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THE SET OF SOLUTIONS OF INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract

In this paper, we first prove an existence theorem for the integrodifferential equation

$$\begin{cases} x'(t) = f\left(t, x(t), \int_0^t k(t, s, x(s))\, ds\right), & t \in I_a = [0, a],\ a \in R_+, \\ x(0) = x_0 \end{cases} \tag{$*$}$$

where f, k, x are functions with values in a Banach space E and the integral is taken in the sense of Henstock–Kurzweil–Pettis. In the second part of the paper we show that the set S of all solutions of the problem ($*$) is compact and connected in $(C(I_d, E), \omega)$, where $I_d \subset I_a$.

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1. Introduction

In this paper, we prove an existence theorem for the problem

$$\begin{cases} x'(t) = f\left(t, x(t), \int_0^t k(t, s, x(s))\, ds\right), & t \in I_a = [0, a],\ a \in R_+,\ x_0 \in E, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where f, k, x are functions with values in a Banach space E and the integral is taken in the sense of Henstock–Kurzweil–Pettis [13].

The Henstock–Kurzweil integral encompasses the Newton, Riemann and Lebesgue integrals [17, 19, 24]. A particular feature of this integral is that the integral of highly oscillating functions such as $F'(t)$, where $F(t) = t^2 \sin t^{-2}$ on $(0, 1]$ and $F(0) = 0$, can be defined. This integral was introduced by Henstock and Kurzweil independently in 1957–58 and has since proved useful in the study of ordinary differential equations [4, 8, 9, 23]. In [12] the authors defined the Henstock–Kurzweil–Pettis integral, which is a generalization of the Henstock–Kurzweil integral and the Pettis integral.

The existence theorem presented in this paper is an extension of previous results, for example [1–3, 14, 20, 21, 25, 27, 28].

Let $(E, \|\cdot\|)$ be a Banach space and let E^* be the dual space. Moreover, let $(C(I_a, E), \omega)$ denote the space of all continuous functions from I_a to E endowed with the topology $\sigma(C(I_a, E), C(I_a, E)^*)$.

In this paper we prove that the set S of all solutions of the integrodifferential equation (1.1) on $I_d = [0, d]$, $0 < d \leq a$, is connected and compact in $(C(I_d, E), \omega)$. This problem was investigated by Cichoń and Kubiaczyk [11, 22], Szufla [30] and others.

Let us recall, that a function $f : I_a \to E$ is said to be *weakly continuous* if it is continuous from I_a to E endowed with its weak topology. A function $g : E \to E_1$, where E and E_1 are Banach spaces, is said to be *weakly–weakly sequentially continuous* if for each weakly convergent sequence $(x_n) \subset E$, the sequence $(g(x_n))$ is weakly convergent in E_1. If a sequence x_n tends weakly to x_0 in E we denote it by $x_n \to^{\omega} x_0$.

The fundamental tool in this paper is the measure of weak noncompactness developed by DeBlasi [6] and Banaś and Rivero [5].

Let A be a bounded nonempty subset of E.

The *measure of weak noncompactness* $\beta(A)$ is defined by

$$\beta(A) = \inf\{t > 0 \mid \text{there exists } C \in K^{\omega} \text{such that } A \subset C + tB_0\},$$

where K^{ω} is a set of weakly compact subsets of E and B_0 is a norm unit ball in E. Some properties of the measure of weak noncompactness $\beta(A)$ are:

(i) if $A \subset B$, then $\beta(A) \leq \beta(B)$;
(ii) $\beta(A) = \beta(\overline{A^w})$, where $\overline{A^w}$ denotes the weak closure of A;
(iii) $\beta(A) = 0$ if and only if A is relatively weakly compact;
(iv) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$;
(v) $\beta(\lambda A) = |\lambda|\beta(A)$ $(\lambda \in R)$;
(vi) $\beta(A + B) \leq \beta(A) + \beta(B)$;
(vii) $\beta(\overline{\text{conv}}(A)) = \beta(\text{conv}(A)) = \beta(A)$, where $\text{conv}(A)$ denotes the convex hull of A.

LEMMA 1.1 [26]. *Let $H \subset C(I_a, E)$ be a family of strongly equicontinuous functions. Let for $t \in I_a$, $H(t) = \{h(t) \in E,\ h \in H\}$. Then $\beta(H(I_a)) = \sup_{t \in I_a} \beta(H(t))$ and the function $t \to \beta(H(t))$ is continuous.*

LEMMA 1.2 [11]. *Let (X, d) be a metric space and let $f : X \to (E, \omega)$ be sequentially continuous. If $A \subset X$ is a connected subset in X, then $f(A)$ is the connected subset in (E, ω).*

Fix $x^* \in E^*$ and consider the problem

$$\begin{cases} (x^*x)' = x^*\Big(f(t, x(t), \int_0^t k(t, s, x(s))\, ds\Big), \\ x(0) = x_0, \end{cases} \qquad t \in I_a,\ x_0 \in E. \tag{1.2}$$

Let us introduce the following definitions.

DEFINITION 1.3 [29]. Let $F : [a, b] \to E$ and let $A \subset [a, b]$. The function $f : A \to E$ is a *pseudoderivative of* F on A if for each $x^* \in E^*$ the real-valued function $x^* F$ is differentiable almost everywhere on A.

It is clear that the left-hand side of (1.2) can be rewritten in the form $x^*(x'(t))$ where x' denotes the pseudoderivative.

DEFINITION 1.4 [17, 24]. A family $\mathbf{F}$ of functions F is said to be *uniformly absolutely continuous* in the restricted sense on A (or uniformly $AC^*(A)$ for short), if for every $\varepsilon > 0$ there is $\eta > 0$ such that for every F in $\mathbf{F}$ and for every finite or infinite sequence of nonoverlapping intervals $\{[a_i, b_i]\}$ with $a_i, b_i \in A$ and satisfying $\sum_i |b_i - a_i| < \eta$, we have $\sum_i \omega(F, [a_i, b_i]) < \varepsilon$ where $\omega(F, [a_i, b_i])$ denotes the oscillation of F over $[a_i, b_i]$ (that is, $\omega(F, [a_i, b_i] = \sup\{|F(r) - F(s)| : r, s \in [a_i, b_i]\})$. A family $\mathbf{F}$ of functions F is said to be *uniformly generalized absolutely continuous* in the restricted sense on $[a, b]$ or uniformly ACG^* on $[a, b]$ if $[a, b]$ is the union of a sequence of closed sets A_i such that on each A_i, the family $\mathbf{F}$ is uniformly $AC^*(A_i)$.

