BULL. AUSTRAL. MATH. SOC. 05B20
VOL. 16 (1977), 297-305. (05B30, 15A24, 64K05)

Using product designs to construct orthogonal designs

Peter J. Robinson

This paper produces new types of designs, called *product designs*, which prove extremely useful for constructing orthogonal designs. An orthogonal design of order 2^t and type

$$(1, 1, 1, 1, 2, 2, 4, 4, \ldots, 2^{t-2}, 2^{t-2})$$

is constructed. This design often meets the Radon bound for the number of variables.

We also show that all orthogonal designs of order 2^t and type $(a, b, c, d, 2^t-a-b-c-d)$, with $0 < a+b+c+d < 2^t$, exist for $t = 5, 6$, and 7.

1. Introduction

An orthogonal design of order n *and type* $(u_1, u_2, \ldots, u_l)$ $(u_i > 0)$ on commuting variables $x_1, x_2, \ldots, x_l$ is a $n \times n$ matrix, A, with entries from $\{0, \pm x_1, \ldots, \pm x_l\}$, which satisfies

$$AA^t = \sum_{i=1}^{l} \left(u_i x_i^2\right) I_n .$$

It was shown in [1] that $l \le \rho(n)$, where $\rho(n)$ (Radon's function) is defined by

$$\rho(n) = 8c + 2^d$$

Received 30 November 1976. Communicated by Jennifer R. Seberry.

where

$$n = 2^a \cdot b \ , \quad b \text{ odd}, \quad a = 4c + d \ , \quad 0 \le d < 4 \ .$$

If A and B are orthogonal designs of order n and types $(u_1, \ldots, u_l)$ and $(v_1, \ldots, v_k)$ respectively, and if

$$AB^t = BA^t \ ,$$

then we say A and B are *amicable orthogonal designs* of order n and types $((u_1, \ldots, u_l); (v_1, \ldots, v_k))$.

In [2], Geramita and Wallis give the following result.

CONSTRUCTION 1.1 (Geramita and Wallis [2]). *Suppose there exist three matrices* R, P , *and* S *of order* n *which give amicable orthogonal designs*

$$S \ , \quad x_1R + x_2P$$

of types $((v_1, \ldots, v_k); (u_1, u_2))$. *Then*

$$\begin{bmatrix} y_1R+y_2P & y_3R+y_4P & S & y_5R+y_6P \\ -y_3R+y_4P & y_1R-y_2P & -y_5R-y_6P & S \\ -S & y_5R-y_6P & y_1R+y_2P & -y_3R+y_4P \\ -y_5R+y_6P & -S & y_3R+y_4P & y_1R-y_2P \end{bmatrix}$$

is an orthogonal design of order $4n$ *and type*

$$(v_1, v_2, \ldots, v_k, u_1, u_1, u_1, u_2, u_2, u_2) \ .$$

We note that the above matrix can be written in the form

$$M_1 \times R + M_2 \times P + N \times S \ ,$$

where M_1, M_2 , and N are the 4×4 matrices of the coefficients of R, P , and S respectively.

In this paper we are interested in generalizing this idea. With this in mind, we give the following definition.

DEFINITION 1.2. Let M_1, M_2 , and N be orthogonal designs of order

n and types $(a_1, \ldots, a_r)$, $(b_1, \ldots, b_h)$, and $(c_1, \ldots, c_j)$ respectively. Then $(M_1; M_2; N)$ are product designs of order n and types $(a_1, \ldots, a_r; b_1, \ldots, b_h; c_1, \ldots, c_j)$ if

(i) $M_1 * N = M_2 * N = 0$ (Hadamard product),

(ii) $M_1 + N$ and $M_2 + N$ are orthogonal designs, and

(iii) $M_1 M_2^t = M_2 M_1^t$.

Construction 1.1 produces product designs of order 4 and types (1, 1, 1; 1, 1, 1; 1).

2. Constructing product designs

Before we give the main results of this section, we produce two examples of product designs. For convenience we write the designs M_1, M_2, and N in the form $M_1 + zN$ and M_2, and replace $-a$ by $\bar{a}$.

