Can. J. Math., Vol. XXX, No. 3, 1978, pp. 518–530

AUTOMATIC CONTINUITY FOR LINEAR FUNCTIONS INTERTWINING CONTINUOUS LINEAR OPERATORS ON FRECHET SPACES

MARC P. THOMAS

Introduction. Many results concerning the automatic continuity of linear functions intertwining continuous linear operators on Banach spaces have been obtained, chiefly by B. E. Johnson and A. M. Sinclair [**1**; **2**; **3**; **5**]. The purpose of this paper is essentially to extend this automatic continuity theory to the situation of Fréchet spaces. Our motive is partly to be able to handle the more general situation, since for example, questions about Fréchet spaces and LF spaces arise in connection with the functional calculus. But also equivalences between (T_n) and (T_n, R_n) theorems easily follow in this more general setting. The first section is mainly devoted to extending the (T_n), (T, R), and (T_n, R_n) theorems to deal with Fréchet spaces. In the second section we apply our results to give necessary and sufficient conditions for a countable spectrum operator on a Fréchet space to possess a discontinuous commuting operator.

1. In all the following X (or X_n) will denote an F-space over **C**, and Y a Fréchet space over **C**. By an *F-space* we mean X is a linear topological space with invariant metric d, which is complete. By a *Fréchet space*, we assume also that the space is locally convex. Hence the topology on Y is given by a countable separating family of seminorms $\{\|\cdot\|_k\}$, and we assume without loss of generality that $\|\cdot\|_{k+1} \geqq \|\cdot\|_k$, all k.

Let V be any subset of Y. We will observe the convention that $\bar{V}$ denotes the closure of V in the Fréchet topology of Y, whereas $\bar{V}^k$ denotes the closure of V in the kth seminorm. It is clear that $\bar{V} \subseteq \bar{V}^k$. Let $B(X)$ denote the vector space of all continuous linear operators on X. Let $B(Y)$ be analogous and let $B(X, Y)$ denote the vector space of all continuous linear operators from X to Y. If S is any linear function from X to Y we define the *separating subspace* $\mathscr{S}(S)$ as follows.

$$\mathscr{S}(S) \equiv \{y \in Y\text{: there is } x_n \to 0 \text{ in } X \text{ and } Sx_n \to y\}.$$

As a consequence of the open mapping theorem for F-spaces, we have the following commonly known results concerning the separating subspace.

1) S is continuous if and only if $\mathscr{S}(S) \equiv (0)$. (It is not necessary for Y to be Fréchet here, only that it be an F-space.)

Received February 17, 1977 and in revised form, July 26, 1977.

2) If Q is a continuous linear operator from Y into some F-space, then $\mathscr{S}(QS) = \overline{Q\mathscr{S}(S)}$.

3) Hence from 1) and 2) we obtain the result that QS is continuous if and only if $Q\mathscr{S}(S) \equiv (0)$.

Consider the following three lemmas:

LEMMA 1.1a. (T_n, R_n) *Let X be an F-space and Y a Fréchet space. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of continuous linear operators on X, i.e. $\{T_n\}_{n=1}^{\infty} \subseteq B(X)$, and let $\{R_n\}_{n=1}^{\infty} \subseteq B(Y)$. Suppose S is a linear function from X to Y satisfying $(ST_n - R_nS) \in B(X, Y)$ for all n. Then given k there exists $n(k)$ such that*

$$\overline{R_1R_2 \ldots R_m\mathscr{S}(S)}^k = \overline{R_1R_2 \ldots R_{n(k)}\mathscr{S}(S)}^k,$$

for all $m \geqq n(k)$.

LEMMA 1.1b. (T, R) *Let X be an F-space and Y a Fréchet space. Let $T \in B(X)$ and $R \in B(Y)$. Suppose S is a linear function from X to Y satisfying $ST = RS$. Then given k there exists $n(k)$ such that*

$$\overline{R^m\mathscr{S}(S)}^k = \overline{R^{n(k)}\mathscr{S}(S)}^k,$$

for all $m \geqq n(k)$.

LEMMA 1.1c. (T_n) *Let $X_0, X_1, X_2 \ldots$ be F-spaces and Y be a Fréchet space. Let $T_n \in B(X_n, X_{n-1})$, $n = 1, 2, 3, \ldots$. Suppose S is a linear function from X_0 to Y. Then given k there exists $n(k)$ such that*

$$\overline{\mathscr{S}(ST_1T_2 \ldots T_m)}^k = \overline{\mathscr{S}(ST_1T_2 \ldots T_{n(k)})}^k,$$

for all $m \geqq n(k)$.

Some remarks are in order. If X and Y are Banach spaces Lemma 1.1a is commonly known as the (T_n, R_n) theorem and is proved by N. Jewell and A. Sinclair in [**6**, Lemma 1]. Of course there is only one seminorm, namely the norm, so their conclusion reads: There exists n such that

$$\overline{R_1R_2 \ldots R_m\mathscr{S}(S)} = \overline{R_1R_2 \ldots R_n\mathscr{S}(S)},$$

for all $m \geqq n$. Lemma 1.1b is obviously a special case of Lemma 1.1a where each $T_n = T$, each $R_n = R$, and $(ST - RS) = 0$, which is certainly a continuous linear operator from X to Y. If all the X_i's and Y are Banach spaces Lemma 1.1c is commonly known as the (T_n) theorem. It is proved by K. Laursen in [**4**, Proposition 2.1], who also notes that the (T_n, R_n) theorem follows from the (T_n) theorem because

$$\mathscr{S}(ST_1T_2 \ldots T_m) = \mathscr{S}(R_1R_2 \ldots R_mS)$$

as a consequence of $(ST_n - R_nS) \in B(X, Y)$. Furthermore by principle 2) above

$$\mathscr{S}(R_1R_2 \ldots R_mS) = \overline{R_1R_2 \ldots R_m\mathscr{S}(S)}.$$

https://doi.org/10.4153/CJM-1978-047-x Published online by Cambridge University Press