2. Henstock–Kurzweil–Pettis integrals in Banach spaces

In this section we present a definition of the Henstock–Kurzweil–Pettis integral, which is a generalization of both Pettis and Henstock–Kurzweil integrals. For basic definitions we refer the reader to [17] or [24].

DEFINITION 2.1 [17, 24]. Let δ be a positive function defined on the interval $[a, b]$. A tagged interval $(x, [c, d])$ consists of an interval $[c, d] \subseteq [a, b]$ and a point $x \in [c, d]$. The tagged interval $(x, [c, d])$ is subordinate to δ if $[c, d] \subseteq (x - \delta(x), x + \delta(x))$.

Let $P = \{(s_i, [c_i, d_i]) \mid 1 \le i \le n,\ n \in N\}$ be such a collection in $[a, b]$. Then:

(i) the points $\{s_i \mid 1 \le i \le n\}$ are called the *tags* of P;
(ii) the intervals $\{[c_i, d_i] \mid 1 \le i \le n\}$ are called the *intervals* of P;
(iii) if $\{(s_i, [c_i, d_i]) \mid 1 \le i \le n\}$ is subordinate to δ for each i, then we write P is *sub-*δ;
(iv) if $[a, b] = \bigcup_{i=1}^n [c_i, d_i]$, then P is called a *tagged partition* of $[a, b]$;
(v) if P is a tagged partition of $[a, b]$ and if P is sub-δ, then we write P is *sub-*δ *on* $[a, b]$;
(vi) if $f : [a, b] \to E$, then $f(P) = \sum_{i=1}^n f(s_i)(d_i - c_i)$;
(vii) if F is defined on the subintervals of $[a, b]$, then

$$F(P) = \sum_{i=1}^n F([c_i, d_i]) = \sum_{i=1}^n [F(d_i) - F(c_i)].$$

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If $F:[a,b]\to E$, then F can be treated as a function of intervals by defining $F([c,d]) = F(d) - F(c)$. For such a function, $F(P) = F(b) - F(a)$ if P is a tagged partition of $[a,b]$.

DEFINITION 2.2 [17, 24]. A function $f:[a,b]\to R$ is Henstock–Kurzweil integrable on $[a,b]$ if there exists a real number L with the following property: for each $\varepsilon > 0$ there exists a positive function δ on $[a,b]$ such that $|f(P) - L| < \varepsilon$ whenever P is a tagged partition of $[a,b]$ that is subordinate to δ.

The function f is Henstock–Kurzweil integrable on a measurable set $A\subset[a,b]$ if $f\chi_A$ is Henstock–Kurzweil integrable on $[a,b]$. The number L is called the *Henstock–Kurzweil integral* of f. We denote this integral by $(HK)\int_a^b f(t)\,dt$.

DEFINITION 2.3 [7]. A function $f:[a,b]\to E$ is *Henstock–Kurzweil integrable* on $[a,b]$ ($f\in HK([a,b],E)$) if there exists a vector $z\in E$ with the following property: for every $\varepsilon > 0$ there exists a positive function δ on $[a,b]$ such that $\|f(P) - z\| < \varepsilon$ whenever P is a tagged partition of $[a,b]$ sub-δ. The function f is Henstock–Kurzweil integrable on a measurable set $A\subset[a,b]$ if $f\chi_A$ is Henstock–Kurzweil integrable on $[a,b]$. The vector z is the *Henstock–Kurzweil integral of f*.

Note that this definition includes the generalized Riemann integral defined by Gordon [18].

DEFINITION 2.4 [7]. A function $f:[a,b]\to E$ is *HL integrable* on $[a,b]$ ($f\in HL([a,b],E)$) if there exists a function $F:[a,b]\to E$, defined on the subintervals of $[a,b]$, satisfying the following property: given $\varepsilon>0$ there exists a positive function δ on $[a,b]$ such that if $P=\{(s_i,[c_i,d_i])\mid 1\le i\le n\}$ is a tagged partition of $[a,b]$ sub-δ, then

$$\sum_{i=1}^{n}\|f(s_i)(d_i-c_i) - F([c_i,d_i])\| < \varepsilon.$$

REMARK 2.5. We note that by the triangle inequality:

$$f\in HL([a,b],E) \quad \text{implies } f\in HK([a,b],E).$$

In general, the converse is not true. For real-valued functions, the two integrals are equivalent.

DEFINITION 2.6 [29]. The function $f: I_a\to E$ is *Pettis integrable* (P integrable for short) if:

(i) for all $x^*\in E^*$, x^*f is Lebesgue integrable on I_α; and
(ii) for all $A\subset I_a$, A-measurable, there exists $g\in E$ for all $x^*\in E^*$ such that $x^*g = (L)\int_A x^*f(s)\,ds$.

Now we present a definition of the integral which is a generalization both Pettis and Henstock–Kurzweil integrals.

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DEFINITION 2.7 [13]. The function $f : I_a \to E$ is *Henstock–Kurzweil–Pettis integrable* (HKP integrable for short) if there exists a function $g : I_a \to E$ with the following properties:

(i) for all $x^* \in E^*$, $x^* f$ is Henstock–Kurzweil integrable on I_a; and
(ii) for all $t \in I_a$ and all $x^* \in E^*$, $x^* g(t) = (HK) \int_0^t x^* f(s)\, ds$.

This function g is called a *primitive of* f and by $g(a) = \int_0^a f(t)\, dt$ we denote the *Henstock–Kurzweil–Pettis integral of* f on the interval I_a.

REMARK 2.8. Each function which is HL integrable is integrable in the sense of Henstock–Kurzweil–Pettis. This notion of an integral is essentially more general than previous notions (in Banach spaces).