EXAMPLE 2.1. The following matrices give product designs of order 8 and types (1, 1, 1; 1, 1, 1; 5);

$$\begin{bmatrix} x_1 & x_2 & z & x_3 & z & z & z & \bar{z} \\ \bar{x}_2 & x_1 & \bar{x}_3 & z & z & \bar{z} & z & z \\ \bar{z} & x_3 & x_1 & \bar{x}_2 & z & \bar{z} & \bar{z} & \bar{z} \\ \bar{x}_3 & \bar{z} & x_2 & x_1 & z & z & \bar{z} & z \\ \bar{z} & \bar{z} & \bar{z} & \bar{z} & x_1 & x_2 & z & \bar{x}_3 \\ \bar{z} & z & z & \bar{z} & \bar{x}_2 & x_1 & x_3 & z \\ \bar{z} & \bar{z} & z & z & \bar{z} & \bar{x}_3 & x_1 & \bar{x}_2 \\ z & \bar{z} & z & \bar{z} & x_3 & \bar{z} & x_2 & x_1 \end{bmatrix} \quad \begin{bmatrix} y_1 & y_2 & 0 & y_3 & & & & \\ y_2 & \bar{y}_1 & \bar{y}_3 & 0 & & & 0 & \\ 0 & \bar{y}_3 & y_1 & y_2 & & & & \\ y_3 & 0 & y_2 & \bar{y}_1 & & & & \\ & & & & y_2 & y_3 & 0 & \bar{y}_1 \\ & & & & y_3 & \bar{y}_2 & y_1 & 0 \\ & 0 & & & 0 & y_1 & y_2 & y_3 \\ & & & & \bar{y}_1 & 0 & y_3 & \bar{y}_2 \end{bmatrix} .$$

EXAMPLE 2.2. The following matrices give product designs of order 12 and types (1, 1, 1; 1, 1, 1; 9).

https://doi.org/10.1017/S0004972700023327 Published online by Cambridge University Press

$$\begin{bmatrix}
x_1 & x_2 & x_3 & \bar{z} & z & z & z & \bar{z} & z & \bar{z} & z & \bar{z} \\
\bar{x}_2 & x_1 & \bar{z} & \bar{x}_3 & z & \bar{z} & \bar{z} & \bar{z} & \bar{z} & \bar{z} & \bar{z} & \bar{z} \\
\bar{x}_3 & z & x_1 & x_2 & z & z & \bar{z} & z & z & z & \bar{z} & \bar{z} \\
z & x_3 & \bar{x}_2 & x_1 & z & \bar{z} & z & z & z & \bar{z} & \bar{z} & z \\
\bar{z} & \bar{z} & \bar{z} & \bar{z} & x_1 & x_2 & x_3 & \bar{z} & z & z & \bar{z} & z \\
\bar{z} & z & \bar{z} & z & \bar{x}_2 & x_1 & \bar{z} & \bar{x}_3 & z & \bar{z} & z & z \\
\bar{z} & z & z & \bar{z} & \bar{x}_3 & z & x_1 & x_2 & \bar{z} & \bar{z} & \bar{z} & z \\
z & z & \bar{z} & \bar{z} & z & x_3 & \bar{x}_2 & x_1 & \bar{z} & z & z & z \\
\bar{z} & z & \bar{z} & \bar{z} & \bar{z} & \bar{z} & z & z & x_1 & x_2 & x_3 & \bar{z} \\
z & z & \bar{z} & z & \bar{z} & z & z & \bar{z} & \bar{x}_2 & x_1 & \bar{z} & \bar{x}_3 \\
\bar{z} & z & z & z & z & \bar{z} & z & \bar{z} & \bar{x}_3 & z & x_1 & x_2 \\
z & z & z & \bar{z} & \bar{z} & \bar{z} & \bar{z} & \bar{z} & z & x_3 & \bar{x}_2 & x_1
\end{bmatrix}
\begin{bmatrix}
\begin{matrix} y_1 & y_2 & y_3 & 0 \\ y_2 & \bar{y}_1 & 0 & \bar{y}_3 \\ y_3 & 0 & \bar{y}_1 & y_2 \\ 0 & \bar{y}_3 & y_2 & y_1 \end{matrix} & 0 & 0 \\
0 & \begin{matrix} y_2 & y_3 & y_1 & 0 \\ y_3 & \bar{y}_2 & 0 & \bar{y}_1 \\ y_1 & 0 & \bar{y}_2 & y_3 \\ 0 & \bar{y}_1 & y_3 & y_2 \end{matrix} & 0 \\
0 & 0 & \begin{matrix} y_3 & \bar{y}_1 & \bar{y}_2 & 0 \\ \bar{y}_1 & \bar{y}_3 & 0 & y_2 \\ \bar{y}_2 & 0 & \bar{y}_3 & \bar{y}_1 \\ 0 & y_2 & \bar{y}_1 & y_3 \end{matrix}
\end{bmatrix} .$$

