

An improved Reynolds technique for approximate solution of linear stochastic differential equations

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Our starting point is a formal linear stochastic differential equation of first order (higher order equations can be transformed to systems of these)

$$\frac{dI(x, \omega)}{dx} = a(x, \omega)I(x, \omega) - W(x, \omega; I), \quad (1)$$

where I, a, W are stochastic functions $I = \bar{I} + I', a = \bar{a} + a', W = \bar{W} + W'$, with $\langle I \rangle = \bar{I}, \langle I' \rangle = 0$, and analogously for a and W . I, a , and W are allowed to depend on the element ω of a set Ω in which a probability measure is defined in the usual way (see e.g. Doob, 1953; de Witt-Morette, 1981). To get a solution of eq.(1) for the mean intensity \bar{I} we treat the problem according the Reynolds averaging technique in the usual manner : The stochastic equation is changed into an infinite hierarchical system of equations for the correlations. At first we take the mean of eq.(1)

$$\frac{d \langle I(x) \rangle}{dx} = \bar{a}(x)\bar{I}(x) + \langle a'(x)I(x) \rangle - \bar{W}(x). \quad (2)$$

Multiplying eq.(1) with $a'(x')$ resp. with $a'(x')a'(x'')$ and so on we get by taking the mean

$$\begin{aligned} \frac{d}{dx} \langle a'(x')I(x) \rangle &= \langle a'(x')(a(x)I(x) - W(x)) \rangle \\ &\dots \\ \frac{d}{dx} \langle a'(x^{(n)}) \dots a'(x')I(x) \rangle &= \langle a'(x^{(n)}) \dots a'(x')(a(x)I(x) - W(x)) \rangle \end{aligned} \quad (3)$$

where $x^{(n)}$ is a n-dashed x (dependence on ω is omitted). The eqs.(2) and (3) form an infinite hierarchical system of differential equations. A (calculable) solution can be obtained by truncating the infinite system. A simple cut off, often used in physics, consists in neglecting higher order correlations ($n > 2$). Thus we get for $n = 2$:

$$\langle a'(x')a'(x)I(x) \rangle = \langle a'(x')a'(x) \rangle \bar{I}(x) \quad (4)$$

($\langle a'a'I' \rangle = 0$). For small deviations from the solution of the completely random (uncorrelated) case, the cut off according to eq.(4) is a valid approximation. **However**, especially for larger fluctuations far away from the totally random case the cut off according to eq.(4) becomes incorrect: It completely neglects higher order correlations. To take into account these higher order correlations via 2- point correlations (as does e.g. a Markov process) resp. via n-point correlations, we approximate the left side of eq.(5) in such a way that the two known limiting cases of

the stochastic process, the solution of the completely random case and the solution $\langle I_\infty \rangle$ of a totally correlated stochastic process, become exactly included:

$$\langle a'(x')a'(x)I(x) \rangle = \langle a'(x')a'(x)I_\infty(x) \rangle, \tag{5a}$$

where I_∞ can be obtained by including ω in the nonstochastic(!) solution I_o of eq.(1): $I_o(x) \rightarrow I_o(x, \omega)$ (solution of eq.(1) in the totally correlated case only!) $\equiv I_\infty$. In the n-th order we get the generalised form

$$\langle a'(x^{(n)}) \dots a'(x)I(x) \rangle = \langle a'(x^{(n)}) \dots a'(x)I_\infty(x) \rangle. \tag{5b}$$

The eqs.(5) are the simplest ansatz to fulfil the above made conditions. A more complex ansatz could give better results.

Cutting off the system of the eqs.(2) and (3) according to eq.(5b) we get

$$\begin{aligned} \bar{I}(x) = \Lambda \langle W(x') \rangle - \Lambda^2 \langle a'(x')W(x'') \rangle + \dots + - \Lambda^n \langle a'(x') \dots a'(x^{(n-1)}) \\ \left[W(x^{(n)}) - a'(x^{(n)})I_\infty(x^{(n)}) \right] \rangle + \Delta_n \end{aligned} \tag{6}$$

with

$$\begin{aligned} \Lambda X(x) &= \exp\left(\int_0^x \bar{a}(x')dx'\right) \int_x^\infty dx' \exp\left(-\int_0^{x'} \bar{a}(x'')dx''\right) X(x') \\ \Lambda^2 X(x') &= \Lambda \left(\exp\left(\int_0^{x'} \bar{a}(x'')dx''\right) \int_{x'}^\infty dx'' \exp\left(-\int_0^{x''} \bar{a}(x''')dx'''\right) X(x'') \right) \end{aligned}$$

and the error term $\Delta_n = \Lambda^n \langle a'(x') \dots a'(x^{(n)}) (I - I_\infty) \rangle$.

If for a special stochastic process any $\Delta_n = 0$ we have found an exact solution. In general there are no closed systems of equations and a Δ_n is assumed to be sufficiently small to cut off the infinite system. From eq.(6) some special cases follow:

a) $\Delta_1 = 0$: On the one hand it is: $\bar{I} = \Lambda(\bar{W} - \langle a'I_\infty \rangle)$, a totally correlated stochastic process and on the other hand $\bar{I} = \Lambda\bar{W}$, a completely randomly stochastic process.

b) $\Delta_2 = 0$: $\bar{I}(x) = \Lambda\bar{W}(x') - \Lambda^2 \langle a'(x')(W(x'') - a'(x'')I_\infty(x'')) \rangle$. (7)

Eq.(7) is (by including of finite correlations) the simplest solution of eq.(1) in the frame of the proposed cut off. It is comparable with solutions in the frame of a Markov process (MP) but is not restricted to special velocity fields as does the MP. Eq.(6) is not only applicable to small perturbations caused by a random process. By taking into account even higher order momentum the accuracy increases like a power law : $D^n/n!$, where $D = (f(v_{mi}) - f(v_{ma}))/f(v_o)$. If $D > 1$ then the validity decreases up to a limited n and then increases! For $n \geq D$ this series expansion is absolutely convergent. Eq.(7) as well as eq.(6) become exact for completely random and totally correlated stochastic processes.

References

De Witt-Morette, C., Elworthy, K.D.: 1981, *Stochastic Differential Equations, Proceedings of the "5-Tage-Kurs", Bielefeld, University*, .
 Doob, J.L.: 1953, *Stochastic Processes, Wiley, New York*, .