

SOLUTION OF GENERAL INHOMOGENEOUS LINEAR DIFFERENCE EQUATIONS

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Abstract

The general solution of the r th order inhomogeneous linear difference equation is given in the form

$$u_m = a_1^{(n-r)}(n)u_{r-1} + \dots + a_r^{(n-r)}(n)u_0 + b^{(n-r)}(n).$$

The coefficients $a_i^{(n-r)}(n)$, $i = 2, \dots, r$, and $b^{(n-r)}(n)$ can be evaluated from n values $a_1^{(k)}(n)$, $k = 0, \dots, n-1$, which satisfy an r th order homogeneous linear difference equation. In the r th order homogeneous case and if $n \geq 2r$, the method requires the evaluation of r determinants of successive orders $n-2r+1, n-2r+2, \dots, n-r$. If $r \leq n \leq 2r-1$, only $n-r$ determinants are required, with orders varying from 1 to $n-r$. In the second order inhomogeneous case, $a_1^{(k)}(n)$ can be evaluated from a continued fraction and a simple product.

1. Introduction

The r th order homogeneous linear difference equation

$$\sum_{i=0}^r a_i(n)u_{n-i} = 0, \quad n \geq r, \tag{1.1}$$

where the coefficients $a_i(n)$, $i = 0, 1, \dots, r$, with $a_0(n) \neq 0$, are arbitrary functions of n , has been studied by Singh in a recent paper [3]. Singh has given two forms of the solution, namely the matrix form and the determinantal one. The latter, obtained by Cramer's rule, is as follows:

$$u_n = (-1)^{n-r+1} p_n^{-1} \sum_{s=1}^r D_s^n u_{r-s}, \quad n \geq r, \tag{1.2}$$

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where

$$p_n = \prod_{l=r}^n a_0(l)$$

and

$$D_s^n = \begin{vmatrix} a_s(r) & a_0(r) & 0 & \dots & 0 & \dots & 0 \\ a_{s+1}(r+1) & a_1(r+1) & a_0(r+1) & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_r(2r-s) & a_{r-s}(2r-s) & \vdots & & a_0(2r-s) & \dots & 0 \\ 0 & \vdots & \vdots & \dots & \vdots & \dots & 0 \\ \vdots & a_r(2r) & a_{r-1}(2r) & \dots & \vdots & \dots & 0 \\ \vdots & 0 & a_r(2r+1) & \dots & \vdots & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & a_0(n-1) \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & a_1(n) \end{vmatrix} \quad (1.3)$$

The method requires the evaluation of r determinants of order $n - r + 1$.

Singh's result generalizes the solution given by Brown [1] in case of the second order homogeneous equation

$$H(n)u_{n+1} + G(n)u_n + F(n)u_{n-1} = 0, \quad (1.4)$$

namely

$$u_{n+1} = (-1)^n \{u_0 F(1)C_2^n + u_1 C_1^n\} / \prod_{s=1}^n H(s), \quad (1.5)$$

where C_1^n and C_2^n are tridiagonal determinants of orders n and $n - 1$ defined by

$$C_m^n = \begin{vmatrix} G(m) & H(m) & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ F(m+1) & G(m+1) & H(m+1) & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & F(m+2) & G(m+2) & H(m+2) & 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \dots & F(n-2) & G(n-2) & H(n-2) & 0 \\ \vdots & \dots & \dots & \dots & \dots & 0 & F(n-1) & G(n-1) & H(n-1) \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & F(n) & G(n) \end{vmatrix} \quad (1.6)$$

For the case (1.4), in a recent paper, Love [2] has proposed an alternative method for evaluating the solution from a continued fraction and a simple product, as follows:

$$u_{n+1} = u_0 \prod_{s=0}^n p_s \tag{1.7}$$

where

$$p_s = \frac{u_{s+1}}{u_s} = -\frac{G(s)}{H(s)} - \frac{F(s)}{H(s)} \frac{-\frac{G(s-1)}{H(s-1)} - \frac{F(s-1)}{H(s-1)}}{-\frac{G(s-2)}{H(s-2)} - \dots} \dots - \frac{F(1)}{H(1)} \frac{u_1}{u_0} \tag{1.8}$$

Love’s method does not apply to the general case of (1.1), nor to the second order inhomogeneous linear equation.

In the following section, we consider the general r th order inhomogeneous linear difference equation reduced to the form

$$u_n = \sum_{i=1}^r a_i(n)u_{n-i} + b(n), \quad n \geq r, \tag{1.9}$$

where $a_i(n), i = 1, 2, \dots, r$, are arbitrary functions of n , and $a_r(n) \neq 0$.

In the homogeneous case, $b(n) \equiv 0$, our method contributes another approach to the problem, different from Singh’s. Moreover, if $n \geq 2r$, instead of r determinants of order $n - r + 1$ as in Singh’s method, our approach requires r determinants of successive orders $n - 2r + 1, n - 2r + 2, \dots, n - r$. If $r \leq n \leq 2r - 1$, only $n - r$ determinants are required, with orders varying from 1 to $n - r$.