It is easily seen that the above argument in the context of F-spaces and Fréchet spaces is still valid and shows that Lemma 1.1a follows from Lemma 1.1c. However, in the context of F-spaces and Fréchet spaces, it is rather surprising that all these lemmas are *equivalent*, as follows. It suffices to show that Lemma 1.1b implies Lemma 1.1c. So, given the hypotheses of Lemma 1.1c form new spaces

$$X \equiv \prod_{i=0}^{\infty} X_i = \{(x_0, x_1, x_2, \ldots) : x_i \in X_i\}$$

and

$$Z \equiv \prod_{i=0}^{\infty} Y = \{(y_0, y_1, y_2, \ldots) : y_i \in Y\}.$$

It is trivial that the countable direct product of F-spaces is an F-space with coordinate-wise convergence. Likewise the countable direct product of Fréchet spaces is a Fréchet space. In the latter case we may take as seminorms:

$$||(y_0, y_1, y_2, \ldots)||_k \equiv \sum_{i=0}^{k} ||y_i||_k, \quad k = 1, 2, 3, \ldots.$$

Hence X is an F-space and Z is a Fréchet space. Let π_n be the canonical projection of Z onto the nth coordinate, so $\pi_n \in B(Z, Y)$. Define

$$\tilde{T}(x_0, x_1, x_2, \ldots) \equiv (T_1x_1, T_2x_2, T_3x_3, \ldots)$$
$$\tilde{S}(x_0, x_1, x_2, \ldots) \equiv (Sx_0, ST_1x_1, ST_1T_2x_2, \ldots)$$
$$\tilde{R}(y_0, y_1, y_2, \ldots) \equiv (y_1, y_2, y_3, \ldots).$$

It is then easily verified that $\tilde{T} \in B(X)$, $\tilde{R} \in B(Z)$, and $\tilde{S}\tilde{T} = \tilde{R}\tilde{S}$. Also

$$\mathscr{S}(\tilde{S}) = \{(y_0, y_1, y_2, \ldots) : (x_{i0}, x_{i1}, x_{i2}, \ldots) \to 0 \quad \text{in } X, \text{ and}$$
$$\tilde{S}(x_{i0}, x_{i1}, x_{i2}, \ldots) \to (y_0, y_1, y_2, \ldots)\}.$$

Then $Sx_{i0} \to y_0$, $ST_1x_{i1} \to y_1$, $ST_1T_2x_{i2} \to y_2 \ldots$ etc. So $\pi_n\mathscr{S}(\tilde{S}) \subseteq \mathscr{S}(ST_1T_2 \ldots T_n)$. But if $y \in \mathscr{S}(ST_1T_2 \ldots T_n)$ then $(0, 0, 0, \ldots, 0, y, 0, \ldots) \in \mathscr{S}(\tilde{S})$ where y is in the nth coordinate. Thus $\pi_n\mathscr{S}(\tilde{S}) = \mathscr{S}(ST_1T_2 \ldots T_n)$. It also easily follows that $\pi_0\tilde{R}^n\mathscr{S}(\tilde{S}) = \pi_n\mathscr{S}(\tilde{S})$. Hence

$$\mathscr{S}(ST_1T_2 \ldots T_n) = \pi_0\tilde{R}^n\mathscr{S}(\tilde{S}).$$

An application of Lemma 1.1b implies the result, in view of the coordinate-wise convergence on Z.

Note that this technique applies only if we use the more general concept of F-spaces and Fréchet spaces, since the countable direct product of Banach spaces is a Fréchet space but not a Banach space. We will now obtain all three lemmas by proving only Lemma 1.1c. In the proof it will become clear from the role the seminorms play, why we require Y to be a Fréchet space, and not just an F-space.

Proof. (of Lemma 1.1c) It is trivial that $\mathscr{S}(ST_1T_2 \ldots T_m) \supseteq \mathscr{S}(ST_1T_2 \ldots T_{m+1})$, all m. Suppose the result fails for some fixed k. Then there exists a sequence of increasing positive integers $\{m(i)\}$ such that $m(0) = 0$ and

$$\overline{\mathscr{S}(ST_1T_2 \ldots T_{m(i+1)})}^k \subsetneqq \overline{\mathscr{S}(ST_1T_2 \ldots T_{m(i)})}^k,$$

for all i. But we may let $T_1' = T_1T_2 \ldots T_{m(1)}$, $T_2' = T_{m(1)+1} \ldots T_{m(2)}, \ldots$, $T_i' = T_{m(i-1)+1} \ldots T_{m(i)}, \ldots$. Also letting $X_0' = X_0$, $X_1' = X_{m(1)}, \ldots$, $X_i' = X_{m(i)}, \ldots$, we have $T_i' \in B(X_i', X_{i-1}')$, $i = 1, 2, 3, \ldots$. Hence without loss of generality we may "drop the primes" and suppose

$$\overline{\mathscr{S}(ST_1T_2 \ldots T_{i+1})}^k \subsetneqq \overline{\mathscr{S}(ST_1T_2 \ldots T_i)}^k,$$

