A Determinantal Expansion for a Class of Definite Integral

Part 4.

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1. We shall show in this part the relation of generalised C.F.'s to ordinary C.F.'s, in the main confining our attention to Stieltjes type fractions. Moreover we shall bring out the part played by Parseval's theorem in our development of the subject, and a property of extremal solutions of the Stieltjes moment problem given by M. Riesz.[1]

2. The Stieltjes moment problem[2] (S.M.P.) concerns itself with finding a bounded non-decreasing function $\psi(x)$ in $(0, \infty)$ such that

$$\int_0^\infty x^n d\psi(x) = \mu_n, \; n = 0, 1, 2, \ldots,$$

where the μ's are real. The solution offered by Stieltjes depends upon the characteristics of the C.F. *associated* with the formal expansion

$$F(z, \psi) = \int_0^\infty \frac{d\psi(x)}{z+x} \sim \frac{\mu_0}{z} - \frac{\mu}{z^2} + \frac{\mu_2}{z^3} - \ldots, \tag{1}$$

namely

$$F(z, \psi) \sim \frac{1}{az_1} + \frac{1}{a_2} + \frac{1}{a_3 z} + \frac{1}{a_4} + \frac{1}{a_5 z} + \cdots \tag{2}$$

and the *corresponding* C.F. (obtained by contraction)

$$F(z, \psi) \sim \frac{\lambda_1}{z+C_1} - \frac{\lambda_2}{z+C_2} - \frac{\lambda_3}{z+C_3} - \cdots \tag{3}$$

A necessary and sufficient condition for the existence of a solution of the S.M.P. is that $a_j > 0$, $j = 1, 2, 3, \ldots$; the solution is unique if $\sum_1^\infty a_j$ diverges. If $\sum_1^\infty a_j$ converges there may be an

[1] M. Riesz, "Sur le problème des moments," *Arkiv for matematik, astronomi och fysik*, 16 (12), 1-21 ; 16 (19), 1-21 ; 17 (16) 1-52.

[2] See, for example, J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society Surveys No. 1, 1943).

infinity of solutions; of these there are two called extremal solutions, which have the property that $\{(\bar{\omega}_{2s}(-z)\}$ and $\{\bar{\omega}_{2s+1}(-z)\}$ are orthogonal systems with respect to them, where $\bar{\chi}_s(z)/\bar{\omega}_s(z)$ is the s^{th} convergent of (2).

The moments μ_j and the elements a_j are related as follows:

$$a_{2j} = \frac{\Delta_j^2}{\Delta^1_{j-1}\,\Delta^1_j}, \qquad a_{2j+1} = \frac{\Delta'^2_j}{\Delta_j\,\Delta_{j+1}} \tag{4}$$

where $\begin{cases} \Delta_j = \mid \mu_0, \mu_2, \ldots \mu_{2j-2} \mid, & \Delta'_j = \mid \mu_1, \mu_3, \ldots \mu_{2j-1} \mid \\ \quad > 0. & \quad > 0 \\ \Delta_0 = 1, \quad \Delta_1 = \mu_0, & \Delta'_0 = 1, \quad \Delta'_1 = \mu_1. \end{cases}$

Stieltjes showed that when there is a solution of the S.M.P. then

$$F(z, \psi) - \bar{\chi}_{2s}(z)/\bar{\omega}_{2s}(z) = \min_{p_s} \int_0^\infty \frac{p_s^2(x)d\psi(x)}{z+x}$$

$$z\{\bar{\chi}_{2s+1}(z)/\bar{\omega}_{2s+1}(z) - F(z, \psi)\} = \min_{p_s} \int_0^\infty \frac{p_s^2(x)xd\psi(x)}{z+x}$$

where $p_s(-z) = 1$, $z > 0$, and the minimum is taken over all polynomials of degree less than or equal to s. For our present discussion it is important to write these in the form

$$F(z, \psi) - \bar{\chi}_{2s+2}(z)/\bar{\omega}_{2s+2}(z) = \min_{\Pi_s} \int_0^\infty (z+x)d\psi(x)\left\{\frac{1}{z+x} - \Pi_s(x)\right\}^2 \tag{5}$$

$$z\{\bar{\chi}_{2s+3}(z)/\bar{\omega}_{2s+3}(z) - F(z, \psi)\} = \min_{\Pi_s} \int_0^\infty x(z+x)d\psi(x)\left\{\frac{1}{z+x} - \Pi_s(x)\right\}^2, \tag{6}$$

the forms (5) and (6) showing the obvious relation to Parseval's theorem. Indeed from a formal point of view all we have to do to find the even part of (2) is to write down the Parseval expansion for

$$\int_0^\infty f(x)^2 d\bar{\psi}(x) \quad \text{where } f(x) = (z+x)^{-1}, \quad \bar{\psi}(x) = \int_0^x (z+t)d\psi(t)$$

and the Tchebicheffian orthogonal polynomials with respect to $d\psi(x)$ are related to those for $d\bar{\psi}(x)$ by the theorem of Christoffel (Part 1, 14). Similarly the odd part of (2) follows from $\int_0^\infty f(x)^2 d\hat{\psi}(x)$ where $\hat{\psi}(x) = \int_0^x t(z+t)d\psi(t)$.

