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On the Structure of the Spreading Models of a Banach Space

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Abstract. We study some questions concerning the structure of the set of spreading models of a separable infinite-dimensional Banach space X. In particular we give an example of a reflexive X so that all spreading models of X contain ℓ_1 but none of them is isomorphic to ℓ_1. We also prove that for any countable set C of spreading models generated by weakly null sequences there is a spreading model generated by a weakly null sequence which dominates each element of C. In certain cases this ensures that X admits, for each $\alpha < \omega_1$, a spreading model $(\tilde{x}_i^{(\alpha)})_i$ such that if $\alpha < \beta$ then $(\tilde{x}_i^{(\alpha)})_i$ is dominated by (and not equivalent to) $(\tilde{x}_i^{(\beta)})_i$. Some applications of these ideas are used to give sufficient conditions on a Banach space for the existence of a subspace and an operator defined on the subspace, which is not a compact perturbation of a multiple of the inclusion map.

1 Introduction

It is known that for every seminormalized basic sequence (y_i) in a Banach space and for every $\varepsilon_n \searrow 0$ there exists a subsequence (x_i) and a seminormalized basic sequence $(\tilde{x}_i)$ such that: for all $n \in \mathbb{N}$, $(a_i)_{i=1}^n \in [-1,1]^n$ and $n \le k_1 < \cdots < k_n$,

$$\Big| \big\| \sum_{i=1}^n a_i x_{k_i} \big\| - \big\| \sum_{i=1}^n a_i \tilde{x}_i \big\| \Big| < \varepsilon_n. \tag{1}$$

The sequence $(\tilde{x}_i)$ is called the *spreading model of* (x_i) and it is a suppression-1 unconditional basic sequence if (y_i) is weakly null (see [4, 5]; see also [3, I.3. Proposition 2] and [21] for more about spreading models). This, in conjunction with Rosenthal's ℓ_1 theorem [26], yields that every separable infinite dimensional Banach space X admits a suppression 1-unconditional spreading model $(\tilde{x}_i)$. In fact one can always find a 1-unconditional spreading model [27]. It is natural to ask if one can always say more. What types of spreading models must always exist? Sometimes we refer to the closed linear span of $(\tilde{x}_i)$, as the spreading model of (x_i). By James' well known theorem [12], every such X thus admits a spreading model $\widetilde{X}$ which is either reflexive or contains an isomorph of c_0 or ℓ_1. It was once speculated that for all such X some spreading model $(\tilde{x}_i)$ must be equivalent to the unit vector basis of c_0 or ℓ_p for some $1 \le p < \infty$, but this was proved to be false [22]. A replacement conjecture was brought to our attention by V. D. Milman: Must every separable space X admit a

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spreading model which is either isomorphic to c_0 or ℓ_1 or is reflexive? In Section 2 we show this to be false by constructing a space X so that for all spreading models $\widetilde{X}$ of X, $\widetilde{X}$ contains ℓ_1 but $\widetilde{X}$ is never isomorphic to ℓ_1. The example borrows some of the intuition behind the example of [22]. That space had the property that amongst the ℓ_p and c_0 spaces only ℓ_1 could be block finitely representable in any spreading model $(\tilde{x}_i)$, yet no spreading model could contain ℓ_1.

The motivation behind our example comes from the "Schreierized" version $S(d_{w,1})$ of the Lorentz space $d_{w,1}$. Let $1 = w_1 > w_2 > \cdots$ with $w_n \to 0$ and $\sum_{n=}^{\infty} w_n = \infty$. Then $d_{w,1}$ is the sequence space whose norm is given by

$$\|x\| = \sum_n w_n x_n^*,$$

where x is the sequence (x_n) and (x_n^*) is the decreasing rearrangement of $(|x_n|)$. One could then define the sequence space $S(d_{w,1})$ as the completion of c_{00} (the linear span of finitely supported sequences of reals) under

$$\|x\| = \sup_{\substack{n\in\mathbb{N} \\ n\le k_1<k_2<\cdots<k_n}} \sum_{i=1}^{n} w_i x_{k_i}^*.$$

In this case the unit vector basis (e_i) has a spreading model, namely the unit vector basis of $d_{w,1}$, which is not an ℓ_1 basis but whose span is hereditarily ℓ_1. Since $S(d_{w,1})$ is hereditarily c_0, it does not solve Milman's question. In order to avoid c_0 one may also define the "Tsirelsonized" version $T(d_{w,1})$ of $d_{w,1}$. Let $T(d_{w,1})$ be the completion of c_{00} under the implicit equation

$$\|x\| = \max\Bigl(\|x\|_\infty, \sup \sum_{i=1}^{n} w_i \|E_i x\|^*\Bigr),$$

where the supremum is taken over all integers n, and all *admissible* sets $(E_i)_{i=1}^n$ *i.e.*, $n \le E_1 < \cdots < E_n$ (this means $n \le \min E_1 \le \max E_1 < \min E_2 \le \cdots$) and $E_i x$ is the restriction of x to the set E_i. It may well be that $T(d_{w,1})$ has the properties we desire but we were unable to show this. Thus we were forced to "layer" the norm in a certain sense (see Section 2 below).

In Section 3 we consider in a wider context $SP_\omega(X)$, the partially ordered set of all spreading models $(\tilde{x}_i)$ generated by weakly null sequences in X. The partial order is defined by domination: we write $(\tilde{x}_i) \ge (\tilde{y}_i)$ if for some $C < \infty$ we have $C\|\sum a_i\tilde{x}_i\| \ge \|\sum a_i\tilde{y}_i\|$ for all scalars (a_i). We identify $(\tilde{x}_i)$ and $(\tilde{y}_i)$ in $SP_\omega(X)$ if $(\tilde{x}_i) \ge (\tilde{y}_i) \ge (\tilde{x}_i)$. We prove (in Proposition 3.2) that if $C \subseteq SP_\omega(X)$ is countable, then there exists $(\tilde{x}_i) \in SP_\omega(X)$ which dominates all members of C. This enables us to prove that in certain cases one can produce an uncountable chain $\{(\tilde{x}_i^{(\alpha)})_i\}_{\alpha<\omega_1}$ with $(\tilde{x}_i^{(\alpha)})_i < (\tilde{x}_i^{(\beta)})_i$ if $\alpha < \beta < \omega_1$. This yields a solution to a uniformity question raised by H. Rosenthal. The question (and a dual version) are as follows: let a separable Banach space Z have the property that for all spreading models $(\tilde{x}_i)$ of normalized

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basic sequences

$$\lim_n \Big\| \sum_{i=1}^n \tilde{x}_i \Big\| \Big/ n = 0 \quad \Big(\text{respectively, } \lim_n \Big\| \sum_{i=1}^n \tilde{x}_i \Big\| = \infty\Big).$$

Does there exist (λ_n) with $\lim_n \lambda_n/n = 0$ (respectively $\lim_n \lambda_n = \infty$) such that for all spreading models $(\tilde{x}_i)$ of normalized basic sequences in Z

$$\lim_n \Big\| \sum_{i=1}^n \tilde{x}_i \Big\| \Big/ \lambda_n = 0 \quad \Big(\text{respectively, } \lim_n \Big\| \sum_{i=1}^n \tilde{x}_i \Big\| \Big/ \lambda_n = \infty\Big)?$$

We give negative answers to these questions. The example that solves the first question is the space X of Section 2. Moreover every subspace of X fails to admit such a sequence (λ_n). We do not know of a hereditary solution to the second question.

In Section 5 we consider the problem: if $|SP_\omega(X)| = 1$, *i.e.*, if X has a unique spreading model up to equivalence, must this spreading model be equivalent to the unit vector basis in c_0 or ℓ_p for some $1 \le p < \infty$? The question was asked of us by S. A. Argyros. It is easy to see that the answer is positive if the spreading models are uniformly isomorphic. We show that the answer is positive if 1 belongs to the "Krivine set" of some spreading model.

Definition 1.1 Let (x_i) be a 1-spreading basic sequence (see (2)). The *Krivine set* of (x_i) is the set of p's ($1 \le p \le \infty$) with the following property: for all $\varepsilon > 0$ and $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $(\lambda_k)_{k=1}^m \subset \mathbb{R}$, such that for all $(a_i)_1^n \subseteq \mathbb{R}$,

$$\frac{1}{1+\varepsilon} \|(a_i)_{i=1}^n\|_p \le \Big\| \sum_{i=1}^n a_i y_i \Big\| \le (1+\varepsilon) \|(a_i)_{i=1}^n\|_p$$

where $y_i = \sum_{k=1}^m \lambda_k x_{(i-1)m+k}$ for $i = 1, \dots, n$, and $\|\cdot\|_p$ denotes the norm of the space ℓ_p.

The proof of Krivine's theorem [14] as modified by H. Lemberg [15]) (see also [9, Remark II.5.14] and [19]), shows that for every 1-spreading basic sequence (x_i) the Krivine set of (x_i) is non-empty. It is important to note that our definition of a Krivine p requires not merely that ℓ_p be block finitely representable in $[x_i : i \in \mathbb{N}]$ but each ℓ_p^n unit vector basis is obtainable by means of an identically distributed block basis.

An immediate consequence of the fact that the Krivine set of a spreading model is non-empty is the following:

Remark 1.2 Assume that (x_i) is a seminormalized basic sequence in a Banach space X which has a spreading model $(\tilde{x}_i)$. We can assume that for some decreasing to zero sequence (ε_i) (1) is satisfied. Then there is a $p \in [1, \infty]$ such that for all n and all

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$\varepsilon > 0$ there exists a finite sequence $(\lambda_i)_{i=1}^m \subset \mathbb{R}$ so that any block (y_i) of (x_i) of the form

$$y_i = \sum_{j=1}^m \lambda_j x_{n(i,j)},$$

$$\text{with } n(1,1) < n(1,2) < \cdots < n(1,m) < n(2,1) < \cdots n(2,m) < n(3,1) \cdots$$

has a spreading model $(\tilde{y}_i)$ which is isometric to the sequence $(\sum_{j=1}^m \lambda_j \tilde{x}_{(i-1)m+j})_{i\in\mathbb{N}}$ and has the property that its first n elements are $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^n. For $i_0 \in \mathbb{N}$ large enough (or passing to an appropriate subsequence of (x_i)) we also observe that $(y_{k_j})_{j=1}^n$ is $(1+2\varepsilon)$-equivalent to the ℓ_p^n unit basis whenever $i_0 < k_1 < \cdots k_n$.

In Section 6 we give sufficient conditions on a Banach space X for the existence of a subspace Y of X and an operator $T\colon Y \to X$ which is not a compact perturbation of the inclusion map. W. T. Gowers [9] proved that there exists a subspace Y of the Gowers–Maurey space GM (constructed in [10]) and there exists an operator $T\colon Y \to GM$ which is not a compact perturbation of the inclusion map. Here we extend the work of Gowers to a more general setting. For example, suppose that X admits a spreading model $(\tilde{x}_i)$ which is not equivalent to the unit vector basis in ℓ_1 but such that 1 is in the Krivine set of $(\tilde{x}_i)$. Then (Theorem 6.1) there exists a subspace W of X and a bounded operator $T\colon W \to W$ such that $p(T)$ is not a compact perturbation of the identity, for any polynomial p.

Our terminology is standard as may be found in [16, 17]. All our Banach spaces will be considered spaces over the real field $\mathbb{R}$. If $A \subset X$, where X is a Banach space, then $\mathrm{span}(A)$ is the linear span of A and $[A] = \overline{\mathrm{span}}(A)$ is the closed linear span of A. If S is a set, $c_{00}(S)$ denotes the vector space of finitely supported real valued functions on S. If $S = \mathbb{N}$ we write $c_{00} = c_{00}(\mathbb{N})$. S_X is the unit sphere of X and B_X is the unit ball of X. A basic sequence (x_i) is *block finitely represented* in (y_i) if for all $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists a block basis $(z_i)_{i=1}^n$ of (y_i) satisfying

$$(1+\varepsilon)^{-1}\Big\| \sum_1^n a_i x_i \Big\| \le \Big\| \sum_1^n a_i z_i \Big\| \le (1+\varepsilon)\Big\| \sum_1^n a_i x_i \Big\|$$

for all $(a_i)_1^n \subseteq \mathbb{R}$. We say *$\ell_p$ is block finitely represented in (y_i)* if the unit vector basis of ℓ_p is block finitely represented in (y_i).

Let (x_i) be a basic sequence and $C \ge 1$. Then (x_i) is called *C-spreading* if for all $(a_i) \in c_{00}$ and all choices of $n_1 < n_2 < \cdots$ in $\mathbb{N}$,

$$\frac{1}{C}\Big\| \sum_{i=1}^\infty a_i x_i \Big\| \le \Big\| \sum_{i=1}^\infty a_i x_{n_i} \Big\| \le C\Big\| \sum_{i=1}^\infty a_i x_i \Big\|, \tag{2}$$

and (x_i) is called *C-suppression unconditional* if for all $(a_i) \in c_{00}$ and $A \subset \mathbb{N}$.

$$\Big\| \sum_{i\in A} a_i x_i \Big\| \le C\Big\| \sum_{i=1}^\infty a_i x_i \Big\|.$$

We say that (x_i) is *C-subsymmetric* if it is C-spreading and C-suppression unconditional. Here we slightly deviate from the notions in [16, 17], where *C-subsymmetric* is defined to be C-spreading and C-unconditional (with respect to changes of signs). We say that (x_i) is *spreading, unconditional, or subsymmetric*, if for some $C \geq 1$, (x_i) is C-spreading, C-unconditional, or C-subsymmetric, respectively.

2 Spreading Models Containing ℓ_1 Which Are Not ℓ_1

Let us start with an observation which will be used several times through out the paper.

Proposition 2.1 *Assume that (f_i) is a normalized subsymmetric basic sequence. The following conditions are equivalent.*

(a) *(f_i) is equivalent to the unit vector basis of ℓ_1.*
(b) *There is an $r > 0$ so that $\left\| \sum_{i=1}^n f_i \right\| \geq rn$, for all $n \in \mathbb{N}$.*
(c) *There is a $C > 0$ such that for all $\rho > 0$ there exists an $(a_i^{(\rho)}) \in c_{00} \cap [-\rho, \rho]^{\mathbb{N}}$, so that*

$$\Big\| \sum_{i=1}^{\infty} a_i^{(\rho)} f_i \Big\| = 1 \text{ and } \sum_{i=1}^{\infty} |a_i^{(\rho)}| \leq C.$$

Proof Clearly (a)⇒(b)⇒(c). To prove the converse, we first assume, without loss of generality, that (f_i) is 1-subsymmetric. Let f_i^*, $i \in \mathbb{N}$, be the coordinate functionals. Since (f_i^*) is also a 1-subsymmetric basic sequence, we only need to show that the partial sums $(\sum_{i=1}^n f_i^*)_n$ are bounded in the dual norm.

