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ON THE POLE ORDER AND HODGE FILTRATIONS OF ISOLATED HYPERSURFACE SINGULARITIES

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ABSTRACT. Unlike for a smooth projective hypersurface, for an isolated hypersurface singularity, the pole order and Hodge filtrations do not in general coincide. This note studies the difference between the two.

Introduction. Let $V \subset \mathbb{P}^{n+1} = P^{n+1}(\mathbb{C})$ be a smooth hypersurface. According to Grothendieck [G], $H^*(\mathbb{P}^{n+1} \setminus V)$ is the cohomology of the complex of rational forms on $\mathbb{P}^{n+1}$ with poles along V. Furthermore, Griffiths [Gr] proved that the Hodge filtration on $H^{n+1}(\mathbb{P}^{n+1} \setminus V)$ is induced by the pole order filtration on this complex. Karpishpan [K] and Dimca [D] have looked for an analogous result for isolated hypersurface singularities.

Suppose $f \in \mathbb{C}[z_0, \ldots, z_n]$ has an isolated critical point at 0 with $f(0) = 0$. Choosing $\eta \ll \varepsilon \ll 1$, set

$$S = \left\{t \in \mathbb{C} \;\middle|\; |t| < \eta\right\}, \quad \left\{X = z \in \mathbb{C}^{n+1} \;\middle|\; |z| < \varepsilon, |f(z)| < \eta\right\},$$
$$S^* = S \setminus \{0\}, \quad X_0 = f^{-1}(0) \cap X, \quad X^* = X \setminus X_0 = f^{-1}(S^*).$$

Then $H^k(X^*) = 0$ unless $k = 0, n$, or $n+1$. In [K], 0.3 and [D], 2.5 it is shown that the Hodge filtration is included in the pole order filtration in $H^{n+1}(X^*)$ and vice versa in $H^n(X^*)$. In general, they do not coincide.

To make this more precise, recall that $f\colon X^* \longrightarrow S^*$ is the Milnor fibration, with generic fibre X_∞ (*cf.* [Sch-St]). Steenbrink [St] has shown that there is an exact sequence of mixed Hodge structures

$$(1) \qquad 0 \longrightarrow H^n(X^*) \xrightarrow{\text{co}} H^n(X_\infty)_1(-1) \xrightarrow{\text{sp}} H^{n+1}(X^*) \longrightarrow 0$$

Here N is the logarithm of the unipotent part of the monodromy and $H^n(X_\infty)_1$ is the eigenspace of the semisimple part belonging to the eigenvalue 1. Let

$$\Omega_f^\bullet = \Omega_{X,0}^\bullet(*X_0)$$

and let

$$P^k\Omega_f^l = \Omega_{X,0}^l\big((l-k+1)X_0\big)$$

be its pole order filtration. Grothendieck's result also holds for $H^\bullet(X^*)$:

$$H^\bullet(X^*) = H^\bullet(\Omega_f^\bullet)$$

and $P^\bullet$ induces a filtration on $H^\bullet(X^*)$ (*cf.* [K], 2.1). The results of Karpishpan ((i) and (ii)) and Dimca (ii) can be described as follows:

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THEOREM. *(i)* co: $\big(H^n(X^*), P^\bullet\big) \longrightarrow \big(H^n(X_\infty)_1, F^\bullet\big)$
(ii) sp: $\big(H^n(X_\infty)_1, F(-1)^\bullet\big) \longrightarrow \big(H^{n+1}(X^*), P^\bullet\big)$.

The purpose of this note is to give conditions which explain when these mappings will fail to be strict. The formalism of [Sch-St] is used. The results and examples can easily be translated into Varchenko's formalism of asymptotic expansions (*cf.* [AGV]).

§**1.** We first recall the result of Karpishpan ([K], 2.2–2.6) and make the mappings in it more explicit. From the short exact sequence

$$0 \longrightarrow \Omega^\bullet_{X\times S/S,0}(*\,\mathrm{graph} f) \xrightarrow{\times t} \Omega^\bullet_{X\times S/S,0}(*\,\mathrm{graph} f) \longrightarrow \Omega^\bullet_f \longrightarrow 0$$

he obtains the sequence ((**) in [K])

$$0 \longrightarrow H^n(\Omega^\bullet_f) \xrightarrow{\mathrm{cob}} \mathcal{H} \xrightarrow{t} \mathcal{H} \xrightarrow{\mathrm{sp}} H^{n+1}(\Omega^\bullet_f) \longrightarrow 0 \tag{2}$$

where

$$\mathcal{H} = H^{n+1}\big(\Omega^\bullet_{X\times S/S,0}(*\,\mathrm{graph} f)\big)$$

