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WEIGHTED SHIFTS AND COMMUTING NORMAL EXTENSION

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Abstract

The main result of this paper shows that the existence of commuting normal extension (c.n.e.) for an arbitrary family of commuting subnormal operators can be determined by considering appropriate families of multivariable weighted shifts. In proving this some known criteria for c.n.e. are generalized. It is also shown that a family of jointly quasi-normal operators has c.n.e.

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1. Introduction

Multivariable weighted shifts have been used to study systems of commuting contractions in Hilbert space and have been especially useful in the study of commuting subnormal operators. In this paper we extend a result of Lambert (1976) to show that the existence of a commuting normal extension (c.n.e.) for arbitrary commuting subnormals can be determined by considering an appropriate family of weighted shifts. In proving this we also extend some results of Embry (1973) and Ito (1958) to give conditions for c.n.e. to exist.

2. Preliminaries

A (bounded linear) operator T acting on a separable Hilbert space H is called subnormal if and only if there exists a normal operator N on some Hilbert space $K \supset H$ such that the restriction $N|_H = T$. It is a basic result of Halmos and Bram (1955), p. 76 that T is subnormal if and only if

$$\sum_{i,j=1}^{n} (T^i x_j, T^j x_i) \geqslant 0 \quad \text{for all } \{x_0, \ldots, x_n\} \text{ finite subsets of } H.$$

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Commuting subnormal operators $T_1, \dots, T_n$ are said to have c.n.e. if and only if there exist commuting normals $N_1, \dots, N_n$ all defined on some $K \supset H$ such that $N_i|_H = T_i$, $i = 1, \dots, n$. Ito (1958), p. 5 modified the techniques of Halmos and Bram to show $T_1, \dots, T_n$ have c.n.e. if and only if

$$\sum_{I,J} (T^I x_J, T^J x_I) \geqslant 0 \quad \text{for all } \{x_I\} \text{ finite subsets of } H.$$

Here, we use capital letter indices to denote a multi-index $I = (i_1, \dots, i_n)$ and $T^I = T_1^{i_1} \dots T_n^{i_n}$. It has recently been shown by Abrahamse (1978) and Lubin (1977, 1978) that there exist commuting subnormals without c.n.e.

If we identify H with l^2, or equivalently H^2, and let $\{z_n : n = 0, 1, \dots\}$ be an orthonormal basis, then given a bounded sequence $\{a_n\}$ of positive numbers, the operator T defined by $Tz_n = a_n z^{n+1}$ is called the weighted shift associated with $\{a_n\}$. An unfamiliar reader can consult Shields (1974) for the basic properties of weighted shifts. For $z = (z_1, \dots, z_n)$ and $\{z^J = z_1^{j_1} \dots z_n^{j_n} : j_i \geqslant 0, i = 1, \dots, n\}$ an orthonormal basis of H, given any bounded net $\{w_{J,k} : j_i \geqslant 0, i, k = 1, \dots, n\}$ of positive numbers, we define

$$T_i z^J = w_{J,i} z_i z^J = w_{J,i} z^{J+e_i}, \quad i = 1, \dots, n,$$

where $e_i = (0, \dots, 1, \dots, 0)$ has 1 in the ith coordinate. If $w_{J,i} w_{J+e_i,k} = w_{J,k} w_{J+e_k,i}$ for all i, k, J, then $T_i T_k = T_k T_i$ and $T_1, \dots, T_n$ are called the commuting weighted shifts associated with $\{w_{J,k}\}$. For the basic theory of these operators, see Jewell and Lubin (1979).

The similarity in Theorems 2.1 and 2.2 (below) motivated Lambert's Theorem 2.3 which follows. We give the analogs for commuting operators in Section 3.

THEOREM 2.1. (Shields (1974), p. 84.) *Let T be the weighted shift associated with $\{a_n\}$, and let $\beta_0 = 1$, $\beta_n = a_0 \dots a_{n-1} = a_{n-1} \beta_{n-1}$. Then T is subnormal if and only if there exists a probability measure μ defined in $[0, a]$ such that $\beta_n^2 = \int_0^a t^{2n} \, d\mu(t)$, $n = 0, 1, \dots$, where $a = \|T\| = \sup_n a_n$.*

THEOREM 2.2. (Embry (1973), p.63.) *An operator T is subnormal if and only if there exists a positive operator valued measure ρ defined on $[0, a]$, $\|a\| = \|T\|$ such that $T^{*n} T^n = \int_0^a t^{2n} \, d\rho(t)$, $n = 0, 1, \dots$.*

Note that the integral converges in the strong operator topology.