Finally, in case of the second order inhomogeneous linear equations, our results make Love’s method applicable.

2. Method of solution

Assume that $n > r$. In (1.9), writing $n - 1$ for n and eliminating u_{n-1} between the obtained equation and (1.9), we have

$$u_n = \sum_{i=1}^r a_i^{(1)}(n)u_{n-i-1} + b^{(1)}(n), \quad n > r, \quad (2.1)$$

where

$$a_i^{(1)}(n) = a_i(n-1)a_1(n) + a_{i+1}(n), \quad i = 1, \dots, r, \quad (2.2)$$

and

$$b^{(1)}(n) = b(n-1)a_1(n) + b(n), \quad (2.3)$$

with the convention $a_i(n) \equiv 0$ if $i > r$. Repeating the above operation $n - r$ times by taking for n in (1.9) successive values $n - 1, n - 2, \dots, r$, we obtain

$$u_n = \sum_{i=1}^r a_i^{(n-r)}(n)u_{r-i} + b^{(n-r)}(n). \quad (2.4)$$

The coefficients $a_i^{(n-r)}(n)$ and $b^{(n-r)}(n)$ satisfy the formulae

$$a_i^{(k)}(n) = a_i(n-k)a_1^{(k-1)}(n) + a_{i+1}^{(k-1)}(n), \quad (2.5)$$

$$b^{(k)}(n) = b(n-k)a_1^{(k-1)}(n) + b^{(k-1)}(n), \quad (2.6)$$

where $1 \leq k \leq n - r; i = 1, \dots, r$ and with the conventions

$$a_i^{(0)}(n) \equiv a_i(n), \quad i = 1, \dots, r, \quad (2.7)$$

$$b^{(0)}(n) \equiv b(n), \quad (2.8)$$

$$a_i^{(k)}(n) \equiv 0 \quad \text{for all } k \quad \text{if } i > r \quad \text{or } i < 1. \quad (2.9)$$

It is easy to see that the problem can be reduced to the evaluation of n coefficients $a_1^{(k)}(n)$, where k varies from 0 to $n - 1$.

Indeed, equation (2.5) can be written as follows

$$a_i^{(k)}(n) = -a_{i-1}(n-k-1)a_1^{(k)}(n) + a_{i-1}^{(k+1)}(n). \quad (2.10)$$

By performing backward iterations with respect to i and down to $i = 1$ and by noting the conventions (2.9) and $a_0(n) \equiv -1$, we can obtain

$$a_1^{(k)}(n) = - \sum_{j=1}^i a_{i-j}(n-k-j)a_1^{(k+j-1)}(n). \quad (2.11)$$

Similarly, we can deduce from (2.6) the relation

$$b^{(k)}(n) = b(n) + \sum_{j=1}^k b(n-k+j-1)a_1^{(k-j)}(n). \quad (2.12)$$

In the case of the homogeneous equation, the number of necessary coefficients $a_1^{(k)}(n)$ reduces to r when k varies from $n - r$ to $n - 1$.

On the other hand, by performing forward iterations of equation (2.5) with respect to i and up to $i = r$ and by making use of the condition (2.9), we can obtain the relation

$$a_i^{(k)}(n) = \sum_{j=1}^{r-i+1} a_{i+j-1}(n - k + j - 1)a_1^{(k-j)}(n), \quad i \leq r \leq k + i - 1, \tag{2.13}$$

which gives the following difference equation for $a_1^{(k)}(n)$ when $i = 1$:

$$a_1^{(k)}(n) = \sum_{j=1}^r a_j(n - k + j - 1)a_1^{(k-j)}(n), \quad k \geq r \geq 1. \tag{2.14}$$

It is surprising that this equation is similar to the homogeneous part of the original equation (1.9), with modified coefficients.

The general solution $a_1^{(k)}(n)$ can be evaluated as functions of the first r values,

$$\left. \begin{aligned} a_1^{(0)}(n) &= a_1(n), \\ a_1^{(1)}(n) &= a_1(n - 1)a_1(n) + a_2(n), \\ a_1^{(2)}(n) &= a_1(n - 2)a_1(n - 1)a_1(n) \\ &\quad + a_1(n - 2)a_2(n) + a_1(n)a_2(n - 1) + a_3(n), \\ &\dots \\ a_1^{(r-1)}(n) &= a_1(n - r + 1)a_1^{(r-2)}(n) + a_2^{(r-2)}(n). \end{aligned} \right\} \tag{2.15}$$

Assume that $k \geq r$. Letting $k = k, k - 1, \dots, r$ in (2.14), we obtain a system of $k - r + 1$ equations which effectively gives $a_1^{(k)}(n)$ in determinantal form.

