# ON THE FREE SPECTRUM OF THE VARIETY GENERATED BY THE COMBINATORIAL COMPLETELY 0-SIMPLE SEMIGROUPS

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**Abstract.** We give an asymptotic bound for the size of the $n$-generated relatively free semigroup in the variety generated by all combinatorial strictly 0-simple semigroups.

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**1. Introduction.** For finite algebras there are strong connections between the structural properties of the algebra and the free spectra. If **G** is a finite group, then the size of the $n$-generated relatively free group in the variety generated by **G** is polynomial in $n$ if and only if **G** is nilpotent and at least doubly-exponential if **G** is not nilpotent ([**3**] and [**8**]).

Let **A** be a $k$-element finite algebra and let $\mathcal{V}$ denote the variety generated by **A**. It is known that the size of the free algebra in $\mathcal{V}$ freely generated by $n$ free elements ($|\mathbf{F}_{\mathcal{V}}(n)|$) is less than $k^{k^n}$. If $k \geq 2$, then this number is greater than $n$. The free spectrum of a variety $\mathcal{V}$ is the sequence of cardinalities $|\mathbf{F}_{\mathcal{V}}(n)|$, $n = 0, 1, 2, \ldots$. For example the free spectrum of Boolean algebras is $|\mathbf{F}_{\mathcal{V}}(n)| = 2^{2^n}$. The first important question about free spectra is the following: within the above bounds what are the possible numbers?

Another theorem on free spectra is Theorem 12.3 in [**4**]. If $\mathcal{V}$ is a nontrivial locally finite congruence distributive variety, then for every $c$ such that $0 < c < 1$, and for every large $n$, the free spectrum of $\mathcal{V}$ is bounded below by $2^{2^{cn}}$. There are so called gap theorems for the free spectra, as well. At the lower end, for example, there is Theorem 12.2 in [**4**] which states the following. Let $\mathcal{V}$ be a variety generated by a $k$-element algebra. Then either $|\mathbf{F}_{\mathcal{V}}(n)| \leq cn^k$ for some finite $c$, or else $|\mathbf{F}_{\mathcal{V}}(n)| \geq 2^{n-k}$ for all $n$.

For simple algebras there is a characterization of possible free spectra using tame congruence theory. The tame congruence types are denoted by **1**, **2**, **3**, **4** or **5** and called unary, affine, Boolean, lattice or semilattice type, respectively. The next result of Joel Berman is taken from [**1**]:

THEOREM 1.1. *For each $k \geq 2$ there exist positive constants $d_1, \ldots, d_5$ and $c_1$, $c_2$, $c_4$ such that if $\mathbf{A}$ is a $k$-element simple algebra and $\mathcal{V}$ is generated by $\mathbf{A}$, then for every sufficiently large $n$,*

(1) *if* $\mathrm{typ}(\mathbf{A}) = \mathbf{1}$, *then* $d_1 n \leq |\mathbf{F}_{\mathcal{V}}(n)| \leq c_1 n^{\log_2 k}$;

(2) *if* $\mathrm{typ}(\mathbf{A}) = \mathbf{2}$, *then* $d_2 k^n \leq |\mathbf{F}_{\mathcal{V}}(n)| \leq c_2 k^{(k-1)n}$;

(3) *if* $\mathrm{typ}(\mathbf{A}) = \mathbf{3}$, *then* $k^{d_3 k^n} \leq |\mathbf{F}_{\mathcal{V}}(n)| \leq k^{k^n}$;

(4) *if* $\mathrm{typ}(\mathbf{A}) = \mathbf{4}$, *then* $k^{d_4 k^n / \sqrt{n}} \leq |\mathbf{F}_{\mathcal{V}}(n)| \leq k^{c_4 k^n / \sqrt{n}}$;

(5) *if* $\mathrm{typ}(\mathbf{A}) = \mathbf{5}$, *then* $d_5 k^n \leq |\mathbf{F}_{\mathcal{V}}(n)| \leq k^{\sigma(n)}$ *for*

$$\sigma(n) = \frac{nk}{n - k(k-1)^3} \binom{n}{(k-1)^3} (k-1)^{n-(k-1)^3}.$$

As we see, for type 5 there is a huge gap between the two bounds. A very interesting class of type 5 algebras is the class of completely 0-simple semigroups. Completely 0-simple semigroups are one of the basic building blocks for semigroups like simple groups for groups. A semigroup is called combinatorial if it contains no nontrivial proper subgroups. The first natural step in investigating free spectra of semigroups is to find the free spectrum of the variety generated by the combinatorial completely 0-simple semigroups. In this paper we prove that the free spectrum of this variety is asymptotically $2^{n^2 + 2\log n}$.