Let $\rho > 0$ be arbitrary but fixed. Choose $x_\rho^* = \sum b_i^{(\rho)} f_i^* \in S_{\mathrm{span}(f_i^*: i \in \mathbb{N})}$ so that

$$x_\rho^* \Big(\sum a_i^{(\rho)} f_i \Big) = \sum b_i^{(\rho)} a_i^{(\rho)} = 1.$$

By unconditionality we can assume that $\mathrm{sign}(b_i^{(\rho)}) = \mathrm{sign}(a_i^{(\rho)}) = +$, for $i = 1, 2, \ldots$ and deduce that

$$\rho \cdot \Big| \Big\{ i : b_i^{(\rho)} \geq \frac{1}{2C} \Big\} \Big| \geq \sum_{\substack{i \in \mathbb{N} \\ b_i^{(\rho)} > 1/(2C)}} a_i^{(\rho)} b_i^{(\rho)} = 1 - \sum_{\substack{i \in \mathbb{N} \\ b_i^{(\rho)} \leq 1/(2C)}} a_i^{(\rho)} b_i^{(\rho)} \geq \frac{1}{2}.$$

This implies, again by the fact that (f_i^*) is 1-subsymmetric that

$$\Big\| \sum_{i=1}^{\lfloor \frac{1}{2\rho} \rfloor} \frac{1}{2C} f_i^* \Big\| \leq \Big\| \sum_{\substack{i \in \mathbb{N} \\ b_i^{(\rho)} \geq 1/(2C)}} b_i^{(\rho)} f_i^* \Big\| \leq \|x_\rho^*\| = 1,$$

and finishes the proof, if we let $\rho \to 0$. ■

Theorem 2.2 *There exists a reflexive Banach space X with an unconditional basis such that the spreading model of any normalized basic sequence in X is not isomorphic to c_0 or ℓ_1 and is not reflexive.*

For $x = (x_i)_i \in c_{00}$ we write $\operatorname{supp} x = \{i : x_i \neq 0\}$. For $x, y \in c_{00}$ and an integer k we say that $x < y$ if $\max \operatorname{supp} x < \min \operatorname{supp} y$, and we write $k < x$ if $k < \min \operatorname{supp} x$. Let (e_i) denote the unit vector basis of c_{00}.

In order to prove Theorem 2.2 we will construct a space X which has certain properties as stated in the following result, which will easily imply Theorem 2.2.

Theorem 2.3 *There is a space X with the following properties:*

(a) *X has a normalized 1-unconditional basis (e_i).*
(b) *For any normalized block basis of (e_i) having a spreading model $(\tilde{x}_i)$ we have that $(\tilde{x}_i)$ is not equivalent to the unit vector basis of ℓ_1.*
(c) *For any normalized block basis of (e_i) having a spreading model $(\tilde{x}_i)$ we have that ℓ_1 embeds into* $\operatorname{span}(\{\tilde{x}_i : i \in \mathbb{N}\})$.

Proof of Theorem 2.2 Let X be chosen as in Theorem 2.3. Since X has an unconditional basis and does not contain a subspace isomorphic to ℓ_1 or c_0 (otherwise a block basis of (e_i) would be equivalent to either the unit vector basis of ℓ_1 or c_0, both of which are excluded by (b) and (c)), X must be reflexive.

Since X is reflexive, every normalized basic sequence in X has a subsequence which is equivalent to a block basis of (e_i). Therefore (b) and (c), and the fact that ℓ_1 has a unique subsymmetric basis, imply that all the spreading models of normalized basic sequences in X are neither reflexive nor isomorphic to c_0 or ℓ_1. ■

Construction of the space X: First we choose an increasing sequence of integers (n_i) such that

$$\frac{1}{(n_1 + n_2 + \cdots + n_k)^{1/p}} \sum_{i=1}^{k} \frac{n_i}{3^i} \xrightarrow[k\to\infty]{} \infty \quad \text{for all } p > 1. \tag{3}$$

In order to choose a sequence (n_i) satisfying (3), first choose a sequence $(p_k)_k$ with $p_k \searrow 1$ and then inductively on $k \in \mathbb{N}$ pick (n_k) to satisfy

$$\frac{1}{(n_1 + n_2 + \cdots + n_k)^{1/p_k}} \sum_{i=1}^{k} \frac{n_i}{3^i} > k$$

for all $k \in \mathbb{N}$. Now we choose a norm $\|\cdot\|$ on c_{00} to satisfy the following Tsirelson type equation (see [24]):

$$\|x\| = \|x\|_\infty \vee \sup_{\substack{k\in\mathbb{N} \\ k \le E_1^{(i)} < E_2^{(i)} < \cdots < E_{n_i}^{(i)}, \\ \text{for } i \le k}} \sum_{i=1}^{k} \frac{1}{3^i} \sum_{j=1}^{n_i} \|E_j^{(i)} x\|.$$

Note that we do not require that $E_j^{(s)} \cap E_{j'}^{(t)} = \varnothing$ if $s \neq t$. Henceforth in this section X will denote the completion of c_{00} under this norm. It is easy to see that the unit vector basis (e_i) is a normalized 1-unconditional basis for X. It will be useful to introduce the sequence of equivalent norms $\|\cdot\|_i$, for $i \in \mathbb{N}$, as follows:

$$\|x\|_i = \sup_{E_1 < E_2 < \cdots < E_{n_i}} \sum_{j=1}^{n_i} \|E_j x\|.$$

Note that we have

$$\|x\| = \|x\|_\infty \vee \sup_{k \in \mathbb{N}} \sum_{i=1}^{k} \frac{1}{3^i} \|[k,\infty)x\|_i.$$

Proof of Theorem 2.3 (a) is immediate.
(b) We need the following auxiliary results. We postpone the proofs.

Lemma 2.4 *For any normalized block basis (y_i) of (e_i) and for any $\varepsilon > 0$ there exists a subsequence (x_i) and $i_0 \in \mathbb{N}$ such that for any $N \in \mathbb{N}$ and integers k, $j_1, \dots, j_N$ with $i_0 \le k \le j_1 < j_2 < \cdots < j_N$ we have that*

$$\sum_{i=i_0}^{k} \frac{1}{3^i} \Big\| [k,\infty)\Big(\frac{1}{N}\sum_{s=1}^{N} x_{j_s}\Big) \Big\|_i < \varepsilon. \tag{4}$$

Lemma 2.5 *Let (y_i) be a normalized block basis of (e_i) in X which has a spreading model $(\tilde{y}_i)$ and suppose that $N \in \mathbb{N}$ satisfies*

$$0.99 \le \Big\| \frac{1}{2N}(\tilde{y}_1 + \cdots + \tilde{y}_{2N}) \Big\|. \tag{5}$$

Then there exists $k \in \mathbb{N}$ and a subsequence (x_i) of (y_i) such that for all $j_1 < j_2 < \cdots < j_N$,

$$0.96 < \sum_{i=1}^{k} \frac{1}{3^i} \Big\| [k,\infty)\Big(\frac{1}{N}\sum_{s=1}^{N} x_{j_s}\Big) \Big\|_i. \tag{6}$$

For the proof of (b) assume to the contrary that there exists a normalized block basis (y_i) of (e_i) whose spreading model $(\tilde{y}_i)$ is equivalent to the unit vector basis of ℓ_1. Without loss of generality [3, Proposition 4 in Ch. II §2], we can assume that (5) is valid for all $N \in \mathbb{N}$. For $\varepsilon = 0.01$ choose $i_0 \in \mathbb{N}$ and a subsequence of (y_i) which satisfies the conclusion of Lemma 2.4. Choose $N \in \mathbb{N}$ with

$$\frac{2}{N} \sum_{i=1}^{i_0 - 1} n_i < 0.01.$$

Since (5) is valid, by Lemma 2.5 there exists $k \in \mathbb{N}$ and a further subsequence (x_i) which satisfies (6). Now let $j_1 < j_2 < \cdots < j_N$ with $k \le j_1$ and let $x =$

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$(1/N)\sum_{s=1}^{N} x_{j_s}$. We will first estimate for $i \in \mathbb{N}$ the value of $\|x\|_i$. Choose $E_1 < E_2 < \cdots < E_{n_i}$ so that

$$\|x\|_i = \sum_{i=1}^{n_i} \|E_j(x)\|.$$

Since (e_j) is 1-unconditional, we can assume that the E_j's are intervals in $\mathbb{N}$, that $\min E_1 = 1$, and that $\max E_j = \min E_{j+1} - 1$, for $j = 1, \ldots, n_i - 1$.

For $\ell = 1, 2, \ldots, n_i$ put $I_\ell = \{s \le N : \operatorname{supp}(x_{j_s}) \subset E_\ell\}$ and $I_0 = \{1, 2, \ldots, N\} \setminus \bigcup_{\ell=1}^{n_i} I_\ell$ and note that $I_0 = \{s \le N : \exists \ell_1, \ell_2 \le n_i, \ell_1 \ne \ell_2, \operatorname{supp}(x_{j_s}) \cap E_{\ell_t} \ne \varnothing, t = 1, 2\}$, and that $\sum_{\ell \le n_i} |I_\ell| \le N$. Moreover note that each E_ℓ can only have a non empty intersection with the support of at most two x_{j_s}'s, $s \in I_0$. Therefore we deduce

$$\|x\|_i = \sum_{\ell=1}^{n_i} \|E_\ell(x)\| \le \frac{1}{N} \sum_{\ell=1}^{n_i} \Big[\sum_{s \in I_\ell} \|x_s\| + \Big\| E_\ell\Big(\sum_{s \in I_0} x_s \Big) \Big\| \Big] \le 1 + \frac{2n_i}{N}. \tag{7}$$

By Lemma 2.5 we have (the second term below on the right disappears if $k < i_0$)

$$\begin{aligned} 0.96 &< \sum_{i=1}^{i_0-1} \frac{1}{3^i} \|[k,\infty)x\|_i + \sum_{i=i_0}^{k} \frac{1}{3^i} \|[k,\infty)x\|_i \\ &\le \sum_{i=1}^{i_0-1} \frac{1}{3^i} \|x\|_i + 0.01 && \text{(by (4) since } k \le j_1) \\ &\le \sum_{i=1}^{i_0-1} \frac{1}{3^i} \frac{2n_i + N}{N} + 0.01 < 0.01 + 0.5 + 0.01 = 0.52 && \text{(by (7))} \end{aligned}$$

which is a contradiction.
(c) Here we need the following result whose proof is again postponed:

Lemma 2.6 *Let (z_i) be a normalized block basis of (e_i) with spreading model $(\tilde z_i)$. Then for every $K_1 \in \mathbb{N}$ there exists a $K_2 > K_1$ and (w_i), an identically distributed block basis of (z_i), which has a spreading model $(\widetilde w_i)$ (which is a block basis of $(\tilde z_i)$) such that for all $\ell \in \mathbb{N}$: $0.98 \le \|w_\ell\| \le 1$ and*

$$\sum_{i=K_1+1}^{K_2} \frac{1}{3^i} \|[K_2,\infty)w_\ell\|_i > 0.4\,. \tag{8}$$

Let (z_i) be a normalized block basis of (e_i) having a spreading model $(\tilde z_i)$. By passing to a subsequence if necessary we can assume that (1) is satisfied for some sequence (ε_n) which converges to 0. By applying Lemma 2.6 repeatedly, there exists an increasing sequence of integers (K_n), ($K_1 = 0$), and for every $n \in \mathbb{N}$ there exists an identically distributed block basis $(w_i^{(n)})_i$ of (z_i) having spreading model $(\widetilde w_i^{(n)})_i$, which is also a block basis of $(\tilde z_i)$, such that for all $n, \ell \in \mathbb{N}$, $0.98 \le \|w_\ell^{(n)}\| \le 1$ and

$$\sum_{i=K_n+1}^{K_{n+1}} \frac{1}{3^i} \|[K_{n+1},\infty)w_\ell^{(n)}\|_i > 0.4. \tag{9}$$

Choose a sequence (m_i) of integers such that $(\widetilde{w}^{(i)}_{m_i})$ is a block sequence of $(\tilde{z}_i)_i$. We claim that $(\widetilde{w}^{(i)}_{m_i})$ is equivalent to the unit vector basis of ℓ_1. We show that for $N \in \mathbb{N}$ and $(a_i)_{i=1}^N \subseteq \mathbb{R}$,

$$\Big\| \sum_{i=1}^N a_i \widetilde{w}^{(i)}_{m_i} \Big\| > 0.4 \sum_{i=1}^N |a_i|.$$

Let $j_1 < j_2 < \cdots$ be such that $w^{(1)}_{j_1} < w^{(2)}_{j_2} < \cdots < w^{(N)}_{j_N} < w^{(1)}_{j_{N+1}} < w^{(2)}_{j_{N+2}} < \cdots < w^{(N)}_{j_{2N}} < w^{(1)}_{j_{2N+1}} < \cdots$. Then, since (z_i) satisfies (1), it follows that

$$\Big\| \sum_{i=1}^N a_i \widetilde{w}^{(i)}_{m_i} \Big\| = \lim_{\ell} \Big\| \sum_{n=1}^N a_n w^{(n)}_{j_{(\ell-1)N+n}} \Big\|.$$

If we choose ℓ such that $\sum_{n=1}^N w^{(n)}_{j_{(\ell-1)N+n}}$ is supported on $[K_{N+1}, \infty)$ then

$$\begin{aligned}
(10) \qquad \Big\| \sum_{n=1}^N a_n w^{(n)}_{j_{(\ell-1)N+n}} \Big\| &\geq \sum_{i=1}^{K_{N+1}} \frac{1}{3^i} \Big\| [K_{N+1}, \infty) \sum_{n=1}^N a_n w^{(n)}_{j_{(\ell-1)N+n}} \Big\|_i \\
&\geq \sum_{n=1}^N |a_n| \sum_{i=K_n+1}^{K_{n+1}} \frac{1}{3^i} \| [K_{n+1}, \infty) w^{(n)}_{j_{(\ell-1)N+n}} \|_i \\
&> 0.4 \sum_{n=1}^N |a_n| \qquad \text{(by (9))}.
\end{aligned}$$

■

Proof of Lemma 2.4 Since for all i and j we have $1 \leq \|y_j\|_i \leq n_i$, by a simple compactness and diagonalization argument there exists a subsequence (x_i) of (y_i) such that

$$(11) \qquad \big| \|x_i\|_i - \|x_j\|_i \big| \leq 1 \quad \text{for all } i \leq j.$$