THEOREM 2.3. (Lambert (1976), p. 478.) *Let T be an injective operator on H and for each $0 \neq x \in H$, let T_x be the weighted shift corresponding to weight sequence $\{\|T^{n+1} x\| / \|T^n x\|\}$. Then T is subnormal if and only if each T_x is subnormal.*

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3. Shifts and c.n.e.

THEOREM 3.1. (Lubin (1977), p. 841.) *Let $T_1, \ldots, T_n$ be commuting weighted shifts associated with the net $\{w_{J,k}\}$. Define $\beta_0 = 1$, and β_J by the equation $T^J 1 = \beta_J z^J$. Then $T_1, \ldots, T_n$ have c.n.e. if and only if there exists a probability measure μ defined on the n-dimensional rectangle $R = [0, a_1] \times [0, a_2] \times \ldots \times [0, a_n]$, $a_i = \|T_i\|$, such that $\int_R t_1^{2j_1} \ldots t_n^{2j_n} d\mu(t) = \int t^{2J} d\mu(t) = \beta_J$ for all J.*

THEOREM 3.2. *$T_1, \ldots, T_n$ have c.n.e. if and only if there exists a positive operator valued measure p defined on some n-dimensional rectangle R such that $T^{*J} T^J = \int_R t^{2J} d\rho(t)$ for all J.*

The proof of 3.2 will be given in Section 4.

THEOREM 3.3. *Let $T_1, \ldots, T_n$ be commuting injective operators on H. For each $0 \neq x \in H$, let $T_{1,x}, \ldots, T_{n,x}$ be the commuting weighted shifts corresponding to the net $\{w_{J,i} = (\|T^{J+e_i} x\| / \|T^J x\|)\}$. Then $T_1, \ldots, T_n$ have c.n.e. if and only if $T_{1,x} \ldots, T_{n,x}$ have c.n.e. for each x.*

PROOF. Note that if $T_1, \ldots, T_n$ are not all injective, T_i can be replaced by $(T_i - \lambda I)$ without affecting the existence of c.n.e. Also, since each $T_{i,x}$ is a direct sum of one variable weighted shifts, T_i is subnormal if and only if $T_{i,x}$ is subnormal for all x by Theorem 2.3. Our proof below is almost identical to Lambert's.

Suppose $T_1, \ldots, T_n$ have c.n.e. By 3.2, there exists $\rho(t)$ such that

$$T^{*J} T^J = \int_R t^{2J} d\rho(t) \quad \text{for all } J.$$

If $\|x\| = 1$, then

$$\begin{aligned} \|T^J x\|^2 &= (T^{*J} T^J x, x) \\ &= \int_R t^{2J} d(\rho(t) x, x) \\ &= \int_R t^{2J} d\mu_x(t), \end{aligned}$$

where $d\mu_x(t) = d(\rho(t)x, x)$. Since $\|T^J x\| = T^J 1 = T_{1,x}^{j_1}, \ldots, T_{n,x}^{j_n} 1$, 3.1 shows that $T_{1,x}, \ldots, T_{n,x} 1$, have c.n.e. for each unit vector x, and hence for all x.

Conversely, suppose $T_{1,x}, \ldots, T_{n,x}$ have c.n.e. for all x. For an arbitrary unit vector x, by 3.1 again, there exists a probability measure μ_x on a rectangle R such that

$$\|T^J x\|^2 = \int_R t^{2J} d\mu_x(t) \quad \text{for all } J.$$

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For $S \subset R$, we define $(\rho(S)x, x) = \mu_x(S)$. Then $\rho(S)$ is a positive operator valued measure on R, and

$$(T^{*J}T^J x, x) = \|T^J x\|^2$$
$$= \int_R t^{2J}\, d(\rho(t)x, x),$$

so

$$T^{*J}T^J = \int_R t^{2J}\, d\rho(t)$$

and by 3.2 $T_1, \ldots, T_n$ have c.n.e.

