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A NOTE ON PERIODIC SOLUTIONS OF A FORCED LIÉNARD-TYPE EQUATION

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Abstract

Criteria for guaranteeing the existence, uniqueness and asymptotic stability (in the sense of Liapunov) of periodic solutions of a forced Liénard-type equation under certain assumptions are presented. These criteria are obtained by application of the Manásevich–Mawhin continuation theorem, Floquet theory, Liapunov stability theory and some analysis techniques. An example is provided to demonstrate the applicability of our results.

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1. Introduction

In this note we consider the existence, uniqueness and asymptotic stability of periodic solutions of the forced Liénard-type equation

$$x''(t) + f(x(t))x'(t) + g(t, x(t)) = e(t), \tag{1.1}$$

where $f \in C^1(\mathbb{R}, \mathbb{R})$, $e \in C(\mathbb{R}, \mathbb{R})$, $g \in C^1(\mathbb{R}^2, \mathbb{R})$ and $e(t)$ and $g(t, x)$ are T-periodic functions in t with $T > 0$.

In 1927, the Dutch physical scientist van der Pol [53] described self-excited oscillations in an electrical circuit with a triode tube with resistive properties that change with the current. He originally introduced an equation of the form

$$x''(t) + a(x(t)^2 - 1)x'(t) + x(t) = 0 \tag{1.2}$$

to model the oscillations in an electrical circuit with a triode tube, and proved that (1.2) has a unique nontrivial periodic solution which is stable. This result plays an important role in radio. In 1928, motivated by the work of van der Pol [53], the French

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mathematical physicist Liénard [34] studied problems of oscillations in nonlinear mechanics and gave some criteria for guaranteeing the existence, uniqueness and stability of periodic solutions of a general class of equations as follows, for which the van der Pol equation is a special case:

$$x''(t) + f(x(t))x'(t) + x(t) = 0. \tag{1.3}$$

Equation (1.3) is referred to as a Liénard equation. In 1942, Levinson and Smith [33] investigated the existence of periodic solutions of a more general class of equations

$$x''(t) + f(x(t))x'(t) + g(x(t)) = 0, \tag{1.4}$$

which arise in many fields, such as physics, mechanics and engineering. Equation (1.4) is sometimes referred to as a Liénard-type equation or generalized Liénard equation. There has been much subsequent development. For instance, Écalle [15] and Ilyashenko [27] proved that (1.4) has finitely many limit cycles when f and g are polynomial; Zhang *et al.* [61] studied the number of limit cycles of (1.4) by applying the Poincaré–Bendixson theorem without requiring that f and g are polynomial. For more literature, see, for example, [1, 3, 4, 7, 12–14, 21, 22] and the references therein. On the other hand, some authors discussed the global asymptotic stability of the zero solution (a trivial periodic solution) of (1.4). For details, we refer the reader to [20, 25, 28, 29, 46, 49–52, 57, 60] and the references therein.

In 1943, Lefschetz [32] studied the forced Liénard-type equation

$$x''(t) + f(x(t))x'(t) + g(x(t)) = e(t), \tag{1.5}$$

and gave an existence theorem for periodic solutions, under some dissipativity conditions on f. Then, many researchers contributed to the theory of this equation with respect to the existence of periodic solutions, and systematically developed the results in [32]. For details, we refer to [5, 23, 47, 48] and the references therein.

In 1968, Lazer [30] proved the existence of periodic solutions of (1.5) with f a constant function by applying nonlinear functional analysis tools and avoiding dissipativity conditions on f. In 1972, Lazer's result was extended to (1.5) by Mawhin [41], who abandoned any restrictive condition on f except continuity. In 1977, Gaines and Mawhin [17] introduced some continuation theorems and applied them to discussing the existence of solutions of certain ordinary differential equations. A specific example was provided in [17, Page 99] on how T-periodic solutions could be obtained by means of these theorems. In the course of the derivations, it is realized that once *a priori* bounds for the T-periodic solutions of the homotopic equations are known, then standard procedures will allow these theorems to imply the existence of T-periodic solutions of the original equation. Applying these approaches, several authors [37, 39, 59, 62] discussed (1.1) with one or two variations and obtained many new results on the existence of T-periodic solutions of (1.5), and developed the results in [5, 23, 30, 32, 41, 47, 48]. In 1998, Manásevich and Mawhin [40] studied the existence of T-periodic solutions of the nonlinear system with p-Laplacian operators

$$(\varphi_p(x'(t)))' = h(t, x, x'), \tag{1.6}$$

where $p > 1$, $\varphi_p : \mathbb{R} \to \mathbb{R}$ and $\varphi_p(s) = |s|^{p-2}s$ is a one-dimensional p-Laplacian. The general boundary value problem of (1.6) was originally derived from non-Newtonian fluid mechanics theory, the turbulent flow of a gas in a porous medium and generalized reaction–diffusion theory; for details, see, for example, [16, 26]. Manásevich and Mawhin [40] discussed the periodic boundary value problem of (1.6), that is, the existence of T-periodic solutions. They proved an existence theorem, namely, the Manásevich–Mawhin continuation theorem. Recently, many authors have studied a class of Liénard-type p-Laplacian equations with a deviating argument by applying this continuation theorem, obtaining some criteria for guaranteeing the existence of T-periodic solutions of these equations; for details, see [8, 9, 18, 19, 35, 36, 38]. In addition, some authors have investigated the stability of periodic solutions of the Duffing-type equations

$$x''(t) + bx'(t) + g(t, x(t)) = 0 \tag{1.7}$$

and

$$x''(t) + bx'(t) + g(t, x(t)) = e(t). \tag{1.8}$$

Ortega [45] studied the stability of periodic solutions of (1.7) and obtained some stability results through use of a topological index. Lazer and McKenna [31] obtained stability results by converting (1.7) to a fixed-point problem and using the linearization technique. Subsequently, Chen and Li [6] studied the rate of decay of stable periodic solutions of (1.8), and determined a sharp rate of exponential decay for a solution near the unique periodic solution. In addition, Zitan and Ortega [63] investigated the existence of asymptotically stable periodic solutions of (1.1) with $e(t) = 0$ by applying degree theory and an upper and lower solutions technique. They proved an existence theorem as follows.