(i) Pettis integral: by the definition of the Pettis integral and since each Lebesgue integrable function is HK integrable, a P integrable function is clearly HKP integrable;
(ii) Bochner, Riemann, and Riemann–Pettis integrals [18];
(iii) MsShane integral [16];
(iv) Henstock–Kurzweil (HL) integral [7].

We present below an example of function which is HKP integrable but neither HL integrable nor P integrable.

EXAMPLE. Let $f : [0, 1] \to (L^\infty[0, 1], \|\cdot\|_\infty)$ and let $f(t) = \chi_{[0,t]} + A(t) \cdot F'(t)$, where

$$F(t) = t^2 \sin t^{-2}, \quad t \in (0, 1], \quad F(0) = 0, \quad \chi_{[0,t]}(\tau) = \begin{cases} 1, & \tau \in [0, t], \\ 0, & \tau \notin [0, t], \end{cases} \quad t, \tau \in [0, 1],$$

and $A(t)(\tau) = 1$ for $\tau, t \in [0, 1]$.

Put $f_1(t) = \chi_{[0,t]}$, $f_2(t) = A(t) \cdot F'(t)$.

We show that a function $f(t) = f_1(t) + f_2(t)$ is integrable in the sense of Henstock–Kurzweil–Pettis.

Observe that

$$x^*(f(t)) = x^*(f_1(t) + f_2(t)) = x^*(f_1(t)) + x^*(f_2(t)).$$

Moreover, the function $x^*(f_1(t))$ is Lebesgue integrable (in fact f_1 is P integrable [15], so is Henstock–Kurzweil integrable, and the function $x^*(f_2(t))$is Henstock–Kurzweil integrable by Definition 2.2.

For each $x^* \in E^*$ the function $x^* f$ is not Lebesgue integrable because $x^* f_2$ is not Lebesgue integrable. Thus, f is not P integrable. Moreover, the function f_1 is not strongly measurable [15] and the function f_2 is strongly measurable. Thus, their sum f is not strongly measurable. Then, by [7, Theorem 9], f is not HL integrable.

Now we list some properties of the HKP integral which are important in the next sections of our paper.

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THEOREM 2.9. *Let $f : [a, b] \to E$ be HKP integrable on $[a, b]$ and let*

$$F(x) = \int_a^x f(s)\,ds, \quad x \in [a, b].$$

Then:

(i) *for each x^* in E^* the function $x^* f$ is HK integrable on $[a, b]$ and*

$$(HK)\int_a^x x^*(f(s))\,ds = x^*(F(x));$$

(ii) *the function F is weakly continuous on $[a, b]$ and f is a pseudoderivative of F on $[a, b]$.*

THEOREM 2.10 [13]. *Let $f : [a, b] \to E$. If $f = 0$ almost everywhere on $[a, b]$, then f is HKP integrable on $[a, b]$ and $\int_a^b f(t)\,dt = 0$.*

THEOREM 2.11 ([13] Mean value theorem for the HKP integral). *If the function $f : I_a \to E$ is HKP integrable, then*

$$\int_I f(t)\,dt \in |I| \cdot \overline{\text{conv}}\, f(I),$$

where I is an arbitrary subinterval of I_a and $|I|$ is the length of I.

THEOREM 2.12 [10]. *Let $f : I_a \to E$ and assume that $f_n : I_a \to E$, $n \in N$ are HKP integrable on I_a. Let F_n be a primitive of f_n. If we assume that:*

(i) *for all $x^* \in E^*$, $x^*(f_n(t)) \to x^*(f(t))$ almost everywhere on I_a;*
(ii) *for each $x^* \in E^*$ the family $G = \{x^* F_n \mid n = 1, 2, \ldots\}$ is uniformly ACG^* on I_a (that is, weakly uniformly ACG^* on I_a);*
(iii) *for each $x^* \in E^*$ the set G is equicontinuous on I_a;*

then f is HKP integrable on I_a and $\int_0^t f_n(s)\,ds$ tends weakly in E to $\int_0^t f(s)\,ds$ for each $t \in I_a$.

3. An existence result for integrodifferential equations in the weak sense

In this section, we prove an existence theorem for problem (1.1).

Let

$$B = \{x \in E : \|x\| \le \|x_0\| + b,\ b > 0\},$$
$$\widetilde{B} = \{x \in (C(I_a, E), \omega) : x(0) = x_0,\ \|x\| \le \|x_0\| + b,\ b > 0\}.$$

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Moreover, let

$$F(x)(t) = x_0 + \int_0^t f\left(z, x(z), \int_0^z k(z, s, x(s))\, ds\right) dz,\ t \in I_a,$$
$$K = \{F(x) \mid x \in \widetilde{B}\},$$
$$K_1 = \left\{ \int_0^z k(z, s, x(s))\, ds \mid z \in [0, t],\ t \in [0, a],\ x \in \widetilde{B}\right\}.$$

We consider the problem

$$x(t) = x_0 + \int_0^t f\left(z, x(z), \int_0^z k(z, s, x(s))\, ds\right) dz, \quad t \in I_a, \tag{3.1}$$

where the integrals are taken in the sense of HKP.

To obtain the existence result and to investigate the structure of a solution set for our problem it is necessary to define the notion of a solution.

DEFINITION 3.1 [29]. A function $x : I_a \to E$ is said to be *a pseudo-solution* of the problem (1) if it satisfies the following conditions:

(i) $x(\cdot)$ is ACG_* function;
(ii) $x(0) = x_0$;
(iii) for each $x^* \in E^*$ there exists a set $A(x^*)$ with Lebesgue measure zero, such that for each $t \notin A(x^*)$,

$$(x^*x)'(t) = x^*\left(f\left(t, x(t), \int_0^t k(t, s, x(s))\, ds\right)\right).$$

Here $'$ denotes the pseudoderivative.

In the proof of the main theorem we shall apply the following result.

THEOREM 3.2 [22]. *Let E be a metrizable locally convex topological vector space. Let D be a closed convex subset of E, and let F be a weakly–weakly sequentially continuous map of D into itself. If, for some $x \in D$, the implication*

$$\overline{V} = \overline{\text{conv}}(\{x\} \cup F(V)) \text{ implies } V \text{ is relatively weakly compact}, \tag{3.2}$$

holds for every subset V of D, then F has a fixed point.