for all i. If V is a subset of Y, then $(\bar{V})^k = \bar{V}^k$, thus

$$\mathscr{S}(ST_1T_2 \ldots T_{i+1}) \subsetneqq \mathscr{S}(ST_1T_2 \ldots T_i),$$

for all i. Let Q_i be the canonical quotient map of Y onto $Y/\mathscr{S}(ST_1T_2 \ldots T_i)$, which is also a Fréchet space. Then $Q_{i+1}\mathscr{S}(ST_1T_2 \ldots T_{i+1}) \equiv (0)$ whereas $Q_{i+1}\mathscr{S}(ST_1T_2 \ldots T_i) \not\equiv (0)$. From our previous remarks, this implies that $Q_{i+1}ST_1T_2 \ldots T_{i+1}$ is continuous whereas $Q_{i+1}ST_1T_2 \ldots T_i$ is not. Let $|||\cdot|||_{i+1}$ be the quotient seminorm: $\inf \|\cdot + \mathscr{S}(ST_1T_2 \ldots T_{i+1})\|_k$ on $Y/\mathscr{S}(ST_1T_2 \ldots T_{i+1})$. Let d_i be an invariant metric for X_i. We claim that given i, $\delta > 0$ and N a positive integer, there exists $x \in X_i$ satisfying $d_i(x, 0) < \delta$ but

$$|||Q_{i+1}ST_1T_2 \ldots T_ix|||_{i+1} \geqq N.$$

To see this, choose $y \in \mathscr{S}(ST_1T_2 \ldots T_i)$ with $|||Q_{i+1}y|||_{i+1} = N + 1$ using the fact that $\overline{\mathscr{S}(ST_1T_2 \ldots T_i)}^k \supsetneqq \mathscr{S}(ST_1T_2 \ldots T_{i+1})$. There is $x_n \to 0$ in X_i with $ST_1T_2 \ldots T_ix_n \to y$. Hence $d_i(x_n, 0) \to 0$ and $|||Q_{i+1}ST_1T_2 \ldots T_ix_n|||_{i+1} \to N + 1$, thus an x_n with n sufficiently large will suffice. We may also choose a sequence of positive reals $\{\epsilon(i)\}$ such that $d_i(x, 0) \leqq \epsilon(i)$ implies

$$|||Q_iST_1T_2 \ldots T_ix|||_i \leqq 1, \quad i = 1, 2, 3 \ldots.$$

We may assume $\epsilon(i) < 2^{-i}$ and that $\epsilon(i)$ decreases to 0 as $i \to \infty$. Hence we may inductively form a sequence of elements $x_i \in X_i$ satisfying:

i) $d_{j-1}(T_jT_{j+1} \ldots T_ix_i, 0) < \epsilon(i)2^{-i}$ for all $1 \leqq j \leqq i$.

ii) $d_i(x_i, 0) < \epsilon(i)2^{-i}$.

iii) $|||Q_{i+1}ST_1T_2 \ldots T_ix_i|||_{i+1} > i + \left|\left|\left|Q_{i+1}S \sum_{n=1}^{i-1} T_1T_2 \ldots T_nx_n\right|\right|\right|_{i+1}.$

Let $x = \sum_{n=1}^{\infty} T_1T_2 \ldots T_nx_n \in X_0$ which converges absolutely in X_0 since

$d_0(T_1T_2 \ldots T_nx_n, 0) < \epsilon(n)2^{-n} \leqq 2^{-n}$. It follows that

$$\begin{aligned}||Sx||_k &\geqq |||Q_{N+1}Sx|||_{N+1} \\ &= \left|\left|\left| Q_{N+1}S \sum_{n=1}^{N-1} T_1T_2 \ldots T_nx_n + Q_{N+1}ST_1T_2 \ldots T_Nx_N \right.\right.\right. \\ &\qquad \left.\left.\left. + Q_{N+1}S \sum_{n=N+1}^{\infty} T_1T_2 \ldots T_nx_n \right|\right|\right|_{N+1} \\ &\geqq N - |||Q_{N+1}ST_1T_2 \ldots T_{N+1}y_N|||_{N+1},\end{aligned}$$

where $y_N = \sum_{m=N+1}^{\infty} T_{N+2} \ldots T_mx_m$ which converges absolutely by i) above. But

$$\begin{aligned}d_{N+1}(y_N, 0) &\leqq \sum_{m=N+1}^{\infty} \epsilon(m)2^{-m} \\ &\leqq \epsilon(N+1) \sum_{m=N+1}^{\infty} 2^{-m} \\ &\leqq \epsilon(N+1).\end{aligned}$$

Thus $|||Q_{N+1}ST_1T_2 \ldots T_{N+1}y_N|||_{N+1} \leqq 1$ which implies $||Sx||_k \geqq N - 1$ for all N, a contradiction and the result follows.

We now concentrate on the situation in Lemma 1.1a.

We remark that if $S \in B(X, Y)$, and V is an open convex set in Y, then $S^{-1}(V)$ is an open convex set in X. Hence if X is an F-space with no open convex sets other than $\emptyset$ and X (e.g. L^p, $0 < p < 1$), then S is the zero map. This is well known, but serves to illustrate the difference in situations when S is not assumed to be continuous. There do exist discontinuous intertwining maps from such F-spaces into Fréchet and Banach spaces.