4. Conditions for c.n.e.

In this section we prove some technical results necessary for Theorem 3.2 and also of independent interest; these results are basically combinations of results of Embry (1973) and Ito (1958). The proofs modify the methods of Embry and Ito but the main ideas trace back to Bram (1955).

LEMMA 4.1. *Suppose for every finite subset of H $\{x_I : i_1, i_2, \ldots, i_n = 0, 1, \ldots, M\}$, we have*

$$(S) \qquad \sum_{I,J} (T^{I+J}x_I, T^{I+J}x_J) \geqslant 0,$$

where $T_1, \ldots, T_n$ are commuting operators. Then for all multi-indices K,

$$\sum_{I,J} (T^{I+J+K}x_I, T^{I+J+K}x_J) \leqslant \|T_1\|^{2k_1} \ldots \|T_n\|^{2k_n} \sum_{I,J} (T^{I+J}x_{,I}\, T^{I+J}x_J).$$

PROOF. Let $\varepsilon > 0$ and $A_i = T_i/(\|T_i\| + \varepsilon)$. Let Y be the Hilbert space direct sum

$$Y = \oplus \sum_I H_I \quad \text{where } H_I = H \quad \text{for all } I.$$

For $\bar{x} = \{x_I\} \in Y$ such that all but finitely many x_I are zero, define

$$S\bar{x} = \bar{y} = \{y_I\} \quad \text{where} \quad y_I = \sum_J A^{*I+J}A^{I+J}x_J.$$

Then

$$\|S\bar{x}\|^2 = \sum_I \|\sum_J A^{*I+J}A^{I+J}x_J\|^2$$
$$\leqslant \sum_I (\sum_J \|A^{*I+J}A^{I+J}\|\, \|x_J\|)^2$$
$$\leqslant \sum_I \sum_J \|A^{*I+J}A^{I+J}\|^2 \|\bar{x}\|^2$$
$$\leqslant (\sum_I \sum_J \|A_1\|^4 \ldots \|A_n\|^{4(I+J)}) \|\bar{x}\|^2$$
$$= (1 - \|A_1\|^4)^{-2} \ldots (1 - \|A_n\|^4)^{-2} \|\bar{x}\|^2,$$

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so S is a bounded linear operator defined on a dense subset of Y. Further,

$$(S\bar{x},\bar{x}) = \sum_I \Big(\sum_J A^{*I+J}A^{I+J}x_J, x_I\Big)$$
$$= \sum_{I,J}(A^{I+J}x_J, A^{I+J}x_I) \geqslant 0 \quad \text{by } (S).$$

We define

$$R\bar{x} = \bar{z} = \{z_I\} \quad \text{where} \quad z_I = \sum_J A^{*I+J+K}x_J$$

and note that

$$\|R\bar{x}\|^2 = \sum_I \Big\|\sum_J A^{*(I+K)+J}A^{(I+K)+J}x_J\Big\|^2$$
$$\leqslant \sum_I \Big\|\sum_J A^{*I+J}A^{I+J}x_J\Big\|^2 = \|S\bar{x}\|^2.$$

Therefore, $R \leqslant S$ and we have

$$\sum_{I,J}(T^{I+J+K}x_I, T^{I+J+K}x_J) \leqslant (\|T_1\|+\varepsilon)^{2k_1}\ldots(\|T_n\|+\varepsilon)^{2k_n}\sum_{I,J}(T^{I+J}x_I, T^{I+J}x_J).$$

Since ε is arbitrary, the lemma follows.

Let Γ be an abelian semigroup with identity 0 and for each $\gamma \in \Gamma$, let T_γ be an operator on H such that

$$T_{\gamma_1}T_{\gamma_2} = T_{(\gamma_1+\gamma_2)} \quad \text{for } \gamma_1, \gamma_2 \in \Gamma$$

and

$$T_0 = I.$$

Then $\{T_\gamma\}$ is called a representation of Γ.