On the other hand it has been proved by Riesz (*loc. cit.*) that if $\psi(x)$ is an extremal solution of the S.M.P. (or the unique solution) then Parseval's theorem applies to all $f(x) \epsilon L^2_\psi$. An extension of this which we require is the following: If $\psi(x)$ is an extremal solution of the S.M.P., then Parseval's theorem applies to $f(x) \epsilon L^2{}_{\hat{\psi}}$ where $\hat{\psi}(x) = \int_0^x \Pi(t) d\psi(t)$ and $\Pi(t)$ is a non-negative polynomial of fixed degree which is not identically zero. Clearly $\hat{\psi}(x)$ is bounded and non-decreasing and the moments are given by

$$\mu'_s = \int_0^\infty x^s d\hat{\psi}(x) = \int^\infty x^s \Pi(x) d\psi(x)$$
$$= \sum_{(r)} \gamma_r \mu_{r+s} \text{ where } \Pi(x) = \sum_{(r)} \gamma_r x^r.$$

Thus if $\psi(x)$ is the unique solution for a given sequence $\{\mu_s\}$ (assuming such a solution exists) then $\int_0^x \Pi(t) d\psi(t)$ is the unique solution for the sequence $\{\Pi(\mu)\}$, provided $\Pi(x)$ is a non-negative polynomial. It is of interest to recall that the S.M.P. is determined in the particular cases [1]

$$\mu_s = \int_0^\infty x^s x^{b-1} \exp(-kx^a)\, dx,\ b > 0, \quad k > 0,\ a \geqq \tfrac{1}{2} \tag{7}$$

$$\mu_s = \int_0^\infty x^s f(x)\, x^{b-1} \exp(-kx^a) dx \tag{8}$$

where $f(x)$ is a positive bounded function on (a, ∞), $a > 0$. Moreover quite apart from the theory of continued fractions, Hardy [2] proved that the S.M.P. is determined for $\psi(t) = \int_0^t \phi(x) dx$, $\phi(x) \geqq 0$, provided $\int_0^\infty [\phi(t)]^q e^{\delta\sqrt{t}} dt < \infty$ for $q \geqq 1$, $\delta > 0$.

It will be noticed that the uniqueness of the moment problem $\mu_n = \int_0^\infty x^n \phi(x) dx$ depends upon the order of magnitude of $\phi(x)$ for large positive x. Thus $\phi(x) = \exp - x^{\frac{1}{4}}$ does not approach zero rapidly enough for $x \to \infty$, and the Stieltjes C.F. corresponding to $\int_0^\infty \frac{\exp - x^{\frac{1}{4}} dx}{z + x} = F(z)$ diverges by oscillation. But making the substitu-

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[1] T. J. Stieltjes, *Oeuvres Complètes*, Vol. 2, pp. 505-506, 518-520.

[2] G. H. Hardy, "On Stieltjes' 'problème des moments,'" *Messenger of Mathematics*., 46 (1917), 175-182; 47 (1917), 81-88.

tion $x = t^2$ we have $F(z) = \int_0^\infty \frac{e^{-\sqrt{t}}dt^2}{z+t^2}$ and we shall show in the sequel that the second order C.F. for $F(z)$ converges. In general it may be remarked that if $\psi(x)$ is the solution of a determined S.M.P. then the C.F. (2), with elements given by (4) in terms of the moments, converges for $z > 0$; but the Stieltjes C.F. corresponding to $\int_0^\infty \frac{d\psi(x)}{z+x^s} = F(z)$ may not converge, s being a positive integer greater than unity. However, the s^{th} order C.F. corresponding to $F(z)$ does converge in this case.

2.0 We now state some properties of the convergents of Stieltjes C.F.'s which we require. We consider the expansions

$$F(z) = \int_0^\infty \frac{d\psi(x)}{x+z} = \frac{b_1}{z} \frac{b_2}{+1} + \frac{b_3}{z} \frac{b_4}{+1} + \cdots \quad \left(= \frac{\chi_s(z)}{\omega_s(z)} \text{ as } s \to \infty\right) \tag{9}$$

$$= \frac{b_1}{z+b_2} - \frac{b_2 b_3}{z+b_3+b_4} - \frac{b_4 b_5}{z+b_5+b_6} - \cdots \quad \left(= \frac{\chi_{2s}(z)}{\omega_{2s}(z)} \text{ as } s \to \infty\right) \tag{10}$$

assumed to be convergent for $z > 0$. Then we have the recurrence relations

$$\begin{cases} \omega_{2s}(z) = (z + b_{2s-1} + b_{2s})\,\omega_{2s-2}(z) - b_{2s-1}b_{2s-2}\omega_{2s-4}(z) \\ \omega_{2s+1}(z) = (z+b_{2s}+b_{2s+1})\omega_{2s-1}(z) - b_{2s}\, b_{2s-1}\omega_{2s-3}(z) \quad s=2,3,\ldots \end{cases} \tag{11}$$

and similarly for $\chi_s(z)$ with $\chi_0 = 0$, $\chi_1 = b_1$; $\omega_0 = 1$, $\omega_1 = z$, from which we derive the determinantal relations [1]

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[1] The following abbreviated notation for alternant types of determinants will be used throughout:

$$| A_r(z_1), \quad B_s(z_2), \quad C_t(z_3) \overset{+}{|} = \begin{vmatrix} A_r(z_1) & B_r(z_2) & C_r(z_3) \\ A_s(z_1) & B_s(z_2) & C_s(z_3) \\ A_t(z_1) & B_t(z_2) & C_t(z_3) \end{vmatrix}$$

where any functional symbol cannot be separated from its argument.
Thus

$$| \chi_{2r}(z_1), \omega_{2r+2}(z_2) \overset{+}{|} = \begin{vmatrix} \chi_{2r}(z_1) & \omega_{2r}(z_2) \\ \chi_{2r+2}(z_1) & \omega_{2r+2}(z_2) \end{vmatrix}$$

but $| \omega_{2r}(z_1), \omega^{2r+2}(z_2) |$ is unambiguous. Similarly when the symbol of functionality is tied to its suffix we shall write

$$\overset{+}{|} A_r(z_1), B_s(z_2), C_t(z_3) | = \begin{vmatrix} A_r(z_1) & B_s(z_1) & C_t(z_1) \\ A_r(z_2) & B_s(z_2) & C_t(z_2) \\ A_r(z_3) & B_s(z_3) & C_t(z_3) \end{vmatrix}.$$

Thus $$\overset{+}{|} f_l(x), p_r(z) | = \begin{vmatrix} f_l(x) & p_r(x) \\ f_l(z) & p_r(z) \end{vmatrix}.$$