We further specialize to the case where $T_nT_m = T_mT_n$ and $R_rR_m = R_mR_n$, all n and m, and we say $\{T_n\}$ and $\{R_n\}$ are *commuting sequences of continuous linear operators on X and Y*, respectively. We have a preliminary lemma on projective limits which generalizes the Mittag-Leffler theorem:

LEMMA 1.1d. *Let $\{X_n\}$ be algebraic vector subspaces of a Fréchet space X. Let $\{T_n\}$ be any sequence of commuting operators in $B(X)$ such that $T_nX_{n+1} \subseteq X_n$ for all n. Let $\{t(n)\}$ be any increasing sequence of natural numbers with $t(n) \to \infty$. Let* **P** *be the following projective limit*

$$\bar{X}_1 \xleftarrow{T_1} \bar{X}_2 \xleftarrow{T_2} \bar{X}_3 \xleftarrow{T_3} \bar{X}_4 \xleftarrow{T_4} \ldots.$$

Letting $\{||\cdot||_l: ||\cdot||_l \leqq ||\cdot||_{l+1}\}$ be a family of seminorms which determines the Fréchet topology of X, suppose $\{l(n)\}$ has been chosen to satisfy the following:

i) $l(n) \geqq t(n)$

https://doi.org/10.4153/CJM-1978-047-x Published online by Cambridge University Press

ii) *there is* M_n *such that*

$$||T_{n-1}T_{n-2}\dots T_m x||_{l(n)} \leqq M_n||x||_{l(n)}$$

for $m = 1, 2, \dots, n-1$ *for all* $x \in X$. *Then if*

$$\overline{T_n X_{n+1}}^{l(n)} \supseteq X_n \quad \textit{for all } n,$$

we also have that

$$\overline{\pi_n \mathbf{P}}^{l(n)} \supseteq X_n \quad \textit{for all } n,$$

where $\pi_n : \mathbf{P} \to \bar{X}_n$ *is the canonical projection into* $\bar{X}_n$.

Proof. We note that such a sequence $\{l(n)\}$ can always be chosen, since any finite set of operators in $B(X)$ is equi-continuous. Fix n and let $\epsilon > 0$. Let $x_n \in X_n$. Choose $x_{n+1} \in X_{n+1}$ so that $||T_n x_{n+1} - x_n||_{l(n)} < \epsilon/2^n M_n$. Continue inductively choosing $x_{n+p+1} \in X_{n+p+1}$ so that

$$||T_{n+p}x_{n+p+1} - x_{n+p}||_{l(n+p)} < \epsilon/2^{n+p}M_{n+p}, \quad p = 1, 2, 3, \dots.$$

Given any non-negative integer j, observe that

$$\begin{aligned}
&\sum_{p=j+1}^{\infty} ||T_{n+p}\dots T_{n+j}x_{n+p+1} - T_{n+p-1}\dots T_{n+j}x_{n+p}||_{l(n+p)} \\
&\quad = \sum_{p=j+1}^{\infty} ||(T_{n+p-1}\dots T_{n+j})(T_{n+p}x_{n+p+1} - x_{n+p})||_{l(n+p)} \\
&\quad \leqq \sum_{p=j+i}^{\infty} M_{n+p}||T_{n+p}x_{n+p+1} - x_{n+p}||_{l(n+p)} \\
&\quad \leqq \sum_{p=j+1}^{\infty} \epsilon/2^{n+p}
\end{aligned}$$

which converges. Since $\{||\cdot||_{l(n+p)}\}$ also determine the Fréchet topology of X we have that $\{T_{n+p}T_{n+p-1}\dots T_{n+j}x_{n+p+1}\}_{p=j+1}^{\infty}$ is Cauchy in X and hence there is $s_{n+j} \in \bar{X}_{n+j}$ such that

$$T_{n+p}\dots T_{n+j}x_{n+p+1} \to s_{n+j} \quad \text{as } p \to \infty, j = 0, 1, 2, \dots.$$

But if we define $s_{n-1} = T_{n-1}s_n$, $s_{n-2} = T_{n-2}s_{n-1}\dots s_1 = T_1 s_2$, then it is clear that $(s_i) \in \mathbf{P}$ and $\pi_n(s_i) = s_n$. We also have that

$$\begin{aligned}
&||T_{n+m}\dots T_n x_{n+m+1} - x_n||_{l(n)} \\
&\leqq ||T_n x_{n+1} - x_n||_{l(n)} + \sum_{p=1}^{m} ||T_{n+p}\dots T_n x_{n+p+1} - T_{n+p-1}\dots T_n x_{n+p}||_{l(n} \\
&\leqq ||T_n x_{n+1} - x_n||_{l(n)} + \sum_{p=1}^{m} ||(T_{n+p-1}\dots T_n)(T_{n+p}x_{n+p+1} - x_{n+p})||_{l(n+p)} \\
&\leqq \sum_{p=0}^{m} M_{n+p}||T_{n+p}x_{n+p+1} - x_{n+p}||_{l(n+p)} \leqq \sum_{p=0}^{m} \epsilon/2^{n+p} \leqq \epsilon.
\end{aligned}$$

Since $T_{n+m}T_{n+m-1}\ldots T_n x_{n+m+1} \to s_n$ as $m \to \infty$ this implies that $\|s_n - x_n\|_{t(n)} \leqq \epsilon$. Thus we have shown that $\overline{\pi_n \mathbf{P}}^{t(n)} \supseteq X_n$ since ϵ was arbitrary.

For technical reasons we will now add the hypothesis that each R_n appears an infinite number of times in the sequence. If a subspace Z is invariant under each R_n, there is thus a largest algebraic subspace D of Z such that $R_n D = D$, all n. We shall generally denote this D by $D(\{R_n|Z: n = 1, 2, 3\ldots\})$. Note $\mathscr{S}(S)$ is closed and invariant under each R_n. We have the following lemma.