COROLLARY 4.2. *Suppose for all* $\{x_1, \ldots, x_{n-1}\} \subset H$ *and* $\{\gamma_1, \ldots, \gamma_{n-1}\}$ Γ, *we have*

$$(S_\gamma) \qquad \sum_{i,j=1}^{n-1}(T_{\gamma_i+\gamma_j}x_i, T_{\gamma_i+\gamma_j}x_j) \geqslant 0.$$

Then for any $\beta \in \Gamma$,

$$\sum_{i,j}(T_{\gamma_i+\gamma_j+\beta}x_i, T_{\gamma_i+\gamma_j+\beta}x_j) \leqslant \|T_\beta\|^2 \sum_{i,j}(T_{\gamma_i+\gamma_j}x_i, T_{\gamma_i+\gamma_j}x_j).$$

PROOF. We apply Lemma 4.1 letting

$$T_i = T_{\gamma_i}, \quad i = 1, \ldots, n-1, \quad T_n = T_\beta,$$
$$x_1 = x_k \quad \text{if } I = e_k = (0, \ldots, 1, \ldots, 0), \quad k = 1, \ldots, n-1,$$
$$x_I = 0 \quad \text{otherwise}$$

and

$$K = e_n = (0, \ldots, 0, 1).$$

Note that (S_γ) implies, by reindexing, that $T_1, \ldots, T_n$ satisfy (S_1). Also, by using the corollary, we can replace $(\|T_1\|^{2K_1}\ldots\|T_n\|^{2K_n})$ by $\|T^K\|^2$ in the conclusion of 4.1.

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THEOREM 4.3. *Let* $\{T_\gamma: \gamma\in\Gamma\}$ *be a representation of* Γ *in* $B(H)$. *There exists a representation* $\{N_\gamma: \gamma\in\Gamma\}$ *of normal operators on some* $K\supset H$ *such that* $N_\gamma|_H = T_\gamma$, *that is* $\{T_\gamma\}$ *has c.n.e., if and only if* $\{T_\gamma\}$ *satisfies* (S_γ).

PROOF. Necessity is clear. To prove sufficiency let $X = \Pi H_\gamma$ be the Cartesian product over Γ of Hilbert spaces H_γ each identified with H, and consider $D = \{\bar{x} = \{x_\gamma\}\in X: x_\gamma = 0$ for all but finitely many $\gamma\}$. On the linear manifold D, define the bilinear form

$$(\bar{x},\bar{y}) = \sum_{\beta,\gamma\in\Gamma} (T_{\beta+\gamma}x_\beta, T_{\beta+\gamma}y_\gamma).$$

Let Y be the set of equivalences classes obtained in X by identifying $\bar{x}$ with 0 if $(\bar{x},\bar{x}) = 0$; Y then becomes an inner product space since (S_γ) holds.

For $\alpha\in\Gamma$, define Q_α on X by

$$Q_\alpha\bar{x} = \bar{y}, \quad \text{where } y_\gamma = T_\alpha x_\gamma \quad \text{for all } \gamma\in\Gamma.$$

By 4.2, we have that

$$\begin{aligned}(Q_\alpha\bar{x}, Q_\alpha\bar{x}) &= \sum_{\beta,\gamma} (T_{\beta+\gamma+\alpha}x_\beta, T_{\beta+\gamma+\alpha}x_\gamma)\\ &\leqslant \|T_\alpha\|^2(\bar{x},\bar{x}).\end{aligned}$$

We can consider Q_α to be a continuous linear operator on Y. We define E_α on D by

$$E_\alpha\bar{y} = \bar{z} \quad \text{where} \quad z_\gamma = \sum_{\delta+\alpha=\gamma} y_\delta,$$

the sum being 0 if no such δ exists. Note that $\{\delta: \delta+\alpha = \gamma\}$ may be infinite, but z_γ is well defined for $y\in D$. Then

$$\begin{aligned}(Q_\alpha\bar{x}, Q_\alpha\bar{y}) &= \sum_{\beta,\delta} (T_{\beta+\delta+\alpha}x_\beta, T_{\beta+\delta+\alpha}y_\delta)\\ &= \sum_{\beta,\gamma} (T_{\beta+\gamma}x_\beta, T_{\beta+\gamma}\sum_{\delta+\alpha=\gamma} y_\delta)\\ &= \sum_{\beta,\gamma} (T_{\beta+\gamma}x_\beta, T_{\beta+\gamma}z_\gamma)\\ &= (\bar{x}, E_\alpha\bar{y}).\end{aligned}$$