The next theorem gives us a way of obtaining product designs from product designs of smaller orders.

THEOREM 2.3. *Let* $(M_1;\ y_1M_3+y_2M_4;\ N)$ *be product designs of order* n *and types* $(a_1, \ldots, a_r;\ b_1, b_2;\ c_1, \ldots, c_j)$ *and let* S *and* $x_1R + x_2P$ *be amicable orthogonal designs of order* m *and types* $((u);\ (v, w))$. *Then*

$$(x_1P \times M_3 + R \times M_1;\ y_1S \times M_3 + y_2R \times M_4;\ z_1P \times M_4 + S \times N)$$

are product designs of order mn *and types*

$$(wb_1, va_1, \ldots, va_r;\ ub_1, vb_2;\ wb_2, uc_1, \ldots, uc_j) .$$

Proof. By straightforward verification.

A very useful form of this theorem is obtained by using the amicable orthogonal designs of order 2 and types $((1, 1);\ (2))$ (see [1]). We state this particular case in the following corollary.

COROLLARY 2.4. *Let* $(M_1;\ y_1M_3+y_2M_4;\ N)$ *be product designs of order* n *and types* $(a_1, \ldots, a_r;\ b_1, b_2;\ c_1, \ldots, c_j)$. *Then there are product designs of order* $2n$ *and types*

https://doi.org/10.1017/S0004972700023327 Published online by Cambridge University Press

$$\left(b_1, a_1, \ldots, a_r; 2b_1, b_2; b_2, 2c_1, \ldots, 2c_j\right) .$$

In Theorem 2.3 and Corollary 2.4 we may have M_3 or M_4 equal to zero. In this case, however, the next theorem gives a better result.

THEOREM 2.5. *If* $(M_1; M_2; N)$ *are product designs of order* n *and types* $(a_1, \ldots, a_r; b; c_1, \ldots, c_j)$ *and if* S *and* $y_1R + P$ *are amicable orthogonal designs of order* m *and types*

$$\left((u_1, \ldots, u_l); (v, w_1, \ldots, w_k)\right)$$

then there are product designs of order mn *and the following types:*

(i) $(va_1, \ldots, va_r; vb; cu_1, \ldots, cu_l, bw_1, \ldots, bw_k)$, *and*

(ii) $(va_1, \ldots, va_r; vb; uc_1, \ldots, uc_j, bw_1, \ldots, bw_k)$,

where u *and* c *are the sums of the* u_i*'s and* c_i*'s respectively.*

Proof. We consider

$$\left(R \times M_1; R \times M_2; S \times N + P \times M_2\right)$$

with the appropriate variables equated.

The next result gives us a way of obtaining product designs from amicable orthogonal designs.

THEOREM 2.6. *If* S_1 *and* $x_1R_1 + x_2P_1$ *are amicable orthogonal designs of order* n *and types* $\left((u_1, \ldots, u_l); (v, w)\right)$, *and* S_2 *and* $y_1R_2 + y_2P_2$ *are amicable orthogonal designs of order* m *and types* $\left((s_1, \ldots, s_k); (q, p)\right)$, *then*

$$\left(S_1 \times R_2 + xR_1 \times P_2; R_1 \times S_2 + yP_1 \times R_2; P_1 \times P_2\right)$$

are product designs of order mn *and types*

$$\left(qu_1, \ldots, qu_l, vp; vs_1, \ldots, vs_k, wq; wp\right) .$$

Proof. By straightforward verification.