Now we claim that

$$(12) \qquad \sum_{i=1}^\infty \frac{1}{3^i} \|x_i\|_i \leq \frac{3}{2}.$$

Indeed, otherwise there exists $k \in \mathbb{N}$ such that

$$(13) \qquad \sum_{i=1}^k \frac{1}{3^i} \|x_i\|_i > \frac{3}{2}.$$

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Choose $j \geq k$ such that x_j is supported on $[k, \infty)$. Then

$$\begin{aligned}
\|x_j\| &\geq \sum_{i=1}^{k} \frac{1}{3^i} \|[k,\infty)x_j\|_i \\
&= \sum_{i=1}^{k} \frac{1}{3^i} \|x_j\|_i && \text{(since } x_j \text{ is supported on } [k,\infty)) \\
&\geq \sum_{i=1}^{k} \frac{1}{3^i} (\|x_i\|_i - 1) && \text{(by (11), since } j \geq k) \\
&> \frac{3}{2} - \sum_{i=1}^{k} \frac{1}{3^i} > 1 && \text{(by (13))}
\end{aligned}$$

which is a contradiction. Thus (12) is established. Now choose $i_0 \in \mathbb{N}$ such that

$$\sum_{i=i_0}^{\infty} \frac{1}{3^i} \|x_i\|_i + \sum_{i=i_0}^{\infty} \frac{1}{3^i} < \varepsilon. \tag{14}$$

Let $k, j_1, \dots, j_N \in \mathbb{N}$ with $i_0 \leq k \leq j_1 < j_2 < \cdots < j_N$. We have

$$\begin{aligned}
\sum_{i=i_0}^{k} \frac{1}{3^i} \Big\| [k,\infty) \sum_{s=1}^{N} x_{j_s} \Big\|_i &\leq \sum_{s=1}^{N} \sum_{i=i_0}^{k} \frac{1}{3^i} \|x_{j_s}\|_i \\
&\leq \sum_{s=1}^{N} \sum_{i=i_0}^{k} \frac{1}{3^i} (\|x_i\|_i + 1) && \text{(by (11), since } k \leq j_1) \\
&< N\varepsilon && \text{(by (14)).}
\end{aligned}$$

■

Proof of Lemma 2.5 From (5) there exists a subsequence (z_i) of (y_i) such that

$$0.98 < \Big\| \frac{1}{2N} (z_1 + z_2 + \cdots + z_N + z_{j_1} + z_{j_2} + \cdots + z_{j_N}) \Big\| \tag{15}$$

for all $N < j_1 < j_2 < \cdots < j_N$. Let K be the maximum element in the support of z_N. Now for $j_1 < j_2 < \cdots < j_N$ let $u = (z_1 + \cdots + z_N)/N$, $v = (z_{j_1} + \cdots + z_{j_N})/N$ and $w = (u + v)/2$. By the definition of the norm of X there exists $k' \in \mathbb{N}$, which depends on $j_1, \dots, j_N$, such that

$$\|w\| = \sum_{i=1}^{k'} \frac{1}{3^i} \|[k',\infty)w\|_i.$$

By (15) we have that $0.98 < \|w\|$ and thus $k' \le K$. By the triangle inequality we obtain

$$0.98 < \frac{1}{2}\sum_{i=1}^{k'} \frac{1}{3^i}\|[k',\infty)u\|_i + \frac{1}{2}\sum_{i=1}^{k'} \frac{1}{3^i}\|[k',\infty)v\|_i$$
$$\le \frac{1}{2}\|u\| + \frac{1}{2}\sum_{i=1}^{k'} \frac{1}{3^i}\|[k',\infty)v\|_i \le 0.5 + \frac{1}{2}\sum_{i=1}^{k'} \frac{1}{3^i}\|[k',\infty)v\|_i.$$

Thus

$$0.96 < \sum_{i=1}^{k'} \frac{1}{3^i}\|[k',\infty)v\|_i.$$

Now by Ramsey's theorem [25] (see also [21]) there exists a subsequence (x_i) of $(z_i)_{i\ge N}$ and $k \le K$ such that $k'(j_1, j_2, \dots, j_N) = k$ for all choices of $j_1 < j_2 < \cdots < j_N$, and, thus (6) is valid for all $j_1 < j_2 < \cdots < j_N$. ■

Proof of Lemma 2.6 Let us first note that neither ℓ_p, $p > 1$, nor c_0 are finitely block represented in X. Indeed, if (x_i) for $i = 1, \dots, n_1 + \cdots + n_k$ (for some $k \in \mathbb{N}$) is a normalized block basis of (e_i) which is 2-equivalent to the first $n_1 + \cdots + n_k$ unit basic vectors of ℓ_p for some $p > 1$, then if $\operatorname{supp} x_1 > k$, it follows that

$$2(n_1 + \cdots + n_k)^{1/p} \ge \Big\|\sum_{i=1}^{n_1+\cdots+n_k} x_i\Big\|$$
$$\ge \sum_{i=1}^{k} \frac{1}{3^i}\sum_{j=1}^{n_i}\|x_j\| \qquad \text{(by definition of the norm)}$$
$$= \sum_{i=1}^{k} \frac{1}{3^i} n_i$$

which contradicts (3). Similarly the case $p = \infty$ is excluded and thus the conclusions of Remark 1.2 hold only for $p = 1$.

Let (z_i) be a normalized block sequence in X having a spreading model $(\tilde{z}_i)$, and let $K_1 \in \mathbb{N}$. Choose $N \in \mathbb{N}$ such that

$$\frac{2}{N}\sum_{i=1}^{K_1} n_i < 0.01. \tag{16}$$

By Remark 1.2 there exists an identically distributed block basis (y_i) of (z_i) having spreading model $(\tilde{y}_i)$ which satisfies (5) and $(\tilde{y}_\ell)$ is a block basis of $(\tilde{z}_i)$. Thus by Lemma 2.5 there exists $K_2 \in \mathbb{N}$ and a subsequence (x_i) of (y_i) such that (6) is satisfied for $k = K_2$ and for all $j_1 < j_2 < \cdots < j_N$. Let

$$w_\ell = \frac{1}{N}\sum_{j=1}^{N} x_{N(\ell-1)+j} \quad \text{for } \ell \in \mathbb{N}.$$

Since (5) is satisfied, by passing to a subsequence we can assume that $0.98 \le \|w_\ell\| \le 1$ for all ℓ. Let $(\tilde{w}_i)$ be the spreading model of (w_i). Then for all $\ell \in \mathbb{N}$,

$$\widetilde{w}_\ell = \frac{1}{N}\sum_{j=1}^{N} \tilde{y}_{N(\ell-1)+j}.$$

Thus $(\widetilde{w}_\ell)$ is a block basis of $(\tilde{z}_i)$ and

$$0.96 < \sum_{i=1}^{K_2} \frac{1}{3^i}\|[K_2,\infty)w_\ell\|_i. \tag{17}$$

Note also that by with the same argument as in the proof of (7),

$$\sum_{i=1}^{K_1-1} \frac{1}{3^i}\|[K_2,\infty)w_\ell\|_i \le \sum_{i=1}^{K_1-1} \frac{1}{3^i}\frac{2n_i+N}{N} < 0.01 + 0.5 = 0.51. \quad \text{(by (16))} \tag{18}$$

Now (17) and (18) immediately give (8). ■

3 The Set of Spreading Models of X

We recall the standard

Definition 3.1 Let (x_i) and (y_i) be basic sequences and $C \ge 1$. We say that (x_i) C-dominates (y_i), if $C\|\sum_i a_i x_i\| \ge \|\sum a_i y_i\|$ for all $(a_i) \in c_{00}$. We say that (x_i) dominates (y_i), denoted by $(x_i) \ge (y_i)$, if (x_i) C-dominates (y_i) for some $C \ge 1$. We write $(x_i) > (y_i)$, if $(x_i) \ge (y_i)$ and $(y_i) \not\ge (x_i)$. If $\mathcal{B}$ is a set of basic sequences and (z_i) is a basic sequence, then we say that (z_i) uniformly dominates $\mathcal{B}$ if there exists $C \ge 1$ such that (z_i) C-dominates every element of $\mathcal{B}$.

The set $SP(X)$ of all spreading models generated by normalized basic sequences in X is partially ordered by domination, provided that we identify equivalent spreading models. $SP_\omega(X)$ denotes the subset of those spreading models generated by weakly null sequences.

Our first result in this section shows that every countable subset of $SP_\omega(X)$ admits an upper bound in $SP_\omega(X)$.

Proposition 3.2 *Let $(C_n) \subset (0,+\infty)$ be such that $\sum C_n^{-1} < \infty$, and for $n \in \mathbb{N}$ let $(x_i^{(n)})_i$ be a normalized weakly null sequence in some Banach space X having spreading model $(\tilde{x}_i^{(n)})_i$.*

Then there exists a seminormalized weakly null basic sequence (y_i) in X with a spreading model $(\tilde{y}_i)$ having the following properties.

(a) *$(\tilde{y}_i)$ C_n-dominates $(\tilde{x}_i^{(n)})_i$ for all $n \in \mathbb{N}$.*

(b) *If for no $n \in \mathbb{N}$, $(\tilde{x}_i^{(n)})_i$ is equivalent to the unit vector basis of ℓ_1, then $(\tilde{y}_i)$ is not equivalent to the unit vector basis of ℓ_1.*

(c) *If (z_i) is a basic sequence which uniformly dominates $(\tilde{x}_i^{(n)})_i$ for all $n \in \mathbb{N}$, then (z_i) dominates $(\tilde{y}_i)$.*

In order to prove Proposition 3.2 we first need to generalize the fact that spreading models of normalized weakly null sequences exist and are suppression 1-unconditional.

Lemma 3.3 is actually a special case of a more general situation [11]. The results could also be phrased in terms of countably branching trees of order mn and proved much like the arguments in [13].

Lemma 3.3 *Let $n, m \in \mathbb{N}$ and $\varepsilon > 0$. Let $(x_i^{(1)})_i, (x_i^{(2)})_i, \dots, (x_i^{(n)})_i$ be normalized weakly null sequences in a Banach space X. Then there exists a subsequence L of $\mathbb{N}$ so that for all families of integers $(k_j^{(i)})_{i=1,j=1}^{n,m}$ and $(\ell_j^{(i)})_{i=1,j=1}^{n,m}$ in L, with $k_1^{(1)} < k_1^{(2)} < \cdots < k_1^{(n)} < k_2^{(1)} < \cdots < k_2^{(n)} < \cdots < k_m^{(1)} < \cdots < k_m^{(n)}$ and $\ell_1^{(1)} < \ell_1^{(2)} < \cdots < \ell_1^{(n)} < \ell_2^{(1)} < \cdots < \ell_m^{(n)}$, and $(a_i^{(j)})_{i=1,j=1}^{m,n} \subseteq [-1,1]$ we have*

$$\Big| \Big\| \sum_{i=1}^{m} \sum_{j=1}^{n} a_i^{(j)} x^{(j)}_{\ell_i^{(j)}} \Big\| - \Big\| \sum_{i=1}^{m} \sum_{j=1}^{n} a_i^{(j)} x^{(j)}_{k_i^{(j)}} \Big\| \Big| \leq \varepsilon.$$

Proof This follows easily by Ramsey's theorem. Let $(a_i^{(j)})_{i=1,j=1}^{m,n} \subseteq [-1,1]$. Partition $[0, mn]$ into finitely many intervals of length less than $\varepsilon/2$. Partition the sequences of length mn, $k_1^{(1)} < k_1^{(2)} < \cdots < k_1^{(n)} < k_2^{(1)} < \cdots < k_m^{(n)}$ of $\mathbb{N}$, according to which interval $\| \sum_{i=1}^{m} \sum_{j=1}^{n} a_i^{(j)} x^{(j)}{}_{k_i^{(j)}} \|$ belongs. Thus by Ramsey's theorem for some infinite subsequence L of $\mathbb{N}$ these expressions belong to the same interval, if $k_i^{(j)} \in L$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. We repeat this for a finite $\varepsilon/4$-net of $[-1,1]^{mn}$ endowed with the ℓ_1^{mn} norm. ■

Lemma 3.4 *Let $n, m \in \mathbb{N}$, $\varepsilon > 0$ and $(x_i^{(1)})_i, \dots, (x_i^{(n)})_i$ be normalized weakly null sequences in a Banach space X. Then there exists a subsequence L of $\mathbb{N}$ so that for all integers in L, $k_1^{(1)} < k_1^{(2)} < \cdots < k_1^{(n)} < k_2^{(1)} < \cdots < k_2^{(n)} < \cdots < k_m^{(1)} < \cdots < k_m^{(n)}$, the vectors $(x^{(j)}_{k_i^{(j)}})_{i=1,j=1}^{m,n}$ form a suppression $(1+\varepsilon)$-unconditional basic sequence.*

Proof By passing to subsequences, if necessary, we may assume that the sequence $(x_i^{(j)})_{j=1,i=1}^{n,\infty}$ satisfies the conclusion of Lemma 3.3 for ε replaced by $\varepsilon/2$ and $L = \mathbb{N}$. Let $\delta = \varepsilon/(2nm)$. We claim that for every $j_0 \leq n$ and $i_0 \in \mathbb{N}$ there exists $i_1 > i_0$ such that for every functional $f \in X^*$ of norm 1 there exists $i \in [i_0, i_1]$ with $|f(x_i^{(j_0)})| < \delta$. Indeed, assume that such an $i_1 > i_0$ did not exist. Then we could find for each $i_1 > i_0$ an $f_{i_1}^* \in S_{X^*}$ such that $|f_{i_1}^*(x_i^{(j_0)})| \geq \delta$ for all $i \in \{i_0, i_0+1, \dots i_1\}$. Let f^* be a w*-accumulation point of the set $\{f_{i_1}^* : i_1 \geq i_0\}$. It follows that $|f^*(x_i^{(j_0)})| \geq \delta$ for all $i \geq i_0$, which contradicts the assumption that $(x_i^{(j_0)})$ is weakly null, and proves the claim.

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Iterating this claim we can pass to an infinite subsequence L of $\mathbb{N}$ with the following property: for $k_1^{(1)} < k_1^{(2)} < \cdots < k_1^{(n)} < k_2^{(1)} < \cdots < k_m^{(n)}$ in L,

$$F \subseteq \{k_1^{(1)}, k_1^{(2)}, \ldots, k_1^{(n)}, k_2^{(1)}, \ldots, k_m^{(n)}\},$$

and $f \in X^*$ of norm 1, there exist $\ell_1^{(1)} < \ell_1^{(2)} < \cdots < \ell_1^{(n)} < \ell_2^{(1)} < \cdots < \ell_m^{(n)}$ in $\mathbb{N}$ with $\ell_i^{(j)} = k_i^{(j)}$ if $k_i^{(j)} \in F$ and $|f(x^{(j)}_{\ell_i^{(j)}})| < \delta$ if $k_i^{(j)} \notin F$. Let $k_1^{(1)} < k_1^{(2)} < \cdots < k_1^{(n)} < k_2^{(1)} < \cdots < k_m^{(n)}$ in L, $F \subseteq \{k_1^{(1)}, k_1^{(2)}, \ldots, k_1^{(n)}, k_2^{(1)}, \ldots, k_m^{(n)}\}$, and $(a_i^{(j)})_{i=1,j=1}^{m,n} \subseteq [-1,1]$ with $\|\sum_{i=1}^m \sum_{j=1}^n a_i^{(j)} x^{(j)}_{k_i^{(j)}}\| = 1$. There exists $f \in X^*$ of norm 1 such that

$$\begin{aligned}
\Big\| \sum_{\{(i,j):k_i^{(j)}\in F\}} a_i^{(j)} x^{(j)}_{k_i^{(j)}} \Big\| &= f\Big(\sum_{\{(i,j):k_i\in F\}} a_i^{(j)} x^{(j)}_{k_i^{(j)}} \Big), \qquad \text{and choosing } (\ell_i^{(j)}) \text{ as above,}\\
&\le f\Big(\sum_{i=1}^m \sum_{j=1}^n a_i^{(j)} x^{(j)}_{\ell_i^{(j)}} \Big) + \delta nm\\
&\le \Big\| \sum_{i=1}^m \sum_{j=1}^n a_i^{(j)} x^{(j)}_{\ell_i^{(j)}} \Big\| + \frac{\varepsilon}{2} \le (1+\frac{\varepsilon}{2}) + \frac{\varepsilon}{2} = 1+\varepsilon.
\end{aligned}$$

■

Proof of Proposition 3.2 Using Lemma 3.4, a diagonal argument and relabelling, we can assume that for all ℓ and all choices of $\ell \le k_1^{(1)} < k_1^{(2)} < \cdots < k_1^{(\ell)} < k_2^{(1)} < \cdots < k_\ell^{(\ell)}$ the vectors $(x^{(j)}_{k_i^{(j)}})1 \le i, j \le \ell$ are suppression 2-unconditional.