LEMMA 1.2. *Under the same hypotheses as Lemma* 1.1a, *further suppose that* $\{T_n\}$ *and* $\{R_n\}$ *are commuting sequences of continuous linear operators on X and Y respectively. If each* R_n *appears an infinite number of times in* $\{R_n\}$, *then for each k there is* $n(k)$ *satisfying*

$$\overline{R_m R_{m-1}\ldots R_1 \mathscr{S}(S)}^k = \overline{R_{n(k)}\ldots R_2 R_1 \mathscr{S}(S)}^k$$

$$= \overline{D(\{R_n|\mathscr{S}(S) : n = 1, 2, 3\ldots\})}^k, \quad m \geqq n(k).$$

Proof. Let k be given. Let $n(k)$ be as in Lemma 1.1a. Since the R_n's commute we have that

$$\overline{R_m R_{m-1}\ldots R_1 \mathscr{S}(S)}^k = \overline{R_{n(k)}\ldots R_2 R_1 \mathscr{S}(S)}^k,$$

for all $m \geqq n(k)$. We construct two sequences as follows. Let $l(1) = t(1) = k$. It is clear that $\|x\|_{t(1)} \leqq \|x\|_{l(1)}$ for all $x \in Y$. Let $t(p + 1) = k + p$, $p = 1, 2, 3\ldots$. Choose $l(p + 1)$, $p = 1, 2, 3, \ldots$, to satisfy the following:

i) $l(p + 1) > l(p)$, for all p.

ii) $l(p + 1) \geqq t(p + 1)$, for all p.

iii) there is M_{p+1} such that $\|R_p R_{p-1}\ldots R_m x\|_{t(p+1)} \leqq M_{p+1}\|x\|_{l(p+1)}$,

for all $x \in Y$ and $m = 1, 2, 3\ldots$ p.

The $l(p + 1)$'s are chosen inductively and iii) is if course possible since $\{R_n\} \subseteq B(Y)$. Again by Lemma 1.1a we may choose $n(l(p + 1))$ strictly increasing in p so that

$$\overline{R_m R_{m-1}\ldots R_1 \mathscr{S}(S)}^{l(p+1)} = \overline{R_{n(l(p+1))}\ldots R_2 R_1 \mathscr{S}(S)}^{l(p+1)}$$

for all $m \geqq n(l(p + 1))$. Let $X_1 = R_{n(l(1))}\ldots R_2 R_1 \mathscr{S}(S)$, and let $X_{p+1} = R_{n(l(p+1))}\ldots R_2 R_1 \mathscr{S}(S)$, $p = 1, 2, 3\ldots$. Since $n(l(p + 1))$ are increasing and $R_p \mathscr{S}(S) \subseteq \mathscr{S}(S)$, it is clear that $R_p X_{p+1} \subseteq X_p$, for all p. Given a fixed p, there is some $m > n(l(p + 1))$ such that $R_p = R_m$. Hence

$$\begin{aligned}\overline{R_p X_{p+1}}^{l(p)} &= \overline{R_m R_{n(l(p+1))}\ldots R_2 R_1 \mathscr{S}(S)}^{l(p)}\\ &\supseteq \overline{R_m R_{m-1}\ldots R_{n(l(p+1))}\ldots R_{n(l(p))}\ldots R_2 R_1 \mathscr{S}(S)}^{l(p)}\\ &= \overline{R_{n(l(p))}\ldots R_2 R_1 \mathscr{S}(S)}^{l(p)}\\ &\supseteq X_p,\end{aligned}$$

for all p. Let $\mathbf{P}$ be the following projective limit:

$$\bar{X}_1 \xleftarrow{R_1} \bar{X}_2 \xleftarrow{R_2} \bar{X}_3 \xleftarrow{R_3} \bar{X}_4 \dots .$$

Let $\pi_p : \mathbf{P} \to \bar{X}_p$ be canonical for each n. By Lemma 1.1d, it follows that $\overline{\pi_p \mathbf{P}}^{t(p)} \supseteq X_p$, for all p. In particular $\overline{\pi_1 \mathbf{P}}^k \supseteq R_{n(k)} \dots R_2 R_1 \mathscr{S}(S)$ since $t(1) = k$ and $n(l(1)) = n(k)$. Note $\pi_1(\mathbf{P})$ is divisible by all the R_n's and $\pi_1(\mathbf{P}) \subseteq \bar{X}_1 \subseteq \mathscr{S}(S)$, as $\mathscr{S}(S)$ is closed. Thus

$$\pi_1(\mathbf{P}) \subseteq D(\{R_n | \mathscr{S}(S) : n = 1, 2, 3 \dots\}).$$

Hence

$$\overline{D(\{R_n | \mathscr{S}(S) : n = 1, 2, 3 \dots\})}^k \supseteq \overline{R_{n(k)} \dots R_2 R_1 \mathscr{S}(S)}^k.$$

Since the reverse containment is trivial, the lemma follows.

We remark that in the proof we actually showed the somewhat stronger result that if

$$\overline{R_m R_{m-1} \dots R_1 \mathscr{S}(S)}^k = \overline{R_n R_{n-1} \dots R_1 \mathscr{S}(S)}^k,$$

for all $m \geqq n$, then

$$\overline{R_n R_{n-1} \dots R_1 \mathscr{S}(S)}^k = \overline{D(\{R_n | \mathscr{S}(S) : n = 1, 2, 3 \dots\})}^k.$$

The above is useful most often in the form of the following corollary, in which $D(\{R_n : n = 1, 2, 3, \dots\})$ is of course the largest subspace D in Y such that $R_n D = D$, all n.

