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A Factorization Theorem for Multiplier Algebras of Reproducing Kernel Hilbert Spaces

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Abstract. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and let $H \subset L^2(X, \mu)$ be a separable reproducing kernel Hilbert space on X. We show that the multiplier algebra of H has property $(A_1(1))$.

Introduction

In this paper we show that the multiplier algebra of any separable reproducing kernel Hilbert space of square integrable functions has property $(A_1(1))$. The idea of the proof is to embed this algebra, via the Berezin transform, into a dual operator space that is completely isometric to an abelian von Neumann algebra. We then employ dual algebra techniques to get the A_1-factorization property.

1 Reproducing Kernel Hilbert Spaces and the Berezin Transform

In this paper we shall use some very elementary properties of the reproducing kernel Hilbert spaces. The classical reference on this topic is [3]. Its vast connections with interpolation theory are treated in [1].

Let H be a reproducing kernel Hilbert space of complex valued functions on some nonempty set X and let $B(H)$ be the algebra of all bounded operators on H. For each $\lambda \in X$ let $e_\lambda \in H$ be the unique function in H for which

$$(x, e_\lambda) = x(\lambda), \quad \forall x \in H.$$

This function is called the reproducing kernel for the point λ. For each $\lambda \in X$ with $e_\lambda \neq 0$ let $k_\lambda = e_\lambda / \|e_\lambda\|$. This function is called the normalized reproducing kernel for λ. A scalar valued function f on X is called a multiplier if $fx \in H$ for every $x \in H$. The set of all multipliers of H will be denoted by $M(H)$ and is called the multiplier algebra of H. Every $f \in M(H)$ induces a bounded operator T_f on H defined by $T_f(x) = fx, x \in H$. The set of all operators T_f with $f \in M(H)$ is a weakly closed commutative subalgebra of $B(H)$ that is also called the multiplier algebra of H. It will be clear from the context whether we refer to functions or operators. An operator $T \in B(H)$ is a multiplier if and only if each e_λ is an eigenvector for T^*.

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Suppose now that there exist a σ-algebra $\mathcal{B}$ of subsets of X and a σ-finite measure μ on $\mathcal{B}$ such that every function $x \in H$ is measurable, square integrable, and

$$\|x\|_H = \|x\|_{L^2(X,\mu)}.$$

If H is a reproducing kernel Hilbert space with these properties, we shall write $H \subset L^2(X, \mu)$. For the remainder of this section we shall assume that H is a separable reproducing kernel Hilbert space satisfying these assumptions relative to a σ-finite measure space $(X, \mathcal{B}, \mu)$.

The function $\lambda \to \|e_\lambda\|$ is measurable on X, therefore the set $\{\lambda \in X : e_\lambda = 0\}$ is measurable. From now on we shall discard this set by considering H as a reproducing kernel Hilbert space on the complement of this set and also by restricting the measure to the measurable subsets of the complement. The set of multiplication operators T_f remains untouched by this change. Under this condition, it is easy to see that every multiplier $f \in M(H)$ is measurable and essentially bounded.

If $A \in B(H)$, then its Berezin transform is a scalar valued function on X defined by $\widehat{A}(\lambda) = (Ak_\lambda, k_\lambda)$, $\lambda \in X$. This concept was introduced in [9] and plays an important role in the operator theory on function spaces; see [18, Chapter 6]. The function $\widehat{A}$ is measurable and essentially bounded and $\|\widehat{A}\|_\infty \leq \|A\|$ for every $A \in B(H)$, where $\|\cdot\|_\infty$ holds for the norm in $L^\infty(X, \mu)$, the Banach algebra of all measurable and essentially bounded functions on X. The mapping $A \mapsto \widehat{A}$ is easily seen to be completely contractive , hence completely positive as well.

If $f \in L^\infty(X, \mu)$, then the Toeplitz operator T_f on H is defined by

$$T_f(x) = P_H(fx) \quad x \in H,$$

where P_H is the orthogonal projection of $L^2(X, \mu)$ onto H. When $f \in M(H)$, then the Toeplitz operator with symbol f equals the multiplication operator induced by f. The map $f \mapsto T_f$ is completely positive, contractive and weak* continuous on $L^\infty(X, \mu)$.