Let $m_1 = 0$ and for $i \in \mathbb{N}$ let $m_{i+1} = m_i + i$. Let $(C_i) \subset (0,\infty)$ such that $\sum C_i^{-1} < \infty$. By passing to the same subsequences of $(x_i^{(n)})_i$, for each $n \in \mathbb{N}$, we can assume in addition that the seminormalized sequence (y_j), where

$$y_j = \sum_{i=1}^{j} 16\, C_i^{-1} x^{(i)}_{m_j+i} \quad \text{for all } j \in \mathbb{N},$$

has a spreading model $(\tilde{y}_j)$. It is easy to check that (y_j) is weakly null since each $(x_i^{(n)})_{i=1}^\infty$ is weakly null for each $n \in \mathbb{N}$.

Let $i_0, m \in \mathbb{N}$ and $(a_j)_{j=1}^m \subset \mathbb{R}$. Let $n \in \mathbb{N}$ such that $\max(m, i_0) \le n$ and

$$8 \sum_{j=1}^m |a_j| \sum_{i=n+1}^{\infty} C_i^{-1} \le C_{i_0}^{-1} \Big\| \sum_{j=1}^m a_j \tilde{x}_j^{(i_0)} \Big\|.$$

In addition choose n so that $\frac{1}{2}\| \sum_{j=1}^m a_j x^{(i_0)}_{\ell_j}\| \le \|\sum_{j=1}^m a_j \tilde{x}_j^{(i_0)}\| \le 2\|\sum_{j=1}^m a_j x^{(i_0)}_{\ell_j}\|$ and $\frac{1}{2}\|\sum_{j=1}^m a_j y_{\ell_j}\| \le \|\sum_{j=1}^m a_j \tilde{y}_j\| \le 2\|\sum_{j=1}^m a_j y_{\ell_j}\|$ for all choices of $n \le \ell_1 <$

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$\ell_2 < \cdots < \ell_m$. If $n \le k_1 < k_2 < \cdots < k_m$, we have from Lemma 3.4 and our inequalities,

$$
\begin{aligned}
(19)\qquad \Big\|\sum_{j=1}^m a_j\tilde{y}_j\Big\| &\ge \frac12\Big\|\sum_{j=1}^m a_j y_{k_j}\Big\|\\
&= \frac12\Big\|\sum_{j=1}^m\sum_{i=1}^{k_j} a_j 16C_i^{-1}x^{(i)}_{m_{k_j}+i}\Big\|\\
&\ge \frac12\Big\|\sum_{j=1}^m\sum_{i=1}^{n} a_j 16C_i^{-1}x^{(i)}_{m_{k_j}+i}\Big\| - \frac12\sum_{j=1}^m\sum_{i=n+1}^{k_j}|a_j|16C_i^{-1}\\
&\ge \frac14\Big\|\sum_{j=1}^m a_j 16C_{i_0}^{-1}x^{(i_0)}_{m_{k_j}+i_0}\Big\| - C_{i_0}^{-1}\Big\|\sum_{j=1}^m a_j\tilde{x}_j^{(i_0)}\Big\|\\
&\ge \frac18\Big\|\sum_{j=1}^m a_j 16C_{i_0}^{-1}\tilde{x}_j^{(i_0)}\Big\| - C_{i_0}^{-1}\Big\|\sum_{j=1}^m a_j\tilde{x}_j^{(i_0)}\Big\|\\
&= C_{i_0}^{-1}\Big\|\sum_{j=1}^m a_j\tilde{x}_j^{(i_0)}\Big\|.
\end{aligned}
$$

This proves part (a) of the proposition.

In order to show the remaining parts let $m \in \mathbb{N}$ and $(a_j)_{j=1}^m \subseteq \mathbb{R}$, and first note that

$$
\begin{aligned}
(20)\qquad \Big\|\sum_{j=1}^m a_j\tilde{y}_j\Big\| &= \lim_{j_1\to\infty}\cdots\lim_{j_m\to\infty}\Big\|\sum_{s=1}^m a_s y_{j_s}\Big\|\\
&= \lim_{j_1\to\infty}\cdots\lim_{j_m\to\infty}\Big\|\sum_{s=1}^m a_s\sum_{i=1}^{j_s}16C_i^{-1}x^{(i)}_{m_{j_s}+i}\Big\|\\
&\le \limsup_{j_1\to\infty}\cdots\limsup_{j_m\to\infty}\sum_{i=1}^\infty 16\,C_i^{-1}\Big\|\sum_{s=1}^m a_s x^{(i)}_{m_{j_s}+i}\Big\|\\
&= \sum_{i=1}^\infty 16\,C_i^{-1}\Big\|\sum_{s=1}^m a_s\tilde{x}_s^{(i)}\Big\|.
\end{aligned}
$$

Part (c) now follows from (20) and the assumption that (C_i^{-1}) is summable. In order to show part (b), assume that for any $n \in \mathbb{N}$ $(\tilde{x}^{(n)})_i$ is not equivalent to ℓ_1 and let $\delta > 0$. First choose i_0 so that $16\sum_{i=i_0+1}^\infty C_i^{-1} < \delta/2$. Then choose N large enough so that $\frac1N\|\sum_{j=1}^N\tilde{x}_j^{(i)}\| \le \delta/(2i_0)$, for $i = 1,\dots,i_0$ (using Proposition 2.1), and finally apply (20) for $m = N$, $a_j = \frac1N$, to obtain $\frac1N\|\sum_{j=1}^N\tilde{y}_j\| < \delta$. ∎

Remark 3.5 Using a similar argument we can prove the following:

(a) Let $C = \{(\tilde{x}_i^{(n)})_i\}_{n\in\mathbb{N}}$ be a strictly increasing chain in $SP_\omega(X)$. Suppose that $(\tilde{z}_i)_i \in SP_\omega(X)$ is an upper bound for C. Then there exists an upper bound $(\tilde{x}_i)_i \in SP_\omega(X)$ for which $(\tilde{x}_i)_i < (\tilde{z}_i)_i$.

(b) If $(\tilde{x}_i^{(n)})_i \in SP_\omega(X)$ for $n \le m \in \mathbb{N}$ then there exists $(\tilde{x}_i)_i \in SP_\omega(X)$ which is equivalent to the norm given by

$$\|(a_i)\| = \max_{n\le m}\Big\|\sum_i a_i\tilde{x}_i^{(n)}\Big\|.$$

An analogous result for asymptotic structure of spaces with a shrinking basis is obtained in [20, Proposition 5.1].

Proposition 3.6 *Suppose that (x_i) is a normalized weakly null sequence in a Banach space X which has a spreading model $(\tilde{x}_i)$ which is not equivalent to the unit vector basis of ℓ_1. Assume that 1 belongs to the Krivine set of $(\tilde{x}_i)$. Then for all sequences $(\lambda_n) \subset \mathbb{R}$, with $\lambda_n \nearrow \infty$ and $\lim_n n/\lambda_n = \infty$, there is a normalized block sequence (y_n) of (x_n) having a spreading model $(\tilde{y}_n)$ which satisfies:*

$$\limsup_n \frac{n}{\big\|\sum_{i=1}^n \tilde{y}_i\big\|} = \limsup_n \frac{\big\|\sum_{i=1}^n \tilde{y}_i\big\|}{\lambda_n} = \infty.$$

Moreover, the set of all spreading models in X which are not equivalent to the unit vector basis of ℓ_1, and are generated by weakly null sequences, has no maximal element (with respect to domination).

Note that the space X constructed in Section 2 is reflexive and satisfies the hypothesis of the proposition (as does every subspace of X).

Proof Using $\lim n/\lambda_n = \infty$, choose a subsequence (n_k) of $\mathbb{N}$ such that $n_k/\lambda_{n_k} \ge 2^{k+1}k$ for all k. Since 1 belongs to the Krivine set of $(\tilde{x}_i)$, for every $n \in \mathbb{N}$ there exists a block sequence $(x_i^{(n)})_i$ of (x_i) which is identically distributed with respect to (x_i) and it has a normalized spreading model $(\tilde{x}_i^{(n)})_i$ as given in Remark 1.2 (for $p = 1$ and $\varepsilon = 1$) satisfying

$$\frac{n_k}{2} \le \Big\|\sum_{i=1}^{n_k} \tilde{x}_i^{(k)}\Big\|.$$

Since (x_i) is weakly null and $(\tilde{x}_i)$ not equivalent to the unit vector basis of ℓ_1, we have that for all $n \in \mathbb{N}$, $(x_i^{(n)})_i$ is weakly null and $(\tilde{x}_i^{(n)})_i$ is not equivalent to the unit vector basis of ℓ_1. We can also assume without loss of generality that $(x_i^{(n)})_i$ is normalized. Let (y_i) be the sequence which is provided by Proposition 3.2 for $C_k = 2^{-k}$. By Proposition 3.2(b) we have that $(\tilde{y}_i)$ is not equivalent to the unit vector basis of ℓ_1 thus

$$\limsup_n \frac{n}{\big\|\sum_{i=1}^n \tilde{y}_i\big\|} = \infty$$

by Proposition 2.1. Also, by Proposition 3.2 we have that

$$2^k \Big\| \sum_{j=1}^{n_k} \tilde{y}_j \Big\| \geq \Big\| \sum_{j=1}^{n_k} \tilde{x}_i^{(k)} \Big\| \geq \frac{n_k}{2}.$$

Thus for all $k \in \mathbb{N}$,

$$\frac{\| \sum_{j=1}^{n_k} \tilde{y}_j \|}{\lambda_{n_k}} \geq \frac{n_k}{2^{k+1} \lambda_{n_k}} \geq k,$$

which shows that

$$\limsup_n \frac{\| \sum_{j=1}^{n} \tilde{y}_j \|}{\lambda_n} = \infty,$$

and finishes the proof of the first part of Proposition 3.6 once we normalize (y_n).

To prove the "moreover" part, given a spreading model $(\tilde{z}_i) \in SP_\omega(X)$ not equivalent to the unit vector basis of ℓ_1 use the first part of the Proposition to get $(\tilde{y}_i)$ with (choose $\lambda_k = \| \sum_{i=1}^{n} \tilde{z}_i \|$, for $k \in \mathbb{N}$)

$$\limsup_n \frac{\| \sum_{i=1}^{n} \tilde{y}_i \|}{\| \sum_{i=1}^{n} \tilde{z}_i \|} = \infty.$$

Therefore $(\tilde{z}_i)$ is not maximal. ■

In some circumstances we will be able to conclude that $SP_\omega(X)$ admits a transfinite strictly increasing chain. The logical part of the argument is a simple proposition.

Proposition 3.7 *Let X be a separable infinite dimensional Banach space. Let $C \subseteq SP_\omega(X)$ be a non-empty set satisfying the following two conditions:*

(i) *C does not have a maximal element with respect to domination;*

(ii) *for every $(\tilde{X}_n)_{n\in\mathbb{N}} \subseteq C$ there exists $\tilde{X} \in C$ such that $\tilde{X}_n \leq \tilde{X}$ for every $n \in \mathbb{N}$.*

Then for all $\alpha < \omega_1$ there exists $\tilde{X}^{(\alpha)} \in C$ such that if $\alpha < \beta < \omega_1$ then $\tilde{X}^{(\alpha)} < \tilde{X}^{(\beta)}$.

Proof We use transfinite induction. Suppose that $\tilde{X}^{(\alpha)}$ have been constructed for $\alpha < \beta < \omega_1$. Then $\tilde{X}^{(\beta)}$ is chosen using (i) and (ii) if β is a successor ordinal and (ii) if β is a limit ordinal. ■

Remark 3.8 (1) The set $C = SP_\omega(X)$ satisfies condition (ii) by virtue of Proposition 3.2. Hence if $SP_\omega(X)$ does not have a maximal element, then it contains an uncountable increasing chain.

(2) Suppose $SP_\omega(X)$ contains $(\tilde{x}_i)$ such that 1 is in the Krivine set of $(\tilde{x}_i)$ but $(\tilde{x}_i)$ is not equivalent to the unit vector basis of ℓ_1. Let C be the set of all elements of $SP_\omega(X)$ which are not equivalent to the unit vector basis in ℓ_1. Then it satisfies (ii) by Proposition 3.2 and (i) by Proposition 3.6. Therefore C contains an uncountable increasing chain. Examples of such a space X are the space constructed in Section 2, Gowers–Maurey space *GM* [10] and Schlumprecht's space *S* [28].