In the following lemma, we give an example of product designs which will be used in the next section to produce a very useful orthogonal

https://doi.org/10.1017/S0004972700023327 Published online by Cambridge University Press

design.

LEMMA 2.7. *There are product designs of order* 2^t , $t \geq 4$, *and types*

$$\left(1, 1, 1, 1, 2, 4, \ldots, 2^{t-3}; 2, 2^{t-2}; 2, 4, \ldots, 2^{t-3}, 2^{t-2}, 2^{t-2}\right) .$$

Proof. The product designs of order 4 and types (1, 1, 1; 1, 2; 1) produce product designs of order 8 and types (1, 1, 1, 1; 2, 2; 2, 2) (Corollary 2.4). This design in turn produces product designs of order 16 and types (1, 1, 1, 1, 2; 2, 4; 2, 4, 4) .

By repeated use of Corollary 2.4 we obtain the required result.

In [3] we give a list of product designs of orders 4, 8, 16, 32 , and 64 which are obtained by using the results given in this section.

3. Constructing orthogonal designs from product designs

We now produce a generalization of Construction 1.1.

THEOREM 3.1. *Let* S *and* $y_1R + P$ *be amicable orthogonal designs of order* m *and types* $\left(\left(u_1, \ldots, u_l\right); \left(v, w_1, \ldots, w_k\right)\right)$ *and let* $\left(M_1; M_2; N\right)$ *be product designs of order* n *and types* $\left(a_1, \ldots, a_r; b_1, \ldots, b_h; c_1, \ldots, c_j\right)$. *Then there exist orthogonal designs of order* mn *and types*

(i) $\left(va_1, \ldots, va_r, wb_1, \ldots, wb_h, uc_1, \ldots, uc_j\right)$,

(ii) $\left(va_1, \ldots, va_r, wb_1, \ldots, wb_h, u_1c, \ldots, u_lc\right)$,

(iii) $\left(va_1, \ldots, va_r, w_1b, \ldots, w_kb, uc_1, \ldots, uc_j\right)$,

(iv) $\left(va_1, \ldots, va_r, w_1b, \ldots, w_kb, u_1c, \ldots, u_lc\right)$,

where b, c, u , *and* w *are the sums of the* b_i*'s*, c_i*'s*, u_i*'s* , *and* w_i*'s respectively.*

Proof. We consider

$$M_1 \times R + M_2 \times P + N \times S$$

with the appropriate variables equated.

https://doi.org/10.1017/S0004972700023327 Published online by Cambridge University Press

As an example of the use of this theorem, we give the following lemma.

LEMMA 3.2. *There is an orthogonal design of order* 2^t , $t \geq 2$, *and type* $(1, 1, 1, 1, 2, 2, 4, 4, \ldots, 2^{t-2}, 2^{t-2})$.

Proof. If $t \geq 5$ we apply the above theorem with the product designs of order 2^{t-1} and types

$$(1, 1, 1, 1, 2, 4, \ldots, 2^{t-4}; 2, 2^{t-3}; 2, 4, \ldots, 2^{t-4}, 2^{t-3}, 2^{t-3})$$

(Lemma 2.7) and amicable orthogonal designs of order 2 and types $((1, 1); (2))$.

The orthogonal design of order 16 and type $(1, 1, 1, 1, 2, 2, 4, 4)$ may be obtained in a similar manner by using the product designs of order 8 and types $(1, 1, 1, 1; 2, 2; 2, 2)$ (see Proof of Lemma 2.7).

The design of order 4 and type $(1, 1, 1, 1)$ is given in [1] and the design of type $(1, 1, 1, 1, 2, 2)$ is obtained from the orthogonal design of order 8 and type $(1, 1, 1, 1, 1, 1, 1, 1)$ (see [1]).

We note that the above orthogonal design has $2t$ variables. If $t = 4k + 1$, $\rho(t) = 8k + 2 = 2t$, and if $t = 4k + 2$, $\rho(t) = 8k + 4 = 2t$. Therefore, if $t = 4k + 1$ or $4k + 2$, the above design has the maximum number of variables allowed. We also note that the above design is full. That is, the design contains no zeros.