Every *completely 0-simple semigroup* is isomorphic to a so called Rees matrix semigroup. A *Rees matrix semigroup* over the group $G$ adjoined with a 0, $G^0 (= G \cup \{0\})$, is constructed in the following way: Let $I$, $\Lambda$ be non-empty sets, and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix over $G^0$, such that each row and column contains at least one non-zero element. The matrix $P$ is called a sandwich matrix. The multiplication rule in the Rees matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P] = (I \times G \times \Lambda) \cup \{0\}$ is

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, a p_{\lambda i} b, \mu) & \text{if } p_{\lambda i} \neq 0, \\ 0 & \text{if } p_{\lambda i} = 0, \end{cases}$$

$$(i, a, \lambda)0 = 0(i, a, \lambda) = 00 = 0.$$

A semigroup is called *combinatorial* if it contains no nontrivial subgroup. A completely 0-simple semigroup is combinatorial if $G = \{1\}$. Here, 1 denotes the identity element of $G$. In the case of combinatorial completely 0-simple semigroups for the triple $(i, 1, \lambda)$ we write $[i, \lambda]$. The two semigroups playing the most important roles among completely 0-simple semigroups are the Brandt-semigroup $\mathbf{B_2} = \mathcal{M}^0[G; \{1, 2\}, \{1, 2\}; E]$, where $E$ is the $2 \times 2$ identity matrix and

$$\mathbf{A_2} = \mathcal{M}^0[G; \{1, 2\}, \{1, 2\}; P], \text{ where } P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Both semigroups are simple of type 5. The Brandt semigroup is both completely 0-simple and a so called 'inverse semigroup'. The authors proved in [6] that $\log |\mathbf{F}_{\mathcal{B}}(n)| \sim 2n \log n$, where $\mathcal{B}$ denotes the variety generated by the Brandt semigroup. Let $\mathcal{A}$ denote the variety generated by $\mathbf{A_2}$. Then $\mathcal{A}$ contains all combinatorial 0-simple semigroups. So the variety generated by all combinatorial completely 0-simple semigroups is generated by $\mathbf{A_2}$. In this paper we investigate the free spectra of $\mathcal{A}$. We prove that $|\mathbf{F}_{\mathcal{A}}(n)| \sim 2^{n^2 + 2\log n}$. We do it via associating directed graphs to terms over $\mathbf{A_2}$.
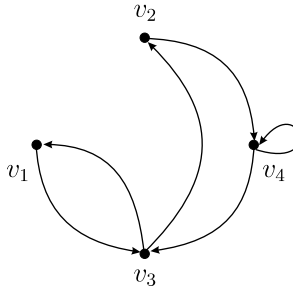
Figure 1. $t = x_1 x_3 x_2 x_4 x_4 x_3 x_1 x_3 x_2$.

**2. Preliminaries.** A *term* on an algebra $\mathbf{A}$ is an expression that can be obtained using (iterated) compositions of the basic operations and projections. The *projections* are the trivial operations $p_i^n(x_1, \ldots, x_n) = x_i$.

Two terms $t_1$ and $t_2$ are called *equivalent* ($t_1(x_1, \ldots, x_n) \equiv t_2(x_1, \ldots, x_n)$ or shortly $t_1 \equiv t_2$) over an algebra $\mathbf{A}$ if the term operations $t_1^{\mathbf{A}}$ and $t_2^{\mathbf{A}}$ are equal, i.e., for every $\overline{a} \in A^n$, $t_1^{\mathbf{A}}(\overline{a}) = t_2^{\mathbf{A}}(\overline{a})$.