THEOREM A ([63, Theorem 4.1]). *Assume that there exist* $\Gamma \in L^\infty(\mathbb{R}/T\mathbb{Z})$, $m, M, K > 0$ *such that:*

(A1) $0 < m = \inf f \le \sup f = M < \infty$;

(A2) $g'_x(t, x) \le \alpha - \beta(\gamma + \alpha^{1/2})$ *and* $\alpha - \beta(\gamma + \alpha^{1/2}) > 0$, *for almost every* $t \in \mathbb{R}$, *for all* $x \in \mathbb{R}$;

(A3) $|g'_x(t, x)| \le \Gamma(t)$, *for almost every* $t \in \mathbb{R}$, *for all* $x \in \mathbb{R}$;

(A4) *one of the following conditions holds:*

   (1) $\int_0^T g(t, \varphi(t))\,dt > 0$, *for each* $\varphi \in C(\mathbb{R}/T\mathbb{Z})$ *with* $\min \varphi \ge K$;

   (2) $\int_0^T g(t, \varphi(t))\,dt < 0$, *for each* $\varphi \in C(\mathbb{R}/T\mathbb{Z})$ *with* $\max \varphi \le -K$;

(A5) *the number of* T*-periodic solutions of* (1.1) *with* $e(t) = 0$ *is finite, where* $\beta = (M - m)/2$, $\gamma = (M + m)/2$ *and* $\alpha = (\pi/T)^2 + \gamma^2/4$.

Then (1.1) *with* $e(t) = 0$ *has at least one asymptotically stable* T*-periodic solution.*

To the best of our knowledge, few authors have considered the existence, uniqueness and asymptotic stability of T-periodic solutions of the forced Liénard-type equations as in (1.1) without the restrictive conditions (A1)–(A5) in Theorem A. Therefore, it is essential to continue to study the periodic solutions of (1.1) in this case.

In this paper, we are keen to dispel any perception that the mathematical proofs of existence, uniqueness and stability that we present are merely verifying facts which might already be obvious in other disciplines, based on purely physical considerations. In particular, in many nonlinear problems arising in practical dynamical systems, physical reasoning alone is not sufficient or fully convincing. In these cases questions of existence, uniqueness and stability are of importance in understanding the full range of solution behaviour possible, and represent a genuine mathematical challenge. The answers to these mathematical questions then provide the basis for obtaining the best numerical solutions to these problems, and determining other important practical aspects of the solution behaviour.

In the machine vibration field, some vibration phenomena can be modeled by (1.1) according to Newton's second law, where $e(t)$ is the external input acceleration, $f(x)x'(t)$ is the nonlinear damping force, and $g(t, x)$ is the nonlinear restoring force; see, for example, [11, 43, 58]. Subsequently, a question naturally arises: *under what conditions does the vibration system present a regular and steady motion, that is, a stable periodic motion?* This is important for physicists and engineers since irregular and unsteady motion is not expected. Therefore, it is necessary to discuss the existence, uniqueness and asymptotic stability of periodic solutions of (1.1). Figure 1 shows the various applications of the Liénard equation (1.1).

The main purpose of this paper is to establish some new criteria for guaranteeing the existence, uniqueness and asymptotic stability of T-periodic solutions of (1.1), by applying the Manásevich–Mawhin continuation theorem, Floquet theory, Liapunov stability theory and some analysis techniques. We remark that, in contrast to Theorem A, we do not need to impose that the infimum of f is positive. Furthermore, conditions (A2)–(A5) can be replaced by different ones.

The following notation will be used throughout the rest of this paper:

$$|x|_\infty = \max_{t\in[0,T]} |x(t)|, \quad |x'|_\infty = \max_{t\in[0,T]} |x'(t)|, \quad \bar{e} = \frac{1}{T}\int_0^T e(t)\,dt$$
$$|x|_k = \left(\int_0^T |x(t)|^k\,dt\right)^{1/k} \quad \text{for } k \in \mathbb{N}.$$

Consider the Banach spaces with associated norms

$$C_T^1 := \{x \in C^1(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t)\}, \quad \|x\|_{C_T^1} = \max\{|x|_\infty, |x'|_\infty\},$$

and

$$C_T := \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t)\}, \quad \|x\|_{C_T} = |x|_\infty.$$