COROLLARY 1.3. *Under the same hypotheses as Lemma* 1.1a, *further suppose that* $\{T_n\}$ *and* $\{R_n\}$ *are commuting sequences of continuous linear operators on* X *and* Y *respectively. Suppose each* R_n *appears an infinite number of times in* $\{R_n\}$ *and that* $D(\{R_n : n = 1, 2, 3 \dots\}) \equiv (0)$. *Let* Q_k *be the canonical quotient map of* Y *with null space* $\{y \in Y : \|y\|_k = 0\}$. *Then for each* k *there is* $n(k)$ *such that* $Q_k R_{n(k)} \dots R_2 R_1 S$ *is continuous.*

Proof. Observe that

$$D(\{R_n | \mathscr{S}(S) : n = 1, 2, 3, \dots\}) \subseteq D(\{R_n : n = 1, 2, 3 \dots\}) \equiv (0).$$

Hence Lemma 1.2 implies there is $n(k)$ such that

$$Q_k R_{n(k)} \dots R_2 R_1 \mathscr{S}(S) \equiv (0),$$

and thus $Q_k R_{n(k)} \dots R_2 R_1 S$ is continuous.

In the next section we apply the above results to questions of automatic continuity.

2. We next develop some sufficient conditions for automatic continuity of a linear function S from X to Y. We could continue to proceed in the generality of the previous section, considering commuting sequences $\{T_n\} \subseteq B(X)$ and $\{R_n\} \subseteq B(Y)$ where X is an F-space, Y a Fréchet space and $ST_n - R_nS \in B(X, Y)$ all n. However, in this setting the hypotheses get rather technical and instead we choose to specialize to the following case. Let both X and Y be Fréchet spaces. Let $T \in B(X)$ and $R \in B(Y)$ with $ST - RS \in B(X, Y)$. If $\sigma(R)$ is countable it may be reformed into a sequence $\{\lambda_n\}$ such that each element of $\sigma(R)$ appears an infinite number of times. It is elementary then that if we let $T_n = T - \lambda_n$ and $R_n = R - \lambda_n$, we have $ST_n - R_nS \in B(X, Y)$ for all n. Hence we can apply our previous results in Section 1. We need some definitions.

Definition 2.1. We say a complex number λ is in the *generalized point spectrum of* R and we will write $\lambda \in \sigma_{gp}(R)$ provided the following hold:

i) $\lambda \in \sigma(R) \equiv \{\mu: (R - \mu) \text{ is not bijective}\}$.

ii) There is a non-zero vector $y \in Y$ such that for fixed k, $||(R - \lambda)^n y||_k = 0$ for all but finitely many n.

Some observations are in order. Note that ii) can occur without i) occurring. For example, let $Y = \prod_{n=-\infty}^{\infty} \mathbf{C}$, the Fréchet direct product of $\mathbf{C}$. Let R be the left shift. Then R^{-1} is the right shift so $0 \notin \sigma(R)$, but $y = (\ldots 0, 0, 0, 1, 0, 0, 0 \ldots)$ satisfies ii) with $\lambda = 0$. We also note that

$$\sigma_p(R) = \{\lambda: (R - \lambda)y = 0 \text{ for some } y \neq 0\} \subseteq \sigma_{gp}(R).$$

If Y is actually a Banach space, $\sigma_p(R) = \sigma_{gp}(R)$. Finally if $\lambda \in \sigma_{gp}(R)$ and y is as above then $\sum_{n=1}^{\infty} \alpha_n (R - \lambda)^n y$ converges in Y for every sequence $\{\alpha_n\}$ of complex numbers.

Definition 2.2. We say that *Y has no non-trivial R divisible subspaces* if whenever D is an algebraic subspace with $(R - \lambda)D = D$ all $\lambda \in \mathbf{C}$, then $D \equiv (0)$. Equivalently,

$$D(\{R - \lambda: \lambda \in \mathbf{C}\}) \equiv (0).$$

However, if $\sigma(R)$ is countable, it is easily seen that $\sigma(R) = \{\lambda_i\}$ implies

$$D(\{R - \lambda_i: n = 1, 2, 3 \ldots\}) = D(\{R - \lambda: \lambda \in \mathbf{C}\}).$$

Definition 2.3. Let $\lambda \in \mathbf{C}$. We say that $(T - \lambda)X$ *has finite codimension in* X provided $X/(T - \lambda)X$ is a finite dimensional vector space over $\mathbf{C}$. We shall use the following notation

$$[X: (T - \lambda)X] < +\infty.$$

This implies $(T - \lambda)X$ is closed in X since $X \cong (T - \lambda)X \oplus F$ as vector spaces for some finite dimensional subspace F. The product topology of the range space topology on $(T - \lambda)X$ and the relative topology on F is also

Fréchet and stronger than the original topology on X. Hence, the open mapping theorem shows they are homeomorphic which forces $(T - \lambda)X$ to be closed. Also $(T - \lambda)$ is then an open map of X onto $(T - \lambda)X$.

THEOREM 2.4. *Let X and Y be Fréchet spaces. Let $T \in B(X)$ and $R \in B(Y)$ with $\sigma(R)$ countable. Suppose S is a linear function from X to Y satisfying $(ST - RS) \in B(X, Y)$. If*

i) *Y has no non-trivial R divisible subspaces, and*

ii) *$\lambda \in \sigma_{gp}(R)$ implies $[X: (T - \lambda)X] < +\infty$,*

then S is continuous.