THEOREM 3.3. *Assume that for each uniformly ACG^* function $x : I_a \to E$, the functions $k(\cdot, s, x(s))$, $f(\cdot, x(\cdot), \int_0^{(\cdot)} k(\cdot, s, x(s))\, ds)$ are HKP integrable and $k(t, s, \cdot)$, $f(t, \cdot, \cdot)$ are weakly–weakly sequentially continuous functions. Suppose that there exists constants $c_1, c_2, c_3 > 0$ such that*

$$\beta(f(I, A, C)) \le c_1 \cdot \beta(A) + c_2 \cdot \beta(C) \quad \text{for each } A, C \subset B,\ I \subset I_a, \tag{3.3}$$
$$\beta(k(I, I, X)) \le c_3 \cdot \beta(X) \quad \text{for each } X \subset B,\ I \subset I_a, \tag{3.4}$$

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where

$$f(I, A, C) = \{f(t, x_1, x_2) \mid (t, x_1, x_2) \in I \times A \times C\},$$
$$k(I, I, X) = \{k(t, s, x) \mid (t, s, x) \in I \times I \times X\}$$

and β denotes the measure of weak noncompactness of DeBlasi.

Moreover, let K and K_1 be equicontinuous and uniformly ACG on I_a. Then there exists a pseudo-solution of the problem (1.1) on I_d, for some $0 < d \le a$, $0 < d \cdot c_1 + d^2 \cdot c_2 \cdot c_3 < 1$.*

PROOF. Fix an arbitrary $b \ge 0$. Put

$$B = \{x \in E : \|x\| \le \|x_0\| + b,\ b > 0\},$$
$$\widetilde{B} = \{x \in (C(I_d, E), \omega) : x(0) = x_0,\ \|x\| \le \|x_0\| + b,\ b > 0\},$$

where d is given below.

Recall that a continuous function $F(x) \in K$ defined on $[0, a]$ is equicontinuous on $[0, a]$, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|F(x)(t) - F(x)(\tau)\| < \varepsilon$, for all $x \in \widetilde{B}$, whenever $|t - \tau| < \delta$ and $t, \tau \in [0, a]$. Thus, for each $\varepsilon > 0$ there exists $\delta > 0$, such that

$$\left\| \int_\tau^t f\left(z, x(z), \int_0^z k(z, s, x(s))\, ds\right) dz \right\| < \varepsilon,$$

for all $x \in \widetilde{B}$, whenever $|t - \tau| < \delta$ and $t, \tau \in [0, a]$. As a result, there exists a number d, $0 < d \le a$, such that

$$\left\| \int_0^t f\left(z, x(z), \int_0^z k(z, s, x(s))\, ds\right) dz \right\| \le b, \qquad t \in I_d \text{ and } x \in \widetilde{B}.$$

We now show that the operator F is well defined and maps $\widetilde{B}$ into $\widetilde{B}$. To see this note that, for any $x^* \in E^*$, such that $\|x^*\| \le 1$, for any $x \in \widetilde{B}$ and $t \in I_d$

$$\begin{aligned}
|x^* F(x)(t)| &\le |x^* x_0| + \left| x^* \int_0^t f\left(z, x(z), \int_0^z k(z, s, x(s))\, ds\right) dz \right| \\
&\le \|x^*\|\|x_0\| + \|x^*\| \left\| \int_0^t f\left(z, x(z), \int_0^z k(z, s, x(s))\, ds\right) dz \right\| \\
&\le \|x_0\| + b,
\end{aligned}$$

so

$$\sup\{|x^* F(x)(t)| :\ x^* \in E^*,\ \|x^*\| \le 1\} \le \|x_0\| + b$$

and as a result

$$\|F(x)(t)\| \le \|x_0\| + b,$$

so $F(x)(t) \in \widetilde{B}$.

We shall show that the operator F is weakly–weakly sequentially continuous. By [26, Lemma 9] a sequence $x_n(\cdot)$ is weakly convergent in $C(I_d, E)$ to $x(\cdot)$ if

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and only if $x_n(t)$ tends weakly to $x(t)$, for each $t \in I_d$, so if $x_n \xrightarrow{\omega} x$ in $C(I_d, E)$, then $k(t, s, x_n(s)) \xrightarrow{\omega} k(t, s, x(s))$ in E for $t \in I_d$ and by Theorem 2.12 (see our assumptions on K_1) we have

$$\lim_{n\to\infty} \int_0^t k(z, s, x_n(s))\, ds = \int_0^t k(z, s, x(s))\, ds$$

weakly in E, for each $t \in I_d$. Moreover, because f is weakly–weakly sequentially continuous,

$$f\left(t, x_n(t), \int_0^t k(t, s, x_n(s))\, ds\right) \xrightarrow{\omega} f\left(t, x(t), \int_0^t k(t, s, x(s))\, ds\right)$$

in E, for each $t \in I_d$. Thus, Theorem 2.12 (see our assumptions on K) implies $F(x_n)(t) \to F(x)(t)$ weakly in E, for each $t \in I_d$, so [26, Lemma 9] guarantees that $F(x_n) \to F(x)$ in $(C(I_d, E), \omega)$.

Suppose that $V \subset \widetilde{B}$ satisfies the condition $\overline{V} = \overline{\text{conv}}(\{x\} \cup F(V))$. We shall prove that V is relatively weakly compact and so (3.2) is satisfied. Since $V \subset \widetilde{B}$, $F(V) \subset K$. Then $V \subset \overline{V} = \overline{\text{conv}}(\{x\} \cup F(V))$ is equicontinuous. By Lemma 1.1, $t \mapsto v(t) = \beta(V(t))$ is continuous on I_d.

For fixed $t \in I_d$ we divide the interval $[0, t]$ into m parts: $0 = t_0 < t_1 < \cdots < t_m = t$, where $t_i = it/m$, $i = 0, 1, \ldots, m$ and for fixed $z \in [0, t]$ we divide the interval $[0, z]$ into m parts: $0 = z_0 < z_1 < \cdots < z_m = z$, where $z_j = jz/m$, $j = 0, 1, \ldots, m$.