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$$
\begin{aligned}
&\left| \chi_{2s+2}(z),\ \omega_{2s}(z) \right|^{+} = \prod_{\nu=1}^{2s+1} b_\nu \\
&\left| \chi_{2s+1}(z),\ \omega_{2s+3}(z) \right|^{+} = z \prod_{\nu=1}^{2s+2} b_\nu \qquad (12)\\
&\left| \chi_{2s+4}(z),\ \omega_{2s}(z) \right|^{+} = (z + b_{2s+3} + b_{2s+4}) \prod_{\nu=1}^{2s+1} b_\nu \\
&\left| \chi_{2s+1}(z),\ \omega_{2s+5}(z) \right|^{+} = z(z + b_{2s+4} + b_{2s+5}) \prod_{\nu=1}^{2s+2} b_\nu\ .
\end{aligned}
$$

The relation between the even and odd convergents is given by[1]

$$
\begin{aligned}
&\left| \omega_{2s}(0),\ \chi_{2s+2}(z) \right|^{+} = \omega_{2s}(0)\chi_{2s+1}(z), \qquad \omega_{2s}(0) = \prod_{r=1}^{s} b_{2r}\,, \qquad (13)\\
&\left| \omega_{2s}(0),\ \omega_{2s+2}(z) \right|^{+} = \omega_{2s}(0)\omega_{2s+1}(z), \qquad \chi_{2s+1}(0) = \prod_{r=0}^{s} b_{2r+1}\,.
\end{aligned}
$$

From (12) it is easily proved that

$$
\left\{
\begin{aligned}
&\left| \chi_{2r}(z_1),\ \omega_{2r+2}(z_1),\ \omega_{2r+4}(z_2) \right|^{+} = (z_1 - z_2)\ \omega_{2r+2}(z_2) \prod_{s=1}^{2r+1} b_s \\
&\left| \chi_{2r+1}(z_1),\ \omega_{2r+3}(z_1),\ \omega_{2r+5}(z_2) \right|^{+} = z_1(z_2 - z_1)\ \omega_{2r+3}(z_2) \prod_{s=1}^{2r+2} b_s.
\end{aligned}
\right. \qquad (14)
$$

2.1 *The orthonormal polynomials.* We introduce the system $\{p_r(x)\}$ where

$$
\int_0^\infty p_r(x)p_s(x)\,d\psi(x) = \delta_{rs}, \qquad (15)
$$

with recurrence relation

$$
\begin{aligned}
&p_r(x) = (A_r x - B_r)\,p_{r-1}(x) - C_r p_{r-2}(x),\ p_{-1} = 0, \qquad r = 1, 2, \ldots, \\
&p_0(x) = k_0, \quad p_1(x) = (A_1 x - B_1)k_0 \qquad (16)
\end{aligned}
$$

and $A_r = k_r/k_{r-1} > 0, \quad C_r = A_r/A_{r-1} > 0,$
where $k_r > 0$ is the highest coefficient in $p_r(x)$. Clearly $B_r > 0$, for

$$
B_r = A_r \int^\infty x p^2_{r-1}(x) d\psi(x).
$$

Moreover

$$
\left\{
\begin{aligned}
&B_r/A_r = b_{2r} + b_{2r-1}, \qquad && B_1/A_1 = b_2 \\
&A_r^{-2} = b_{2r+1}b_{2r}, && A_1^{-2} = b_3 b_2 \\
&k_r^{-2} = \prod_{s=1}^{2r+1} b_s
\end{aligned}
\right. \qquad (17)
$$

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[1] See J. Shohat, "On Stieltjes Continued Fractions," *American Journal of Math.*, LIV. (1932), 79-84.

by comparison with (11). We also require

$$\begin{cases} k_r\omega_{2r}(z) = (-)^r p_r(-z) \\ k_r\chi_{2r}(z) = (-)^r \int_0^\infty \dfrac{p_r(-z) - p_r(x)}{z+x}\, d\psi(x) \end{cases} \qquad (18)$$

$$r = 0, 1, 2, \ldots .$$

3. *A fundamental identity.* We shall now prove an identity which relates the generalised convergents of

$$F(z_1, z_2, \ldots z_n) = \int_0^\infty \frac{d\psi(x)}{f_n(x)},$$

where $f_n(x) = \prod_{\lambda=1}^{n} (x + z_\lambda)$, to those of $F(z)$.

We consider

$$\Delta_j(x) = \overset{+}{|}\, f_j(x),\ p_r(-z_1),\ p_{r+1}(-z_2), \ldots p_{r+n-1}(-z_n) \,| \qquad (19)$$

$$= f_n(x) \sum_{s=0}^{r-1} \bar{A}_{sj}(r) q_s(x) \qquad j = l, m;\ r = 1, 2, \ldots .,$$

where

$$f_l(x) = \prod_{\lambda=1}^{l} (x + x_\lambda), \qquad f_m(x) = \prod_{\lambda=1}^{m} (x + y_\lambda), \qquad l < n,\ m \leqslant n,$$

$f_l(x), f_m(x), f_n(x)$ are polynomials in x with real coefficients, $f_n(x) > 0$ for $x \geqslant 0$ with distinct roots, and

$$\int_0^\infty q_r(x) q_s(x) f_n(x) d\psi(x) = 0 \qquad r \neq s$$

$$= \phi_r \qquad r = s.$$

But from Part 1, paragraph 4, we have

$$q_r(x) = (-)^n \frac{k_{r+n}}{k_r} \sum_{s=0}^{r} |\, p_s(-z_1), p_{r+1}(-z_2), \ldots p_{r+n-1}(-z_n) \,| p_s(x) \quad (20)$$

$$\phi_r = (-)^n \frac{k_{r+n}}{k_r} |\, p_r(-z_1), \ldots p_{r+n-1}(-z_n) \,|\, |\, p_{r+1}(-z_1), \ldots p_{r+n}(-z_n) \,|. \quad (21)$$

