THE THICKNESS OF THE COMPLETE GRAPH

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1. The problem. A *graph* G consists of a finite set of p *points* and q *lines* joining pairs of these points. Each line joins two distinct points and no pair of points is joined by more than one line. A *subgraph* of G is a graph whose points and lines are also in G. If every pair of points of a graph is joined by a line, the graph is called *complete* and is denoted by K_p. A *planar* graph can be embedded in the plane, that is, drawn in the plane in such a way that none of its lines intersect.

It has long been known **(5)** that the complete graph K_5 is non-planar. Only recently has it been proved **(1; 6)** that the complete graph K_9 cannot be expressed as the union of two planar subgraphs. The *thickness* $t(G)$ of a graph G is defined as the minimum number of planar subgraphs whose union is G; this term was proposed by Tutte **(7)**. From the above remarks and some constructions, it follows that $t(K_5) = 2$ and $t(K_9) = 3$. Moreover, any other graphs with those thicknesses have more points and lines. The object of this paper is to determine the thickness of five-sixths of all complete graphs, that is, the evaluation of $t(K_p)$ for five of every six consecutive integers p. We first establish a lower bound for the thickness of any complete graph.

THEOREM 1. $t(K_p) \geqslant [\frac{1}{6}(p+7)]$.

Proof. This result makes direct use of Euler's polyhedron formula, which states that in a connected graph embedded in the plane, the number of faces is two more than the number of edges minus the number of vertices. With that formula, it is easy to show **(4)** that in a planar graph with p points and q lines, $q \leqslant 3(p-2)$. Therefore, for any graph G, $q \leqslant 3(p-2)t(G)$, an inequality generalized in **(3)**. In particular, $\frac{1}{2}p(p-1) \leqslant 3(p-2)t(K_p)$. Since the thickness of a graph is always an integer,

$$t(K_p) \geqslant \left[\frac{\frac{1}{2}p(p-1) + 3(p-2) - 1}{3(p-2)}\right],$$

from which the theorem follows directly.

2. The device. To prove the main theorem, an outline of which was given in **(2)**, we require several auxiliary results that will assist in constructing some planar graphs. With the help of these graphs, we shall then construct a minimum collection of planar graphs whose union is K_p, where $p = 6n+2$.

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The main device to be used is the matrix defined below. Let $A = (a_{ij})$ be the $n \times n$ symmetric matrix whose entries are residue classes (mod n) defined by

$$a_{ij} \equiv (-1)^i[\tfrac{1}{2}i] + (-1)^j[\tfrac{1}{2}j] \pmod{n}.$$

For the values $n = 6$ and $n = 7$, these matrices are as follows:

$$\begin{bmatrix} 0 & 1 & 5 & 2 & 4 & 3 \\ 1 & 2 & 0 & 3 & 5 & 4 \\ 5 & 0 & 4 & 1 & 3 & 2 \\ 2 & 3 & 1 & 4 & 0 & 5 \\ 4 & 5 & 3 & 0 & 2 & 1 \\ 3 & 4 & 2 & 5 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 6 & 2 & 5 & 3 & 4 \\ 1 & 2 & 0 & 3 & 6 & 4 & 5 \\ 6 & 0 & 5 & 1 & 4 & 2 & 3 \\ 2 & 3 & 1 & 4 & 0 & 5 & 6 \\ 5 & 6 & 4 & 0 & 3 & 1 & 2 \\ 3 & 4 & 2 & 5 & 1 & 6 & 0 \\ 4 & 5 & 3 & 6 & 2 & 0 & 1 \end{bmatrix}$$

THEOREM 2. *Each integer* $0, 1, \ldots, n-1$ *appears once in every row and every column of* A.

Proof. Since the matrix A is symmetric, it is sufficient to show that all entries of an arbitrary column are different. Suppose $a_{ij} = a_{kj}$ with $i \neq k$. Then

$$(-1)^i[\tfrac{1}{2}i] \equiv (-1)^k[\tfrac{1}{2}k] \pmod{n}.$$

Clearly i and k cannot have the same parity. But if, say, i is even and k is odd, then $\frac{1}{2}i = n - \frac{1}{2}(k-1)$. Since $i \leqslant n$ and $k \leqslant n$, this is impossible. Hence, no two distinct entries of any column are equal.

