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TENSION FIELD AND INDEX FORM OF ENERGY-TYPE FUNCTIONALS

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Abstract. We derive variational formulae for natural first order energy functionals and obtain criteria for the stability of isometric immersions. This generalizes known results for the classical energy, the p-energy and the exponential energy.

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1. Introduction. By an energy-type functional defined on smooth maps $f:(M^n,g)\to(V^k,h)$ of compact Riemannian manifolds we mean a functional obtained by integration of a first order differential operator $\phi(df)$ where $df\in\Gamma(T^*M\otimes f^*TV)$ denotes the differential of f and $\phi:M(\mathbb{R},n\times k)\to\mathbb{R}_0^+$ is invariant under the action of $O(n)\times O(k)$. Especially ϕ yields a parallel function $T^*M\otimes f^*TV\to\mathbb{R}_0^+$. We can rewrite $\phi(df)=\Phi(df^*df)$ for some function $\Phi:M(\mathbb{R},n\times n)^+\to\mathbb{R}$ on nonnegative symmetric matrices which is invariant under conjugation by $O(n)$. The functionals in question take the form

$$E_\Phi(f):=\int_M\Phi(df^*df)\,d\mathrm{vol}_g,$$

where we have used the Riemannian metrics to identify $T^*M=TM$ and $T^*V=TV$ to get the endomorphism df^*df of TM.

Famous examples of this construction are the classical energy, $\Phi(A)=\mathrm{Tr}A$, the exponential energy, $\Phi(A)=\exp(\mathrm{Tr}A)$ as in [**7**], the p-energy, $\Phi(A)=(\mathrm{Tr}A)^p$ but also the volume, where $\Phi(A)=(\det A)^{1/2}$. Results similiar to ours in the case where Φ is a function of the Trace, $\Phi(A)=F(\mathrm{Tr}A)$, have been obtained in [**1**]. In particular the exponential energy was treated in [**2**] and the p-energy in [**3**]. There is a vast literature for the classical energy, see e.g. the survey papers [**5**, **6**]. For a discussion of stability results in this case we refer to [**9**] and the references there.

Here we will derive the first and second variational formulae for the Φ-energy functional. The Bochner formula for vector fields then implies that isometries are Φ-stable under certain conditions on the first and second derivative of Φ. As in the classical case, (see [**4**], [**9**]) there is also a range of maps Φ such that the identity on the sphere S^n is unstable for the Φ-energy.

2. Variation formulae for the Φ-Energy. In order to derive variational formulae we will restrict ourselves to functionals which can be expressed with smooth Φ, i.e. we work with Φ rather than ϕ. This has the advantage that the domain $TM^*\otimes TM$ of Φ

is independent of f. For polynomial (or even analytic) ϕ this is no loss of generality by the remark at the end of this section. In the sequel we will always assume M compact or at least that the variations are compactly supported. Consider a 2-parameter variation of f, i.e. a map

$$F : I \times J \times M \to V(s,t,m) \mapsto f_{s,t}(m)$$

where I, J are intervalls around 0. Denote by ∇ the Riemannian connections on the bundles TM, F^*TV and f^*TV and let $v := dF(\frac{\partial}{\partial t}) = \frac{\partial}{\partial t} f_{s,t}(m)$, $w := dF(\frac{\partial}{\partial s}) = \frac{\partial}{\partial s} f_{s,t}(m)$ be the variation vector fields along $f = f_0 = f_{0,0}, f_t = f_{0,t}$. We compute the variation at a point $p \in M$. Let $e_1, \ldots, e_n$ be a local orthonormal framing of TM in a vicinity of p with $\nabla_{e_i} e_j = 0$ at p. Note that for the commutators we have $[e_i, \frac{\partial}{\partial s}] = 0$, $[e_i, \frac{\partial}{\partial t}] = 0$ and $[e_i, e_j](p) = 0$. We also write $\bar{\partial}_{i,j}\Phi := \partial_{i,j}\Phi + \partial_{j,i}\Phi$. In the subsequent calculations summation over the indices i, j, k, l is tacitely assumed. For the first variation of the Φ-energy density we obtain

$$\begin{aligned}
\frac{d}{dt}\Phi(df_t^* df_t) &= d\Phi(\nabla df \otimes df + df \otimes \nabla df) \\
&= \bar{\partial}_{i,j}\Phi(df^* df)\big\langle \nabla_{\frac{\partial}{\partial t}} dFe_i \,\big|\, dFe_j \big\rangle \\
&= \bar{\partial}_{i,j}\Phi(df^* df)\big\langle \nabla_{e_i} v \,\big|\, df e_j \big\rangle \\
&= e_i(\bar{\partial}_{i,j}\Phi(df^* df)\langle v \mid df e_j\rangle) - \big\langle v \,\big|\, \nabla_{e_i}(\bar{\partial}_{i,j}\Phi(df^* df) df e_j)\big\rangle \\
&= \operatorname{div}((\bar{\partial}_{i,j}\Phi(df^* df)\langle v \mid df e_j\rangle) e_i) - \langle v \mid \tau_\Phi(f)\rangle.
\end{aligned}$$