If now $\psi(x)$ is a solution of a determined S.M.P., and the integrals $\int_0^\infty \frac{(f_l(x))^2}{f_n(x)} d\psi(x)$ and $\int_0^\infty \frac{(f_m(x))^2}{f_n(x)} d\psi(x)$ converge, then Parseval's theorem applies to the functions $f_l(x)/f_n(x)$ and $f_m(x)/f_n(x)$ giving, with respect to the distribution function $\int_0^x f_n(t) d\psi(t)$,

$$F(z_1, z_2, \ldots z_n) = \int_0^\infty \frac{f_l(x) f_m(x)}{f_n(x)} d\psi(x) = \sum_{j=0}^{\infty} A_{lj} A_{mj} \phi_j \qquad (22)$$

where

$$A_{kj}\phi_j = \int_0^\infty f_k(x) q_j(x) d\psi(x), \qquad k = l, m, \qquad (23)$$

$$j = 0, 1, 2, \ldots .$$

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We also write

$$\sum_{j=0}^{r-1} A_{lj}A_{mj}\,\phi_j = \frac{\chi_r(z_1, z_2, \ldots z_n)}{\omega_r(z_1, z_2, \ldots z_n)} \equiv \frac{\chi_r(z_\lambda)_1^n}{\omega_r(z_\lambda)_1^n} \tag{24}$$

and (24) gives the r^{th} convergent of the n^{th} order C.F. for $F(z_1, z_2, \ldots z_n)$.

But from (19)

$$\int_0^\infty \frac{\Delta_l(x)\Delta_m(x)}{f_n(x)}\,d\psi(x) = \sum_{j=0}^{r-1} \bar{A}_{lj}(r)\bar{A}_{mj}(r)\phi_j, \qquad r = 1, 2, \ldots. \tag{25}$$

where $\bar{A}_{kj}(r) = \mid p_r(-z_1)\ldots\ldots p_{r+n-1}(-z_n) \mid A_{kj} \qquad k = l, m,$

$j = 0, 1, \ldots r-1.$

Hence from (24)

$$\frac{\chi_r(z_\lambda)_1^n}{\omega_r(z_\lambda)_1^n} = \mid p_r(-z_1), \ldots p_{r+n-1}(-z_n) \mid^{-1} \int_0^\infty \frac{\Delta_m(x)f_l(x)}{f_n(x)}\,d\psi(x). \tag{26}$$

But since $l < n$ and the roots of $f_n(x)$ are distinct, we have

$$f_l(x)/f_n(x) = (-)^{n-1}\frac{\overset{+}{\left| z_1^0, z_2^1, \ldots z_{n-1}^{n-2}, \dfrac{f_l(-z_n)}{z_n+x} \right|}}{\left| z_1^0, z_2^1, \ldots\ldots z_n^{n-1} \right|}. \tag{27}$$

Using (27) and (18) in (26) we have

$$\frac{\chi_r(z_\lambda)_1^n}{\omega_r(z_\lambda)_1^n} = \begin{vmatrix} g & f_m(-z_1) & f_m(-z_2) & \ldots & f_m(-z_n) \\ g_r & \omega_{2r}(z_1) & \omega_{2r}(z_2) & \ldots & \omega_{2r}(z_n) \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ g_{r+n-1} & \omega_{2r+2n-2}(z_1) & \omega_{2r+2n-2}(z_2) & \ldots & \omega_{2r+2n-2}(z_n) \end{vmatrix} \tag{28}$$

where

$$\begin{cases} g = (-)^n \overset{+}{\mid} z_1^0, z_2^1, \ldots z_{n-1}^{n-2}, F_m(z_n)f_l(-z_n) \mid / \mid z_1^0, z_2^1, \ldots z_n^{n-1} \mid \\ g_s = (-)^n \overset{+}{\mid} z_1^0, z_2^1, \ldots z_{n-1}^{n-2}, \chi_{2s}(z_n)f_l(-z_n) \mid / \mid z_1^0, z_2^1, \ldots z_n^{n-1} \mid, \\ \qquad\qquad s = r, r+1, \ldots r+n-1 \\ F_m(z_s) = \displaystyle\int_0^\infty \frac{f_m(-z_s) - f_m(x)}{z_s + x}\,d\psi(x), \qquad s = 1, 2, \ldots n \end{cases}$$

and (28) consists of the ratio of the determinant of order $n+1$ and the determinant obtained from this by deleting its first row and column. An alternative to (28) appears by using the partial fraction form of the elements in the first column, and we find

$$\frac{\chi_r(z_\lambda)_1^n}{\omega_r(z_\lambda)_1^n} = - \left| \begin{array}{c|cccc} \displaystyle\sum_{s=1}^{n} \frac{F_m(z_s) f_l(-z_s)}{\prod\limits_{r=1}^{n}{}' \, (z_r - z_s)} & f_m(-z_1) & f_m(-z_2) & \cdots & f_m(-z_n) \\ \hline \displaystyle\sum_{s=1}^{n} \frac{\chi_{2r}(z_s) f_l(-z_s)}{\prod\limits_{r=1}^{n}{}' \, (z_r - z_s)} & \omega_{2r}(z_1) & \omega_{2r}(z_2) & \cdots & \omega_{2r}(z_n) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \displaystyle\sum_{s=1}^{n} \frac{\chi_{2r+2n-2}(z_s) f_l(-z_s)}{\prod\limits_{r=1}^{n}{}' \, (z_r - z_s)} & \omega_{2r+2n-2}(z_1) & \omega_{2r+2n-2}(z_2) & & \omega_{2r+2n-2}(z_n) \end{array} \right| \tag{29}$$

If we take $\omega_r(z_\lambda)_1^n = \mid \omega_{2r}(z_1), \ldots \omega_{2r+2n-2}(z_n) \mid / \mid z_1^0, z_2^1, \ldots z_n^{n-1} \mid$ then (28) and (29) give expressions for $\chi_r(z_\lambda)_1^n$ in terms of $\chi_{2s}(z_\lambda)$ and $\omega_{2s}(z_\lambda)$, $s = r$ to $r + n - 1$, $\lambda = 1$ to n. We note that if in addition to $l < n$, we have $m < n$, then l and m may be interchanged in (28) and (29) yielding an identity between two forms of the numerator $\chi_r(z_\lambda)_1^n$.