For an algebra $\mathbf{A}$ the set of $n$-ary term operations is $\mathrm{Clo}_n \mathbf{A}$. If $\mathcal{V}$ is the variety generated by $\mathbf{A}$ then $|\mathbf{F}_{\mathcal{V}}(n)| = |\mathrm{Clo}_n \mathbf{A}|$ for all $n$.

Let $t = t(x_1, \ldots, x_n)$ be an $n$-ary term. A term operation $t^{\mathbf{A}}$ is said to be *essentially $n$-ary*, if it depends on all of its variables, i.e. if for all $1 \le i \le n$ there exist $a_1, \ldots, a_{i-1}, a, b, a_{i+1}, \ldots, a_n \in A$ such that

$$t(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) \ne t(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

For $n \ge 1$, denote the set of essentially $n$-ary term operations over $\mathbf{A}$ by $E_n(\mathbf{A})$, while $E_0(\mathbf{A})$ denotes the set of all constant unary term operations of $\mathbf{A}$. Now we define $p_n(\mathbf{A}) = |E_n(\mathbf{A})|$. Thus the equations $|\mathbf{F}_{\mathcal{V}}(n)| = |\mathrm{Clo}_n \mathbf{A}| = \sum_{k=0}^{n} \binom{n}{k} p_k(\mathbf{A})$ hold.

By the word *graph* we shall mean a directed graph without multiple edges. For a graph $G$ we shall denote the vertices and edges by $V(G)$ and $E(G)$, respectively. We say that a graph $G$ contains an *Eulerian walk* if there exists a directed walk in $G$ which contains all edges, possibly several times.

**3. Term equivalence over $\mathbf{A_2}$.** Recall that the sandwich matrix of the combinatorial 0-simple semigroup $\mathbf{A_2}$ is

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

the index sets are $\Lambda = I = \{1, 2\}$ and $[i, \lambda][j, \gamma] = 0$ if and only if $\lambda = j = 2$.

Let $t(x_1, x_2, \ldots, x_k) = t$ be a term over $\mathbf{A_2}$. Let $G(t)$ denote the following directed graph: $G(t)$ has $k$ many vertices $(v_1, v_2, \ldots, v_k)$. There is an edge between $v_i$ and $v_j$ if $x_j$ follows $x_i$ somewhere in $t$ that is, if $x_i x_j$ is a subword of $t$. For example, we get Figure 1 for the term $t = x_1 x_3 x_2 x_4 x_4 x_3 x_1 x_3 x_2$. Constructions of graphs related to semigroup terms can be found in [2], [5], [7], [10], [12],

Proposition 3.1 is formulated in [12], we omit its proof.

PROPOSITION 3.1. *Let $t_1(x_1, x_2, \ldots, x_k)$ and $t_2(x_1, x_2, \ldots, x_k)$ be two terms over $\mathbf{A_2}$, such that $t_1 = y_1 y_2 \ldots y_m$ and $t_2 = z_1 z_2 \ldots z_n$, where $y_i, z_j \in \{x_1, x_2, \ldots, x_k\}$, for every $1 \le i \le m$ and $1 \le j \le n$. Then $t_1 \equiv t_2$ if and only if the following hold:*

(1) $G(t_1) = G(t_2)$;
(2) $y_1 = z_1$;
(3) $y_m = z_n$.

**4. The free spectrum of $\mathcal{A}$.** The next lemma is consequence of Proposition 3.1.

LEMMA 4.1. *Let $G$ be a directed graph with n vertices and without multiple edges. If $G$ contains a closed Eulerian walk then there are exactly $n^2$ many non-equivalent essentially n-ary terms inducing the same graph.*

*Proof.* Let $v_1, \ldots, v_n$ be the vertices of $G$. As there is a closed Eulerian walk in $G$, there is a closed Eulerian walk starting at $v_j$: $p_1 = v_j \ldots v_j$ and a walk from $v_j$ to $v_i$: $p_2 = v_j \ldots v_i$. Then $p = x_j \ldots x_i$, the concatenation of the walks $p_1$ and $p_2$,

(1) covers all edges of $G$, as $p_1$ covers all edges of $G$;
(2) start by $v_j$;
(3) ends by $v_i$.