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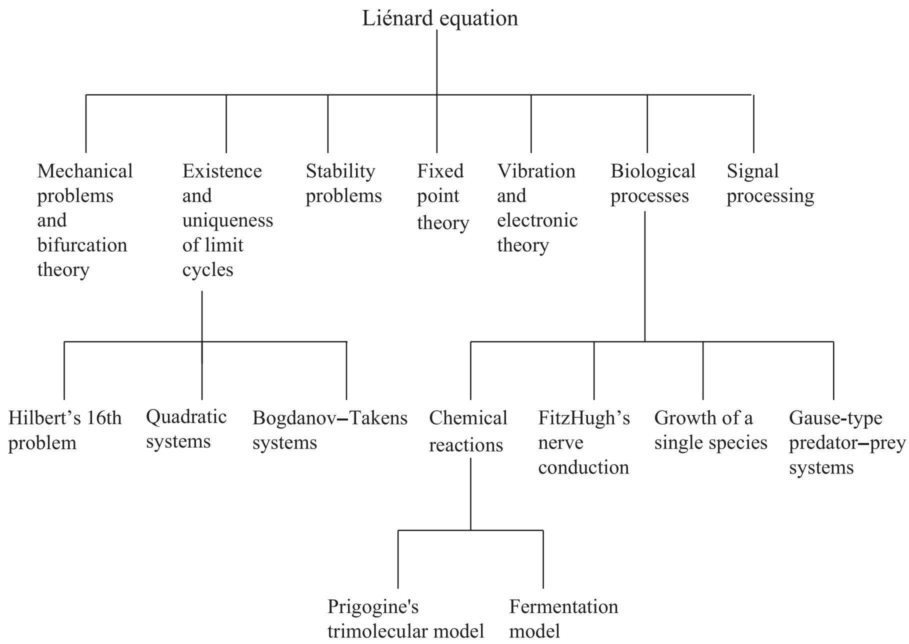

FIGURE 1. Diagrammatic representation for applications of the Liénard equation (1.1); sometimes $e(t) = 0$ and $g(t, x) = g(x)$.

We can now state our main theorem.

THEOREM 1.1. *Assume that there exist constants* $L_1, L_2, K, D \geq 0$ *such that:*

(H1) $|f(u) - f(v)| \leq L_1|u - v|$ *and* $|f(x)| \leq K$, *for all* $u, v, x \in \mathbb{R}$*;*

(H2) $x(g(t, x) - \bar{e}) > 0$, *for all* $|x| > D$ *and* $t \in \mathbb{R}$*;*

(H3) $|g(t, u) - g(t, v)| \leq L_2|u - v|$, *for all* $u, v, t \in \mathbb{R}$*;*

(H4)
$$L_1 M_2 \frac{T^2}{4\pi} + K\frac{T}{2\pi} + L_2\frac{T^2}{4\pi} < 1;$$

(H5) $[g(t, u) - g(t, v)](u - v) > 0$, *for all* $u, v, t \in \mathbb{R}$*;*

(H6) $\int_0^T [g'_x(t, x) - \frac{1}{4} f^2(x)]\, dt > 0$ *and* $f(x) > 0$, *for all* $x \in I_1$*;*

(H7) $g'_x(t, x) + \frac{1}{2} f'(x)y - \frac{1}{4} f^2(x) < 1/T^2$, *for all* $x \in I_1$, $y \in I_2$, $t \in \mathbb{R}$*;* *where*

$$G = \max\{|g(t, 0)| : 0 \leq t \leq T\}, \qquad M_0 = \frac{(L_2 D + G + |e|_\infty)\sqrt{T}}{1 - K(T/2\pi) - L_2(T^2/4\pi)},$$
$$M_1 = D + \frac{\sqrt{T^3}}{4\pi} M_0, \qquad M_2 = \frac{\sqrt{T}}{2} M_0,$$
$$I_1 = [-M_1, M_1], \qquad I_2 = [-M_2, M_2].$$

Then (1.1) *has a unique* T*-periodic solution which is asymptotically stable.*

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REMARK 1. In our recent papers [54–56], we investigated the existence and uniqueness of periodic solutions for some second-order ordinary differential equations. In [56], we studied a Liénard equation with two deviating arguments and obtained some better results than those from the literature; in [55], a Duffing-type p-Laplacian equation was investigated, and an improved sufficient condition and a necessary and sufficient condition were presented; in [54], a Rayleigh equation with a deviating argument was studied, and two sufficient conditions were obtained. We can see that all of these do not consider the stability of periodic solutions, but Theorem 1.1 above also contains some results of stability.

The outline of this paper is as follows. In Section 2 we introduce some lemmas which will help us to obtain the main theorem. Section 3 gives the proof of Theorem 1.1. In Section 4 an illustrative example is provided to demonstrate the applicability of our results.

2. Lemmas

In this section, we will introduce some lemmas which can help us to get our main theorem. First, let us recall the Manásevich–Mawhin continuation theorem, which is useful in obtaining the existence of T-periodic solutions of (1.1).

For the periodic boundary value problem

$$(\varphi_p(x'(t)))' = h(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T), \tag{2.1}$$

where $p > 1$, $\varphi_p : \mathbb{R} \to \mathbb{R}$, $\varphi_p(s) = |s|^{p-2}s$ is a one-dimensional p-Laplacian, $h \in C(\mathbb{R}^3, \mathbb{R})$ is T-periodic in the first variable and we have the following result.

LEMMA 2.1 ([40]). *Let Ω be an open bounded set in C^1_T. If:*

(i) *for each $\lambda \in (0, 1)$ the problem*

$$(\varphi_p(x'(t)))' = \lambda h(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T)$$

has no solution on $\partial\Omega$;

(ii) *the equation*

$$H(a) := \frac{1}{T}\int_0^T h(t, a, 0)\, dt = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}$;

(iii) *the Brouwer degree of H is*

$$\deg(H, \Omega \cap \mathbb{R}, 0) \neq 0;$$

then the periodic boundary value problem (2.1) has at least one T-periodic solution on $\overline{\Omega}$.