Thus, $E_\alpha = Q_\alpha^* Q_\alpha$ is well defined on $\tilde{Y}$, the Hilbert space completion of Y. It is easy to see that

$$\begin{aligned}Q_\alpha Q_\beta &= Q_{\alpha+\beta} = Q_\beta Q_\alpha,\\ E_\alpha E_\beta &= E_{\alpha+\beta} = E_\beta E_\alpha\end{aligned}$$

and

$$Q_\alpha E_\beta = E_\beta Q_\alpha \quad \text{for } \alpha,\beta\in\Gamma.$$

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By 4.5 below, it follows that there exists a normal semigroup $\{N_\alpha\}$ on $K \supset \tilde{Y}$ such that $N_\gamma|_{\tilde{Y}} = Q_\gamma$. Hence, $N_\gamma|_H = T_\gamma$.

DEFINITION 4.4. A set of operators $\{T_\gamma : \gamma \in \Gamma\} \subset B(H)$ is called *jointly quasinormal* if and only if $\{T_\gamma, T_\gamma^* T_\gamma : \gamma \in \Gamma\}$ forms a set of commuting operators. Recall that for $\Gamma = \{\gamma_0\}$, our definition is the standard definition of quasinormal.

THEOREM 4.5. *Every jointly quasinormal set of operators has c.n.e.*

PROOF. Let $\{T_\gamma : \gamma \in \Gamma\}$ be jointly quasinormal with $T_\gamma = U_\gamma P_\gamma$ the unique polar decomposition of T_γ, that is, $P_\gamma = (T_\gamma^* T_\gamma)^{\frac{1}{2}}$ and U_γ a partial isometry with $\ker(U_\alpha) = \ker(P_\alpha)$. From quasinormality we have that $U_\alpha P_\alpha = P_\alpha U_\alpha$, and since joint quasinormality requires that

$$T_\alpha T_\beta = T_\beta T_\alpha, \quad T_\alpha P_\beta^2 = P_\beta^2 T_\alpha, \quad P_\alpha^2 P_\beta^2 = P_\beta^2 P_\alpha^2 \quad \text{for all } \alpha, \beta \in \Gamma,$$

the standard polynomial approximation argument implies that

$$T_\alpha P_\beta = P_\beta T_\alpha \quad \text{and} \quad P_\alpha P_\beta = P_\beta P_\alpha \quad \text{for all } \alpha, \beta \in \Gamma.$$

Since P_β is self-adjoint, we also have $T_\alpha^* P_\beta = P_\beta T_*^\alpha$, and therefore the entire von Neumann algebra generated by $\{I, T_\alpha\}$, which contains U_α, is contained in the commutant of P_α. Hence

$$U_\alpha P_\beta = P_\beta U_\alpha \quad \text{for all } \alpha, \beta \in \Gamma.$$

Let $P_\beta x = y$. By the commutativity established above, we have

$$\begin{aligned} P_\alpha U_\alpha U_\beta y &= P_\alpha U_\alpha U_\beta P_\beta x \\ &= T_\alpha T_\beta x = T_\beta T_\alpha x \\ &= U_\beta P_\beta P_\alpha U_\alpha x \\ &= P_\alpha U_\beta U_\alpha P_\beta x \\ &= P_\alpha U_\beta U_\alpha y, \end{aligned}$$

and therefore $P_\alpha U_\alpha U_\beta = P_\alpha U_\beta U_\alpha$ on the range of P_β. Since $(\text{range } (P_\beta))^\perp = \ker P_\beta = \ker U_\beta$ is invariant under U_α, due to the fact that $U_\alpha P_\beta = P_\beta U_\alpha$, we have

$$P_\alpha U_\alpha U_\beta = P_\alpha U_\beta U_\alpha \quad \text{on } H,$$

that is,

$$U_\alpha U_\beta P_\alpha = U_\beta U_\alpha P_\alpha.$$

Thus, $U_\alpha U_\beta = U_\beta U_\alpha$ on range (P_α) and as above, $U_\alpha U_\beta = U_\beta U_\alpha = 0$ on $(\text{range } (P_\alpha))^\perp = \ker(P_\alpha)$, so $U_\alpha U_\beta = U_\beta U_\alpha$ on H. Thus, $\{U_\gamma, P_\gamma : \gamma \in \Gamma\}$ forms a set of commuting operators.