The confluent case of (28) — (29) in its general form is complicated, but particular cases may be obtained from first principles. Thus if $z_1 = z_2 \ldots\ldots = z_j$, $j \leqslant n$, then the limiting form appears by letting $z_2 \to z_1$, $z_3 \to z_1$, ... $z_j \to z_1$ in succession, and subtracting appropriate columns. The complication arises from the fact that $f_l(x)$ and $f_m(x)$ may be functions of z_λ, $\lambda = 1$ to n.

4.0 We shall now consider generalised continued fraction expansions for $F(z_1, z_2, \ldots z_n)$ of four kinds, namely

(i) convergent increasing sequences,

(ii) convergent decreasing sequences,

(iii) convergent sequences,

(iv) convergent sequences involving an arbitrary parameter.

In the main we shall confine our attention to second order C.F.'s for $F(z_1, z_2)$. We assume that $\psi(x)$ is the unique solution of a S.M.P.

4.1 *Increasing sequences.* In (29) take $f_l = f_m = 1$, $f_n(x) = (x+z_1)(x+z_2) > 0$, $z_1 \neq z_2$ ($z_2 = \bar{z}_1$ if z_1 is complex) so that $F_m(z_s) = 0$, and

$$F(z_1, z_2) = \int_0^\infty \frac{d\psi(x)}{(x+z_1)(x+z_2)}$$

$$= \lim_{r\to\infty} \frac{-1}{z_1 - z_2} \left| \begin{matrix} 0 & 1 & 1 \\ \chi_{2r}(z_2) - \chi_{2r}(z_1) & \omega_{2r}(z_1) & \omega_{2r}(z_2) \\ \chi_{2r+2}(z_2) - \chi_{2r+2}(z_1) & \omega_{2r+2}(z_1) & \omega_{2r+2}(z_2) \end{matrix} \right| \Bigg/ \left| \begin{matrix} \omega_{2r}(z_1) & \omega_{2r}(z_2) \\ \omega_{2r+2}(z_1) & \omega_{2r+2}(z_2) \end{matrix} \right|$$

where the expansion, in view of (22), is an increasing sequence. By (12) this may be written

$$F(z_1, z_2) = \underset{r\to\infty}{\text{l.i.s.}} \frac{\overset{+}{|\chi_{2r}(z_1), \omega_{2r+2}(z_2)|} + \overset{+}{|\chi_{2r}(z_2), \omega_{2r+2}(z_1)|} + 2\prod_{s=1}^{2r+1} b_s}{(z_1 - z_2)\,|\omega_{2r}(z_2), \omega_{2r+2}(z_1)|} \tag{30}$$

where we use the abbreviation l.i.s. for *limit of the increasing sequence.*[1]

If in particular $z_1 = z_2 = z$ then by letting $z_2 \to z_1$ in (30) we have

$$F(z, z) = \int_0^\infty \frac{d\psi(x)}{(z+x)^2} = \underset{r\to\infty}{\text{l.i.s.}} \frac{\overset{+}{|\omega'_{2r}(z), \chi'_{2r+2}(z)|}}{|\omega_{2r}(z), \omega'_{2r+2}(z)|} \tag{31}$$

$$= -\frac{dF(z)}{dz}, \qquad z > 0.$$

The general formula of this type is found similarly from (28) and gives

$$\int_0^\infty \frac{d\psi(x)}{(z+x)^{n+1}} = \underset{r\to\infty}{\text{l.i.s.}} \frac{(-)^n}{n!} \frac{\overset{+}{|\chi_{2r}^{(n)}(z), \omega_{2r+2}^{(1)}(z), \omega_{2r+4}^{(2)}(z), \ldots \omega_{2r+n}^{(n)}(z)|}}{\overset{+}{|\omega_{2r}(z), \omega_{2r+2}^{(1)}(z), \omega_{2r+4}^{(2)}(z) \ldots \omega_{2r+2n}^{(n)}(z)|} \tag{32}$$

$$z > 0.$$

---

[1] As a particular example suppose that by using (4) and an equivalence transformation we find the convergent expansion

$$\tfrac{1}{2}\int_0^\infty \frac{xe^{-\sqrt{x}}dx}{x+z} = \lim_{r\to\infty} \frac{\chi_r(z)}{\omega_r(z)} = \frac{b_1}{z+}\,\frac{b_2}{1+}\,\frac{b_3}{z+}\,\frac{b_4}{1+}\ldots.$$

$$\sim \frac{3!}{z} - \frac{5!}{z^2} + \frac{7!}{z^3} - \frac{9!}{z^4} + \ldots.$$

Then by (30) with $z_1 = it = -z_2$, $t > 0$, we have a convergent expansion for

$$F(it, -it) = \tfrac{1}{2}\int_0^\infty \frac{xe^{-\sqrt{x}}dx}{x^2+t^2} \sim \frac{3!}{t^2} - \frac{7!}{t^4} + \frac{11!}{t^6} - \frac{15!}{t^8} + \ldots$$

But the Stieltjes C.F for $F(it, -it) = \frac{1}{4}\int_0^\infty \frac{\exp^{-x^{\frac{1}{4}}}dx}{x+t^2}$ diverges by oscillation.

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There are of course other forms of increasing sequences for $F(z_1, z_2)$; for example we could use $f_l = x$, $f_m = x$, $f_n = x^2(x+z_1)(x+z_2)$ in (28). But (30) seems to be the simplest of this type.