is the stalk at 0 of the Gauss-Manin system of f ([Sch-St], §3, [P], p. 157). The coboundary map cob is given as follows: let

$$\frac{\varphi}{f^k} \in \Omega^n_f$$

be closed, *i.e.*

$$0 = d\Big(\frac{\varphi}{f^k}\Big) = \frac{fd\varphi - kdf \wedge \varphi}{f^{k+1}},$$

or equivalently,

$$fd\varphi = kdf \wedge \varphi. \tag{3}$$

Lift $\frac{\varphi}{f^k}$ to $\frac{\varphi}{(f-t)^k} \in \Omega^n_{X\times S/S,0}(*\,\mathrm{graph} f)$. Then

$$d\Big(\frac{\varphi}{(f-t)^k}\Big) = \frac{(f-t)d\varphi - kdf \wedge \varphi}{(f-t)^{k+1}} = \frac{-td\varphi}{(f-t)^{k+1}}$$

because of (3). Therefore

$$\mathrm{cob}\Big[\frac{\varphi}{f^k}\Big] = -\Big[\frac{d\varphi}{(f-t)^{k+1}}\Big] = -\frac{1}{k!}\partial_t^k[d\varphi] \in \mathcal{H}$$

where $[d\varphi] \in H''$ (*cf.* [Sch-St], §3). Thus

$$\mathrm{cob}\, P^{n-k+1} \subseteq F^{n-k}.$$

The specialization map sp is just

$$\mathrm{sp}\Big[\frac{\omega}{(f-t)^{k+1}}\Big] = \Big[\frac{\omega}{f^{k+1}}\Big]$$

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for $[\frac{\omega}{(f-t)^{k+1}}] \in \mathcal{H}$. So

$$\operatorname{sp} F^{n-k} \subseteq P^{n-k+1}.$$

Now in order to compare (2) with (1) it helps to change (2) slightly as in [K], 2.5–2.6. Since ∂_t is invertible on $\mathcal{H}$ (3.5 in [Sch-St]), we can replace cob by co $:= \partial_t^{-1} \circ$ cob and t by $-2\pi i t\partial_t$, to obtain the sequence

$$0 \longrightarrow H^n(\Omega_f^\bullet) \xrightarrow{\text{co}} \mathcal{H} \xrightarrow{-2\pi i t\partial_t} \mathcal{H} \xrightarrow{\text{sp}} H^{n+1}(\Omega_f^\bullet) \longrightarrow 0. \tag{4}$$

Explicitly,

$$\operatorname{co}\left[\frac{\varphi}{f^k}\right] = -\frac{1}{k!}\partial_t^{k-1}[d\varphi]$$

so that

$$\operatorname{co} P^{n-k+1} \subseteq F^{n-k+1}.$$

With the above formulas for co and sp it is easy to verify the exactness of (4) directly. In fact exactness at the second term is equivalent to (3).

To get (1) out of (4), we replace $\mathcal{H}$ by $\operatorname{Gr}_V^0 \mathcal{H} = V^0/V^{>0} \cong C_0$ (*cf.* §4, [Sch-St] for definitions) as in [K]:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H^n(\Omega_f^\bullet) & \xrightarrow{\text{co}} & \operatorname{Gr}_V^0 \mathcal{H} & \xrightarrow{-2\pi i t\partial_t} & \operatorname{Gr}_V^0 \mathcal{H} & \xrightarrow{\text{sp}} & H^{n+1}(\Omega_f^\bullet) & \longrightarrow & 0 \\
 & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
0 & \longrightarrow & H^n(X^*) & \xrightarrow{\text{co}} & H^n(X_\infty)_1 & \xrightarrow{N} & H^n(X_\infty)_1 & \xrightarrow{\text{sp}} & H^{n+1}(X^*) & \longrightarrow & 0.
\end{array}$$

§**2.** We now want to study when co fails to be strict. Suppose

$$\operatorname{co}\left[\frac{\varphi}{f^k}\right] = -\frac{1}{k!}\partial_t^{k-1}[d\varphi] = -\frac{1}{k!}\partial_t^k[df \wedge \varphi] \in F^{n-k+j}(\operatorname{Gr}_V^0 \mathcal{H})$$

for some j, $1 \leq j \leq k$. So

$$\partial_t^k[df \wedge \varphi] \equiv \partial_t^{k-j}[\psi] \pmod{V^{>0}}$$

for some $[\psi] \in H''$. Then

$$\partial_t^j[df \wedge \varphi] \equiv [\psi] \pmod{V^{>k-j}}$$

and

$$t^j\partial_t^j[df \wedge \varphi] \equiv [f^j\psi] \pmod{V^{>k}}.$$

But from (3)

$$t^j\partial_t^j[df \wedge \varphi] = t\partial_t(t\partial_t - 1)\cdots(t\partial_t - j + 1)[df \wedge \varphi] = c_j[df \wedge \varphi]$$

where $c_j = k\cdots(k-j+1)$. So we have

$$[df \wedge \varphi] \equiv \frac{1}{c_j}[f^j\psi] \pmod{V^{>k}}.$$

Using the identity $\partial_t^j t^j = \partial_t t \cdots (\partial_t t + j - 1)$ it is easy to reverse this argument, which gives us