Proof. Let $\mathcal{S}(S)$ be the separating subspace of S. Let $\lambda \in \sigma(R)$. Reform $\sigma(R) \sim \{\lambda\}$ into a sequence $\{\mu_n\}$ such that each element appears an infinite number of times. We have two cases

Case 1) $D(\{(R - \mu_n)|\mathcal{S}(S): n = 1, 2 \ldots\}) \equiv (0)$.
Case 2) $D(\{(R - \mu_n)|\mathcal{S}(S): n = 1, 2 \ldots\}) \not\equiv (0)$.

Suppose the first case occurs. Lemma 1.2 implies that for each k there exists $n(k)$ such that $m \geqq n(k)$ implies

$$||(R - \mu_m) \ldots (R - \mu_1)s||_k = 0,$$

for all $s \in \mathcal{S}(S)$. Let $p_n(x) = (x - \mu_n) \ldots (x - \mu_1) \in \mathbf{C}[x]$. If $\{\alpha_n\}$ is any sequence in $\mathbf{C}$ and $y \in \mathcal{S}(S)$ we see that $\sum_{n=1}^{\infty} \alpha_n p_n(R)y$ always converges in Y. Let $\alpha_0 = -(\mu_1 - \lambda)^{-1}, \alpha_1 = -(\mu_2 - \lambda)^{-1}\alpha_0, \ldots, \alpha_n = -(\mu_{n+1} - \lambda)^{-1}\alpha_{n-1} \ldots$ for all n. Let $y \in \mathcal{S}(S)$. Then

$$\begin{aligned}(R - \lambda)[\alpha_0 y + \sum_{n=1}^{\infty} \alpha_n p_n(R)y] &= [(R - \mu_1) + (\mu_1 - \lambda)]\alpha_0 y \\ &\quad + \sum_{n=1}^{\infty} [(R - \mu_{n+1}) + (\mu_{n+1} - \lambda)]\alpha_n p_n(R)y \\ &= -y + \alpha_0(R - \mu_1)y + \sum_{n=1}^{\infty} \alpha_n p_{n+1}(R)y - \alpha_{n-1}p_n(R)y \\ &= -y.\end{aligned}$$

Thus $(R - \lambda)\mathcal{S}(S) = \mathcal{S}(S)$ in Case 1). Suppose now that Case 2) occurs. Let y be any non-zero element of $D(\{(R - \mu_n)|\mathcal{S}(S): n = 1, 2, 3 \ldots\})$. Now Y has no non-trivial divisible subspaces and hence $D(\{R - \mu: \mu \in \sigma(R)\}) \equiv (0)$. Thus

$$D(\{(R - \mu_n)|\mathcal{S}(S): n = 1, 2, 3 \ldots\} \cup \{(R - \lambda)|\mathcal{S}(S)\}) \equiv (0).$$

Let k be fixed. Applying lemma 1.2 we see that there is a polynomial p with all roots from $\{\mu_n\}$ and a positive integer m such that $||(R - \lambda)^m p(R)s||_k = 0$ for all $s \in \mathcal{S}(S)$. Since $(R - \lambda)\mathcal{S}(S) \subseteq \mathcal{S}(S)$ this implies that

https://doi.org/10.4153/CJM-1978-047-x Published online by Cambridge University Press

$||(R - \lambda)^l p(R)s||_k = 0$ for all $s \in \mathscr{S}(S)$ and $l \geqq m$. But $y = p(R)z$ for some $z \in \mathscr{S}(S)$ and hence $||(R - \lambda)^l p(R)z||_k = ||(R - \lambda)^l y||_k = 0$ for all $l \geqq m$. Since k was arbitrary at the start and $y \neq 0$ in $D(\{(R - \mu_n)|\mathscr{S}(S): n = 1, 2, 3 \ldots\})$, this implies that $\lambda \in \sigma_{gp}(R)$. Hence if Case 2) occurs we have that $[X: (T - \lambda)X] < +\infty$. So for each $\lambda \in \sigma(R)$ we either have that $(R - \lambda)\mathscr{S}(S) = \mathscr{S}(S)$ or $[X: (T - \lambda)X] < +\infty$. Form $\sigma(R)$ into a sequence $\{\lambda_n\}$ in which each element occurs an infinite number of times. Let $Q_k: Y \to Y/\{y \in Y: ||y||_k = 0\}$ be canonical. Since $D(\{R - \lambda_n: n = 1, 2, 3 \ldots\}) \equiv (0)$, Corollary 1.3 implies that for each k there is $n(k)$ such that $Q_k(R - \lambda_{n(k)}) \ldots (R - \lambda_2)(R - \lambda_1)\mathscr{S}(S) \equiv (0)$. If $[X: (T - \lambda_i)X] = +\infty$ then $(R - \lambda_i)\mathscr{S}(S) = \mathscr{S}(S)$ and this term may be deleted. Hence there is a polynomial q_k such that $Q_k q_k(R)\mathscr{S}(S) \equiv (0)$ and λ a root of q_k implies $[X: (T - \lambda)X] < +\infty$. Hence $[X: q_k(T)X] < +\infty$ also. Now $Q_k q_k(R)S$ is continuous and equals $Q_k S q_k(T)$ plus some continuous operator. Hence $Q_k S q_k(T)$ is also continuous.

But $q_k(T)X$ is closed, hence $Q_k S$ is continuous on $q_k(T)X$ by the open mapping theorem. But $q_k(T)X$ has finite codimension, and so $Q_k S$ is continuous on all of X. Thus

$$\mathscr{S}(S) \subseteq \{y \in Y: ||y||_k = 0\},$$

for all k. Hence $\mathscr{S}(S) \equiv (0)$ and S is continuous.