THEOREM 3. *Any two distinct integers* r *and* s *in* A *are consecutive entries in two columns. Moreover, in one of these columns they are both on or above the main diagonal, and in the other they are on or below it.*

Proof. Clearly the difference $a_{i,j} - a_{i+1,j}$ (mod n) is independent of j, and we call this residue class d_i. Then

$$d_i \equiv (-1)^i[\tfrac{1}{2}i] - (-1)^{i+1}[\tfrac{1}{2}(i+1)] = (-1)^i\{[\tfrac{1}{2}i] + [\tfrac{1}{2}(i+1)]\} \pmod{n},$$

so

$$d_i = \begin{cases} i & \text{if } i \text{ is even,} \\ n - i & \text{if } i \text{ is odd.} \end{cases}$$

Case 1. Assume n is even. Then each integer $1, 2, \ldots, n-1$ equals some d_i. Choose i so that $d_i \equiv r - s \pmod{n}$. The column j with $a_{i,j} = r$ is then one column in which r and s are consecutive, since $a_{i+1,j} = s$. Similarly, choose h so that $d_h \equiv s - r \pmod{n}$. Then the column k with $a_{h,k} = s$ is another in which r and s are consecutive, since $a_{h+1,k} = r$. Because r and s occur in different order in these two columns, they are indeed different columns.

By routine calculations, we find that, for any integer i,

$$(-1)^i[\tfrac{1}{2}i] \equiv (-1)^{n+1-i}[\tfrac{1}{2}(n+1-i)] + \tfrac{1}{2}n \pmod{n}$$

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if and only if

$$(-1)^i[\tfrac{1}{2}i] \equiv (-1)^{1-i}\{\tfrac{1}{2}n + [\tfrac{1}{2}(1-i)]\} + \tfrac{1}{2}n \pmod{n},$$

which holds if and only if

$$(-1)^i[\tfrac{1}{2}i] + (-1)^i[\tfrac{1}{2}(1-i)] \equiv 0 \pmod{n},$$

which is easily seen to be true. Therefore, by applying the first congruence to both i and j, we have

$$(-1)^i[\tfrac{1}{2}i] + (-1)^j[\tfrac{1}{2}j] \equiv (-1)^{n+1-i}[\tfrac{1}{2}(n+1-i)] + (-1)^{n+1-j}[\tfrac{1}{2}(n+1-j)] \pmod{n},$$

that is, $a_{i,j} = a_{n+1-i,n+1-j}$. Thus, if $r = a_{i,j}$ and $s = a_{i+1,j}$, then $r = a_{n+1-i,n+1-j}$ and $s = a_{n-i,n+1-j}$. Now $i \leqslant j$ if and only if $n - i > n + 1 - j$ so that r and s are consecutive above the diagonal in either column j or column $n + 1 - j$ and below it in the other.

Case 2. Assume n is odd. In this case, each d_i is even, and each even integer $2, 4, \ldots, n-1$ is the value of two different d_i. Moreover, $d_i = d_{n-i}$ for all i. Either $r - s \pmod{n}$ or $s - r \pmod{n}$ is even; without loss of generality, assume the former. Then if $d_i \equiv r - s \pmod{n}$, r and s are consecutive in those columns j and k in which r appears in rows i and $n - i$.