THEOREM 1. $\operatorname{co}[\frac{\varphi}{f^k}] \in F^{n-k+j}(\mathrm{Gr}_V^0 \mathcal{H})$ *if and only if*

$$[df \wedge \varphi] \equiv [f^j \omega] \pmod{V^{>k}}, \text{ in } H''$$

for some $[\omega] \in H''$.

REMARK. $j = 1$ always holds: $[df \wedge \varphi] \equiv [f\omega]$, for some ω, follows from (3).

It is not easy to compute $H^n(\Omega_f^\bullet)$ directly (*cf.* [D]). Calculating $\ker(t\partial_t) \subset \mathrm{Gr}_V^0 \mathcal{H}$ is easy. But once a class $[\omega] \in \ker(t\partial_t)$ has been found, it must be lifted to an element $\omega + \psi \in C_0$, where $\psi \in V^{>0}$. The formula for co (or §2.6 in [K]) makes it clear that

$$\left(H^n(\Omega_f^\bullet), P^\bullet\right) \equiv \left(\ker(t\partial_t) \subset C_0, F^\bullet\right)$$

So suppose $\omega \in F^{n-k+1}$. Can one put a condition on ω, depending on $j \geq 1$, so that $\omega + \psi \in F^{n-(k+j)+1}$? One possibility is the following:

Suppose

$$\partial_t^{k-1}[\omega] \in \ker(t\partial_t) \subset \mathrm{Gr}_V^0 \mathcal{H},$$

with $[\omega] \in H''$, and $[f^{j+1}\omega] \in H'$. Then

$$\partial_t^{k-1}[\omega] \in \operatorname{co} P^{n-(k+j)+1}.$$

The condition

$$[f^{j+1}\omega] \equiv [df \wedge \varphi] \pmod{V^{>k+j}}$$

for some $[df \wedge \varphi] \in H'$, which Theorem 1 would suggest, is not sufficient as the example below shows. In fact we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}_V^{k-1}(H''/H') & \xrightarrow{f} & \mathrm{Gr}_V^k(H''/H') \\ \partial_t^{k-1} \downarrow \cong & & \partial_t^k \downarrow \cong \\ \mathrm{Gr}_F^{n-k+1} \mathrm{Gr}_V^0 \mathcal{H} & \xrightarrow{t\partial_t} & \mathrm{Gr}_F^{n-k} \mathrm{Gr}_V^0 \mathcal{H} \end{array}$$

as in [Sch-St], §7.1. $[\omega]$ defines a non-zero element of $\mathrm{Gr}_V^{k-1}(H''/H')$ and since $(t\partial_t)\partial_t^{k-1}[\omega] = 0$,

$$[f\omega] \equiv [df \wedge \varphi] \pmod{V^{>k+1}}$$

for some $[df \wedge \varphi] \in H'$. So such congruence conditions are always satisfied.

§3. We now derive a condition for $\operatorname{sp}[\frac{\omega}{(f-t)^{k+1}}] \in P^{n-k+j}, j \geq 1$.

THEOREM 2. *For* $\omega \in H''$, $\operatorname{sp}[\frac{\omega}{(f-t)^{k+1}}] = \frac{\omega}{f^{k+1}} \in P^{n-k+j}$, *for* $j \geq 1$, *if and only if*

$$\partial_t^j[\omega] \equiv [\psi] \pmod{t\mathcal{H}}$$

for some $[\psi] \in H''$.

PROOF. Suppose

$$\left[\frac{\omega}{f^{k+1}}\right] = \left[\frac{\psi}{f^{k+1-j}}\right]$$

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in $H^{n+1}(\Omega_f^\bullet)$.This is equivalent to

$$\frac{1}{k!}\partial_t^k[\omega] \equiv \frac{1}{(k-j)!}\partial_t^{k-j}[\psi] \pmod{t\mathcal{H}}$$

or

$$\partial_t^j[\omega] \equiv \frac{k!}{(k-j)!}[\psi] \pmod{t\mathcal{H}}.$$

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