_i', w_{i+2}'\}$ also colored $x_{i-1}x_{i+2}$, and $\{w_i, w_i'\}$ colored $x_{i-1}x_i$. If the linking number is 1 then Δ_{H_0} is obtained from that graph by replacing, say, the edges $\{w_0, w_2\}$ and $\{w_0', w_2'\}$ by $\{w_0, w_2'\}$ and $\{w_0', w_2\}$. In either case, the branch set of $M_1 \to M_0$ is a Hamiltonian circuit in Δ_{H_0}, and the edges not on this circuit are $\{w_3, w_3'\}$, $\{w_1, w_1'\}$, $\{w_0, w_0'\}$, $\{w_2, w_4\}$ and $\{w_2', w_4'\}$. Therefore M_1 is a $\mathbb{Z}_2$ homology sphere and the branch set of $\tilde{M} \to M_1$ is a link L^1 of 5 components. We number the components as L_0^1, L_1^1, L_2^1, L_{31}^1 and L_{32}^1, where L_0^1 has color x_2x_3, L_1^1 has color x_0x_1, L_2^1 has color x_4x_0, and L_{31}^1 and L_{32}^1 have color x_1x_4. By Lemma 7.10, we have $\mathrm{Lk}(L_i^1, L_{3j}^1) = 1$ for $0 \le i \le 2$ and $j = 1$ or 2, and the linking number of any other pair of components except for $\{L_{31}^1, L_{32}^1\}$ is 0. Now the branch set of $M_2 \to M_1$ is L_0^1, so M_2 is a $\mathbb{Z}_2$ homology sphere. For $i = 1$ or 2, L_i^1 is covered by two simple closed curves L_{i1}^2 and L_{i2}^2 in M_2, while L_{3i}^1 is covered by a single curve L_{3i}^2. The branch set of $\tilde{M} \to M_2$ is the link with these six components. There is a surface F in M_1 with $\partial F = L_1^1$ and disjoint from L_2^1. Its inverse image in M_2 shows that $\mathrm{Lk}(L_{11}^2, L_{21}^2) = \mathrm{Lk}(L_{12}^2, L_{21}^2)$ and $\mathrm{Lk}(L_{11}^2, L_{22}^2) = \mathrm{Lk}(L_{12}^2, L_{22}^2)$. Switching the roles of L_1^1 and L_2^1 shows that all four of these linking numbers are equal. Now, for i, $j = 1$ or 2, there is a surface F' in M_1 with $\partial F' = L_{3j}^1$ that meets L_i^1 in a single point. Its inverse image shows that $\mathrm{Lk}(L_{i1}^2, L_{3j}^2) = \mathrm{Lk}(L_{i2}^2, L_{3j}^2) = 1$. These linking numbers determine the matrix Λ of Lemma 7.4 for the covering $\tilde{M} \to M_2$; it has rank 2, and it follows that $H_1(\tilde{M}; \mathbb{Z}_2) \cong \mathbb{Z}_2^7$. Hence $a + b = 6$, so $a = 4$ and $b = 2$, and we are done. ∎

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