By equating variables in the above design we obtain:

COROLLARY 3.3. *All orthogonal designs of type* $(1, 1, a, b, c)$, $a + b + c = 2^t - 2$, *exist in order* 2^t , $t \geq 3$.

We now give another example of the use of product designs in the form of a lemma.

LEMMA 3.4. *There is an orthogonal design of order* 64 *and type* $(2, 2, 3, 4, 5, 5, 9, 9, 10, 15)$.

Proof. Theorem 2.6 gives product designs of order 32 and types $(3, 5, 5, 10; 2, 2, 4, 15; 9)$ by considering the amicable pairs $((1, 1, 2); (1, 3))$ and $((2, 2, 4); (5, 3))$. The result is then obtained by using this product design with the amicable orthogonal design of order 2 and type $((1, 1); (1, 1))$ in Theorem 3.1.

We now give two results to show how product designs may be used to obtain orthogonal designs in orders other than powers of 2 .

LEMMA 3.5. *There are product designs of order* 12 *and types* (1, 1, 1; 1, 1, 4; 4), (1, 1, 4; 1, 1, 1; 1), (1, 1, 4; 1, 4, 4; 1), (1, 1, 4; 1, 1, 4; 4), (1, 4, 4; 1, 4, 4; 1) , *and* (1, 1, 1; 1, 1, 1; 9) .

Proof. Wolfe [4] gives amicable orthogonal designs of types $\big((1, 1); (1, 4)\big)$ and $\big((1, 4); (1, 4)\big)$ in order 6 . By using these designs in Theorem 2.6, we obtain the first 5 designs. The last is given in Example 2.2.

By using Theorem 3.1 with the above designs, we obtain:

COROLLARY 3.6. *There are orthogonal designs of order* 24 *and types*

(1, 1, 1, 1, 1, 1, 9, 9) , (1, 1, 1, 1, 4, 4, 4, 4) ,
(1, 1, 1, 1, 1, 4, 4, 4) , (1, 1, 1, 1, 1, 1, 4, 4) .

4. Applications

In [3] we give a list of orthogonal designs of orders 32, 64 , and 128 . By using these designs, we are able to obtain the following results.

LEMMA 4.1. *All 6-tuples of the form* $(a, b, c, d, e, 32-a-b-c-d-e)$, $0 < a+b+c+d+e < 32$, *are the types of orthogonal designs of order* 32 *except possibly* (1, 1, 1, 1, 1, 27) .

COROLLARY 4.2. *All 5-tuples, except possibly* (1, 1, 1, 1, 27) , *are the types of orthogonal designs of order* 32 .

LEMMA 4.3. *All 5-tuples of the form* $\left(a, b, c, d, 2^t-a-b-c-d\right)$, $0 < a+b+c+d < 2^t$, *are the types of orthogonal designs of order* 2^t , $t = 6$ *and* 7 .

COROLLARY 4.4. *All possible* n*-tuples,* $n = 1, 2, 3, 4$, *are the types of orthogonal designs of orders* 32, 64 , *and* 128 .

References

[1] Anthony V. Geramita, Joan Murphy Geramita and Jennifer Seberry Wallis, "Orthogonal designs", *Linear and Multilinear Algebra* 3 (1975/1976), 231-306.

[2] Anthony V. Geramita and Jennifer Seberry Wallis, "Some new constructions for orthogonal designs", *Combinatorial Mathematics IV*, 46-54 (Proc. Fourth Austral. Conf., University of Adelaide, 1975. Lecture Notes in Mathematics, 560. Springer-Verlag, Berlin, Heidelberg, New York, 1976).

[3] Peter J. Robinson, "Concerning the existence and construction of orthogonal designs", (PhD thesis, Australian National University, Canberra, 1977).

[4] Warren W. Wolfe, "Orthogonal designs - amicable orthogonal designs - some algebraic and combinatorial techniques", (PhD Dissertation, Queen's University, Kingston, Ontario, 1975).

Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra, ACT.

https://doi.org/10.1017/S0004972700023327 Published online by Cambridge University Press