The following result is a strengthening of Proposition 3.6. First recall that if $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ ($n \in \mathbb{N}$) are two basic sequences then the basis-distance between them is defined by

$$d_b((x_i)_{i=1}^n, (y_i)_{i=1}^n) = \sup\Big\{ \frac{\|\sum_{i=1}^n a_i x_i\|}{\|\sum_{i=1}^n b_i x_i\|} : \Big\|\sum_{i=1}^n a_i y_i\Big\| = \Big\|\sum_{i=1}^n b_i y_i\Big\| = 1 \Big\}.$$

Proposition 3.9 *Let (z_i) be a normalized basis and $C < \infty$. Let X be an infinite dimensional Banach space. Assume that for all $n \in \mathbb{N}$ there exists a normalized weakly null sequence $(x_i^n)_i$ in X with spreading model $(\tilde{x}_i^{(n)})_i$ such that $(\tilde{x}_i^{(n)})_{i=1}^n$ C-dominates $(z_i)_{i=1}^n$ for all $n \in \mathbb{N}$. Assume also that (z_i) C-dominates $(\tilde{x}_i^{(n)})$ for each $n \in \mathbb{N}$. Then for every $\lambda_n \nearrow \infty$ there exists a normalized weakly null sequence (y_i) in X with spreading model $(\tilde{y}_i)$ so that*

$$\liminf_n \frac{d_b\big((\tilde{y}_i)_{i=1}^n, (z_i)_{i=1}^n\big)}{\lambda_n} = 0.$$

Proof Since $\lambda_n \nearrow \infty$, we can choose a sequence (n_k) of integers such that $k2^k \le \lambda_{n_k}$ for all k. Apply Proposition 3.2 to obtain a seminormalized weakly null sequence (y_i) in X with a spreading model $(\tilde{y}_i)$ such that $(\tilde{y}_i)$ 2^k-dominates $(\tilde{x}_i^{n_k})$, for all $k \in \mathbb{N}$. By part (c) of Proposition 3.2 we also have that there exists $C' < \infty$ such that (z_i) C'-dominates $(\tilde{y}_i)$. Let $k \in \mathbb{N}$ and $(b_i)_{i=1}^{n_k}$ be a sequence of scalars. Then

$$\Big\|\sum_{i=1}^{n_k} b_i \tilde{y}_i\Big\| \ge 2^{-k}\Big\|\sum_{i=1}^{n_k} b_i \tilde{x}_i^{n_k}\Big\| \ge 2^{-k}C^{-1}\Big\|\sum_{i=1}^{n_k} b_i z_i\Big\|.$$

Thus for $k \in \mathbb{N}$, if $(a_i)_{i=1}^{n_k}$ and $(b_i)_{i=1}^{n_k}$ are finite sequences of scalars satisfying

$$\Big\|\sum_{i=1}^{n_k} a_i z_i\Big\| = \Big\|\sum_{i=1}^{n_k} b_i z_i\Big\| = 1,$$

then

$$\frac{\|\sum_{i=1}^{n_k} a_i \tilde{y}_i\|}{\|\sum_{i=1}^{n_k} b_i \tilde{y}_i\|} \le \frac{C'}{2^{-k}C^{-1}} = CC'2^k \le \frac{\lambda_{n_k}}{k} CC'.$$

Hence $d_b\big((\tilde{y}_i)_{i=1}^{n_k}, (z_i)_{i=1}^{n_k}\big)/\lambda_{n_k} \le k^{-1}CC'$ which tends to zero. The result follows by normalizing (y_i). ■

Propositions 3.6 and 3.9 motivate the following:

Question 3.10 Which normalized subsymmetric bases (y_i) (if any) have the following property: If X is a separable infinite dimensional Banach space so that no spreading model of X is equivalent to (y_i) then there exists $\lambda_n \nearrow \infty$ and a subspace Y of X such that for all spreading models $(\tilde{x}_i)$ of normalized basic sequences in Y,

$$\liminf_j d_b\big((\tilde{x}_i)_{i=1}^j, (y_i)_{i=1}^j\big)/\lambda_j > 0.$$

This question is a generalization of the following problem raised by Rosenthal (which is solved by Proposition 3.6).

Question 3.11 Let Z be a separable infinite dimensional Banach space so that whenever $(\tilde{x}_i)$ is the spreading model of a normalized basic sequence in Z then

$$\lim_n \Big\| \sum_{i=1}^n \tilde{x}_i \Big\| \Big/ n = 0.$$

(*i.e.*, by Proposition 2.1, no spreading model in Z is equivalent to the unit vector basis of ℓ_1). Does there exist $\lambda_n \nearrow \infty$ such that $\lim_n \lambda_n / n = 0$ and for all spreading models $(\tilde{x}_i)$ of normalized basic sequences in Z

$$\lim_n \Big\| \sum_{i=1}^n \tilde{x}_i \Big\| \Big/ \lambda_n = 0?$$

The question asks whether all spreading models of Z must be uniformly distancing themselves from ℓ_1 for large enough dimensions.

Question 3.11 just asks if one could take (y_i) in Question 3.10 to be the unit vector basis of ℓ_1. Proposition 3.6 shows that this is not true, even hereditarily.

The version of Question 3.10 for the unit vector basis of c_0 is the following question. We will give an answer in the next section.

Question 3.12 Let Z be a separable infinite dimensional Banach space so that whenever $(\tilde{x}_i)$ is a spreading model of a normalized basic sequence in Z then

$$\lim_n \Big\| \sum_{i=1}^n \tilde{x}_i \Big\| = \infty.$$

Does there exist a sequence (λ_n) with $\lambda_n \nearrow \infty$ such that for all spreading models $(\tilde{x}_i)$ of normalized basic sequences in Z

$$\lim_n \Big\| \sum_{i=1}^n \tilde{x}_i \Big\| \Big/ \lambda_n = \infty?$$

The hypothesis of this question is equivalent to: no spreading model of Z is isomorphic to c_0. Indeed, suppose $(\tilde{x}_i)$ is a spreading model of a normalized basic sequence (x_i) with $\liminf_n \| \sum_{i=1}^n \tilde{x}_i \| < \infty$. We then obtain, since $(\tilde{x}_i)$ is basic, that $\sup_n \| \sum_{i=1}^n \tilde{x}_i \| \le K$ for some $K < \infty$. In particular, (x_i) must be weakly null and hence $(\tilde{x}_i)$ is unconditional. Thus $(\tilde{x}_i)$ is equivalent to the unit vector basis of c_0. Conversely, if some spreading model $(\tilde{x}_i)$ is a basis for c_0, then

$$\left(\frac{\tilde{x}_{2i+1} - \tilde{x}_{2i}}{\|\tilde{x}_{2i+1} - \tilde{x}_{2i}\|} \right)$$

is equivalent to the unit vector basis of c_0 and is a spreading model of

$$\left(\frac{x_{2i+1} - x_{2i}}{\|x_{2i+1} - x_{2i}\|} \right).$$

4 A Space Having Spreading Models Close to c_0

In this section we give an example which solves Question 3.12 negatively.

Theorem 4.1 *There is a Banach space X with a normalized basis (e_n) so that:*

(a) *For every sequence $(\lambda_n) \subset (0,\infty)$, with $\lim_{n\to\infty} \lambda_n = \infty$ there is a subsequence (e_{n_k}) of (e_n) which has a spreading model $(\tilde{x}_k)$ for which*

$$\lim_{m\to\infty} \Big\| \sum_{i=1}^m \tilde{x}_i \Big\| / \lambda_m = 0.$$

(b) *For every spreading model $(\tilde{x}_n)$ of a normalized block basis (x_n) of (e_n)*

$$\lim_{n\to\infty} \Big\| \sum_{i=1}^n \tilde{x}_i \Big\| = \infty.$$

Before defining X we need some notation. Let $\mathcal{D} = \bigcup_{n=0}^\infty \{0,1\}^n$ be the dyadic tree ordered by extension: $s = (s_i)_1^m \preceq t = (t_i)_1^n$ iff $m \le n$ and $s_i = t_i$ for $i \le m$. If $s = (s_i)_1^m \in \mathcal{D}$ we set $|s| = m$, $|\varnothing| = 0$ and if $s \preceq t$, $[s,t]$ denotes the *segment* $\{\alpha \in \mathcal{D} : s \preceq \alpha \preceq t\}$. A *branch* β in $\mathcal{D}$ is a maximal linearly ordered subset. If $(\beta_i)_{i=1}^\infty \in \{0,1\}^{\mathbb{N}}$ we write $\beta = (\beta_i)$ to denote the branch $(\beta^n)_{n=1}^\infty$ where $\beta^n = (\beta_i)_{i=1}^n$.

Lemma 4.2 *Let $(t_n)_{n=1}^\infty$ be distinct elements of $\mathcal{D}$. Then there exists a subsequence (t'_n) of (t_n) and a sequence (s_n) in $\mathcal{D}$ so that $s_1 \prec s_2 \prec \cdots$ and $([s_n, t'_n])_{n=1}^\infty$ are disjoint segments.*

Proof By passing to a subsequence (*e.g.,* using Ramsey's theorem) we may assume that either $t_1 \prec t_2 \prec \cdots$, in which case we take $s_i = t_i$ for all i, or t_i and t_j are incomparable for all $i \neq j$. In the latter case we let $s_1 = \varnothing$, $t'_1 = t_1$ and choose s_2 with $|s_2| = |t'_1|$ so that $\{t_i : s_2 \prec t_i\}$ is infinite. We let t'_2 be one of these t_i's and select s_3 with $|s_3| = |t'_2|$ so that $\{t_i : s_3 \prec t_i\}$ is infinite and proceed in this fashion. ■

Proof of Theorem 4.1 For each $s \in \mathcal{D}$ we shall define a decreasing sequence $V_s = (V_s(i))_{i=1}^\infty$ in $(0,1]$. If $s = \varnothing$, $V_s(i) = 1$ for all i. If $s = (\varepsilon_i)_{i=1}^m$ let $\{n : \varepsilon_n = 1, n \le m\} = (n_i)_{i=1}^k$ written in increasing order. If $\varepsilon_i = 0$ for $i \le m$ we let $V_s(i) = 1$ for all i. Otherwise for $i \le n_1$, $V_s(i) = 1/n_1$. If $n_j < i \le n_{j+1}$, set $V_s(i) = V_s(n_j) \wedge 1/(n_{j+1} - n_j)$. If $i > n_k$ set $V_s(i) = V_s(n_k)$. If $\beta = (\beta_i)_1^\infty$ is a branch, naturally identified as a sequence of 0's and 1's, V_β is defined similarly. Clearly $\sum_{i=1}^\infty V_\beta(i) = \infty$ for all branches β.

If $x \in c_{00}(\mathcal{D})$ we set

$$\|x\| = \sup \sum_{i=1}^n V_{s_i}(i) |x(t_i)|$$

where the sup is taken over all $n \in \mathbb{N}$, and disjoint segments $[s_1, t_1], \ldots, [s_n, t_n]$ such that $|s_1| \le |s_2| \le \cdots \le |s_n|$. Then $\|x\| \ge \|x\|_\infty$ follows by considering $[\varnothing, t]$. The

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motivation for defining the norm in this manner comes from Lemma 4.2 and Case 1 below.

The unit vector basis $(e_\alpha)_{\alpha\in\mathcal{D}}$ forms a normalized 1-unconditional basis for X, the completion of $(c_{00}(\mathcal{D}), \|\cdot\|)$. We verify (a). Let $\lambda_j \uparrow \infty$ and choose integers $n_1 < n_2 < \cdots$ so that

$$\lambda_{n_j} > j^2 \tag{21}$$

$$\frac{1}{n_{j+1}-n_j} < \frac{1}{n_j - n_{j-1}} \quad \text{for all } j \tag{22}$$

(with $n_0 = 0$).

Let $\beta_{n_j} = 1$ for all j, $\beta_i = 0$ if $i \notin \{n_1, n_2, \dots\}$ and $\beta = (\beta_i)$. Let $\beta^i = (\beta_j)_{j=1}^i$ and let $x_i = e_{\beta^i}$ for $i \in \mathbb{N}$. Let $m \in \mathbb{N}$. We will prove that if $m \in (n_{j_0-1}, n_{j_0}]$ and $n_{j_0} < k_1 < \cdots < k_m$ then

$$\Big\| \sum_{i=1}^m x_{k_i} \Big\| \le j_0 + 1\,. \tag{23}$$

Thus if $(\tilde{x}_i)$ is any spreading model of a subsequence of (x_i), by (21),

$$\frac{\| \sum_{i=1}^m \tilde{x}_i \|}{\lambda_m} < \frac{j_0+1}{(j_0-1)^2}$$

and this yields (a).

Let $[s_1, t_1], \dots, [s_n, t_n]$ be disjoint segments with $|s_1| \le |s_2| \le \cdots \le |s_n|$ such that for $x = \sum_{i=1}^m x_{k_i}$,

$$\|x\| = \sum_{i=1}^n V_{s_i}(i) |x(t_i)|\,.$$

Since each V_s is a decreasing sequence, we may assume that $x(t_i) \ne 0$ for all $i \le n$ and hence $n \le m$. Also each t_i is the support of some x_{k_ℓ} and so the segments must lie all on β. In particular while $|s_1| < k_1$ is possible, $|s_i| \ge k_1$ for $i \ge 2$. Note that $n \le m \le n_{j_0}$. Hence by (22) for $i \ge 2$ the first n elements of V_{s_i} are the first n elements of the sequence

$$\Big(\frac{1}{n_1}\chi_{[1,n_1]}, \frac{1}{n_2-n_1}\chi_{(n_1,n_2]}, \dots, \frac{1}{n_{j_0}-n_{j_0-1}}\chi_{(n_{j_0-1},n_{j_0}]} \Big)\,.$$

Thus $\| \sum_{i=1}^m x_{k_i} \| \le 1 + \sum_{j=1}^{j_0} 1 = j_0 + 1$, and (23) is proved.

To see (b), let $(\tilde{x}_n)$ be the spreading model of a normalized block basis (x_n) of (e_α). By passing to a subsequence of (x_n) we have two cases.

Case 1 There exists $\varepsilon > 0$ so that $\|x_n\|_\infty \ge \varepsilon$ for all n.

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In this case let $|x_i(t_i)| \geq \varepsilon$ for some sequence $(t_i) \subseteq \mathcal{D}$. Passing to a subsequence, using Lemma 4.2, we may assume that there exist $s_1 \prec s_2 \prec \cdots$ with $([s_i, t_i])_{i=1}^{\infty}$ being disjoint segments. It follows that for $k_1 < \cdots < k_m$

$$\Big\| \sum_{i=1}^{m} x_{k_i} \Big\| \geq \sum_{i=1}^{m} V_{s_{k_i}}(i)|x_{k_i}(t_i)| \geq \varepsilon \sum_{i=1}^{m} V_{s_{k_i}}(i).$$

Let β be the branch determined by $(s_i)_{i=1}^{\infty}$. Now $\sum_{i=1}^{\infty} V_{\beta}(i) = \infty$ by our construction and there exists k_0 so that if $k_0 \leq k$ then $V_{s_k}(i) = V_{\beta}(i)$ for $i \leq m$. It follows that

$$\Big\| \sum_{i=1}^{m} \tilde{x}_i \Big\| \geq \varepsilon \sum_{i=1}^{m} V_{\beta}(i)$$

and (b) holds.

Case 2 $\|x_n\|_{\infty} \to 0$.