REMARK 2. If $p = 2$, (2.1) reduces to the periodic boundary value problem

$$x''(t) = h(t, x, x'), \quad x(0) = x(T), \quad x'(0) = x'(T),$$

which implies that Lemma 2.1 can be applied to investigating the existence of T-periodic solutions of (1.1).

Next, another lemma is introduced to help in obtaining the existence and uniqueness of T-periodic solutions of (1.1).

LEMMA 2.2. *If $x \in C^2(\mathbb{R}, \mathbb{R})$ with $x(t+T) = x(t)$, then*

$$|x'|_2^2 \le \left(\frac{T}{2\pi}\right)^2 |x''|_2^2.$$

PROOF. Lemma 2.2 is a direct consequence of the Wirtinger inequality, and one can see [24, 42] for its proof. □

Further, consider the homotopic equation of (1.1):

$$x''(t) + \lambda f(x(t))x'(t) + \lambda g(t, x(t)) = \lambda e(t), \quad \text{for } \lambda \in (0, 1). \tag{2.2}$$

The following lemma will show that the set of all possible T-periodic solutions of (2.2) are bounded in C_T^1 under some restrictive conditions. This result can help us to obtain the existence of T-periodic solutions of (1.1) when applying the above Manásevich–Mawhin continuation theorem.

LEMMA 2.3. *Assume that (H1)–(H3) of Theorem 1.1 hold. Also assume*

(H4′) $$K\frac{T}{2\pi} + L_2\frac{T^2}{4\pi} < 1$$

holds. Then the set of all possible T-periodic solutions of (2.2) is bounded in C_T^1.

PROOF. Let $S \subset C_T^1$ be the set of all possible T-periodic solutions of (2.2). If $S = \emptyset$, the proof is complete.

Suppose $S \ne \emptyset$ and let $x \in S$. Integrating both sides of (2.2) from 0 to T gives

$$\int_0^T [g(t, x(t)) - e(t)]\, dt = \int_0^T [g(t, x(t)) - \bar{e}]\, dt = 0,$$

which implies that there exists $t_1 \in \mathbb{R}$ such that $g(t_1, x(t_1)) - \bar{e} = 0$. It follows from (H2) that $|x(t_1)| \le D$. Hence, for any $t \in [t_1, t_1 + T]$,

$$|x(t)| = \left| x(t_1) + \int_{t_1}^t x'(s)\, ds \right| \le D + \int_{t_1}^t |x'(s)|\, ds \tag{2.3}$$

and

$$|x(t)| = \left| x(t_1 + T) + \int_{t_1+T}^t x'(s)\, ds \right| \le D + \int_t^{t_1+T} |x'(s)|\, ds. \tag{2.4}$$

Combining the inequalities in (2.3) and (2.4), we obtain

$$|x(t)| \le D + \frac{1}{2}\int_0^T |x'(s)|\, ds,$$

which together with the Schwarz inequality and Lemma 2.2 leads to

$$
\begin{aligned}
|x|_\infty = \max_{t\in[t_1,t_1+T]} |x(t)| &\le D + \frac{1}{2}\int_0^T |x'(s)|\,ds \\
&\le D + \frac{1}{2}|1|_2|x'|_2 \le D + \frac{\sqrt{T^3}}{4\pi}|x''|_2.
\end{aligned} \tag{2.5}
$$

Next, multiplying (2.2) by $x''(t)$ and integrating the result from 0 to T, we obtain from Lemma 2.2, (H1), (H3), (2.5) and the Schwarz inequality that

$$
\begin{aligned}
|x''|_2^2 &= -\lambda \int_0^T f(x(t))x'(t)x''(t)\,dt \\
&\qquad - \lambda \int_0^T g(t, x(t))x''(t)\,dt + \lambda \int_0^T e(t)x''(t)\,dt \\
&\le \int_0^T |f(x(t))||x'(t)||x''(t)|\,dt \\
&\qquad + \int_0^T |g(t, x(t))||x''(t)|\,dt + \int_0^T |e(t)||x''(t)|\,dt \\
&\le K\int_0^T |x'(t)||x''(t)|\,dt + \int_0^T [|g(t, x(t)) - g(t, 0)| + |g(t, 0)|]|x''(t)|\,dt \\
&\qquad + \int_0^T |e(t)||x''(t)|\,dt \\
&\le K|x'|_2|x''|_2 + L_2\int_0^T |x(t)||x''(t)|\,dt \\
&\qquad + \int_0^T |g(t, 0)||x''(t)|\,dt + \int_0^T |e(t)||x''(t)|\,dt \\
&\le K\frac{T}{2\pi}|x''|_2^2 + L_2|x|_\infty \int_0^T |x''(t)|\,dt \\
&\qquad + G\int_0^T |x''(t)|\,dt + |e|_\infty \int_0^T |x''(t)|\,dt \\
&\le K\frac{T}{2\pi}|x''|_2^2 + L_2|x|_\infty|1|_2|x''|_2 + G|1|_2|x''|_2 + |e|_\infty|1|_2|x''|_2 \\
&= K\frac{T}{2\pi}|x''|_2^2 + L_2\sqrt{T}|x|_\infty|x''|_2 + G\sqrt{T}|x''|_2 + |e|_\infty\sqrt{T}|x''|_2 \\
&\le \left(K\frac{T}{2\pi} + L_2\frac{T^2}{4\pi}\right)|x''|_2^2 + (L_2 D + G + |e|_\infty)\sqrt{T}|x''|_2,
\end{aligned}
$$

where $G = \max\{|g(t, 0)| : 0 \le t \le T\}$.