Now for each $f \in L^\infty(X, \mu)$ define its Berezin transform by $B(f)(\lambda) = \widehat{T_f}(\lambda)$, $\lambda \in X$. In this way one obtains a unital, completely positive and contractive map

$$B \colon L^\infty(X, \mu) \to L^\infty(X, \mu)$$

which is also weak* continuous because both the Toeplitz map $f \to T_f$ and the Berezin transform $A \to \widehat{A}$ are weak* continuous. Moreover, it is clear from its definition that $B(f) = f$ for every $f \in M(H)$. Since B is positive, we see that the same holds true when $\bar{f} \in M(H)$.

2 Dual Spaces of Operators and Factorizations

In this section H will be a complex separable Hilbert space. We shall now recall several definitions and results from the theory of dual algebras that will be needed in the sequel. A dual algebra is by definition a weak* closed subalgebra of $B(H)$

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containing the identity. For basic definitions and properties of dual algebras we refer to [7].

Let $\mathcal{C}_1(H)$ be the Banach space of all trace-class operators on H endowed with the trace norm $\|\cdot\|_1$. Then $B(H)$ is identified with the dual of $\mathcal{C}_1(H)$ via the pairing

$$\langle T, L\rangle = \mathrm{trace}(TL) \quad T \in B(H), L \in \mathcal{C}_1(H).$$

For every pair of vectors $x, y \in H$, one denotes by $x \otimes y$ the rank-one operator on H defined by $(x \otimes y)(h) = (h, y)x$, $h \in H$.

Let $\mathcal{M} \subset B(H)$ be a weak* closed subspace and let $\mathcal{M}_\perp$ be its preannihilator in $\mathcal{C}_1(H)$. For each $L \in \mathcal{C}_1(H)$ one denotes by $[L]$ its class in the quotient space $Q_{\mathcal{M}} = \mathcal{C}_1(H)/\mathcal{M}_\perp$. Then $\mathcal{M}$ may be identified with the dual of $Q_{\mathcal{M}}$ via the pairing

$$\langle T, [L]\rangle = \mathrm{trace}(TL) \quad T \in \mathcal{M}, [L] \in Q_{\mathcal{M}}.$$

A weak* closed subspace $\mathcal{M} \subset B(H)$ is said to have property $(A_1(r))$ for some $r \geq 1$ if for each $\epsilon > 0$ and for each $[L] \in Q_{\mathcal{M}}$ there exist vectors $x, y \in H$ such that $[L] = [x \otimes y]$ and moreover, $\|x\|, \|y\| \leq ((r+\epsilon)\|[L]\|)^{1/2}$.

A much stronger factorization property is the following. A weak* closed subspace $\mathcal{M} \subset B(H)$ has the property $(A_{\aleph_0}(r))$ for some $r \geq 1$ if for each $\epsilon > 0$ and for each infinite array $\{[L_{ij}]\}_{i,j=0}^{\infty}$ in $Q_{\mathcal{M}}$ such that

$$\sum_{j\geq 0} \|[L_{ij}]\| < \infty \quad \forall i \geq 0,$$

$$\sum_{i\geq 0} \|[L_{ij}]\| < \infty \quad \forall j \geq 0,$$

there exist sequences of vectors $\{x_i\}_{i=0}^{\infty}$and $\{y_j\}_{j=0}^{\infty}$in H such that

$$[L_{ij}] = [x_i \otimes y_j] \quad 0 \leq i, j < \infty$$

and moreover, such that

$$\|x_i\| \leq ((r+\epsilon)\sum_{j\geq 0} \|[L_{ij}]\|)^{1/2} \quad \forall i \geq 0,$$

$$\|y_j\| \leq ((r+\epsilon)\sum_{i\geq 0} \|[L_{ij}]\|)^{1/2} \quad \forall j \geq 0.$$

Given a weak* closed subspace $\mathcal{M} \subset B(H)$ one denotes by $E_0(\mathcal{M})$ the set of all elements $[L] \in Q_{\mathcal{M}}$ for which there exist sequences of vectors $\{x_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ in the unit ball of H, weakly convergent to 0, such that $\lim_{n\to\infty} \|[L] - [x_n \otimes y_n]\| = 0$ and, moreover, $\lim_{n\to\infty}(\|[x_n \otimes z]\| + \|[z \otimes y_n]\|) = 0$, $\forall z \in H$.