Let $V([z_j, z_{j+1}]) = \{u(s) \mid u \in V, z_j \le s \le z_{j+1}\}$, $j = 0, 1, \ldots, m-1$. By Lemma 1.1 and the continuity of v there exists $s_j \in I_j = [z_j, z_{j+1}]$ such that

$$\beta(V([z_j, z_{j+1}])) = \sup\{\beta(V(s)) \mid z_j \le s \le z_{j+1}\} =: v(s_j).$$

By Theorem 2.11 and the properties of the HKP integral we have, for $x \in V$, that

$$\begin{aligned} F(x)(t) &= x_0 + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} f\left(z, x(z), \sum_{j=0}^{m-1} \int_{z_j}^{z_{j+1}} k(z, s, x(s))\, ds\right) dz \\ &\in x_0 + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\overline{\text{conv}}\, f\left(J_i, V(J_i), \sum_{j=0}^{m-1} (z_{j+1} - z_j)\overline{\text{conv}}\, k(I_j, I_j, V(I_j))\right), \end{aligned}$$

where $J_i = [t_i, t_{i+1}]$, $i = 0, 1, \ldots, m-1$.

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Using (3.3), (3.4) and properties of the measure of weak noncompactness we obtain

$$\begin{aligned}
\beta(F(V)(t)) &\le \sum_{i=0}^{m-1}(t_{i+1}-t_i)\beta\Big(f\Big(J_i, V(J_i), \sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{\text{conv}}\, k(I_j, I_j, V(I_j))\Big)\Big)\\
&\le \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot c_1\cdot\beta(V(J_i))\\
&\quad + \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot c_2\cdot\beta\Big(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{\text{conv}}\, k(I_j, I_j, V(I_j))\Big)\\
&\le \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot c_1\cdot\beta(V(I_d))\\
&\quad + \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot c_2\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)\beta(k(I_j, I_j, V(I_j))\\
&\le \beta(V(I_d))\cdot c_1\cdot d + \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot c_2\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)\cdot c_3\cdot\beta(V(I_j))\\
&\le \beta(V(I_d))\cdot c_1\cdot d + \beta(V(I_d))\cdot c_2\cdot c_3\cdot d^2 = \beta(V(I_d))(c_1\cdot d + c_2\cdot c_3\cdot d^2).
\end{aligned}$$

Because $\overline{V} = \overline{\text{conv}}(\{x\}\cup F(V))$, then $\beta(V(t)) = \beta(\overline{\text{conv}}(\{x\}\cup F(V(t))))$, so $\beta(V(t)) \le \beta(V(I_d))(c_1\cdot d + c_2\cdot c_3\cdot d^2)$, for $t\in I_d$.

Using Lemma 1.1 we obtain

$$\beta(V(I_d)) \le \beta(V(I_d))(c_1\cdot d + c_2\cdot c_3\cdot d^2)\beta(V(I_d)) \le \beta(V(I_d))(c_1\cdot d + c_2\cdot c_3\cdot d^2).$$

Since $0 < d\cdot c_1 + d^2\cdot c_2\cdot c_3 < 1$ we obtain $v(t) = \beta(V(t)) = 0$, for $t\in I_d$.

Using Arzela–Ascoli's theorem, we have that V is relatively weakly compact.

By Theorem 3.2 the operator F has a fixed point. This means that there exists a pseudo-solution of the problem (1.1). □

THEOREM 3.4. *Assume that for each uniformly ACG* function $x : I_a \to E$, the functions $k(\cdot, s, x(s))$, $f(\cdot, x(\cdot), \int_0^{(\cdot)} k(\cdot, s, x(s))\,ds)$ are HKP integrable and $k(t, s, \cdot)$, $f(t, \cdot, \cdot)$ are weakly–weakly sequentially continuous functions. Suppose that there exists a constant $c > 0$ and a continuous function $c_1 : I_a \to R_+$ such that*

$$\beta(f(I, A, C)) \le c\cdot\beta(C) \quad \text{for each } A, C\subset B,\ I\subset I_a, \tag{3.5}$$

$$\beta(k(I, I, X)) \le \sup_{s\in I} c_1(s)\beta(X) \quad \text{for each } X\subset B,\ I\subset I_a, \tag{3.6}$$

where

$$f(I, A, C) = \{f(t, x_1, x_2) \mid (t, x_1, x_2)\in I\times A\times C\},$$
$$k(I, I, X) = \{k(t, s, x) \mid (t, s, x)\in I\times I\times X\}$$

and β denotes the measure of weak noncompactness of DeBlasi.

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Moreover, let K and K_1 be equicontinuous and uniformly ACG on I_a. Then there exists a pseudo-solution of the problem* (1.1) *on I_d, for some $0 < d \le a$.*

PROOF. The first part of the proof is the same as in the proof of the previous theorem. It remains to show the relative weak compactness of V, where V is defined in Theorem 3.3. In this case note that for $t \in I_d$ and z_j as in Theorem 3.3

$$\begin{aligned}
\beta(V(t)) &\le \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot c\cdot\beta\Big(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\overline{\operatorname{conv}}\,k(I_j, I_j, V(I_j))\Big)\\
&\le \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot c\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)\beta(k(I_j, I_j, V(I_j)))\\
&\le \sum_{i=0}^{m-1}(t_{i+1}-t_i)\cdot c\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)\sup_{s\in I_j} c_1(s)\beta(V(I_j))\\
&\le c\cdot d\cdot\sum_{j=0}^{m-1}(z_{j+1}-z_j)\cdot c_1(p_j)v(s_j)\\
&= c\cdot d\Big(\sum_{j=0}^{m-1}(z_{j+1}-z_j)\cdot c_1(p_j)v(p_j)\\
&\qquad + \sum_{j=0}^{m-1}(z_{j+1}-z_j)(c_1(p_j)(v(s_j)-v(p_j)))\Big),
\end{aligned}$$

for some $p_j \in I_j$. Fix $\varepsilon > 0$. From the continuity of v we may choose m large enough so that $v(s_j) - v(p_j) < \varepsilon$ and so

$$\begin{aligned}
\beta(V(t)) &\le c\cdot d\Big(\sum_{j=0}^{m-1}(z_{j+1}-z_j)c_1(p_j)v(p_j) + \sum_{j=0}^{m-1}\frac{z}{m}c_1(p_j)\cdot\varepsilon\Big)\\
&\le c\cdot d\Big(\sum_{j=0}^{m-1}(z_{j+1}-z_j)c_1(p_j)v(p_j) + z\cdot\varepsilon\cdot\max_{0\le k\le m-1} c_1(p_k)\Big).
\end{aligned}$$