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It follows from (H4′) that

$$|x''|_2 \le \frac{(L_2 D + G + |e|_\infty)\sqrt{T}}{1 - K(T/2\pi) - L_2(T^2/4\pi)} := M_0. \tag{2.6}$$

Since $x(0) = x(T)$, there exists $t_2 \in [0, T]$ such that $x'(t_2) = 0$. Then, for any $t \in [t_2, t_2 + T]$, we obtain

$$|x'(t)| = \left| x'(t_2) + \int_{t_2}^{t} x''(s)\, ds \right| \le \int_{t_2}^{t} |x''(s)|\, ds \tag{2.7}$$

and

$$\begin{aligned} |x'(t)| &= \left| x'(t_2 + T) + \int_{t_2+T}^{t} x''(s)\, ds \right| \\ &\le \left| -\int_{t}^{t_2+T} x''(s)\, ds \right| \le \int_{t}^{t_2+T} |x''(s)|\, ds. \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8) we obtain

$$|x'(t)| \le \frac{1}{2} \int_0^T |x''(s)|\, ds. \tag{2.9}$$

Using the Schwartz inequality yields

$$|x'|_\infty = \max_{t \in [t_2, t_2+T]} |x'(t)| \le \frac{1}{2} \int_0^T |x''(s)|\, ds \le \frac{1}{2} |1|_2 |x''|_2 = \frac{1}{2} \sqrt{T} |x''|_2. \tag{2.10}$$

Relation (2.10) together with (2.6) yields

$$|x'|_\infty \le \tfrac{1}{2} \sqrt{T} M_0 := M_2. \tag{2.11}$$

Moreover, we can get from (2.5) and (2.6) that

$$|x|_\infty \le D + \frac{\sqrt{T^3}}{4\pi} M_0 := M_1. \tag{2.12}$$

Let $M_3 = \max\{M_1, M_2\}$. Then we know from (2.11) and (2.12) that $|x|_\infty \le M_3$ and $|x'|_\infty \le M_3$. This completes the proof. □

REMARK 3. We can see from the description of condition (H4) of Theorem 1.1 that if the condition (H4′) is replaced by the stronger condition (H4), Lemma 2.3 also holds. This result will be used in Lemma 2.4.

REMARK 4. According to Remark 3 and the proof of Lemma 2.3, we can easily conclude that if $x(t)$ is a T-periodic solution of (1.1), then (2.11) and (2.12) also hold when (H4′) is replaced by (H4).

LEMMA 2.4. *Suppose (H1)–(H5) of Theorem 1.1 hold. Then (1.1) has at most one T-periodic solution.*

PROOF. Suppose that $x_1(t)$ and $x_2(t)$ are T-periodic solutions of (1.1). Set $Z(t) = x_1(t) - x_2(t)$. Then

$$Z''(t) + [f(x_1(t))x_1'(t) - f(x_2(t))x_2'(t)] + [g(t, x_1(t)) - g(t, x_2(t))] = 0. \tag{2.13}$$

Since $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (1.1), integrating (2.13) from 0 to T, we obtain

$$\int_0^T [g(t, x_1(t)) - g(t, x_2(t))]\, dt = 0,$$

which implies that there exists $t_3 \in [0, T]$ such that

$$g(t_3, x_1(t_3)) - g(t_3, x_2(t_3)) = 0. \tag{2.14}$$

Equation (2.14) together with (H5) leads to

$$Z(t_3) = x_1(t_3) - x_2(t_3) = 0.$$

Hence, for any $t \in [t_3, t_3 + T]$, we obtain

$$|Z(t)| = \left| Z(t_3) + \int_{t_3}^{t} Z'(s)\, ds \right| \le \int_{t_3}^{t} |Z'(s)|\, ds \tag{2.15}$$

and

$$\begin{aligned} |Z(t)| &= \left| Z(t_3 + T) + \int_{t_3+T}^{t} Z'(s)\, ds \right| \\ &= \left| -\int_{t}^{t_3+T} Z'(s)\, ds \right| \le \int_{t}^{t_3+T} |Z'(s)|\, ds. \end{aligned} \tag{2.16}$$

Combining (2.15) and (2.16) we obtain

$$|Z(t)| \le \frac{1}{2} \int_0^T |Z'(s)|\, ds. \tag{2.17}$$

Using the Schwartz inequality yields

$$|Z|_\infty = \max_{t\in[t_3,t_3+T]} |Z(t)| \le \frac{1}{2} \int_0^T |Z'(s)|\, ds \le \frac{1}{2}|1|_2|Z'|_2 = \frac{1}{2}\sqrt{T}|Z'|_2. \tag{2.18}$$

Next, multiplying (2.13) by $Z''(t)$ and integrating the result from 0 to T, we obtain, by Lemmas 2.2 and 2.3, Remarks 3 and 4, (H1), (H3), (2.18) and the Schwartz inequality, that

$$\begin{aligned} |Z''|_2^2 = &-\int_0^T [f(x_1(t))x_1'(t) - f(x_2(t))x_2'(t)]Z''(t)\, dt \\ &- \int_0^T [g(t, x_1(t)) - g(t, x_2(t))]Z''(t)\, dt \end{aligned}$$