4. 2 *Decreasing sequences.* When the roots z_1, z_2 are distinct and $(x+z_1)(x+z_2) = x^2 + 2px + q$, $q - p^2 > 0$, then we use the relation

$$(q-p^2)F(z_1, z_2) = b_1 - \int_0^\infty \frac{(x+p)^2 d\psi(x)}{(x+z_1)(x+z_2)}. \tag{33}$$

Taking in (29) $f_l = f_m = x + p$, $f_n = (x+z_1)(x+z_2)$, $F_l(z_s) = -b_1$, we have

$$(q-p^2)\ F(z_1, z_2) = b_1 - \underset{r\to\infty}{\text{l.i.s.}} \left| \begin{array}{c|cc} b_1 & \frac{1}{2}(z_2 - z_1) & \frac{1}{2}(z_1 - z_2) \\ \hline -\frac{1}{2}(\chi_{2r}(z_1) + \chi_{2r}(z_2)) & \omega_{2r}(z_1) & \omega_{2r}(z_2) \\ -\frac{1}{2}(\chi_{2r+2}(z_1) + \chi_{2r+2}(z_2)) & \omega_{2r+2}(z_1) & \omega_{2r+2}(z_2) \end{array} \right|$$

and after using (12) this leads to [1]

$$F(z_1, z_2) = \underset{r\to\infty}{\text{l.d.s}} \frac{\overset{+}{|\chi_{2r}(z_1), \omega_{2r+2}(z_2)|} + \overset{+}{|\chi_{2r}(z_2), \omega_{2r+2}(z_1)|} - 2\prod_{s=1}^{2r+1} b_s}{(z_1 - z_2)\,|\,\omega_{2r}(z_2), \omega_{2r+2}(z_1)\,|}. \tag{34}$$

When $q - p^2 > 0$ it will be seen that the difference between corresponding convergents of (30) and (34) is

$$\frac{-4\prod_{s=1}^{2r+1} b_s}{(z_1 - z_2)\,|\,\omega_{2r}(z_2), \omega_{2r+2}(z_1)\,|} \tag{35}$$

and this exceeds the absolute error in either of them.

Again, taking $f_l = f_m = (x+p)^2$, $f_n = (x+z_1)(x+z_2)(x+p)^2$, $q - p^2 > 0$ in (28) (taking the limiting form with $z_3 = z_4 = p$), we find

$$F(z_1, z_2) = \underset{r\to\infty}{\text{l.d.s.}} \left| \begin{array}{c|cccc} 0 & 1 & 1 & 0 & 0 \\ \hline \dfrac{\chi_{2r}(z_1) - \chi_{2r}(z_2)}{z_1 - z_2} & \omega_{2r}(z_1) & \omega_{2r}(z_2) & \omega_{2r}(p) & \omega'_{2r}(p) \\ \dfrac{\chi_{2r+2}(z_1) - \chi_{2r+2}(z_2)}{z_1 - z_2} & \omega_{2r+2}(z_1) & \omega_{2r+2}(z_2) & \omega_{2r+2}(p) & \omega'_{2r+2}(p) \\ \dfrac{\chi_{2r+4}(z_1) - \chi_{2r+4}(z_2)}{z_1 - z_2} & \omega_{2r+4}(z_1) & \omega_{2r+4}(z_2) & \omega_{2r+4}(p) & \omega'_{2r+4}(p) \\ \dfrac{\chi_{2r+6}(z_1) - \chi_{2r+6}(z_2)}{z_1 - z_2} & \omega_{2r+6}(z_1) & \omega_{2r+6}(z_2) & \omega_{2r+6}(p) & \omega'_{2r+6}(p) \end{array} \right| \tag{36}$$

[1] l.d.s. means *limit of the decreasing sequence.*

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In particular if $p=0$, $q>0$, then after using (12) and (13) we find

$$\int_0^\infty \frac{d\psi(x)}{x^2+q}$$

$$= \operatorname*{l.d.s.}_{r\to\infty} \frac{\left|\chi_{2r+1}(z_1), \omega_{2r+3}(z_2), \omega'_{2r+5}(0)\right|^{+} + \left|\chi_{2r+1}(z_2), \omega_{2r+3}(z_1), \omega'_{2r+5}(0)\right|^{+} - 2q\omega'_{2r+3}(0)\prod_{s=1}^{2r+2} b_s}{(z_2-z_1)\left|\omega_{2r+1}(z_1), \omega_{2r+3}(z_2), \omega'_{2r+5}(0)\right|} \tag{37}$$

Another interesting possibility is to use the expression

$$qF(z_1, z_2) = b_1 - \int_0^\infty \frac{x(x+2p)d\psi(x)}{(x+z_1)(x+z_2)}, \qquad q>0, \tag{38}$$

where $p>0$, $(x+z_1)(x+z_2) = x^2+2px+q$, taking $f_l = f_m = x(x+2p)$, $f_n = x(x+2p)(x^2+2px+q)$ in (29) with $z_1, z_2, z_3 = 2p$, $z_4 = 0$ and $F_l(z_s) = (z_s - 2p)b_1 - a_1$

$$\sum_{s=1}^{4} \frac{F_l(z_s)f_m(-z_s)}{\prod'^{4}_{r=1}(z_r - z_s)} = -b_1, \qquad \sum_{s=1}^{4} \frac{\chi(z_s)f_m(-z_s)}{\prod'^{4}_{r=1}(z_r - z_s)} = \frac{\chi(z_1)-\chi(z_2)}{z_2 - z_1}.$$

Using (12) and (13) we find after some simplification

$$F(z_1, z_2) = \operatorname*{l.d.s.}_{r\to\infty} \frac{\left|\chi_{2r+1}(z_1), \omega_{2r+3}(z_2), \omega_{2r+5}(2p)\right|^{+} + \left|\chi_{2r+1}(z_2), \omega_{2r+3}(z_1), \omega_{2r+5}(2p)\right|^{+} - 2q\omega_{2r+3}(2p)\prod_{s=1}^{2r+2} b_s}{(z_2-z_1)\left|\omega_{2r+1}(z_1), \omega_{2r+3}(z_2), \omega_{2r+5}(2p)\right|} \tag{39}$$

By setting $z_2 = z_1 + h$, $h \to 0$, we have

$$\int_0^\infty \frac{d\psi(x)}{(x+z)^2} = \operatorname*{l.d.s.}_{r\to\infty} - \frac{\left|\chi'_{2r+1}(z), \omega'_{2r+3}(z), \omega_{2r+5}(2z)\right|^{+}}{\left|\omega_{2r+1}(z), \omega'_{2r+3}(z), \omega'_{2r+5}(2z)\right|}, \qquad z>0. \tag{40}$$

4.3 *Convergent sequences.* The approximations considered in 4.1 and 4.2 provide lower and upper bounds, but there are other approximations which merely converge. We shall briefly consider four simple types, derived from (28) – (29).