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A result of Yoshino (1973), p. 269 now implies that $\{T_\alpha, T_\beta\}$ has c.n.e. for any $\alpha, \beta \in \Gamma$, and in fact that $\{T_{\alpha_1}, \ldots, T_{\alpha_n}\}$ has c.n.e. for $\alpha_1, \ldots \alpha_n \in \Gamma$. We prove the general case by transfinite induction. Well-order Γ and suppose that for all $\gamma < \gamma_0$, $\{T_\alpha: \alpha \leqslant \gamma\}$ has c.n.e. It is easy to see from the commutivity established above that $\ker(T_{\gamma_0}) = \ker(P_{\gamma_0}) = \ker(U_{\gamma_0})$ reduces T_α for all $\alpha \in \Gamma$. So we have $H = E \oplus F$ where U_{γ_0} is isometric on E and is 0 on F and E reduces T_α for all $\alpha \in \Gamma$. Considering restrictions to E, $\{T_\alpha|_E: \alpha \leqslant \gamma\}$ has c.n.e. and $P_{\gamma_0}|_E$ is normal, in fact self-adjoint. So $\{T_\alpha|_E, P_{\gamma_0}|_E: \alpha \leqslant \gamma\}$ has c.n.e. and the isometry $U_{\gamma_0}|_E$ extends to an isometry V commuting with these minimal normal extensions by results of Bram (1955), p. 87. We can now extend V to a unitary operator on a larger space, and the normal extensions of $\{T_\alpha|_E, P_{\gamma_0}|_E\}$ will likewise extend. Thus, we have a c.n.e. for $\{T_\alpha|_E$: $\alpha \leqslant \gamma$, $\alpha = \gamma_0\}$. Since $T_{\gamma_0}|_F = 0$, we therefore have normal operators $N_\alpha^{(\gamma)}$ acting on $K^{(\gamma)} \supset H$ with $N_\alpha^{(\gamma)}|_H = T_\alpha$, $\alpha \leqslant \gamma$, $\alpha = \gamma_0$.

If γ_0 is a successor ordinal the above argument suffices and so suppose γ_0 is a limit ordinal. Without loss of generality, we assume each c.n.e. is minimal and hence unique up to isomorphism. Thus, for $\gamma < \gamma_0$ $K^{(\gamma)}$ is the closed linear span of

$$\{N_{\alpha_1}^{(\gamma)*j_1} \ldots N_{\alpha_n}^{(\gamma)*j_n} x: \alpha_i \leqslant \gamma \quad \text{or} \quad \alpha_i = \gamma_0, j_i = 0, 1, \ldots, i = 1, \ldots, n, x \in H\},$$

and hence $K^{(\gamma_1)} \subset K^{(\gamma_2)}$ and $N_\alpha^{(\gamma_2)}|K^{(\gamma_1)} = N_\alpha^{(\gamma_1)}$ if $\alpha \leqslant \gamma_1 < \gamma_2 < \gamma_0$ or $\alpha = \gamma_0$. Let $K = \bigcup_{\gamma < \gamma_0} K^{(\gamma)}$ and for $\alpha \in \Gamma$, define N_α on K by $N_\alpha|_{K^{(\gamma)}} = N_\alpha^{(\gamma)}$ if $\alpha < \gamma$ or $\alpha = \gamma_0$. Then $\{N_\alpha: \alpha \leqslant \gamma_0\}$ on K is a c.n.e. for $\{T_\alpha: \alpha \leqslant \gamma_0\}$.