Since $\varepsilon \to 0$ and $z\cdot\max_{0\le k\le m-1} c_1(p_k)$ is bounded,

$$z\cdot\varepsilon\cdot\max_{0\le k\le m-1} c_1(p_k) \to 0.$$

Therefore

$$v(t) = \beta(V(t)) \le c\cdot d\cdot\int_0^t c_1(s)v(s)\,ds, \qquad t\in[0,d].$$

Using the Gronwall's inequality we have that

$$v(t) = \beta(V(t)) = 0 \quad \text{for } t \in [0, d].$$

Using Arzela–Ascoli's theorem we deduce that V is relatively weakly compact.

By Theorem 3.2 the operator F has a fixed point. This means that there exists a pseudo-solution of the problem (1.1). □

4. Compactness and connectedness

In this section we show that the set S of all solutions of the problem (1.1) on I_d is compact and connected in $(C(I_d, E), \omega)$.

THEOREM 4.1. *Under the assumptions of Theorem 3.3 a set S of all pseudo-solutions of the problem (1.1) on I_d is weakly compact and connected in $(C(I_d, E), \omega)$.*

PROOF. Let S be a set of all solutions of the problem (1.1) on I_d. As $S = F(S)$, by repeating the above argument, with $V = S$ one can show that S is relatively weakly compact in $(C(I_d, E), \omega)$. Since F is weakly continuous on $\overline{S(I_d)}^{\omega}$, S is weakly closed and consequently weakly compact.

Now we prove that S is connected. For any $\eta > 0$, denote by S_η, the set of all functions $u : I_d \to E$ satisfying the following conditions:

(i) $u(0) = x_0$, $u \in \widetilde{B}$,

(ii) $\sup_{t \in I_d} \|u(t) - x_0 - \int_0^t f(z, x(z), \int_0^z k(z, s, x(s))\, ds)\, dz\| < \eta$.

The set S_η is nonempty as $S \subset S_\eta$.

Let $\eta^* < \eta$. By the equicontinuity of K we can choose ρ such that

$$\left\| \int_J f\left(z, x(z), \int_0^z k(z, s, x(s))\, ds\right) dz \right\| \le \eta^* < \eta,$$

for any $x \in (C(I_d, E), \omega)$, $J \subset I_d$ and $|J| < \rho$.

For any $\varepsilon \in (0, d)$, let $v(\cdot, \varepsilon) : I_d \to E$ be defined by the formula:

$$v(t, \varepsilon) = \begin{cases} x_0 & \text{for } 0 \le t \le \varepsilon \\ x_0 + \displaystyle\int_0^{t-\varepsilon} f\left(z, x(z), \int_0^z k(z, s, v(s, \varepsilon))\, ds\right) dz & \text{for } \varepsilon < t \le d. \end{cases}$$

Clearly $v(\cdot, \varepsilon)$ satisfies (i) above. Furthermore, for $0 < \varepsilon \le \min(\rho, d) = l$

$$\left\| v(t, \varepsilon) - x_0 - \int_0^t f\left(z, x(z), \int_0^z k(z, s, v(s, \varepsilon))\, ds\right) dz \right\|$$

$$= \left\{ \begin{matrix} \left\| \displaystyle\int_0^t f\left(z, x(z), \int_0^z k(z, s, v(s, \varepsilon))\, ds\right) dz \right\| & \text{for } 0 \le t \le \varepsilon \\ \left\| \displaystyle\int_{t-\varepsilon}^t f(z, x(z), \int_0^z k(z, s, v(s, \varepsilon))\, ds\Big) dz \right\| & \text{for } \varepsilon < t \le d. \end{matrix} , z \in J \right\} \le \eta^* < \eta,$$

thus $v(\cdot, \varepsilon)$ satisfies (ii) above.

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Now, we prove that S_η is connected. Let us define

$$v_\varepsilon(t) = \begin{cases} x_0, & 0 \le t \le \varepsilon \\ F(v_\varepsilon)(t-\varepsilon), & \varepsilon < t \le d, \end{cases}$$

where $v_\varepsilon = v(\cdot, \varepsilon)$. We show that the mapping $\varepsilon \to v_\varepsilon(\cdot)$ is sequentially continuous from $(0, d)$ into $(C(I_d, E), \omega)$.

Let $0 < \varepsilon < \delta \le d$ (when $\delta \le \varepsilon$ the argument is similar). Let $x^* \in E^*$ be such that $\|x^*\| \le 1$. Now by the definition of $v_\varepsilon(t)$, for $t \in [0, \varepsilon]$

$$|x^*(v_\varepsilon(t) - v_\delta(t))| = 0. \tag{4.1}$$

Next, if $t \in (\varepsilon, \delta]$,

$$\begin{aligned} |x^*(v_\varepsilon(t) - v_\delta(t))| &= \left| x^* \left[\int_0^{t-\varepsilon} f\left(z, x(z), \int_0^z k(z, s, v_\varepsilon(s))\, ds\right) dz \right.\right. \\ &\qquad \left.\left. - \int_0^{t-\delta} f\left(z, x(z), \int_0^z k(z, s, v_\delta(s))\, ds\right) dz \right] \right| \\ &= \left| x^* \int_{t-\delta}^{t-\varepsilon} f\left(z, x(z), \int_0^z k(z, s, v_\varepsilon(s))\, ds\right) dz \right| \\ &= \|x^*\| \left\| \int_{t-\delta}^{t-\varepsilon} f\left(z, x(z), \int_0^z k(z, s, v_\varepsilon(s))\, ds\right) dz \right\| \\ &\le \left\| \int_{t-\delta}^{t-\varepsilon} f\left(z, x(z), \int_0^z k(z, s, v_\varepsilon(s))\, ds\right) dz \right\|. \end{aligned} \tag{4.2}$$

Consequently

$$|x^*(v_\varepsilon(t) - v_\delta(t))| \le \left\| \int_{t-\delta}^{t-\varepsilon} f\left(z, x(z), \int_0^z k(z, s, v_\varepsilon(s))\, ds\right) dz \right\| := A_\delta.$$

Since K is equicontinuous, note that if $\delta \to \varepsilon$, then $A_\delta \to 0$.