(i) $\quad f_l = x$, $f_m = 1$, $f_n = x(x+z_1)(x+z_2)$, $\quad z_1 \neq z_2$, $\quad z_1, z_2, > 0$.

Then $F(z_1, z_2) =$

$$\lim_{r\to\infty} \frac{\left|\chi_{2r+1}(z_1), \omega_{2r+3}(z_2)\right|^{+} + \left|\chi_{2r+1}(z_2), \omega_{2r+3}(z_1)\right|^{+} - (z_1+z_2)\prod_{1}^{2r+2} b_s - (z_1-z_2)(\omega_{2r+2}(z_1) - \omega_{2r+2}(z_2))\prod_{0}^{r} b_{2s+1}}{(z_1-z_2)\left|\omega_{2r+1}(z_2), \omega_{2r+3}(z_1)\right|} \tag{41}$$

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In particular if $z_1 = z_2$, using (12) we find

$$F(z, z) = \lim_{r \to \infty} \frac{\overset{+}{|\chi'_{2r+1}(z), \omega'_{2r+3}(z)|} + \omega'_{2r+2}(z) \prod_0^r b_{2s+1}}{|\omega'_{2r+1}(z), \omega_{2r+3}(z)|}. \qquad (42)$$

(ii) $q - p^2 > 0$. In (33) take $f_m = (x + p)^2, f_l = 1, f_n = x^2 + 2px + q$ and we find an expression which is exactly the same as (30). This brings to light an interesting identity, for we have the two expansions

$$\int_0^\infty \frac{(x + p)^2 d\psi(x)}{x^2 + 2px + q} = \underset{r \to \infty}{\text{l.i.s.}} \left\{ b_1 - \tfrac{1}{4}(z_1 - z_2) \frac{\overset{+}{|\chi_{2r}(z_1), \omega_{2r+2}(z_2)|} + \overset{+}{|\chi_{2r}(z_2), \omega_{2r+2}(z_1)|} - 2 \prod_1^{2r+1} b_s}{|\omega_{2r}(z_1), \omega_{2r+2}(z_2)|} \right\}$$

$$= \lim_{r \to \infty} \left\{ b_1 - \tfrac{1}{4}(z_1 - z_2) \frac{\overset{+}{|\chi_{2r}(z_1), \omega_{2r+2}(z_2)|} + \overset{+}{|\chi_{2r}(z_2), \omega_{2r+2}(z_1)|} + 2 \prod_1^{2r+1} b_s}{|\omega_{2r}(z_1), \omega_{2r+2}(z_2)|} \right\}$$

$$q - p^2 > 0 \text{ or } z_1, z_2 > 0, (z_1 \neq z_2). \qquad (43)$$

The difference between corresponding convergents of the two expansions in (43) is therefore $(z_1 - z_2) \prod_1^{2r+1} b_s / |\omega_{2r}(z_1), \omega_{2r+2}(z_2)|$. In terms of the persymmetric determinants [1] mentioned in Part 3, 3 (a) this comes to

$$\begin{vmatrix} 0 & \mu_0^{(1)} & \mu_1^{(1)} & \cdots & \mu_r^{(1)} \\ \mu_0^{(1)} & \bar{\mu}_0 & \bar{\mu}_1 & \cdots & \bar{\mu}_r \\ \mu_1^{(1)} & \bar{\mu}_1 & \bar{\mu}_2 & \cdots & \bar{\mu}_{r+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \mu_r^{(1)} & \bar{\mu}_r & \bar{\mu}_{r+1} & \cdots & \bar{\mu}_{2r} \end{vmatrix} - \begin{vmatrix} 0 & \mu_0^{(2)} & \mu_1^{(2)} & \cdots & \mu_r^{(2)} \\ \mu_0^{(0)} & \bar{\mu}_0 & \bar{\mu}_1 & \cdots & \bar{\mu}_r \\ \mu_1^{(0)} & \bar{\mu}_1 & \bar{\mu}_2 & \cdots & \bar{\mu}_{r+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \mu_r^{(0)} & \bar{\mu}_r & \bar{\mu}_{r+1} & \cdots & \bar{\mu}_{2r} \end{vmatrix} = \prod_0^{r+1} k_s^{-2} \qquad (44)$$

where

$$\mu_s^{(0)} = \int_0^\infty x^s d\psi(x), \qquad \mu_s^{(1)} = \int_0^\infty (x + p) x^s d\psi(x),$$

$$\mu_s^{(2)} = \int_0^\infty (x + p)^2 x^s d\psi(x), \qquad \bar{\mu}_s = \int_0^\infty (x^2 + 2px + q) x^s d\psi(x)$$

$$k_s^{-2} = \prod_1^{2s+1} b_r$$

and $x^2 + 2px + q$ is non-negative for $0 \leq x < \infty$.

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[1] There is a similar identity for the diagonal determinants given in Part 3, 3 (b).