The space $\mathcal{M}$ is said to have property $X_{0,1}$ if the absolutely convex hull of the set $E_0(\mathcal{M})$ is dense in the closed unit ball of the predual $Q_{\mathcal{M}}$. We shall need the following basic result. This theorem is stated in [7] for dual algebras; however its proof holds for weak* closed spaces as well.

Theorem 2.1 ([7, Theorem 3.6]) *Every weak* closed subspace of $B(H)$ with property $X_{0,1}$ has property $(A_{\aleph_0}(1))$.*

Let $(Y, \mathcal{F}, \nu)$ be a separable, σ-finite measure space and let $\mathcal{D}$ be a separable Hilbert space. Let $L^2(\nu, \mathcal{D})$ be the Hilbert space of all Bochner-measurable and square integrable $\mathcal{D}$-valued functions on Y. For a pair of vectors $x, y \in L^2(\nu, \mathcal{D})$ one denotes by $x \cdot y$ the scalar valued integrable function defined by

$$(x \cdot y)(s) = (x(s), y(s))_{\mathcal{D}} \quad s \in Y,$$

where $(\,\cdot\,,\,\cdot\,)_{\mathcal{D}}$ holds for the scalar product in $\mathcal{D}$.

We shall make use of a fundamental approximate factorization theorem that was proved in [6] under an additional boundedness condition. without, however, separability assumptions on the measure space. That condition was removed in [8] provided that the measure space is separable.

Theorem 2.2 ([6, 8]) *Let $(Y, \mathcal{F}, \nu)$ be a separable, σ-finite measure space and let $\mathcal{D}$ be a separable Hilbert space. Let $H \subset L^2(\nu, \mathcal{D})$ be a closed subspace. Assume that for every measurable subset $\omega \subset Y$ with $0 < \nu(\omega) < \infty$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of unit vectors in H weakly convergent to 0 such that $\|\chi_{Y\setminus\omega} z_n\|_2 \to 0$, where $\chi_{Y\setminus\omega}$ is the characteristic function of the set $Y\setminus\omega$ and $\|\cdot\|_2$ holds for the norm in $L^2(\nu, \mathcal{D})$. Then for every function $f \in L^1(\nu)$ there exist sequences of vectors $\{x_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ in H weakly convergent to 0 such that $\|f - x_n \cdot y_n\|_1 \to 0$ and $\|x_n\|, \|y_n\| \leq \|f\|_1^{1/2}$, $\forall n \geq 1$.*

3 A Factorization Property for Multiplier Algebras

The main result of this paper is the following theorem.

Theorem 3.1 *Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and let $H \subset L^2(X, \mu)$ be a separable reproducing kernel Hilbert space on X. Then the dual algebra of all multiplication operators T_f on H with f in the multiplier algebra $M(H)$ has property $(A_1(1))$. If, moreover, this algebra has no invariant one-dimensional subspaces, then it has property $(A_{\aleph_0}(1))$.*

Proof We shall freely use the notations established above. We may assume that $e_\lambda \neq 0$ for every $\lambda \in X$. Let $\mathcal{T}(B) = \{f \in L^\infty(X, \mu) : Bf = f\}$, where

$$B\colon L^\infty(X, \mu) \to L^\infty(X, \mu)$$

is the Berezin transform. For instance, when H is the Bergman space on the unit disc, then $\mathcal{T}(B)$ is precisely the space of all bounded harmonic functions on the disc [13]. Coming back to the general case, $\mathcal{T}(B)$ is a weak* closed self-adjoint subspace of $L^\infty(X, \mu)$. Moreover, since $\mathcal{T}(B)$ is the fixed point set of a completely positive, unital and weak* continuous mapping on a von Neumann algebra, one can construct, by a standard averaging procedure (see [4]), a completely positive, unital and idempotent map $\Phi\colon L^\infty(X, \mu) \to L^\infty(X, \mu)$ whose range equals $\mathcal{T}(B)$. By a well-known result