Since either i or $n - i$ is even, we can assume i is even. Thus, $r = a_{i,j} = a_{n-i,k}$ and $s = a_{i+1,j} = a_{n-i+1,k}$. To show that r and s are both on or above the diagonal in either column j or k, it is sufficient to show that $i \geqslant j$ if and only if $k \geqslant n - i$. Now,

$$(-1)^{n-i}[\tfrac{1}{2}(n-i)] + (-1)^k[\tfrac{1}{2}k] \equiv (-1)^i[\tfrac{1}{2}i] + (-1)^j[\tfrac{1}{2}j] \pmod{n};$$

so

$$(-1)^k[\tfrac{1}{2}k] \equiv [\tfrac{1}{2}(n-i)] + [\tfrac{1}{2}i] + (-1)^j[\tfrac{1}{2}j] \pmod{n},$$

since i is even. Therefore,

$$(-1)^k[\tfrac{1}{2}k] \equiv [\tfrac{1}{2}(n-1)] + (-1)^j[\tfrac{1}{2}j] \pmod{n}.$$

If $j = n$, then $(-1)^k[\tfrac{1}{2}k] \equiv 0 \pmod{n}$; so $k = 1$. If $j \neq n$ and j is odd, then

$$(-1)^k[\tfrac{1}{2}k] \equiv (\tfrac{1}{2}(n-1) - \tfrac{1}{2}(j-1)) \pmod{n}.$$

From this it follows that k must be even and thus $k = n - j$. If j is even, then k is odd and hence $k = n - j + 2$, since

$$n - \tfrac{1}{2}(k-1) = \tfrac{1}{2}(n - 1 + j).$$

Summarizing,

$$k = \begin{cases} 1 & \text{if } j = n, \\ n - j & \text{if } j \neq n \text{ and } j \text{ is odd}, \\ n - j + 2 & \text{if } j \text{ is even}. \end{cases}$$

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We now consider four cases, recalling that i is even by assumption.

(1) $i \geqslant j$ and j is even. Then $k = n - j + 2$; so $k > n - i$.

(2) $i \geqslant j$ and j is odd. Then $j < i < n$; so $k = n - j$, which implies that $k > n - i$.

(3) $i < j$ and j is odd. Then $k = 1$ or $k = n - j$, and in either case, $k \leqslant n - i$.

(4) $i < j$ and j is even. Then $k = n - j + 2$ and $j \geqslant i + 2$, so that $k \leqslant n - i$.

From these four results, it follows that $i \geqslant j$ if and only if $k > n - i$, which was to be shown.

Let r and s be integers in A. Let j be the column in which r is the first entry and let i be the row in which s is the jth entry, that is, $a_{1,j} = r$ and $a_{i,j} = s$. Define the number

$$m(r, s) = \min\{i, j\} \pmod 2.$$

Since every integer in A appears exactly once in each row and column, the number $m(r, s)$ is defined for every r and s. Note that if h and k are consecutive and on or below the diagonal in the column headed by r, then $m(r, h) = m(r, k)$, whereas if they are consecutive above the diagonal, $m(r, h) \neq m(r, k)$.

THEOREM 4. *Let r and s be integers in A with $r < s$. Then $m(r, s) = 0$ and $m(s, r) = 1$.*

Proof. For this proof, we order the equivalence classes mod n by

$$1 < 2 < \ldots < n - 1 < 0.$$

Note that 0 (congruent to n of course) is taken as the largest element.

We first show that $m(r, s) = 0$. Let $r = a_{1,j}$ and $s = a_{i,j}$. We consider two cases: $r > [\frac{1}{2}n]$ and $r \leqslant [\frac{1}{2}n]$. If $r > [\frac{1}{2}n]$, then j must be odd, because by definition $r \equiv (-1)^j[\frac{1}{2}j] \pmod n$. Since $s > r > [\frac{1}{2}n]$ and $s \equiv r + (-1)^i[\frac{1}{2}i] \pmod n$, it follows that i must be even.

Since $s > [\frac{1}{2}n]$ and $s \equiv \frac{1}{2}i - \frac{1}{2}(j - 1) \pmod n$, it follows that $i < j$. Therefore, in this case, $m(r, s) \equiv \min\{i, j\} \equiv i \equiv 0 \pmod 2$.

Now assume $r \leqslant [\frac{1}{2}n]$. Then j must be even and $r = \frac{1}{2}j$. If i is even or $i > j$, then $m(r, s) = 0$. Suppose i is odd and $i < j$. Then $s = \frac{1}{2}j - \frac{1}{2}(i - 1) < r$, but by hypothesis this is impossible. Therefore $m(r, s) = 0$ for all $r < s$.