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$$
\begin{aligned}
&\le \int_0^T |f(x_1(t))||x_1'(t)-x_2'(t)||Z''(t)|\,dt \\
&\qquad + \int_0^T |f(x_1(t))-f(x_2(t))||x_2'(t)||Z''(t)|\,dt \\
&\qquad + \int_0^T |g(t,x_1(t))-g(t,x_2(t))||Z''(t)|\,dt \\
&\le K\int_0^T |Z'(t)||Z''(t)|\,dt + L_1|x_2'|_\infty \int_0^T |Z(t)||Z''(t)|\,dt \\
&\qquad + L_2\int_0^T |Z(t)||Z''(t)|\,dt \\
&\le K\int_0^T |Z'(t)||Z''(t)|\,dt + L_1M_2 \int_0^T |Z(t)||Z''(t)|\,dt \\
&\qquad + L_2\int_0^T |Z(t)||Z''(t)|\,dt \\
&\le K|Z'|_2|Z''|_2 + L_1M_2|Z|_\infty|1|_2|Z''|_2 + L_2|Z|_\infty|1|_2|Z''|_2 \\
&= K|Z'|_2|Z''|_2 + L_1M_2\sqrt{T}|Z|_\infty|Z''|_2 + L_2\sqrt{T}|Z|_\infty|Z''|_2 \\
&\le \left(L_1M_2\frac{T^2}{4\pi} + K\frac{T}{2\pi} + L_2\frac{T^2}{4\pi}\right)|Z''|_2^2.
\end{aligned}
\tag{2.19}
$$

Since $Z(t)$, $Z'(t)$ and $Z''(t)$ are T-periodic functions, we have, by (H4), Lemma 2.2, (2.18) and (2.19), that

$$Z(t) = Z'(t) = Z''(t) = 0 \quad \forall\, t \in \mathbb{R}.$$

Thus, $x_1(t) \equiv x_2(t)$, for all $t \in \mathbb{R}$. Hence, (1.1) has at most one T-periodic solution. This completes the proof. □

In addition, for convenience of use, we recall a principle of linearized stability for periodic systems; for details, see, for example, [10, Pages 321–322]. Consider the periodic boundary value problem

$$x' = F(t, x), \qquad x(0) = x(T), \tag{2.20}$$

where $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function that is T-periodic in t, and has continuous first-order partial derivative with respect to x. Let p_0 be a T-periodic solution of (2.20); then we associate the T-periodic solution p_0 with the linearized equation

$$y' = F_x'(t, p_0)y. \tag{2.21}$$

The following result due to Liapunov (see, for example, [10, Theorem 2.1, Page 322]), shows the connections between the asymptotic stability of T-periodic solution p_0 of (2.20) and the characteristic exponents of (2.21).

LEMMA 2.5. *If the characteristic exponents associated with (2.21) all have negative real part, then the T-periodic solution p_0 of (2.20) is asymptotically stable.*

3. Proof of Theorem 1.1

We are now in the position to present the proof of Theorem 1.1. We begin with the following existence and uniqueness result, dividing the proof into two steps.

STEP 1. Existence and uniqueness.

According to Remark 2, Lemma 2.1 can be applied to obtain the existence of T-periodic solutions of (1.1).

Set

$$h(t, x, x') = e(t) - f(x)x' - g(t, x);$$

then (2.2) is equivalent to

$$x''(t) = \lambda h(t, x(t), x'(t)), \quad \text{for } \lambda \in (0, 1). \tag{3.1}$$

By Lemma 2.3 and Remark 3, there exists a constant $M_4 > M_3$ such that, for any T-periodic solution $x(t)$ of (2.2) or (3.1),

$$|x|_\infty < M_4 \quad \text{and} \quad |x'|_\infty < M_4.$$

Let

$$\Omega = \{x \in C_T^1 : |x|_\infty < M_4, |x'|_\infty < M_4\}.$$

Then we know that (3.1) has no periodic solution on $\partial\Omega$ as $\lambda \in (0, 1)$, so Lemma 2.1(i) is satisfied.

On the other hand, since

$$H(a) := \frac{1}{T}\int_0^T h(t, a, 0)\, dt,$$

for any $x \in \partial\Omega \cap \mathbb{R}$, $x = M_4$ or $x = -M_4$, we obtain, by (H2), that

$$H(M_4) = \frac{1}{T}\int_0^T [e(t) - g(t, M_4)]\, dt = \frac{1}{T}\int_0^T [\bar{e} - g(t, M_4)]\, dt < 0 \tag{3.2}$$

and

$$H(-M_4) = \frac{1}{T}\int_0^T [e(t) - g(t, -M_4)]\, dt = \frac{1}{T}\int_0^T [\bar{e} - g(t, -M_4)]\, dt > 0, \tag{3.3}$$

which imply that Lemma 2.1(ii) is also satisfied.