Now, for $t \in (\delta, 2\delta]$, we have

$$\begin{aligned} &|x^*(v_\varepsilon(t) - v_\delta(t))| \\ &\quad = \left| x^* \left[\int_0^{t-\varepsilon} f\left(z, x(z), \int_0^z k(z, s, v_\varepsilon(s))\, ds\right) dz \right.\right. \\ &\qquad \left.\left. - \int_0^{t-\delta} f\left(z, x(z), \int_0^z k(z, s, v_\delta(s))\, ds\right) dz \right] \right| \\ &\quad = |x^*(F(v_\varepsilon)(t-\varepsilon) - F(v_\delta)(t-\delta))| \\ &\quad = |x^*[F(v_\varepsilon)(t-\varepsilon) - F(v_\varepsilon)(t-\delta) + F(v_\varepsilon)(t-\delta) - F(v_\delta)(t-\delta)]| \\ &\quad \le |x^*(F(v_\varepsilon)(t-\varepsilon) - F(v_\varepsilon)(t-\delta))| + |x^*(F(v_\varepsilon)(t-\delta) - F(v_\delta)(t-\delta))| \\ &\quad \le \|x^*\| \|F(v_\varepsilon)(t-\varepsilon) - F(v_\varepsilon)(t-\delta)\| + \|x^*\| \|F(v_\varepsilon)(t-\delta) - F(v_\delta)(t-\delta)\|. \end{aligned} \tag{4.3}$$

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Let (δ_n) be a sequence such that $\delta_n \to \varepsilon (\varepsilon \leq \delta_n)$. By (4.1) and (4.2) it follows that $v_{\delta_n}(t)$ converges to $v_\varepsilon(t)$ weakly uniformly for $t \in [0, \delta]$. Thus, $F(v_{\delta_n})(t) \to F(v_\varepsilon)(t)$ weakly on $[0, \delta]$. Now, by (4.3) $v_{\delta_n}(t)$ tends to $v_\varepsilon(t)$ weakly for each $t \in [0, 2\delta]$.

By repeating the above argument and using induction, we obtain that the map $\varepsilon \to v_\varepsilon(\cdot)$ from $(0, d)$ into $(C(I_d, E), \omega)$ is sequentially continuous [26, Lemma 1.9]. Therefore, by Lemma 1.2 the set $V = \{v_\varepsilon(\cdot) \mid 0 < \varepsilon < d\}$ is connected in $(C(I_d, E), \omega)$ (because the interval $[0, d]$ is connected).

Let $x \in S_\eta$. Let us choose $\varepsilon > 0$ such that $0 < \varepsilon < d$ and

$$\sup_{t \in I_d} \left\| x(t) - x_0 - \int_0^t f\left(z, x(z), \int_0^z k(z, s, x(s))\, ds\right) dzt \right\| + \left\| \int_{I_\varepsilon} f\left(z, x(z), \int_0^z k(z, s, x(s))\, ds\right) dz \right\| < \eta. \tag{4.4}$$

For any p, $0 \leq p \leq d$, let $y(\cdot, p) : I_d \to E$ be defined by the formula:

$$[y(t, p) = \begin{cases} x(t), & \text{for } 0 \leq t \leq p \\ x(p) + \dfrac{x_0 - x(p)}{\varepsilon}(t - p), & \text{for } p < t \leq \min(d, p + \varepsilon) \\ x_0 + \displaystyle\int_p^{t-\varepsilon} f\left(z, x(z), \int_0^z k(z, s, y(s, p))\, ds\right) dz, & \text{for } \min(d, p + \varepsilon) < t < d \end{cases} \Bigg\}$$

$y(t, 0) = v(t, \varepsilon)$.

By repeating the above argument with $y(\cdot, p)$ in the place of $v(\cdot, \varepsilon)$ one can show that $y(\cdot, p) \in S_\eta$, for each $p \in [0, d]$ and the mapping $p \to y(\cdot, p)$ from I_d into $(C(I_d, E), \omega)$ is sequentially continuous (for more details see [22, 30]).

Consequently, by Lemma 1.2 the set $T_x = \{y(\cdot, p) \mid 0 \leq p \leq d\}$ is connected in $(C(I_d, E), \omega)$.

Now since $y(\cdot, 0) = v(\cdot, \varepsilon) \in V \cap T_x$, the set $V \cup T_x$ is connected and therefore the set $W = \bigcup_{x \in S_\eta} T_x \cup V$ is connected in $(C(I_d, E), \omega)$.

Moreover, $S_\eta \subset W$, because $x = y(\cdot, p) \in T_x$, for each $x \in S_\eta$. On the other hand $W \subset S_\eta$, since $T_x \subset S_\eta$ and $V \subset S_\eta$. Thus, $S_\eta = W$ is a connected subset of $(C(I_d, E), \omega)$.

Now, suppose that the set S is not connected. As S is weakly compact, there exists nonempty weakly compact sets W_1 and W_2, such that $S = W_1 \cup W_2$ and $S = W_1 \cup W_2$. Consequently, there exist two disjoint weakly open sets U_1, U_2, such that $W_1 \cap W_2 = \emptyset$, $W_2 \subset U_2$. Suppose that, for every $n \in N$, there exists $u_n \in V_n \setminus U$, where $V_n = \overline{S}^{\omega}_{1/n}$ and $U = U_1 \cup U_2$. Note that V_n is a decreasing sequence of nonempty weakly compact connected subsets of $(C(I_d, E), \omega)$.