The proof that $m(s, r) = 1$ is very similar. This time let $s = a_{1,k}$ and $r = a_{h,k}$. We again consider two cases. First, let $s \leqslant [\frac{1}{2}n]$, in which case k is even and $s = \frac{1}{2}k$. Since $r \equiv \frac{1}{2}k + (-1)^h[\frac{1}{2}h] \pmod n$, it follows that h is odd and $h < k$. Hence, $m(s, r) \equiv h \pmod 2 \equiv 1$. Now let $s > [\frac{1}{2}n]$, so k is odd and $s = n - \frac{1}{2}(k - 1)$. If h is odd or $h > k$, then $m(s, r) = 1$. Suppose $h < k$ and h is even. Then $r = n - \frac{1}{2}(k - 1) + \frac{1}{2}h > s$, which is a contradiction. This shows that $m(s, r) = 1$ in all cases, and thus completes the proof of the theorem.

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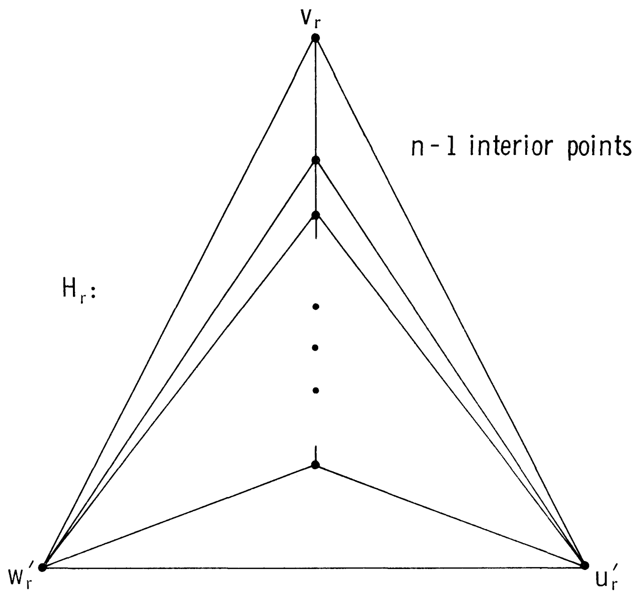

FIGURE 1

We are now in a position to give the construction of planar graphs using matrix A. For $r = 0, 1, \ldots, n-1$, let H_r denote the graph indicated in Figure 1. The $n-1$ points in its interior will be labelled using that column of the matrix A whose first entry is r as follows. Using the hth entry s of that column, label the $(h-1)$-st point down from v_r as v_s or v'_s according as $m(r, s) = 1$ or 0.

Now construct graphs G_r, for $r = 0, 1, \ldots, n-1$, as indicated in Figure 2, where each of the numbered triangles contains a graph isomorphic to H_r. In these triangles, the point v_r corresponds to v_r, u'_r, w_r, v'_r, u_r, w'_r in that order, and w'_r and u'_r to the other two points. The label of each interior point of H_r is modified like that of v_r: for example, v_r is replaced by u'_r in the second triangle of Figure 2, and inside that triangle each v_s is relabelled u'_s and v'_s is relabelled u_s. Thus, each graph G_r contains each of the $6n$ points u_s, v_s, w_s, u'_s, v'_s, w'_s, for $s = 0, 1, \ldots, n-1$.

THEOREM 5. *The graph with* $6n$ *points in which each point is of degree* $6n-2$ *has thickness* n.

Proof. To prove this result, we shall show that the union F of the n planar graphs G_r is the graph described in the theorem. By the symmetry of the graphs, it is sufficient to show that in F the point v_r is adjacent to all other points except one, namely v'_r. This will be done in three steps.

First, in graph G_r, v_r is clearly adjacent to the points u_r, w_r, u'_r, and w'_r.