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(iii) If we use (38) with $f_l = x$. $f_m = x + 2p$, $f_n = (x + z_1)(x + z_2)$ there follows the expansion

$$F(z_1, z_2) = \lim_{r \to \infty} \frac{\overset{+}{| \chi_{2r}(z_1), \omega_{2r+2}(z_2) |} + \overset{+}{| \chi_{2r}(z_2), \omega_{2r+2}(z_1) |} + (z_1 z_2)^{-1}(z_1^2 + z_2^2) \prod_1^{2r+1} b_s}{(z_1 - z_2) \,| \omega_{2r}(z_2), \omega_{2r+2}(z_1) |} \tag{45}$$

$$z_1, z_2 > 0, \qquad z_1 \neq z_2 \qquad \text{or} \qquad q - p^2 > 0,$$

and in particular

$$F(z, z) = \lim_{r \to \infty} \frac{\overset{+}{| \omega'_{2r}(z), \chi'_{2r+2}(z) |} + z^{-2} \prod_{s=1}^{2r+1} b^s}{| \omega'_{2r}(z), \omega_{2r+2}(z) |}. \tag{46}$$

(iv) Using (38) with $f_l = x$, $f_m = x(x + 2p)$, $f_n = x(x^2 + 2px + q)$ we find

$$F(z_1, z_2) = \lim_{r \to \infty} \frac{\overset{+}{| \chi_{2r+1}(z_1), \omega_{2r+3}(z_2) |} + \overset{+}{| \chi_{2r+1}(z_2), \omega_{2r+3}(z_1) |} - (z_1 + z_2) \prod_1^{2r+2} b_s}{(z_1 - z_2) \,| \omega_{2r+1}(z_2), \omega_{2r+3}(z_1) |} \tag{47}$$

$$x^2 + 2px + q > 0 \text{ for } x \geqq 0$$

and in particular

$$F(z, z) = \lim_{r \to \infty} - \frac{\overset{+}{| \chi'_{2r+1}(z), \omega'_{2r+3}(z) |}}{| \omega_{2r+1}(z), \omega'_{2r+3}(z) |}, \qquad z > 0. \tag{48}$$

4.4 *Expansions with arbitrary parameters.* An unusual type of expansion appears if in (29) we take $f_l = f_m = x + p$, $f_n = (x+z)(x+p)^2$ where we assume $z > 0$, and p real. Then

$$F(z) = \int_0^\infty \frac{d\psi(x)}{z + x} = \underset{r \to \infty}{\text{l.i.s.}} \frac{\overset{+}{| \chi_{2r}(z) - \chi_{2r}(p), \omega_{2r+2}(p), \omega'_{2r+4}(p) |} + k_r^{-2}(\omega_{2r+2}(p) - \omega_{2r+2}(z))}{| \omega_{2r}(z), \omega_{2r+2}(p), \omega'_{2r+4}(p) |} \tag{49}$$

for all real $p \neq z$.

If $p = z$ then

$$F(z) = \underset{r \to \infty}{\text{l.i.s.}} \frac{\overset{+}{| \omega_{2r}(z), \omega'_{2r+2}(z), \chi''_{2r+4}(z) |} - k_r^{-2} \omega''_{2r+2}(z)}{| \omega_{2r}(z), \omega'_{2r+2}(z), \omega''_{2r+4}(z) |}, \qquad z > 0. \tag{50}$$

Similarly from $f_l = f_m = x(x + p)$, $f_n = x(x + z)(x + p)^2$ we find

$$F(z) = \underset{r \to \infty}{\text{l.d.s.}} \frac{\overset{+}{| \chi_{2r+1}(z) - \chi_{2r+1}(p), \omega_{2r+3}(p), \omega'_{2r+5}(p) |} - pz^{-1}(z\omega_{2r+3}(p) - p\omega_{2r+3}(z)) \prod_1^{2r+2} b_s}{| \omega_{2r+1}(z), \omega_{2r+3}(p), \omega'_{2r+5}(p) |} \tag{51}$$

$$z > 0, \; p \neq 0, \; p \neq z.$$

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In particular if $p = 0$ then for $z > 0$

$$F(z) = \underset{r\to\infty}{\text{l.d.s.}} \frac{\left|\chi_{2r+1}(z)-\chi_{2r+1}(0),\ \omega'_{2r+3}(0),\ \omega''_{2r+5}(0)\right|^{+} - 2z^{-1}(z\omega'_{2r+3}(0)-\omega_{2r+3}(z))\prod_{1}^{2r+2} b_s}{\left|\omega_{2r+1}(z),\ \omega'_{2r+3}(0),\ \omega''_{2r+5}(0)\right|} \tag{52}$$

and if $p = z > 0$ then

$$F(z) = \underset{r\to\infty}{\text{l.d.s.}} \frac{\left|\omega_{2r+1}(z),\ \chi'_{2r+3}(z),\ \omega''_{2r+5}(z)\right|^{+} + 2(z^{-1}\omega_{2r+3}(z)-\omega'_{2r+3}(z))\prod_{1}^{2r+2} b_s}{\left|\omega_{2r+1}(z),\ \omega'_{2r+3}(z),\ \omega''_{2r+5}(z)\right|}. \tag{53}$$

Calling the rth convergents of (49) and (51) $g_r(z, p)$ and $\hat{g}_r(z, p)$ respectively, we may consider the sequences $\{\max_p g_r(z, p)\}$ and $\{\min_p \hat{g}^r(z, p)\}$ as approximations to $F(z)$, assuming that stationary values exist. That such values do exist is seen from the following asymptotic expansions:

$$g_r(z, p) = \frac{\chi_{2r}(z)}{\omega_{2r}(z)} - \frac{2\prod_{1}^{2r+1} b_s}{p\omega^2_{2r}(z)} + o(p^{-1}) \text{ as } |p| \to \infty \tag{54}$$

$$\hat{g}_r(z, p) = \frac{\chi_{2r+1}(z)}{\omega_{2r+1}(z)} + \frac{2z}{p}\frac{\prod_{1}^{2r+2} b_s}{\omega^2_{2r+1}(z)} + o(p^{-1}) \text{ as } |p| \to \infty. \tag{55}$$

It is evident from (54) that for p large and negative $g_r(z, p)$ is a closer approximation to $F(z)$ than $\chi_{2r}(z)/\omega_{2r}(z)$; similarly for $\hat{g}_r(z, p)$. We shall return to a consideration of these approximations in a later part.

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