We next have some remarks on the necessity of these conditions which have arisen. We first note that A. Sinclair showed [**5**, Theorem 1.2] that if T is not algebraic and R has a non-trivial divisible subspace then there is a discontinuous S such that $ST = RS$. His proof was stated for Banach spaces but the generalization to Fréchet spaces is immediate. If there is $\lambda \in \sigma_p(R)$ and $[X: (T - \lambda)X] = +\infty$, B. Johnson and A. Sinclair showed that again there is a discontinuous S such that $ST = RS$ [**3**, Lemma 2.1]. We generalize this to Fréchet spaces in the following.

LEMMA 2.5. *Let X and Y be Fréchet spaces. Let $T \in B(X)$ and $R \in B(Y)$. Suppose $\lambda \in \sigma_{gp}(R) \sim \sigma_p(T)$ and $[X: (T - \lambda)X] = +\infty$. Then there exists a discontinuous linear function S from X to Y satisfying $ST = RS$.*

Proof. Let y be a non-zero element of Y such that $||(R - \lambda)^n y||_k = 0$ for all but finitely many n when k is fixed. Since $y \neq 0$ there is a first semi-norm such that $||y||_k \neq 0$. Without loss of generality we may assume $||y||_1 \neq 0$. Then there is an N such that $||(R - \lambda)^N y||_1 \neq 0$ but $||(R - \lambda)^n y||_1 = 0$ for $n > N$. Pick a discontinuous linear functional f_0 from X to $\mathbf{C}$ satisfying $f_0(T - \lambda)X \equiv 0$. Define $f_1(T - \lambda)x = f_0 x$ on $(T - \lambda)X$ and extend f_1 to all of X in any way so long as it remains linear. In general after f_n has been chosen, define $f_{n+1}(T - \lambda)x = f_n x$ on $(T - \lambda)X$ and extend to a linear functional on X. Since $\lambda \notin \sigma_p(T)$ it is elementary that the f_n are well defined (discontinuous) linear functionals on X. Define

$$Sx \equiv \sum_{n=0}^{\infty} f_n(x)(R - \lambda)^n y, \quad x \in X,$$

which converges by our previous remarks. It is easily verified that $S(T - \lambda)x = (R - \lambda)Sx$ and hence $ST = RS$. Pick $x_n \to 0$ in X with $|f_0(x_n)| > 1$. Then

$$||(R - \lambda)^N Sx_n||_1 = ||(R - \lambda)^N f_0(x_n) y||_1 > ||(R - \lambda)^N y||_1 \neq 0.$$

Hence $(R - \lambda)^N S$ is discontinuous so S must be also. Observe that this procedure is valid when X is more generally an F-space.

We now specialize to the case where $X = Y$ and $T = R$. Hence we are interested in the case where S commutes with T. If X is a Banach space and $\sigma(T)$ is countable then A. Sinclair's results show that every commuting S is continuous if and only if

i) X has no non-trivial T-divisible subspace, and
ii) $\lambda \in \sigma_p(T)$ implies $[X: (T - \lambda)X] < +\infty$.

This follows from Sinclair's more general theorem [**5**, Theorem 2.2] when $T = R$, since if T is algebraic, there cannot be any non-trivial T divisible subspace. We generalize the above to Fréchet spaces and to the case when $ST - TS \in B(X)$.

THEOREM 2.6. *Let X be a Fréchet space and let $T \in B(X)$ with $\sigma(T)$ countable. Then every linear function S on X, such that $ST - TS \in B(X)$, is continuous if and only if*

i) *X has no non-trivial T-divisible subspace, and*
ii) *$\lambda \in \sigma_{gp}(T)$ implies $[X: (T - \lambda)X] < +\infty$.*

Proof. If i) and ii) hold, Theorem 2.4 implies that such an S must be continuous. If i) fails then T cannot be algebraic and a generalization of [**5**, Theorem 1.2] as noted above implies a discontinuous commuting S exists. If ii) fails we have two cases. If $\lambda \notin \sigma_p(T)$, Lemma 2.5 implies there is a discontinuous commuting S. If $\lambda \in \sigma_p(T)$ and $[X: (T - \lambda)X] = +\infty$, exactly as in [**3**, Lemma 2.1] there is a discontinuous commuting S. Hence in any case, if ii) fails there exist discontinuous commuting functions S, so certainly $ST - TS \in B(X)$. This proves the theorem.

REFERENCES

1. B. E. Johnson, *Continuity of linear operators commuting with continuous linear operators*, Trans. Amer. Math. Soc. *128* (1967), 88–102.

2. ——— *Continuity of linear operators commuting with quasi-nilpotent operators*, Indiana Math. J. *20* (1971), 913–915.

3. B. E. Johnson and A. M. Sinclair, *Continuity of linear operators commuting with continuous linear operators, II*, Trans. Amer. Math. Soc. *146* (1969), 533–540.

4. K. B. Laursen, *Some remarks on automatic continuity*, Spaces of analytic functions, Lecture Notes in Mathematics (Springer-Verlag, Berlin, 1976), 96–108.

5. A. M. Sinclair, *A discontinuous intertwining operator*, Trans. Amer. Math. Soc. *188* (1974), 259–267.

6. A. M. Sinclair and N. P. Jewell, *Epimorphisms and derivations on $L^1[0, 1]$ are continuous*, Bull. London Math. Soc. *8* (1976), 135–139.

University of Texas at Austin,
Austin, Texas