Next, we show that for any $s \neq r$, v_r is adjacent in F to v_s and v'_s. Using

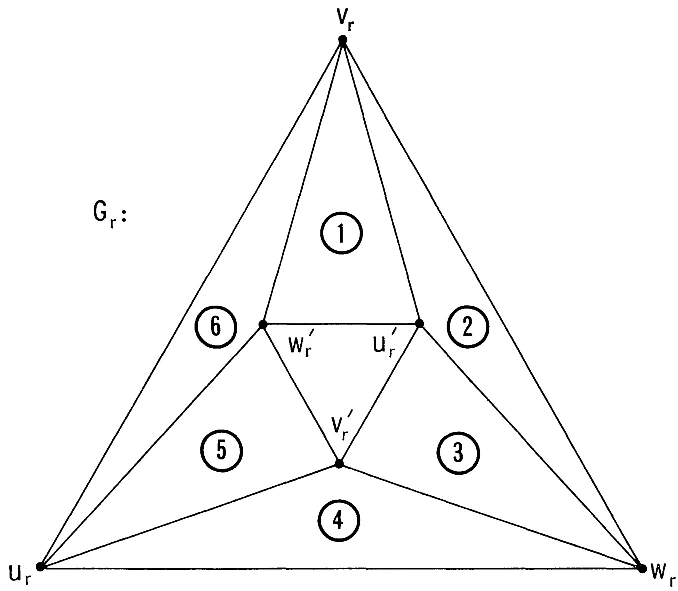

FIGURE 2

Theorem 3, we can assume that r and s are consecutive entries below the diagonal in the column headed by i of the matrix A and above it in the column headed by j. Then in the graph G_i, v_r is adjacent to v_s since

$$m(i, r) = m(i, s)$$

and in graph G_j, v_r is adjacent to v'_s since $m(j, r) \neq m(j, s)$. This follows immediately from the construction of these graphs.

Lastly, we show that v_r is adjacent to u_s, w_s, u'_s, w'_s. In graph G_r, it is adjacent to u'_s and w'_s if $m(r, s) = 1$ and to u_s and w_s if $m(r, s) = 0$. On the other hand, in graph G_s, v_r is adjacent to u'_s and w'_s if $m(s, r) = 1$ and to u_s and w_s if $m(s, r) = 0$. These facts follow directly from our construction and definitions. Since, by Theorem 4, $m(r, s) \neq m(s, r)$, v_r is adjacent to two of these four points in G_r and the other two in G_s.

Thus, except for the point v'_r, v_r is adjacent to each of the other points in exactly one of the n graphs G_i. Since each G_i is a triangulation of the plane, the theorem is proved.

Further constructions are required to provide the planar subgraphs necessary to complete the proof of our main result. We note in passing that a somewhat weaker result, as stated in **(2)**, can be obtained immediately. Form the graph in which u_i and u'_i, v_i and v'_i, and w_i and w'_i are adjacent, and add two more points x and x' adjacent to each other and all the other points. This new graph is clearly planar as indicated in Figure 3. Since the union of this and the n graphs G_r form K_{6n+2}, $t(K_{6n+2}) \leqslant n + 1$. Combined with Theorem 1, this shows that for $m = -1, 0, 1$ or 2, $t(K_{6n+m}) = n + 1$.