First note that there is a function $\delta(m)$, with $\delta(m) \to 0$ as $m \to \infty$ such that the following holds: for an arbitrary $x \in c_{00}(\mathcal{D})$ with $\|x\| = 1$, consider disjoint segments $[s_1, t_1], [s_2, t_2], \ldots, [s_k, t_k]$ with $|s_1| \leq |s_2| \leq \cdots \leq |s_k|$, such that

$$\|x\| = \sum_{j=1}^{k} V_{s_j}(j)|x(t_j)|.$$

Then, whenever $\|x\|_{\infty} \leq \delta(m)$ for some m, then there exists $1 \leq k' \leq k$ such that $|s_{k'}| > m$, $k' > m$ and

$$\sum_{j=k'}^{k} V_{s_j}(j)|x(t_j)| \geq 1/2.$$

Using this fact, since $\|x_i\| = 1$ for all i, and $\|x_i\|_{\infty} \to 0$, we can construct inductively a subsequence (x_{n_i}) of (x_i) (with $n_1 = 1$), and for all i, disjoint segments $[s_1^i, t_1^i], [s_2^i, t_2^i], \ldots, [s_{k_i}^i, t_{k_i}^i]$ with $|s_1^i| \leq |s_2^i| \leq \cdots \leq |s_{k_i}^i|$, and integers k_i' such that

$$\begin{aligned} |s_1^1| \leq \cdots \leq |s_{k_1}^1| \leq |t_{k_1}^1| < |s_1^{n_2}| \leq \cdots \leq |s_{k_{i-1}}^{n_{i-1}}| \leq |t_{k_{i-1}}^{n_{i-1}}| \\ < |s_1^{n_i}| \leq |s_2^{n_i}| \leq \cdots \leq |s_{k_i}^{n_i}| \leq |t_{k_i}^{n_i}| \end{aligned}$$

and

$$1 = \|x_{n_i}\| \geq \sum_{j=1}^{k_i} V_{s_j^i}(k_i' + j)|x_{n_i}(t_j^i)| \geq 1/2 \tag{24}$$

and such that the sequence $k_1', k_1' + 1, \ldots, k_1' + k_1, k_2', k_2' + 1, \ldots, k_2' + k_2, \ldots$ is increasing. Let $i_1 < \cdots < i_m$ be an increasing sequence. Applying (24) for each i_l, $1 \leq l \leq m$ and using the fact that the sequences $V_s(i)$, $i \in \mathbb{N}$ are increasing, we get

$$\sum_{l=1}^{m} \sum_{j=1}^{k_{i_l}} V_{s_j^{i_l}}(k_{i_1} + \cdots + k_{i_{l-1}} + j)|x_{n_{i_l}}(t_j^{i_l})| \geq m/2.$$

Since all segments $[s_j^{i_l}, t_j^{i_l}]$ are disjoint, we deduce $\frac{1}{2}m < \|\sum_{l=1}^{m} x_{n_{i_l}}\|$. Hence the spreading model $(\tilde{x}_n)_n$ must be equivalent to the unit vector basis of ℓ_1. This completes the proof of (b). ■

For all $n \in \mathbb{N}$ it is easy to construct a space X for which the cardinality $|SP(X)| = |SP_\omega(X)| = n$. Indeed, $X = (\sum_{i=1}^{n} \ell_{p_i})_2$ suffices, where the p_i's are distinct elements of $(1, \infty)$. Also if $2 < p_1 < p_2 < \cdots$, then it is not hard to show that $|SP_\omega\big(\big(\sum_{i=1}^{\infty} \ell_{p_i})_2\big)| = \omega$. In this case one obtains an infinite decreasing chain of spreading models.

But we do not know what happens hereditarily. Let us mention some questions (among many) concerning the "hereditary structure of spreading models".

Question 4.3 Does there exist a Banach space such that in every infinite dimensional subspace there exist normalized basic sequences having spreading models equivalent to the unit vector bases of ℓ_1 and ℓ_2? If such a space exists, must it contain more (perhaps uncountably many) mutually non-equivalent spreading models? More generally, does there exist X so that for all subspaces Y of X and $1 \leq p < \infty$, the unit vector basis of ℓ_p (and of c_0) is equivalent to a spreading model of Y? Is the space constructed in [22] or [24] such a space?

In order to answer Question 4.3, the answer to the following question may be useful:

Question 4.4 Can we always isomorphically (or isometrically) stabilize the set of spreading models by passing to appropriate subspaces? That is, for every Banach space X does there exist a subspace Y such that for every normalized basic sequence (y_i) in Y having spreading model $(\tilde{y}_i)$ and for every further subspace Z of Y, there exists a normalized basic sequence (z_i) in Z having spreading model $(\tilde{z}_i)$ such that $(\tilde{z}_i)$ is equivalent (respectively, isometric) to $(\tilde{y}_i)$? Is the space X constructed in Section 2 a counterexample?

Question 4.5 Let $n \in \mathbb{N}$. Does there exist a Banach space so that every subspace has exactly n (isomorphically or isometrically) different spreading models? Does there exist a Banach space so that every subspace has countably infinitely many (isomorphically or isometrically) different spreading models?

Many problems are open concerning the structure of the partially ordered set $SP_\omega(X)$ (in the sense of Definition 3.1). We state a few of these.

Question 4.6 What are the realizable isomorphic structures of the partially ordered set $(SP_\omega(X), \leq)$? In particular, for every finite partially ordered set $(P, \leq)$ such that any two elements admit a least upper bound, does there exist X such that $SP_\omega(X)$ is isomorphic to $(P, \leq)$?

We note that by Proposition 3.2 and Remark 3.5, if $SP_\omega(X)$ is infinite, then one can construct sequences $(\tilde{y}_i^n)_{i=1}^\infty$ and $(\tilde{w}_i^n)_{i=1}^\infty$ in $SP_\omega(X)$ so that

$$(\tilde{y}_i^1) < (\tilde{y}_i^2) < \cdots < (\tilde{w}_i^2) < (\tilde{w}_i^1)\,.$$

Question 4.7 Suppose $SP_\omega(X)$ is finite (or even countable). What can be said about X? Must some spreading model be equivalent to the unit vector basis in c_0 or ℓ_p $(1 \leq p < \infty)$? We address the case $|SP_\omega(X)| = 1$ in Section 5.

5 Spaces with a Unique Spreading Model

The following question was posed to us by Argyros.

Question 5.1 Let X be an infinite dimensional Banach space so that $|SP(X)| = 1$. Must the unique spreading model of X be equivalent to the unit vector basis of ℓ_p for some $1 \leq p < \infty$, or c_0?

One could also raise similar questions by restricting either to those spreading models generated by normalized weakly null basic sequences or, in the case that X has a basis, to those generated by normalized block bases.

We give some partial answers to these questions using our techniques above.

Proposition 5.2 *Let X be an infinite dimensional Banach space so that all spreading models of normalized basic sequences in X are equivalent.*

(a) *If all the spreading models are uniformly equivalent,* i.e., *if there exists $D \in \mathbb{R}$ so that the spreading models of all normalized basic sequences in X are D-equivalent, then all spreading models of X are equivalent to the unit vector basis of ℓ_p for some $1 \leq p < \infty$ or c_0.*
(b) *Let (z_i) be a normalized basic sequence which dominates a (hence every) spreading model of X. Then there exists $C < \infty$ so that (z_i) C-dominates any spreading model of a normalized basic sequence (x_i) in X.*
(c) *If p belongs to the Krivine set of the spreading model $(\tilde{x}_i)$ of some normalized basic sequence (x_i) of X, then $(\tilde{x}_i)$ dominates the unit vector basis of ℓ_p.*
(d) *If 1 belongs to the Krivine set of some spreading model in X, then all spreading models are equivalent to the unit vector basis of ℓ_1.*

Proof If X is not reflexive, then there exists a normalized basic sequence (x_n) in X which dominates the summing basis [12]. By [26], (x_n) has a subsequence (x_{n_k}) which is either equivalent to the unit vector basis of ℓ_1 or it is weak-Cauchy. In the later case $(x_{n_{2k+1}} - x_{n_{2k}})_k$ is weakly null and thus by passing to a subsequence we can assume that it has an unconditional spreading model which dominates the summing basis and hence must be equivalent to the unit vector basis of ℓ_1. Therefore in either case there exists a spreading model in X equivalent to the unit vector basis of ℓ_1, and it is easy to see that (a)–(d) hold. Thus for the proof of (a)–(d) we may assume that X is reflexive.

(a) Let $(\tilde{x}_i)$ be a spreading model of X and let p in the Krivine set of $(\tilde{x}_i)$. By Remark 1.2 for every $n \in \mathbb{N}$ there exists a spreading model $(\tilde{x}_i^{(n)})_i$ of X such that $(\tilde{x}_i^{(n)})_{i=1}^n$ is 2-equivalent to the unit vector basis of ℓ_p^n. Also $(\tilde{x}_i)_{i=1}^n$ is D-equivalent to $(\tilde{x}_i^{(n)})_{i=1}^n$, thus $2D$-equivalent to the unit vector basis of ℓ_p^n.

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(b) Let (z_i) be a normalized basic sequence which dominates all spreading models of X. Assume that the statement is false. Then for every $n \in \mathbb{N}$ there exists a normalized weakly null basic sequence $(x_i^{(n)})$ in X, having spreading model $(\tilde{x}_i^{(n)})$, and there exist scalars $(a_i^{(n)})_i$, such that $\|\sum_i a_i^{(n)}\tilde{x}_i^{(n)}\| = 2^{2n}$ and $\|\sum_i a_i^{(n)} z_i\| = 1$. By Proposition 3.2 there exists a seminormalized weakly null sequence (y_i) in X, having spreading model $(\tilde{y}_i)$ such that $(\tilde{y}_i)$ 2^n-dominates $(\tilde{x}_i^{(n)})$ for all n. Thus $\|\sum_i a_i^{(n)}\tilde{y}_i\| \geq 2^{-n}\|\sum_i a_i^{(n)}\tilde{x}_i^{(n)}\| = 2^n$. Hence (z_i) does not dominate $(\tilde{y}_i)$, which is a contradiction.

(c) This follows from (b) and Remark 1.2.

(d) This follows from (c). ■

Remark 5.3 If X has a basis (e_i) and the hypothesis of Proposition 5.2 is changed to "all spreading models of normalized block bases are equivalent" then one obtains a similar theorem, while the conclusions are restricted to spreading models generated by normalized block bases. The "X is not reflexive" part of the proof is replaced by "(e_i) is not shrinking". If the hypothesis is changed to "all spreading models generated by normalized weakly null basic sequences are equivalent" then one has two cases: Either X is a Schur space, hence X is hereditarily ℓ_1 [26], or X does admit such a spreading model. And the proposition holds in the latter case with the obvious modifications.

If X is a Banach space for which all elements of $SP(X)$ are isometrically isomorphic to each other it follows from Proposition 5.2 that they must all be isometrically isomorphic to ℓ_p, for some $1 \leq p < \infty$, or to c_0. In the case that $p = 1$ or in the c_0 case, it was shown in [23] that X must contain a copy of ℓ_1 or c_0 respectively. But the following question is still open.

Question 5.4 Let $1 < p < \infty$ and assume that all elements of $SP(X)$ are isometrically isomorphic to the unit vector basis of ℓ_p. Does X contain a copy of ℓ_p?

A problem closely related to 4.1 has been considered by V. Ferenczi, A. M. Pelczar and C. Rosendal in [6]: Suppose that X has a basis (e_i) for which every normalized block basis has a subsequence equivalent to (e_i). Must (e_i) be equivalent to the unit vector basis of c_0 or some ℓ_p? The authors obtain results analogous to those in Proposition 5.2.

Many additional questions remain about the structure of the spreading models of a Banach space X.

6 Existence of Non-Trivial Operators on Subspaces of Certain Banach Spaces

In this section we give sufficient conditions on a Banach space X for the existence of a subspace Y of X and an operator $T\colon Y \to X$ which is not a compact perturbation of a multiple of the inclusion map. This property is related to the long standing open problem of whether there exists a Banach space (of infinite dimension) on which every operator is a compact perturbation of a multiple of the identity. Notice that if a Banach space X contains an unconditional basic sequence, then there exists a subspace Y of X and an operator $T\colon Y \to Y$ such that $P(T)$ is not a compact perturbation of a multiple of the identity for all non-constant polynomials P. Indeed Y can be taken to be the closed linear span of the unconditional basic sequence, and T a diagonal operator with infinitely many different eigenvalues, each of of infinite multiplicity. Gowers [9] proved that there exists a subspace Y of the Gowers–Maurey space GM (as defined in [10]), and an operator $T\colon Y \to GM$ which is not a compact perturbation of a multiple of the inclusion. In [1] it is shown that there exists an operator on GM which is not a compact perturbation of a multiple of the identity. It is also known that some of the asymptotic ℓ_1 and hereditary indecomposable spaces constructed by Argyros and I. Deliyanni [2] admit subspaces on which a non trivial operator can be constructed (unpublished work of Argyros and R. Wagner, see also [7, 8]). Our approach generalizes the idea of [9].

Theorem 6.1 *Let X be a Banach space. Assume that there exists a normalized weakly null basic sequence (x_i) in X having spreading model $(\tilde{x}_i)$ which is not equivalent to the unit vector basis of ℓ_1, yet 1 belongs to the Krivine set of $(\tilde{x}_i)$. Then there exists a subspace W of X and a continuous linear operator $T\colon W \to W$ such that $p(T)$ is not a compact perturbation of a multiple of the identity operator on W, for every non-constant polynomial p.*

The proof uses a convenient auxiliary notation. Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ be a family of finite subsets of positive integers. For $(a_i) \in c_{00}$ we set

$$\|(a_i)\|_{\ell_1(\mathcal{F})} = \sup\Big\{ \sum_{i\in F} |a_i| : F \in \mathcal{F}\Big\} .$$

Proof of Theorem 6.1 The main part of the proof is the following:

Claim 1 For every $\ell \in \mathbb{N} \cup \{0\}$ there exists $(w_i^{(\ell)})_i$ a seminormalized sequence in X, an increasing sequence $(M_i^{(\ell+1)})_i$ of positive integers and a sequence $(\delta_i^{(\ell+1)})_i$ of positive numbers with $\sum_i \delta_i^{(\ell+1)} < \infty$, such that $w_1^{(0)}, w_1^{(1)}, w_2^{(0)}, w_1^{(2)}, w_2^{(1)}, w_3^{(0)}, \dots$ is a basic sequence in X, and for every $(a_j^{(\ell)})_{\ell\in\mathbb{N}\cup\{0\}, j\in\mathbb{N}} \in c_{00}((\mathbb{N}\cup\{0\})\times\mathbb{N})$ we have

$$\max_{1\le\ell<\infty} \|(a_j^{(\ell)})_j\|_\ell \le \Big\| \sum_{\ell=0}^\infty \sum_{j=1}^\infty a_j^{(\ell)} w_j^{(\ell)} \Big\| \le \sum_{\ell=0}^\infty \|(a_j^{(\ell)})_j\|_{\ell+1}, \tag{25}$$

where for $\ell \in \mathbb{N}$ and $(a_j)_j \in c_{00}$ we define

$$\|(a_j)_j\|_\ell = \sup_{i\in\mathbb{N}} \delta_i^{(\ell)} \|(a_j)_j\|_{\ell_1(\mathcal{G}_i^{(\ell)})} = \sup_{i\in\mathbb{N}} \sup_{E\in\mathcal{G}_i^{(\ell)}} \delta_i^{(\ell)} \sum_{j\in E} |a_j|, \tag{26}$$

and where for $\ell, i \in \mathbb{N}$ we set $\mathcal{G}_i^{(\ell)} = \{F \subset \mathbb{N} : |F| \le M_i^{(\ell)}\}$.