We distinguish 2 cases:

Case 1: $r \leq k \leq 2r - 1$.

$$a_1^{(k)}(n) = -(-1)^{k-r} \begin{vmatrix} c_{k-r} & a_1(n - k) & \dots & \dots & \dots & a_{k-r}(n - r - 1) \\ c_{k-r-1} & -1 & a_1(n - k + 1) & \dots & \dots & a_{k-r-1}(n - r - 1) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_1 & 0 & \dots & \dots & -1 & a_1(n - r - 1) \\ c_0 & 0 & \dots & \dots & \dots & -1 \end{vmatrix}. \tag{2.16}$$

Case 2: $k \geq 2r$.

$$a_1^{(4)}(n) = -(-1)^{4-r} \times \begin{vmatrix} 0 & a_1(n-k) & \dots & a_1(n-k+r-1) & 0 & \dots & 0 \\ 0 & -1 & a_1(n-k+1) & \dots & a_1(n-k+r) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & \dots & 0 & -1 & a_1(n-2r) & \dots & a_1(n-r-1) \\ c_{r-1} & 0 & \dots & \dots & 0 & -1 & \dots & a_{r-1}(n-r-1) \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots \\ c_1 & 0 & \dots & \dots & \dots & \dots & 0 & -1 & a_1(n-r-1) \\ c_0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 & -1 \end{vmatrix} \tag{2.17}$$

In (2.16) and (2.17), the quantities c_m are defined as

$$c_m = - \sum_{j=m+1}^r a_j(n-r-m+j-1) a_1^{(r+m-j)}(n), \quad 0 \leq m \leq r-1. \tag{2.18}$$

In the r th order inhomogeneous case, the solution requires the evaluation of $n-r$ determinants like (2.16) or (2.17) of orders $1, 2, \dots$, and $n-r$, corresponding to $k = r, r+1, \dots, n-1$. In the r th order homogeneous case, the number of determinants to be evaluated remains the same if $r \leq n \leq 2r-1$ (in this case, $n-r < r$). If $n \geq 2r$, we have to evaluate r determinants of orders $n-2r+1, n-2r+2, \dots, n-r$ corresponding to $k = n-r, \dots, n-1$. In both cases, it is clear that the proposed method reduces either the number or the orders of the determinants to be evaluated in comparison with Singh's method. In the case of the second order homogeneous equation, our results require the evaluation of 2 determinants of orders $n-2$ and $n-3$ (assume that $n \geq 4$) instead of $n-1$ and $n-2$ as in Brown's method.

3. The second order inhomogeneous case and Love's method

Love's method does not apply to the second order inhomogeneous equation

$$u_n = a_1(n)u_{n-1} + a_2(n)u_{n-2} + b(n), \quad n \geq 2. \tag{3.1}$$

In this case, the general solution is given by (2.4) with $r = 2$,

$$u_n = a_1^{(n-2)}(n)u_1 + a_2^{(n-2)}(n)u_0 + b^{(n-2)}(n), \tag{3.2}$$

where

$$a_2^{(n-2)}(n) = -a_1(1)a_1^{(n-2)}(n) + a_1^{(n-1)}(n), \tag{3.3}$$

$$b^{(n-2)}(n) = b(n) + \sum_{j=1}^{n-2} b(j+1)a_1^{(n-j-2)}(n). \tag{3.4}$$

The coefficients $a_1^{(k)}(n)$ can now be evaluated by Love’s method. Indeed, they satisfy the second order homogeneous linear difference equation

$$a_1^{(k)}(n) = a_1(n-k)a_1^{(k-1)}(n) + a_2(n-k+1)a_1^{(k-2)}(n), \tag{3.5}$$

which can be rearranged as

$$\frac{a_1^{(k)}(n)}{a_1^{(k-1)}(n)} = a_1(n-k) + \frac{a_2(n-k+1)}{[a_1^{(k-1)}(n)/a_1^{(k-2)}(n)]}. \tag{3.6}$$

Therefore, the ratios

$$p^{(k)} = \frac{a_1^{(k)}(n)}{a_1^{(k-1)}(n)}$$

can be evaluated in continued fraction form

$$p^{(k)} = a_1(n-k) + \frac{a_2(n-k+1)}{a_1(n-k+1) + \frac{a_2(n-k+2)}{a_1(n-k+2) + \dots}} \dots \frac{a_2(n-2)}{a_1(n-2) + \frac{a_2(n-1)a_1(n)}{a_1(n-1)a_1(n) + a_2(n)}}. \tag{3.7}$$

Finally, $a_1^{(k)}(n)$ can be expressed as a product of ratios of successive $a_1^{(s)}(n)$:

$$\begin{aligned} a_1^{(k)}(n) &= \frac{a_1^{(k)}}{a_1^{(k-1)}} \frac{a_1^{(k-1)}}{a_1^{(k-2)}} \dots \frac{a_1^{(1)}}{a_1^{(0)}} a_1^{(0)} \\ &= a_1(n) \prod_{s=1}^k p^{(s)}. \end{aligned} \tag{3.8}$$

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