Moreover, define

$$\widetilde{H}(x, \mu) = -\mu x + (1 - \mu)\frac{1}{T}\int_0^T [e(t) - g(t, x)]\, dt.$$

In view of (3.2) and (3.3), we get

$$x\widetilde{H}(x, \mu) < 0 \quad \text{for all } x \in \partial\Omega \cap \mathbb{R} \text{ and } \mu \in [0, 1],$$

so $\widetilde{H}(x, \mu)$ is a homotopic transformation. From (3.2) and (3.3) and together with the homotopic invariance theorem,

$$\begin{aligned}\deg(H, \Omega \cap \mathbb{R}, 0) &= \deg\left(\frac{1}{T}\int_0^T [e(t) - g(t, x)]\,dt, \Omega \cap \mathbb{R}, 0\right)\\ &= \deg(-x, \Omega \cap \mathbb{R}, 0) \neq 0.\end{aligned}$$

This implies that Lemma 2.1(iii) is satisfied. Therefore, it follows from Lemma 2.1 and Remark 2 that there exists a T-periodic solution of (1.1). The uniqueness of T-periodic solutions of (1.1) is guaranteed by Lemma 2.4.

STEP 2. Asymptotic stability.

Let $x_0(t)$ be the unique T-periodic solution of (1.1). Then we know from Lemma 2.3 and Remarks 3 and 4 that

$$|x_0|_\infty \leq M_1 \quad \text{and} \quad |x_0'|_\infty \leq M_2. \tag{3.4}$$

Now consider a system equivalent to (1.1) as follows:

$$\begin{cases} u'(t) = v(t),\\ v'(t) = e(t) - f(u(t))v(t) - g(t, u(t)).\end{cases}$$

Then according to (2.20) and (2.21) it is easy to see that the linearized equation of (1.1) is

$$y''(t) + f(x_0(t))y'(t) + [f'(x_0(t))x_0'(t) + g_x'(t, x_0(t))]y(t) = 0. \tag{3.5}$$

In order to show that $x_0(t)$ is asymptotically stable, Lemma 2.5 will be applied.

First, we show that (3.5) does not have real Floquet (or characteristic) multipliers. If not, then there is a real Floquet multiplier α and a nontrivial solution $y(t)$ of (3.5) such that $y(t + T) = \alpha y(t)$. Set $y(t) = \exp[(-1/2)\int_0^t f(x_0(s))\,ds]u(t)$. Then $u(t)$ is a nontrivial solution of

$$u''(t) + [g_x'(t, x_0(t)) + \tfrac{1}{2}f'(x_0(t))x_0'(t) - \tfrac{1}{4}f^2(x_0(t))]u(t) = 0 \tag{3.6}$$

with the Floquet multiplier $\beta = \alpha \exp[(1/2)\int_0^T f(x_0(s))\,ds]$ (that is, $u(t + T) = \beta u(t)$).

Now we show that the following claim is true.

CLAIM. *There exists $\xi \in [0, T]$ such that*

$$u(\xi) = 0. \tag{3.7}$$

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Assume, by way of contradiction, that (3.7) does not hold. Then $u(t) \neq 0$ for all $t \in [0, T]$. Dividing (3.6) by $u(t)$ and integrating the result from 0 to T, noticing that

$$\int_0^T f'(x_0(t))x_0'(t)\,dt = 0 \quad \text{and} \quad u'(0)/u(0) = u'(T)/u(T),$$

we obtain that

$$\int_0^T \left(\frac{u'(t)}{u(t)}\right)^2 dt + \int_0^T \left[g_x'(t, x_0(t)) - \frac{1}{4}f^2(x_0(t))\right] dt = 0,$$

which together with (3.4) contradicts condition (H6). This implies that the claim (3.7) holds.

Therefore we know that this $u(t)$ is a nontrivial solution of the Dirichlet boundary value problem (3.6) with $u(\xi + T) = u(\xi) = 0$. Multiplying (3.6) by $u(t)$ and integrating the result from ξ to $\xi + T$, we have, by (3.4) and (H7), that

$$\begin{aligned} \int_\xi^{\xi+T} u'(t)^2\,dt &= \int_\xi^{\xi+T} \left[g_x'(t, x_0(t)) + \frac{1}{2}f'(x_0(t))x_0'(t) - \frac{1}{4}f^2(x_0(t))\right]u^2(t)\,dt \\ &< \frac{1}{T^2}\int_\xi^{\xi+T} u^2(t)\,dt. \end{aligned} \tag{3.8}$$

For convenience of use, we recall the Dirichlet–Poincaré inequality

$$\int_a^b |q(t)|^2\,dt \leq (b-a)^2 \int_a^b |q'(t)|^2\,dt \quad \text{where } q \in C^1 \text{ and } q(a) = q(b) = 0.$$

Thus we can immediately find that (3.8) contradicts the Dirichlet–Poincaré inequality. Therefore (3.5) does not have real Floquet multipliers.

Next, we show that the characteristic exponents associated with (3.5) all have negative real part. In order to do this, let us consider an equivalent system of (3.5) in

$$X'(t) = A(t)X(t), \tag{3.9}$$

where the vector function $X(t) = (x(t), x'(t))^T$ and $A(t)$ is the matrix function

$$A(t) = \begin{bmatrix} 0 & 1 \\ -f'(x_0(t))x_0'(t) - g_x'(t, x_0(t)) & -f(x_0(t)) \end{bmatrix}.$$

Let $\alpha_1 = e^{T\mu_1}$ and $\alpha_2 = e^{T\mu_2}$ be the Floquet multipliers of (3.9) and μ_1 and μ_2 be the characteristic exponents associated with α_1 and α_2. Then it follows from the above discussion that α_1 and α_2 are a pair of complex conjugates. Applying Liouville's theorem (see, for example, [2, Problem 1, Page 285]),

$$\alpha_1\alpha_2 = \exp\left[\int_0^T \operatorname{trace}(A(t))\,dt\right] = \exp\left[\int_0^T -f(x_0(t))\,dt\right]$$