Once Claim 1 is established, let $\widetilde{W} = \operatorname{span}\{w_j^{(\ell)} : \ell \in \mathbb{N}\cup\{0\},\ j \in \mathbb{N}\}$ and define a linear map $T\colon \widetilde{W} \to \widetilde{W}$ by

$$T(w_j^{(0)}) = 0 \text{ and } T(w_j^{(\ell+1)}) = \frac{1}{2^{\ell+1}} w_j^{(\ell)} \text{ for all } \ell \in \mathbb{N}\cup\{0\} \text{ and } j \in \mathbb{N}.$$

Since $(w_i^{(n)})_{n\in\mathbb{N}\cup\{0\},i\in\mathbb{N}}$ is a basic sequence in X, T is well defined. Let

$$(a_j^{(\ell)})_{\ell\in\mathbb{N}\cup\{0\},j\in\mathbb{N}} \in c_{00}((\mathbb{N}\cup\{0\})\times\mathbb{N})$$

and $x = \sum_{\ell=0}^{\infty}\sum_{j=1}^{\infty} a_j^{(\ell)} w_j^{(\ell)} \in \widetilde{W}$. We have

$$\begin{aligned} \|Tx\| &= \Big\|\sum_{\ell=1}^{\infty}\sum_{j=1}^{\infty} a_j^{(\ell)} \frac{1}{2^\ell} w_j^{(\ell-1)}\Big\| \le \sum_{\ell=1}^{\infty}\frac{1}{2^\ell}\|(a_j^{(\ell)})_j\|_\ell && \text{(by (25))}\\ &\le \max_{1\le\ell<\infty}\|(a_j^{\ell})_j\|_\ell \le \|x\| && \text{(by (25)).} \end{aligned}$$

Thus if W denotes the closure of $\widetilde{W}$ then T extends to a bounded operator on W.

Let $p(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ be a non-constant polynomial. We show that $p(T)$ is not a compact perturbation of a multiple of the identity operator I on W. Indeed, for any $i \in \{1, 2, \ldots, n\}$ and $j \in \mathbb{N}$ we have

$$T^i w_j^{(n)} = T^{i-1}\frac{1}{2^n} w_j^{(n-1)} = T^{i-2}\frac{1}{2^n}\frac{1}{2^{n-1}} w_j^{(n-2)} = \cdots = \prod_{k=n-i+1}^{n}\frac{1}{2^k} w_j^{(n-i)}.$$

Thus for every scalar λ and $j \in \mathbb{N}$ we have

$$(p(T)-\lambda I)w_j^{(n)} = \Big(\sum_{i=1}^{n} a_i T^i - \lambda I\Big) w_j^{(n)} = \sum_{i=1}^{n} a_i\Big(\prod_{k=n-i+1}^{n}\frac{1}{2^k}\Big) w_j^{(n-i)} + (a_0-\lambda)w_j^{(n)}.$$

Since $w_1^{(0)}, w_1^{(1)}, w_2^{(0)}, w_1^{(2)}, w_2^{(1)}, w_3^{(0)}, \cdots$ is a seminormalized basic sequence in X, there exist $j_1 < j_2 < \cdots$ in $\mathbb{N}$ such that $\big(\sum_{i=1}^{n} a_i(\prod_{k=n-i+1}^{n}\frac{1}{2^k}) w_{j_s}^{(n-i)} + (a_0 - \lambda)w_{j_s}^{(n)}\big)_s$ is a seminormalized block sequence of $w_1^{(0)}, w_1^{(1)}, w_2^{(0)}, w_1^{(2)}, w_2^{(1)}, w_3^{(0)}, \cdots$ which proves that $p(T) - \lambda I$ is not a compact operator.

Claim 1 follows from:

Claim 2 There exists a subspace Y of X with a basis and for every $\ell \in \mathbb{N}\cup\{0\}$ there exists a seminormalized weakly null basic sequence $(u_i^{(\ell)})_i$ in Y, an increasing sequence $(M_i^{(\ell+1)})_i$ of positive integers, and a sequence $(\delta_i^{(\ell+1)})_i$ of positive numbers with $\sum_i \delta_i^{(\ell+1)} < \infty$, such that the vectors $u_i^{(n)}$ for $n \in \mathbb{N}\cup\{0\}$ and $i \in \mathbb{N}$ are disjointly supported with respect to the basis in Y, and for every $\ell \in \mathbb{N}\cup\{0\}$ and $(a_j^{(m)})_{m\in\{0,1,\ldots,\ell\},j\in\mathbb{N}} \in c_{00}(\{0,1,\ldots,\ell\}\times\mathbb{N})$ we have that

$$2\max_{1\le m\le\ell}\|(a_j^{(m)})_j\|_m \le \Big\|\sum_{m=0}^{\ell}\sum_{j=1}^{\infty} a_j^{(m)} u_j^{(m)}\Big\| \le (1/2)\sum_{m=0}^{\ell}\|(a_j^{(m)})_j\|_{m+1} \tag{27}$$

(recall that $\|\cdot\|_m$ was defined in (26)).

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Once Claim 2 is established, passing for every $n \in \mathbb{N}$ to a subsequence of $(u_i^{(n)})_i$ (which does not affect the estimates in (27)), and making small perturbations if necessary, we get $(w_i^{(n)})_i$ such that

$$w_1^{(0)}, w_1^{(1)}, w_2^{(0)}, w_1^{(2)}, w_2^{(1)}, w_3^{(0)}, \cdots$$

forms a block basis in Y and for all $(a_j^{(\ell)})_{\ell \in \mathbb{N}\cup\{0\}, j\in\mathbb{N}} \in c_{00}((\mathbb{N}\cup\{0\}) \times \mathbb{N})$ we have

$$\frac{1}{2}\Big\| \sum_{\ell=0}^{\infty}\sum_{j=1}^{\infty} a_j^{(\ell)} u_j^{(\ell)} \Big\| \leq \Big\| \sum_{\ell=0}^{\infty}\sum_{j=1}^{\infty} a_j^{(\ell)} w_j^{(\ell)} \Big\| \leq 2\Big\| \sum_{\ell=0}^{\infty}\sum_{j=1}^{\infty} a_j^{(\ell)} u_j^{(\ell)} \Big\|. \tag{28}$$

Obviously (27) and (28) imply (25) and thus Claim 1 follows.

Now we prove Claim 2. We construct the space Y and inductively on $\ell \in \mathbb{N} \cup \{0\}$ we construct the sequences $(u_i^{(\ell)})_i$, $(M_i^{(\ell+1)})_i$ and $(\delta_i^{(\ell+1)})$ which satisfy (27). The upper and lower estimates are based on the following two lemmas, of independent interest, whose proofs we postpone until the end of the section.

Lemma 6.2 *Let X be a Banach space and $(x_i)_i$ be a normalized weakly null basic sequence in X which has a spreading model $(\tilde{x}_i)$ not equivalent to the unit vector basis of ℓ_1. Then for every $(\delta_n)_{n\geq 2} \subset (0,1)$ there exists a subsequence (x_{m_i}) of (x_i) and an increasing sequence $M_1 < M_2 < \cdots$ of integers, such that for all $(a_i)_i \in c_{00}$ we have (put $\delta_1 = 12$)*

$$\Big\| \sum a_i x_{m_i} \Big\| \leq \sup_{n\in\mathbb{N}} \sup_{F\subset\mathbb{N}, |F|\leq M_n} \delta_n \sum_{i\in F} |a_i|. \tag{29}$$

Lemma 6.3 *Let X be a Banach space and $(z_i)_i$ be a normalized weakly null basic sequence in X which has spreading model $(\tilde{z}_i)_i$ such that 1 belongs to the Krivine set of $(\tilde{z}_i)_i$. There exists a subsequence $(z_i')_i$ of $(z_i)_i$ with the following property.*

Given any infinite subset $J \subseteq \mathbb{N}$, any subsequence $(M_n)_n$ of $\mathbb{N}$, and $(\delta_n)_n \subset (0,\infty)$ with $\sum_{n=1}^{\infty} \delta_n < \infty$, there exists a seminormalized weakly null basic sequence $(y_i)_i$ in the span of $(z_j')_{j\in J}$ which is disjointly supported with respect to $(z_j')_{j\in J}$, such that for all $(a_i) \in c_{00}$ and all y in the span of $(z_j')_{j\notin J}$ we have

$$\sup_{n\in\mathbb{N}} \delta_n \|(a_i)\|_{\ell_1(\mathcal{G}_n)} \leq \|y + \sum a_i y_i\|. \tag{30}$$

with $\mathcal{G}_n := \{A \subset \mathbb{N} : |G| \leq M_n\}$ for $n \in \mathbb{N}$.

Furthermore, if $(\tilde{z}_i)$ is not equivalent to the unit vector basis of ℓ_1 then no spreading model of (y_i) is equivalent to the unit vector basis of ℓ_1.

We now return to the proof of Claim 2.

Since 1 belongs to the Krivine set of $(\tilde{x}_i)$, we can use Lemma 6.3 and assume without loss of generality that (x_i) satifies the conclusion of Lemma 6.3, as stated for (z_i').

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Let $K_0, K_1, K_2, \cdots$ be disjoint infinite sets of positive integers. For all $\ell \in \mathbb{N} \cup \{0\}$ we will construct disjointly supported $u_i^{(\ell)} \in \operatorname{span}\{x_j : j \in K_\ell\}$, $(\delta_i^{(\ell)})_{i\in\mathbb{N}}$ and $(M_i^{(\ell)})_{i\in\mathbb{N}}$ (satisfying the conditions as stated in Claim 2) so that for all

$$(a_j^{(m)})_{m\in\{0,1,\dots,\ell\}, j\in\mathbb{N}} \in c_{00}(\{0,1,\dots,\ell\}\times\mathbb{N})$$

and $y \in \operatorname{span}(x_i : i \in \bigcup_{s>\ell} K_s)$ we have that

$$\Big\| \sum_{m=0}^{\ell} \sum_{j=1}^{\infty} a_j^{(m)} u_j^{(m)} \Big\| \le (1/2) \sum_{m=0}^{\ell} \|(a_j^{(m)})_j\|_{m+1} \tag{31}$$

$$2 \max_{1\le m\le \ell} \|(a_j^{(m)})_j\|_m \le \Big\| y + \sum_{m=0}^{\ell} \sum_{j=1}^{\infty} a_j^{(m)} u_j^m \Big\| \tag{32}$$

(which yields (27) if we put $y = 0$).

Construction of $(u_i^{(0)})_i$, $(\delta_i^{(1)})_{i\in\mathbb{N}}$ *and* $(M_i^{(1)})_{i\in\mathbb{N}}$*:* Let $(\delta_i^{(1)})_{i\ge 2} \subset (0,1)$ such that

$$\sum_{i\ge 2} \delta_i^{(1)} < \infty.$$

Since $(\tilde{x}_i)_i$ is a spreading model of $(x_j)_{j\in K_0}$ which is not equivalent to the unit vector basis of ℓ_1 we may apply Lemma 6.2 to obtain a subsequence (x_{m_i}) of $(x_j)_{j\in K_0}$, an increasing sequence $(M_i^{(1)})_{i\in\mathbb{N}}$ of positive integers, and $\delta_1^{(1)} > 0$ such that for all $(a_i) \in c_{00}$ we have

$$\Big\| \sum a_i x_{m_i} \Big\| \le \frac{1}{2} \|(a_i)\|_1 := \frac{1}{2} \sup_{n\in\mathbb{N}} \delta_n^{(1)} \|(a_i)\|_{\ell_1(\mathcal{G}_n^{(1)})}, \tag{33}$$

where $\mathcal{G}_n^{(1)} = \{G \subset \mathbb{N} : |G| \le M_n^{(1)}\}$. This yields (31) for $\ell = 0$ while (32) is vacuous.

The inductive step — construction of $(u_i^{(\ell)})_i$, $(\delta_i^{(\ell+1)})_i$ *and* $(M_i^{(\ell+1)})$*:* Assume that we have constructed $(u_i^m)_i$, $(M_i^{(m+1)})_i$ and $(\delta_i^{(m+1)})$ for $m = 0, 1, \dots, \ell-1$ so that (31) and (32) are satisfied when ℓ is replaced by $\ell - 1$. Apply Lemma 6.3 for $J = K_\ell$, $(M_i)_i = (M_i^{(\ell)})_i$ and $(\delta_i)_i = (2\delta_i^{(\ell)})_i$ to obtain a disjointly supported seminormalized weakly null basic sequence $(u_i^{(\ell)})_i$ in $\operatorname{span}\{x_j : j \in K_\ell\}$ satisfying for all $(a_i) \in c_{00}$, and $y \in \operatorname{span}\{x_j : j \notin K_\ell\}$ that

$$2 \sup_{n\in\mathbb{N}} \delta_n^{(\ell)} \|(a_i)\|_{\ell_1(\mathcal{G}_n^{(\ell)})} \le \|y + \sum a_i u_i^{(\ell)}\|, \tag{34}$$

where $\mathcal{G}_n^{(\ell)} = \{G \subset \mathbb{N} : |G| \le M_n^{(\ell)}\}$ for $n \in \mathbb{N}$. By passing to a subsequence of $(u_i^{(\ell)})_i$ and relabelling we can assume that $(u^{(\ell_i)})_i$ has a spreading model $(\tilde{u}_i^{(\ell)})$. By the "furthermore" part of Lemma 6.3 we have that $(\tilde{u}_i^{(\ell)})$ is not equivalent to the unit vector basis of ℓ_1. Let $(\delta_i^{(\ell+1)})_{i\ge 2} \subset (0,1)$ such that $\sum_{i\ge 2} \delta_i^{(\ell+1)} < \infty$. Apply Lemma 6.2

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to obtain a subsequence of $(u_i^{(\ell)})_i$ (which we still call $(u_i^{(\ell)})_i$) , an increasing sequence $(M_i^{(\ell+1)})_{i\in\mathbb{N}}$ of positive integers, and $\delta_1^{(\ell+1)} > 0$ such that for all $(a_i) \in c_{00}$ we have

$$\Big\|\sum a_i u_i^{(\ell)}\Big\| \le \frac{1}{2}\|(a_i)\|_{\ell+1} := \frac{1}{2}\sup_{n\in\mathbb{N}} \delta_n^{(\ell+1)}\|(a_i)\|_{\ell_1(\mathcal{G}_n^{\ell+1})}, \tag{35}$$

where $\mathcal{G}_n^{(\ell+1)} = \{G \subset \mathbb{N} : |G| \le M_n^{(\ell+1)}\}$ for $n \in \mathbb{N}$.

We now show that (31) and (32) are satisfied. Let

$$(a_j^{(m)})_{m\in\{0,1,\dots,\ell\}\, j\in\mathbb{N}} \in c_{00}(\{0,1,\dots,\ell\}\times\mathbb{N})$$

and $y \in \operatorname{span}(y_i : i \in \bigcup_{s>\ell} K_s)$.