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from [11] there exist a von Neumann algebra $\mathcal{R}(B)$ and a weak* continuous unital and completely isometric map θ from $\mathcal{T}(B)$ onto $\mathcal{R}(B)$ such that

$$\theta(f)\theta(g) = \theta(\Phi(fg)) \quad f, g \in \mathcal{T}(B).$$

In particular, it follows that $\mathcal{R}(B)$ is abelian. Since $\mathcal{R}(B)$ has separable predual, there exists a separable finite measure space $(Y, \mathcal{F}, \nu)$ such that $\mathcal{R}(B)$ is *-isomorphic with $L^\infty(Y, \nu)$. We may therefore assume that $\mathcal{R}(B) = L^\infty(Y, \nu)$.

Since both the Toeplitz map $f \to T_f$ and the Berezin transform $A \to \widehat{A}$ are completely contractive (see the discussion above), it follows that the map $f \to T_f$ is completely isometric when restricted to $\mathcal{T}(B)$.

We then obtain a completely isometric unital map $\rho\colon L^\infty(Y, \nu) \to B(H)$ such that

$$\rho(\theta(f)) = T_f \quad f \in \mathcal{T}(B).$$

Let $C^*(\mathcal{T}(B))$ be the C^*-subalgebra of $B(H)$ generated by all Toeplitz operators T_f with $f \in \mathcal{T}(B)$. It then follows from [10, Theorem 4.1] that there exists a unital *-homomorphism $\pi\colon C^*(\mathcal{T}(B)) \to L^\infty(Y, \nu)$ such that $\pi(T_f) = \theta(f) \quad f \in \mathcal{T}(B)$.

Let $K(H)$ denote the C^*-algebra of all compact operators on H. Let us assume for the moment that there are no one-dimensional $C^*(\mathcal{T}(B))$-invariant subspaces of H. Under this condition we claim that $\pi(C^*(\mathcal{T}(B)) \cap K(H)) = \{0\}$. In order to prove this, recall first that there exists a family $\{H_j\}_{j\geq 0}$ of closed subspaces of H that are $C^*(\mathcal{T}(B))$-invariant and positive integers $\{m(j)\}_{j\geq 1}$ such that

$$H = H_0 \oplus \Big\{ \bigoplus_{j\geq 1} H_j^{(m(j))} \Big\}$$

and, corresponding to this decomposition,

$$C^*(\mathcal{T}(B)) \cap K(H) = \{0\} \oplus \Big\{ \bigoplus_{j\geq 1} ((K(H_j))^{(m(j))} \Big\},$$

where $H_j^{(m(j))}$ holds for the orthogonal sum of $m(j)$ copies of the space H_j and $(K(H_j))^{(m(j))}$ is the corresponding ampliation of $K(H_j)$.

Suppose now that $\pi((K(H_j))^{(m(j))}) \neq \{0\}$ for some $j \geq 1$. Since π takes values in a commutative C^*-algebra and $K(H_j)$ is simple, it follows that $\dim(H_j) = 1$. However this contradicts our assumption that $C^*(\mathcal{T}(B))$ has no one-dimensional invariant subspaces. This shows that $\pi((K(H_j))^{(m(j))}) = \{0\}$ for all $j \geq 1$, hence

$$\pi(C^*(\mathcal{T}(B)) \cap K(H)) = \{0\}.$$

We then obtain for every $f \in \mathcal{T}(B)$ that

$$\|T_f\| = \|\pi(T_f)\| = \operatorname{dist}(T_f, \ker(\pi)) \leq \operatorname{dist}(T_f, C^*(\mathcal{T}(B)) \cap K(H)) = \|T_f\|_{\mathrm{ess}},$$

where $\|T_f\|_{\mathrm{ess}}$ holds for the essential norm of T_f. Furthermore, using the Lebesgue dominated convergence theorem it follows that the closed unit ball in $\rho(L^\infty(Y, \nu))$ is compact in the strong operator topology.