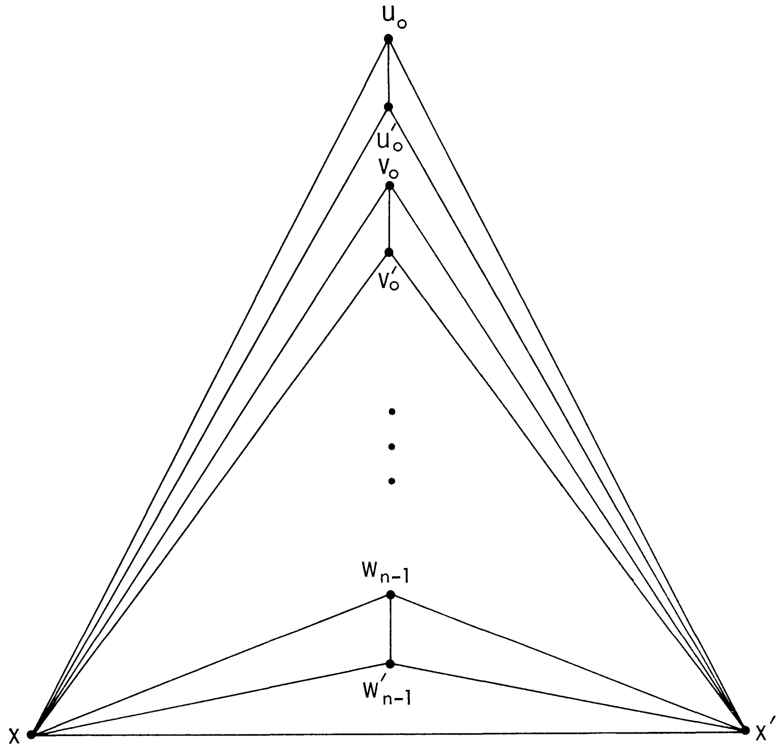

FIGURE 3

Our improved result is a construction showing that this equality also holds for $m = 3$ if $n \geqslant 3$. We first consider the case in which n is even. The following facts pertaining to the graphs G_r are used, where x_i or y_i denotes u_i, v_i, or w_i.

In G_0, $x_{\frac{1}{2}n}$ and $x_{\frac{1}{2}n+1}$ are adjacent.

In $G_{\frac{1}{2}n}$, x_0 and x'_1 are adjacent.

In G_0, x_0, x_1, and y'_0 form a face.

In G_i $(i \neq 0)$, x_i, x'_{i+1}, and y'_i $(y_i \neq x_i)$ form a face, and x'_i, x_{i+1}, and y_i also form a face.

Now modify the graphs G_r by adding three new points u, v, w and some new lines as follows.

Put u adjacent to

$$v_{\frac{1}{2}n} \text{ and } v_{\frac{1}{2}n+1} \text{ in } G_0,$$
$$u_i, u'_{i+1}, v_i \text{ in } G_i \ (i \neq 0).$$

Then, among others, u is adjacent to the points

$$\begin{aligned} &u_i \text{ and } u'_i \text{ for } i \neq 0 \text{ or } 1, \\ &v_i \text{ and } v'_i \text{ for } i = \tfrac{1}{2}n \text{ and } \tfrac{1}{2}n + 1. \end{aligned} \tag{1}$$

Put v adjacent to

$$\begin{array}{ll} v_0,\ v_1, \text{ and } u'_0 \text{ in } G_0, & \\ u_0 \text{ and } u'_1 \text{ in } G_{\frac{1}{2}n}, & \\ v'_i,\ v_{i+1}, \text{ and } u_i \text{ in } G_i & (i = 1, 2, \dots, \tfrac{1}{2}n - 1), \\ v_i,\ v'_{i+1}, \text{ and } u'_i \text{ in } G_i & (i = \tfrac{1}{2}n + 1, \dots, n - 1). \end{array}$$

Then, among others, v is adjacent to the points

$$\begin{array}{l} v_i \text{ and } v'_i \text{ for } i \neq \tfrac{1}{2}n \text{ or } \tfrac{1}{2}n + 1, \\ u_i \text{ and } u'_i \text{ for } i = 0 \text{ and } 1. \end{array} \tag{2}$$

Put w adjacent to

$$\begin{array}{ll} w'_1 \text{ in } G_0, & \\ w_i \text{ and } w'_{i+1} \text{ in } G_i & (i \neq 0). \end{array}$$

Then w is adjacent to the points

$$w_i \text{ and } w'_i \text{ except } w_0. \tag{3}$$

For convenience, we now delete all lines incident with u and v except those listed in (1) and (2) above. Let the graph thus obtained from G_r be denoted by G_r^*. Each G_r^* is planar by virtue of the facts stated above.