Let $H = \overline{\{u_n \mid n \in N\}}^{\omega}$. Note that $u_n - F(u_n) \to 0$ in $(C(I_d, E), \omega)$ as $n \to \infty$ and $H(t) \subset \{u_n(t) - F(u_n)(t) \mid u_n \in H\} + F(H)(t)$. By repeating the argument from the proof of Theorem 3.3, one can show that there exists $u_0 \in H$ such that $u_0 = F(u_0)$, that is, $u_0 \in S$.

Now since $u_n \in V_n \setminus U$ and U is weakly open we have $u_0 \notin U$. This contradicts $u_0 \in S \subset U$.

Therefore, there exists $m \in N$ such that $V_m \subset U = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$. Now since $S \subset V_m$, we have that $U_1 \cap V_m \neq \emptyset \neq U_2 \cap V_m$. Thus, V_m is not connected, a contradiction with the connectedness of each V_n. Consequently, S is connected in $(C(I_d, E), \omega)$. □

References

[1] R. P. Agarwal, M. Meehan and D. O'Regan, 'Positive solutions of singular integral equations—a survey', *Dynam. Systems Appl.* **14**(1) (2005), 1–37.

[2] ———, *Nonlinear Integral Equations and Inclusions* (Nova Science Publishers, Hauppauge, NY, 2001).

[3] R. P. Agarwal and D. O'Regan, 'Existence results for singular integral equations of Fredholm type', *Appl. Math. Lett.* **13**(2) (2000), 27–34.

[4] Z. Artstein, 'Topological dynamics of ordinary differential equations and Kurzweil equations', *J. Differential Equations* **23** (1977), 224–243.

[5] J. Banaś and J. Rivero, 'On measures of weak noncompactness', *Ann. Mat. Pura Appl.* **125** (1987), 213–224.

[6] F. S. DeBlasi, 'On a property of the unit sphere in a Banach space', *Bull. Math. Soc. Sci. Math. R. S. Roumanie* **21** (1977), 259–262.

[7] S. S. Cao, 'The Henstock integral for Banach valued functions', *SEA Bull. Math.* **16** (1992), 36–40.

[8] T. S. Chew, 'On Kurzweil generalized ordinary differential equations', *J. Differential Equations* **76** (1988), 286–293.

[9] T. S. Chew and F. Flordeliza, 'On $x' = f(t, x)$ and Henstock–Kurzweil integrals', *Differential Integral Equations* **4** (1991), 861–868.

[10] M. Cichoń, 'Convergence theorems for the Henstock–Kurzweil–Pettis integral', *Acta Math. Hungarica* **92** (2001), 75–82.

[11] M. Cichoń and I. Kubiaczyk, 'On the set of solutions of the Cauchy problem in Banach spaces', *Arch. Math.* **63** (1994), 251–257.

[12] M. Cichoń, I. Kubiaczyk and A. Sikorska, 'Henstock–Kurzweil and Henstock–Kurzweil–Pettis integrals and some existence theorems', in: *Proc. ISCM Herlany 1999*. 2000, pp. 53–56.

[13] ———, 'The Henstock–Kurzweil–Pettis integrals and existence theorems for the Cauchy problem', *Czechoslovak Math. J.* **54** (2004), 279–289.

[14] F. Cramer, V. Lakshmikantham and A. R. Mitchell, 'On the existence of weak solution of differential equations in nonreflexive Banach spaces', *Nonlinear. Anal., TMA* **2** (1978), 169–177.

[15] R. F. Geitz, 'Pettis integration', *Proc. Amer. Math. Soc.* **82** (1991), 81–86.

[16] R. A. Gordon, 'The McShane integral of Banach-valued functions', *Illinois J. Math.* **34** (1990), 557–567.

[17] ———, *The Integrals of Lebesgue, Denjoy, Perron and Henstock* (American Mathematical Society, Providence, RI, 1994).

[18] ———, 'Riemann integration in Banach spaces', *Rocky Mountain J. Math.* **21** (1991), 923–949.

[19] R. Henstock, *The General Theory of Integration*, Oxford Math. Monographs (Clarendon, Oxford, 1991).

[20] J. Januszewski, 'On the existence of continuous solutions of nonlinear integral equations in Banach spaces', *Comment. Math.* **30** (1990), 85–92.

[21] S. Krzyśka, 'On the existence of continuous solutions of Urysohn and Volterra integral equations in Banach spaces', *Demonstratio Math.* **2** (1995), 353–359.

[22] I. Kubiaczyk, 'Kneser type theorems for ordinary differential equations in Banach spaces', *J. Differential Equations* **45** (1982), 139–146.

[23] I. Kubiaczyk and A. Sikorska, 'Differential equations in Banach spaces and Henstock–Kurzweil integrals', *Discuss. Math. Differ. Incl.* **19** (1999), 35–43.

[24] P. Y. Lee, *Lanzhou Lectures on Henstock Integration*, Ser. Real. Anal., 2 (World Scientific, Singapore, 1989).
[25] M. Meehan and D. O'Regan, 'Existence theory for nonlinear Volterra integrodifferential and integral equations', *Nonlinear Anal. TMA* **31**(3/4) (1998), 317–341.
[26] A. R. Mitchell and Ch. Smith, 'An existence theorem for weak solutions of differential equations in Banach spaces', in: *Nonlinear Equations in Abstract Spaces* (Academic Press, New York, 1978), pp. 387–404.
[27] D. O'Regan, 'Existence results for nonlinear integral equations', *J. Math. Anal. Appl.* **192**(3) (1995), 705–726.
[28] D. O'Regan and M. Meehan, *Existence Theory for Integral and Integrodifferential Equations*, Mathematics and Its Applications, 445 (Kluwer Academic, Dordrecht, 1998).
[29] B. J. Pettis, 'On integration in vector spaces', *Trans. Amer. Math. Soc.* **44** (1938), 277–304.
[30] S. Szufla, 'Kneser's theorem for weak solutions for ordinary differential equations in reflexive Banach spaces', *Bull. Polish Acad. Sci. Math.* **26** (1978), 407–413.

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