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Recall now that we have a completely isometric unital and weak* continuous map

$$\rho\colon L^\infty(Y,\nu)\to B(H).$$

It then follows from the Stinespring Dilation Theorem [17] coupled with a well-known result from von Neumann algebras theory (see [16, Proposition 2.7.4]) that there exist a separable Hibert space $\mathcal{D}$ and an isometry $\Gamma\colon H\to L^2(\nu,\mathcal{D})$ such that for all $g\in L^\infty(Y,\nu)$

$$(\rho(g)x,y)_H=(M_g(\Gamma(x)),\Gamma(y))_{L^2(\nu,\mathcal{D})}\quad \forall x,y\in H,$$

where M_g is the multiplication operator on $L^2(\nu,\mathcal{D})$ induced by g. For more details on this see [15, Lemma 4.1]. Let $\widetilde{H}=\Gamma(H)$, and for any $x\in H$ denote $\widetilde{x}=\Gamma(x)$. Let $\omega\subset Y$ be a measurable subset with $\nu(\omega)>0$. Since ρ is isometric on $L^\infty(Y,\nu)$ and the Calkin map is isometric on $\rho(L^\infty(Y,\nu))$, it then easily follows that there exists a sequence $\{x_n\}_{n=1}^\infty$ of unit vectors in H converging weakly to 0 such that

$$\|\chi_\omega\widetilde{z_n}\|_2\to\|\chi_\omega\|_\infty=1.$$

Equivalently, $\|\chi_{Y\setminus\omega}\widetilde{z_n}\|_2\to 0$.

It now follows from Theorem 2.2 that for every $f\in L^1(\nu)$ there exist sequences of vectors $\{x_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ in H weakly convergent to 0 such that

$$\|f-\widetilde{x_n}\cdot\widetilde{y_n}\|_1\to 0\quad\text{and}\quad \|\widetilde{x_n}\|,\|\widetilde{y_n}\|\le\|f\|_1^{1/2}\quad\forall n\ge 1.$$

This is equivalent to say that for every element $[L]$ in the predual of $\rho(L^\infty(Y,\nu))$ there are sequences of vectors $\{x_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ in H weakly convergent to 0 such that $\|[L]-[x_n\otimes y_n]\|\to 0$ and, moreover, $\|x_n\|,\|y_n\|\le\|[L]\|^{1/2}$, $\forall n\ge 1$. Since the unit ball of $\rho(L^\infty(Y,\nu))$ is compact in the strong topology, one may easily see that

$$\|[x_n\otimes z]\|+\|[z\otimes y_n]\|\to 0\quad\forall z\in H.$$

We may now apply Theorem 2.1 and infer that the space $\{T_f : f\in\mathcal{T}(B)\}$ has property $(A_{\aleph_0}(1))$. In particular, the dual algebra of all multiplication operators has the same property as well.

Now, for the general case, let K_0 be the closed span of all one-dimensional subspaces of H that are T_f-invariant for every $f\in M(H)$ (if any). Since all these operators are subnormal, it follows that K_0 is reducing for every T_f with $f\in M(H)$. Let $K_1=H\ominus K_0$ and assume that $K_1\neq 0$. Then K_1 is again a reproducing kernel Hilbert space on X and for each $f\in M(H)$ the restriction of T_f to K_1 is again a multiplication operator for K_1. In addition there are no one-dimensional subspaces of K_1 that are invariant for all its multiplication operators. It then follows from what we have already proved that the algebra of all multiplication operators of K_1 has property $(A_{\aleph_0}(1))$.

On the other hand, given $f\in M(H)$, the restriction of T_f to K_0 is a normal operator, hence the dual algebra on K_0 generated by all these operators is a subalgebra of an abelian von Neumann algebra therefore it has the property $A_1(1)$.

Finally the dual algebra $\{T_f : f\in M(H)\}$ on H is a subalgebra of the direct sum of two dual algebras with property $A_1(1)$ hence it also has the same property (see [14]). ∎

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In the particular case when H is the Bergman space on the unit disc in the complex plane, the fact that its multiplier algebra has property $(A_{\aleph_0}(1))$ was proved in [2]. In the case of Bergman spaces on multidimensional domains, a similar result was proved in [5].

Multiplier algebras with property $(A_1(1))$ have been recently studied in connection with the Nevanlinna-Pick interpolation problem in reproducing kernel Hilbert space (see [12] and the references therein).

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