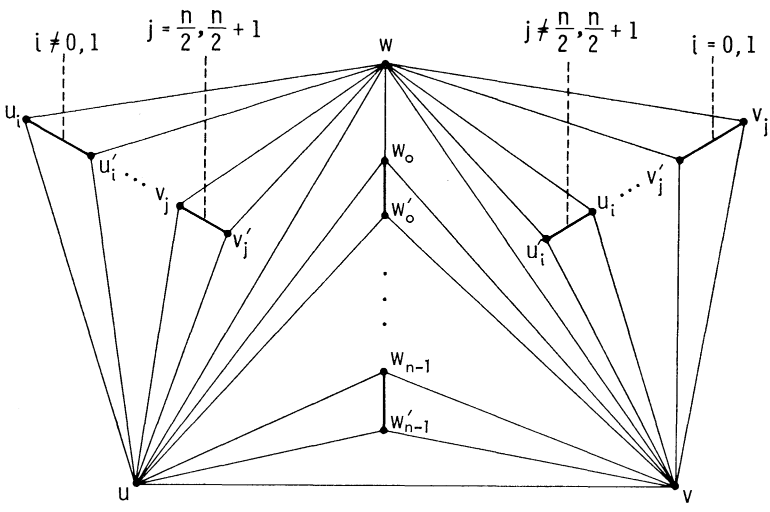

FIGURE 4

The complement G^* of the union of the n graphs G_r^* is itself planar. A construction to show this is given in Figure 4. Here

u_i and u'_i are adjacent, as are v_i and v'_i, and w_i and w'_i (for all i),
u, v, and w are mutually adjacent,
u is adjacent to w_i and w'_i (for all i),
v is adjacent to w_i and w'_i (for all i),

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w is adjacent to u_i, u'_i, v_i, and v'_i (for all i),
u is adjacent to u_i and u'_i for $i = 0, 1$,
u is adjacent to v_i and v'_i for $i \neq \frac{1}{2}n$, $\frac{1}{2}n + 1$,
v is adjacent to v_i and v'_i for $i = \frac{1}{2}n$, $\frac{1}{2}n + 1$,
v is adjacent to u_i and u'_i for $i \neq 0, 1$,
w is adjacent to w_0.

For u, v, w, these cover all cases not in (1), (2), or (3) above. Hence, if n is even, $t(K_{6n+3}) \leqslant n + 1$.

The case in which n is odd is handled in virtually the same way, except for changes necessitated by the subscripts. If each occurrence of $\frac{1}{2}n$ or $\frac{1}{2}n + 1$ is replaced by $\frac{1}{2}(n + 1)$ or $\frac{1}{2}(n - 1)$, respectively, the same argument applies. Hence the details are omitted.

The type of construction used above does not work if $n < 3$. A special construction can be provided to show that $t(K_{15}) \leqslant 3$. Thus for all $n > 1$, $t(K_{6n+3}) \leqslant n + 1$. But by Theorem 1, $t(K_{6n-1}) \geqslant n + 1$. A comparison of these two inequalities, together with some constructions for small values of p, establishes the following theorem.

Main Theorem. *If $p \neq 9$ or $p \not\equiv 4 \pmod 6$, then*

$$t(K_p) = [\tfrac{1}{6}(p + 7)].$$

Equivalently, $t(K_{6n+m}) = n + 1$ *unless* $m = 4$ *or* $6n + m = 9$.

We remarked earlier that $t(K_9) = t(K_{10}) = 3$. Hence there are some values of p for which the equality in the theorem does not hold. On the other hand, there are other values of p, namely 4 and 28, not specified by the theorem, for which the equality is known to hold, since suitable constructions can be provided. The first few values of $t(K_p)$ are shown in the following table:

p	2–4	5–8	9–15	16	17–21	22	23–28	29–33	34
$t(K_p)$	1	2	3	?	4	?	5	6	?

In a future paper, the "toroidal thickness" of all complete graphs will be given, that is, the smallest number of graphs embeddable on a torus whose union is K_p.

Added in proof. In the meantime and independently of us, G. Ringel has also shown in **(8)** that for all p, the toroidal thickness of K_p, denoted $t_1(K_p)$, is given by

$$t_1(K_p) = [\tfrac{1}{6}(p + 4)].$$

The "genus 2" thickness of the complete graph, $t_2(K_p)$, has also been investigated and will be reported elsewhere. The answer, for all p, is

$$t_2(K_p) = [\tfrac{1}{6}(p + 3)].$$

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