Thus the for corresponding term $t = x_j \ldots x_i$ we have

(1) $G(t) = G$;
(2) starts by $x_j$;
(3) ends by $x_i$.

We can repeat this construction for any $1 \le i, j \le n$, hence we have $n^2$ many distinct terms inducing $G$. On the other hand, by Proposition 3.1 we have listed all possible terms inducing $G$. □

PROPOSITION 4.2. *Let $D(n)$ denote the number of directed graphs on n vertices with a closed Eulerian walk. Then $D(n) = o(2^{n^2})$.*

*Proof.* Let $G$ be a digraph and let $|G| = n$. If $G$ does not contain a closed Eulerian walk, then there is a subset $S \subset V(G)$ such that there is no edge from $S$ to $V(G) \setminus S$. Let $S$ be a subset of size $k$. Then there are $2^{k^2}$ many graphs on $S$, $2^{(n-k)^2}$ many graphs on $V(G) \setminus S$ and $2^{(n-k)k}$ possibilities for edges from $V(G) \setminus S$ to $S$. Hence there are $2^{k^2} 2^{(n-k)^2} 2^{(n-k)k}$ digraphs without outgoing edges from $S$.

For $k = 1$ and $k = n - 1$ we have

$$\binom{n}{1} 2^{1^2} 2^{(n-1)^2} 2^{(n-1)1} + \binom{n}{n-1} 2^{(n-1)^2} 2^{(1)^2} 2^{(n-1)1}$$
$$= 2 \cdot n 2^{n^2-n+1} \le n 2^{n^2-n+4}.$$

Moreover $\binom{n}{k} \le 2^n$ implies

$$\binom{n}{k} 2^{k^2} 2^{(n-k)^2} 2^{(n-k)k} \le 2^n 2^{k^2} 2^{(n-k)^2} 2^{(n-k)k} = 2^{n^2+n-nk+k^2}.$$

In case $1 < k < n - 1$

$$2^{n^2+n-nk+k^2} \le 2^{n^2-n+4}.$$

Let $N(n)$ denote the number of directed graphs without a closed Eulerian walk on $n$ vertices. Then

$$N(n) \leq \sum_{1}^{n-1} \binom{n}{k} 2^{k^2} 2^{(n-k)^2} 2^{(n-k)k}.$$

Using the above estimates we obtain the following upper bound for $N(n)$:

$$N(n) \leq n 2^{n^2-n+4} + (n-2) 2^{n^2-n+4} \leq (2n-2) 2^{n^2-n+4}.$$

As $N(n) + D(n) = 2^{n^2}$, we have $D(n) = o(2^{n^2})$. $\qquad\square$

THEOREM 4.3.

$$|\mathbf{F}_{\mathcal{A}}(n)| = o(2^{n^2+2\log n}).$$

*Proof.* By Proposition 3.1 terms correspond to directed graphs and pairs of vertices, hence $|\mathbf{F}_{\mathcal{A}}(n)| \leq n^2 2^{n^2}$ holds. From Lemma 4.1 and from Proposition 4.2 we get $p_n(\mathbf{A_2}) = n^2 \cdot o(2^{n^2})$ for the number of the essentially $n$-ary terms. Since $p_n(\mathbf{A_2}) \leq |\mathbf{F}_{\mathcal{A}}(n)|$ we have:

$$|\mathbf{F}_{\mathcal{A}}(n)| = o(2^{n^2+2\log n}).$$

$\qquad\square$

There is still plenty of work to be done. We gave an asymptotic value for the free spectrum of the variety generated by the combinatorial 0-simple semigroups. One should ask for the free spectra of a variety generated by a (not necessarily combinatorial) 0-simple semigroup in general. It seems that even a very special case looks to be too complicated.

PROBLEM Let $G$ be an arbitrary (abelian, nilpotent) group. Find the free spectra of $\mathcal{M}^0[G; I, \Lambda; P]$ where $P$ is a 0-1 matrix.

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