Finally, to complete 4.3, it remains to show that $\{N_\alpha: \alpha \in \Gamma\}$ is a representation when Γ is a semigroup. We have for any $\alpha, \beta, \gamma_i \in \Gamma$, $i = 1, \ldots, m$, $x \in H$,

$$\begin{aligned} N_\alpha N_\beta(N_{\gamma_1}^{*j_1} \ldots N_{\gamma_n}^{*j_n} x) &= N_\alpha N_\beta(N_\gamma^{*J} x) \\ &= N_\gamma^{*J}(N_\alpha N_\beta x) = N_\gamma^{*J}(T_\alpha T_\beta x) \\ &= N_\gamma^{*J}(T_{\alpha+\beta} x) = N_\gamma^{*J}(N_{\alpha+\beta} x) \\ &= N_{\alpha+\beta}(N_\gamma^{*J} x), \end{aligned}$$

and so $N_\alpha N_\beta = N_{\alpha+\beta}$ and $\{N_\gamma\}$ is a representation. We note that if Γ is a finitely generated semigroup Yoshino's argument suffices verbatim and the induction argument can be eliminated.

COROLLARY 4.6. *$\{T_1, \ldots, T_n\}$ has c.n.e. if and only if for all finite sets $\{X_I\} \subset H$,*

$$\sum_{I,J} (T^{I+J} x_I, T^{I+J} x_J) \geqslant 0.$$

The corollary follows immediately from Theorem 4.3 by reindexing the semigroup $\Gamma = \{I = (i_1, \ldots, i_n): i_j \geqslant 0\}$. We note the general semigroup approach has the advantage of using a summation over simple indices $i, j = 0, \ldots, m$ while the more direct statement in the corollary sums over multindices. We now proceed with the

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PROOF OF 3.2. Suppose $\{T_1, \ldots, T_n\}$ has c.n.e. Then there exist commuting normals $N_1, \ldots, N_n$ on $K \supset H$ with $N_j|_H = T_j$. By the spectral theorem, there exists a spectral measure E on $\mathbf{C}^n$ with

$$N^J = \int_{\mathbf{C}^n} z^J \, dE(z), \quad z = (z_1, \ldots, z_n).$$

We define a spectral measure F on R^n by

$$\begin{aligned} F(S) &= E(S \times T^n) \\ &= E(\{(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) : (r_1, \ldots, r_n) \in S\} \end{aligned}$$

for $S \subset \mathbf{R}^n$. Then F is supported on some n-rectangle R and

$$N^{*J} N^J = \int |z|^{2J} \, dE(z) = \int_R r^{2J} \, dF(r).$$

Letting P project K onto H, we have for $x \in H$,

$$\begin{aligned} (T^{*J} T^J x, x) &= (P N^{*J} N^J x, x) \\ &= \left(P \int r^{2J} \, dF(r) x, x\right) \\ &= \int r^{2J} \, d(PF(r) x, x) \\ &= \int r^{2J} \, d(\rho(r) x, x), \end{aligned}$$

where ρ is the positive operator valued measure on H defined by $\rho(S) = PF(S)$. Thus

$$(E) \qquad T^{*J} T^J = \int t^{2J} \, d\rho(t).$$

Conversely, suppose (E) holds. Then

$$\begin{aligned} &\sum_{I,J} (T^{I+J} x_I, T^{I+J} x^J) \\ &\quad = \sum_{I,J} (T^{*I+J} T^{I+J} x_I, x_J) \\ &\quad = \sum_{I,J} \int t^{I+J} \, d(\rho(t) x_I, x_J) \\ &\quad = \sum_{I,J} \int_R d(\rho(t) t^I x_I, t^J x_J) \\ &\quad = \int_R d\Big(\sum_{I,J} \rho(t)^{\frac{1}{2}} t^I x_I, \rho(t)^{\frac{1}{2}} t^J x_J \Big) \\ &\quad = \int_R d\Big(\Big\| \sum_I \rho(t)^{\frac{1}{2}} t^I x_I \Big\|^2\Big) \geq 0. \end{aligned}$$

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Thus, $\{T_1, \ldots, T_n\}$ has c.n.e. by 4.6. We note that the above proof is a modification of MacNerney (1962), p. 50.

We conclude with the following open question.

QUESTION. If S and T are subnormal operators such that $p(S, T)$ is subnormal for every polynomial p, do S and T have c.n.e.?

Note that our assumption implies that S and T commute. By the results of this paper, it suffices to consider the case of S and T commuting weighted shifts.

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