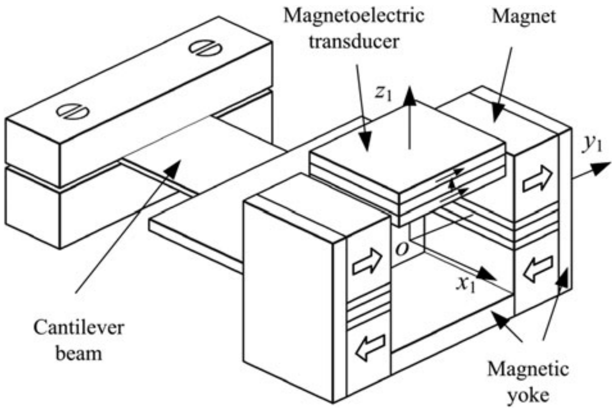

FIGURE 2. Schematic diagram of the proposed vibration energy harvester. Hollow arrows represent the magnetic directions of the magnets.

and

$$\mathrm{Re}(\mu_1) = \mathrm{Re}(\mu_2) = \frac{1}{2}\mathrm{Re}(\mu_1 + \mu_2) = \frac{1}{2T}\ln(\alpha_1\alpha_2) = \frac{1}{2T}\int_0^T -f(x_0(t))\,dt. \tag{3.10}$$

By (3.4) and (H6) that $f(x) > 0$ for $x \in I_1$, (3.10) suggests that $\mathrm{Re}(\mu_1) = \mathrm{Re}(\mu_2) < 0$.

Finally, applying Lemma 2.5 we can get that $x_0(t)$ is asymptotically stable. This completes the proof of Theorem 1.1. □

4. An application to real phenomena

In this section, we present an application of the given results to problems involving real-world phenomena.

In the machine vibration field, a schematic diagram of the proposed vibration energy harvester is shown in Figure 2 [11]. The system consists of a cantilever beam, a magnetic circuit, and a Magnetoelectric transducer. It is well known (see [11, 43, 58] and the references therein) that the motion of the harvester sometimes can be modeled by a form of Liénard equation (1.1) with $f(x) = 1/(\pi^4(1 + x^2))$, $e(t) = (\cos t)/\pi^3$ and $g(t, x) = (\cos^2 t + 1)x(t)/\pi^4$

$$x''(t) + \frac{1}{\pi^4[1 + x^2(t)]}x'(t) + \frac{1}{\pi^4}(\cos^2 t + 1)x(t) = \frac{1}{\pi^3}\cos t \quad \text{where } T = 2\pi. \tag{4.1}$$

Several researchers have analysed (4.1) by employing the Lindstedt–Poincaré method [44], and they have obtained an approximate 2π-periodic solution. They do not state whether this 2π-periodic solution is the unique and stable one, which is relevant for physicists and engineers. We address this here by applying Theorem 1.1.

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PROPOSITION 4.1. *Equation* (4.1) *has a unique stable* 2π*-periodic solution.*

PROOF. Set $K = L_1 = 1/\pi^4$, $L_2 = 2/\pi^4$ and $D = 0$. Then it is easy to check that conditions (H1)–(H3) and (H5) of Theorem 1.1 are satisfied. Now we check that (H4), (H6) and (H7) also hold. According to Theorem 1.1 we can compute that $G = 0$, $M_0 = 0.0876$, $M_1 = M_2 = 0.1098$ and $I_1 = I_2 = [-0.1098, 0.1098]$. Thus an elementary computation shows that (H4) holds.

On the other hand, for any $x \in I_1$,

$$\begin{aligned}\int_0^{2\pi}\left[g'_x(t,x) - \frac{1}{4}f^2(x)\right]dt &= \int_0^{2\pi}\left[\frac{1}{\pi^4}(\cos^2 t + 1) - \frac{1}{4\pi^8(1+x^2)^2}\right]dt\\ &\geq \int_0^{2\pi}\left[\frac{1}{\pi^4}(\cos^2 t + 1) - \frac{1}{4\pi^8}\right]dt\\ &= \frac{3}{\pi^3} - \frac{1}{2\pi^7} > 0\end{aligned}$$

and $f(x) > 0$, which imply that (H6) holds.

Moreover, we obtain for any $x \in I_1$ and $y \in I_2$,

$$\begin{aligned}g'_x(t,x) + \frac{1}{2}f'(x)y - \frac{1}{4}f^2(x) &= \frac{1}{\pi^4}(\cos^2 t + 1) - \frac{x}{\pi^4(1+x^2)^2}y - \frac{1}{4\pi^8(1+x^2)^2}\\ &< \frac{2}{\pi^4} + \frac{|x|}{\pi^4(1+x^2)}|y|\\ &< \frac{2}{\pi^4} + \frac{1}{2\pi^4} \times 0.1098\\ &\approx 0.0212 < \frac{1}{T^2} = \frac{1}{4\pi^2} \approx 0.0253,\end{aligned}$$

which implies that (H7) holds. Therefore (4.1) has a unique 2π-periodic solution which is asymptotically stable. □

REMARK 5. It can be seen that the results in [1, 3–9, 12–15, 18–23, 25, 27–39, 41, 42, 45–53, 57, 59–63] and the references therein do not apply to (4.1) for guaranteeing the existence, uniqueness and asymptotic stability of 2π-periodic solutions, so our results complement those previously known.

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