From (35) and the induction hypothesis it follows that

$$\begin{aligned}\Big\|\sum_{m=0}^{\ell-1}\sum_{j=1}^{\infty} a_j^{(m)}u_j^{(m)} + \sum_{j=1}^{\infty} a_j^{(\ell)}u_j^{(\ell)}\Big\| &\le \Big\|\sum_{m=0}^{\ell-1}\sum_{j=1}^{\infty} a_j^{(m)}u_j^{(m)}\Big\| + \Big\|\sum_{j=1}^{\infty} a_j^{(\ell)}u_j^{(\ell)}\Big\| \\ &\le \frac{1}{2}\sum_{m=0}^{\ell}\|(a_j^{(m)})_j\|_{m+1},\end{aligned}$$

which yields (31). By the inductive hypothesis for y replaced by $y + \sum_{j=1}^{\infty} a_j^{(\ell)}u_j^{(\ell)}$ we can estimate $\|y + \sum_{m=0}^{\ell}\sum_{j=1}^{\infty} a_j^{(m)}u_j^{(m)}\|$ as follows:

$$2\max_{1\le m\le \ell-1}\|(a_j^{(m)})_j\|_m \le \|y + \sum_{j=1}^{\infty} a_j^{(\ell)}u_j^{(\ell)} + \sum_{m=0}^{\ell-1}\sum_{j=1}^{\infty} a_j^{(m)}u_j^{(m)}\|,$$

which, together with (34), implies (32). ∎

If we are interested only in the construction of an operator on a subspace which is not a compact perturbation of a multiple of the inclusion map, then the spreading model assumptions of Theorem 6.1 can be significantly relaxed and the argument would be essentially simpler.

Theorem 6.4 *Let X be a Banach space. Assume that there exist normalized weakly null basic sequences (x_i), (z_i) in X such that (x_i) has spreading model $(\tilde{x}_i)$ which is not equivalent to the unit vector basis of ℓ_1, and (z_i) has spreading model $(\tilde{z}_i)$ such that 1 belongs to the Krivine set of $(\tilde{z}_i)$. Then there exists a subspace Y of X and an operator $T\colon Y \to X$ which is not a compact perturbation of a multiple of the inclusion map.*

Sketch of proof Let $(\delta_n)_{n\ge 2} \subset (0,1)$ such that $\sum_{n=2}^{\infty}\delta_n < \infty$. Using Lemma 6.2 we obtain a subsequence $(x_{m_i})_i$ of (x_i), an increasing sequence $M_1 < M_2 < \cdots$ of integers and $\delta_1 > 0$ such that for all $(a_i) \in c_{00}$ we have

$$\Big\|\sum a_i x_{m_i}\Big\| \le \sup_{n\in\mathbb{N}}\delta_n\|(a_i)_i\|_{\ell_1(\mathcal{G}_n)},$$

where $\mathcal{G}_n = \{G \subset \mathbb{N} : |G| \le M_n\}$ for $n \in \mathbb{N}$. Then by Lemma 6.3 we obtain a seminormalized weakly null basic sequence (y_i) in the span of (z_i) such that for all $(a_i) \in c_{00}$ and $k_1 < k_2 < \cdots$ in $\mathbb{N}$,

$$\Big\| \sum a_i y_{k_i} \Big\| \ge \sup_{n \in \mathbb{N}} \delta_n \|(a_i)\|_{\ell_1(\mathcal{G}_n)}.$$

Thus for every $(a_i) \in c_{00}$ we have $\| \sum a_i x_{m_i} \| \le \| \sum a_i y_i \|$, and passing to subsequences if necessary we may also assume that $x_{m_1}, y_1, x_{m_2}, y_2, \cdots$ is a (seminormalized weakly null) basic sequence. Thus the operator T defined on $\operatorname{span}\{y_i : i \in \mathbb{N}\}$, the closed linear span of (y_i), by $T(y_i) = x_{m_i}$ for all i, is a continuous operator. Also for any scalar λ the operator $T - \lambda I$ (where I denotes the inclusion operator from $\operatorname{span}\{y_i : i \in \mathbb{N}\}$ to X) is non-compact, since $(T - \lambda I)(y_i) = x_{m_i} - \lambda y_i$ which is a seminormalized weakly null sequence. ■

We now give the proofs of Lemmas 6.2 and 6.3.

Proof of Lemma 6.2 (x_n) is weakly null and thus has a subsequence which can be renormed with a 3-equivalent norm to make it bimonotone basic. Therefore if we proved the claim for $\delta'_1 = 4$ and $\delta'_n = \delta_n/3$, for $n \ge 2$, assuming that (x_n) is bimonotone basic, the general claim would follow for $\delta_1 = 12$ and $(\delta_n)_{n \ge 2}$.

Secondly, we can assume that for every $\rho > 0$ there is an $M = M(\rho)$, so that for all $x = \sum_{i=1}^\infty a_i x_i$ of norm 1,

$$|\{i \in \mathbb{N} : |a_i| \ge \rho\}| \le M\,. \tag{36}$$

Otherwise we prove the claim for the sequence (x'_n) (which dominates (x_n)) defined by

$$\Big\| \sum_{n=1}^\infty a_n x'_n \Big\| = \max\Big(\Big[\sum_{n=1}^\infty a_n^2 \Big]^{1/2}, \Big\| \sum_{n=1}^\infty a_n x_n \Big\| \Big) \text{ for } (a_i) \in c_{00}.$$

Thus assume that (x_n) is bimonotone basic and satisfies (36), and let $\delta_1 = 4$.

We choose a sequence $(\varepsilon_j)_{j=1}^\infty \subset (0,1]$ so that

$$\sum_{j=2}^\infty \frac{\varepsilon_{j-1}}{\delta_j} \le \frac{1}{8}. \tag{37}$$

By Proposition 2.1 we may choose a decreasing sequence $(\rho_j) \subset (0,1]$, with $\sum_j \sqrt{\rho_j}(j+1) \le 1/4$ such that

$$\begin{aligned} &\Big\| \sum a_i \tilde{x}_i \Big\| \\ &\le \varepsilon_j \sum |a_j|, \text{ for } (a_i) \in [-\sqrt{\rho_j}, \sqrt{\rho_j}]^{\mathbb{N}} \cap c_{00} \text{ with } \sum a_i \tilde{x}_i \in S_{\operatorname{span}\{\tilde{x}_i : i \in \mathbb{N}\}}. \end{aligned} \tag{38}$$

Finally let $M_j = M(\rho_j)$ satisfy (36).

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Using the definition of spreading models, we also can assume that for all $F \subset \mathbb{N}$, with $j \le F$ and $|F| \le M_j$ and all $(a_i) \in c_{00}$ it follows that

$$\frac{1}{2}\Big\|\sum_{i\in F} a_i x_i\Big\| \le \Big\|\sum_{i\in F} a_i \tilde{x}_i\Big\| \le 2\Big\|\sum_{i\in F} a_i x_i\Big\|. \tag{39}$$

Let $(a_i) \in c_{00}$, with $\|\sum a_i x_i\| = 1$, and let $j \in \mathbb{N}$, $j \ge 2$, and consider the vector $\tilde{y} = \sum_{i>j,\rho_j<|a_i|\le\rho_{j-1}} a_i \tilde{x}_i$. If $\|\tilde{y}\| \ge \sqrt{\rho_{j-1}}$ we can apply (38) to the normalized vector $\tilde{y}/\|\tilde{y}\|$ to obtain

$$\|\tilde{y}\| = \|\tilde{y}\| \cdot \Big\|\frac{\tilde{y}}{\|\tilde{y}\|}\Big\| \le \|\tilde{y}\|\varepsilon_{j-1} \sum_{\substack{i>j\\ \rho_j<|a_i|\le\rho_{j-1}}} \frac{|a_i|}{\|\tilde{y}\|} = \varepsilon_{j-1} \sum_{\substack{i>j\\ \rho_j<|a_i|\le\rho_{j-1}}} |a_i|.$$

Thus, we get, in general (*i.e.*, without the condition $\|\tilde{y}\| \ge \sqrt{\rho_{j-1}}$)

$$\|\tilde{y}\| \le \sqrt{\rho_{j-1}} + \varepsilon_{j-1} \sum_{\substack{i>j\\ \rho_j<|a_i|\le\rho_{j-1}}} |a_i|. \tag{40}$$

Therefore we deduce that (letting $\rho_0 = 1$)

$$\begin{aligned}
1 &= \Big\|\sum a_i x_i\Big\| \\
&\le \sum_{j=1}^{\infty}\Big\| \sum_{\substack{i\\ \rho_j<|a_i|\le\rho_{j-1}}} a_i x_i\Big\| \\
&\le \Big\|\sum_{\substack{i\\ \rho_1<|a_i|\le 1}} a_i x_i\Big\| + \sum_{j=2}^{\infty}\Big\|\sum_{\substack{i\le j\\ \rho_j<|a_i|\le\rho_{j-1}}} a_i x_i\Big\| + \sum_{j=2}^{\infty}\Big\|\sum_{\substack{i>j\\ \rho_j<|a_i|\le\rho_{j-1}}} a_i x_i\Big\| \\
&\le \sup_{\substack{F\subset\mathbb{N}\\ |F|\le M_1}} \sum_{i\in F}|a_i| + \sum_{j=2}^{\infty}\rho_{j-1} j + 2\sum_{j=2}^{\infty}\Big\|\sum_{\substack{i>j\\ \rho_j<|a_i|\le\rho_{j-1}}} a_i\tilde{x}_i\Big\| && \text{(by (39))} \\
&\le \sup_{\substack{F\subset\mathbb{N}\\ |F|\le M_1}} \sum_{i\in F}|a_i| + \frac{1}{4} + 2\sum_{j=2}^{\infty}\sqrt{\rho_{j-1}} + 2\sum_{j=2}^{\infty}\varepsilon_{j-1}\sum_{\substack{i>j\\ \rho_j<|a_i|\le\rho_{j-1}}}|a_i| && \text{(by (40))} \\
&\le \frac{1}{2} + \sup_{\substack{F\subset\mathbb{N}\\ |F|\le M_1}} \sum_{i\in F}|a_i| + \sum_{j=2}^{\infty} 2\frac{\varepsilon_{j-1}}{\delta_j}\delta_j \sum_{\substack{i>j\\ \rho_j<|a_i|\le\rho_{j-1}}}|a_i| \\
&\le \frac{1}{2} + \frac{1}{2}\sup_{j\in\mathbb{N}}\sup_{\substack{F\subset\mathbb{N}\\ |F|\le M_j}} \delta_j\sum_{i\in F}|a_i| && \text{(by (37))},
\end{aligned}$$

which implies the claim. ∎

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Proof of Lemma 6.3 Since 1 belongs to the Krivine set of $(\tilde{z}_i)$, we can use Remark 1.2 and pick for every $n \in \mathbb{N}$ a normalized block sequence consisting of identically distributed vectors $(\tilde{w}_j^{(n)})_j \subset [\tilde{z}_i : i \in \mathbb{N}]$, for $j = 1, 2, \ldots$, such that for any subset $E \subseteq \mathbb{N}$ with $|E| = n$, $(\tilde{w}_j^{(n)})_{j \in E}$ is 2 equivalent to the unit vector basis of ℓ_1^n. We denote the common length of their support by K_n.

Using the Schreier unconditionality theorem ([18], also [3, 21]) we may pass to a subsequence (z_i') of (z_i) such that for any finite subset $F \subseteq \mathbb{N}$ such that $|F| \leq nK_n$ and $n < \min F$, for some $n \in \mathbb{N}$, we have

$$\Big\| \sum_{i \in F} a_i z_i' \Big\| \leq 3 \Big\| \sum_{i \in \mathbb{N}} a_i z_i' \Big\|, \quad \text{for any scalars } (a_i). \tag{41}$$

Fix J, (M_n) and (δ_n) as in the assumptions. For any $n \in \mathbb{N}$, let $(w_j^{(n)})$ be equidistributed vectors in the span of $(z_i')_{i \in J}$ with the same distribution as the elements of $(\tilde{w}_j^{(n)})$, and supported after z_n'. We may also assume that for any subset $E \subseteq \mathbb{N}$ with $|E| = n$, the sequence $(w_j^{(n)})_{j \in E}$ is 3 equivalent to the unit vector basis of ℓ_1^n. We may additionally chose the $w_j^{(n)}$'s so that $w_1^{(1)}, w_1^{(2)}, w_2^{(1)}, w_1^{(3)}, \ldots$ form a block basis with respect to $(z_i')_{i \in J}$ and $\|w_j^{(n)}\|$ is uniformly close to 1.

For j=1, 2, ..., set

$$y_j = \sum_n \delta_n w_j^{(M_n)}.$$

From (41) we have $\frac{1}{4} \max_n \delta_n \leq \|y_j\| \leq 2 \sum_n \delta_n$, for all j. Also (y_j) is clearly weakly null from its construction since $\sum \delta_n < \infty$ and each $(w_j^{(M_n)})_j$ is weakly null.

To prove (30), pick $(a_i) \in c_{00}$ and a vector y supported outside of J. Fix $n \in \mathbb{N}$ and $G \subseteq \mathbb{N}$ with $|G| \leq M_n$. Noting that the supports of ${w_j}^{(M_n)}$'s with respect to (z_i') have cardinality K_{M_n}, by Schreier unconditionality (41) we can isolate the $w_j^{(M_n)}$'s from the expression for y_j to get (note that the support of the vector $\sum_{j \in G} a_j \delta_n {w_j}^{(M_n)}$ with respect to (z_i') has not more than $M_n K_{M_n}$ elements and starts after the M_n-th element)

$$\Big\| y + \sum_j a_j y_j \Big\| \geq (1/3) \Big\| \sum_{j \in G} a_j \delta_n w_j^{(M_n)} \Big\| \geq (1/9) \delta_n \sum_{j \in G} |a_j|.$$

Taking into account the definition of $\mathcal{G}_n$ and of the norm $\|\cdot\|_{\ell_1(\mathcal{G}_n)}$ this completes the proof if we replace the original δ_n's by $9\delta_n$.

To see the "furthermore" statement, note that if $(\tilde{z}_i)$ is not equivalent to the unit vector basis of ℓ_1 then the same is true for the spreading model $(\tilde{w}_i^{(n)})$ of $(w_i^{(n)})$ for $n \in \mathbb{N}$. Using this, Proposition 2.1 and the definition of (y_j), it is easy to verify that (b) holds in Proposition 2.1 for any spreading model of (y_i). ∎

It is proved in [1] that the spreading model of the unit vector basis of the Gowers–Maurey space GM as defined in [10], is isometric to the unit vector basis of Schlumprecht's space S as defined in [28]. Thus Theorem 6.1 immediately gives the following:

Corollary 6.5 *There is a subspace Y of GM and an operator T on Y such that $p(T)$ is not a compact perturbation of a multiple of the identity, for any non-constant polynomial p.*

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