We thus get the

PROPOSITION 2.1. *Define the Φ-tension of a smooth map $f : M \to V$ of compact Riemannian manifolds to be the vector field along f*

$$\begin{aligned}
\tau_\Phi(f) &:= \nabla_{e_i}(\bar{\partial}_{i,j}\Phi(df^* df) df e_j) \\
&= \bar{\partial}_{k,l}\bar{\partial}_{i,j}\Phi(df^* df)\big\langle \nabla_{e_i} df e_k \,\big|\, df e_l \big\rangle df e_j + \bar{\partial}_{i,j}\Phi(df^* df)\nabla_{e_i} df e_j
\end{aligned} \tag{2.2}$$

Then f is Φ-harmonic, i.e. critical for the Φ-energy, if and only if $\tau_\Phi(f) = 0$.

For the second variation we get up to divergence

$$\begin{aligned}
\frac{d^2}{dsdt}\Phi(df_{s,t}^* df_{s,t}) &= -\frac{d}{ds}\langle v | \tau_\Phi(f_s)\rangle \\
&= -\big\langle \nabla_{\frac{\partial}{\partial s}} v \,\big|\, \tau_\Phi(f)\big\rangle - \big\langle v \,\big|\, \nabla_{\frac{\partial}{\partial s}} \tau_\Phi(f_s)\big\rangle \\
&= -\big\langle \nabla_{\frac{\partial}{\partial s}} v \,\big|\, \tau_\Phi(f)\big\rangle - \big\langle v \,\big|\, \nabla_{\frac{\partial}{\partial s}} \nabla_{e_i}(\bar{\partial}_{i,j}\Phi(df_s^* df_s) dFe_j)\big\rangle \\
&= -\big\langle \nabla_{\frac{\partial}{\partial s}} v \,\big|\, \tau_\Phi(f)\big\rangle - \big\langle v \,\big|\, R_{w, df e_i}(\bar{\partial}_{i,j}\Phi(df^* df) df e_j)\big\rangle \\
&\quad - \big\langle v \,\big|\, \nabla_{e_i}\nabla_{\frac{\partial}{\partial s}}(\bar{\partial}_{i,j}\Phi(df_s^* df_s) dFe_j)\big\rangle
\end{aligned}$$

where R denotes the curvature tensor of V. The last term is

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$$
\begin{aligned}
&-\langle v \mid \nabla_{e_i}\nabla_{\frac{\partial}{\partial s}}(\bar{\partial}_{i,j}\Phi(df_s^*df_s)dFe_j)\rangle \\
&= -\left\langle v \mid \nabla_{e_i}\left(\frac{d\bar{\partial}_{i,j}\Phi(df_s^*df_s)}{ds}df\,e_j + \bar{\partial}_{i,j}\Phi(df^*df)\nabla_{\frac{\partial}{\partial s}}dFe_j\right)\right\rangle \\
&= -\langle v \mid \nabla_{e_i}\big(\bar{\partial}_{k,l}\bar{\partial}_{i,j}\Phi(df^*df)\langle\nabla_{\frac{\partial}{\partial s}}dFe_k \mid df\,e_l\rangle df\,e_j + (\bar{\partial}_{i,j}\Phi(df^*df)\nabla_{e_j}w)\big)\rangle \\
&= -\langle v \mid \nabla_{e_i}(\bar{\partial}_{k,l}\bar{\partial}_{i,j}\Phi(df^*df)\langle\nabla_{e_k}w \mid df\,e_l\rangle df\,e_j + \big(\bar{\partial}_{i,j}\Phi(df^*df)\nabla_{e_j}w)\big)\rangle \\
&= +\bar{\partial}_{k,l}\bar{\partial}_{i,j}\Phi(df^*df)\langle\nabla_{e_i}v \mid df\,e_j\rangle\langle\nabla_{e_k}w \mid df\,e_l\rangle + \bar{\partial}_{i,j}\Phi(df^*df)\langle\nabla_{e_i}v \mid \nabla_{e_j}w\rangle
\end{aligned}
$$

where the last identity holds only up to divergence.

PROPOSITION 2.3. *The second variation of the Φ-energy at a Φ-harmonic map f is the integral over*

$$
\begin{aligned}
I_\Phi(f)(v,w) = &-\langle v \mid R_{w,df\,e_i}(\bar{\partial}_{i,j}\Phi(df^*df)df\,e_j)\rangle \\
&+\bar{\partial}_{k,l}\bar{\partial}_{i,j}\Phi(df^*df)\langle\nabla_{e_i}v \mid df\,e_j\rangle\langle\nabla_{e_k}w \mid df\,e_l\rangle \\
&+\bar{\partial}_{i,j}\Phi(df^*df)\langle\nabla_{e_i}v \mid \nabla_{e_j}w\rangle
\end{aligned}
$$

for any vector fields v, w along f.

We finally compute the leading symbol of the second variation. We have

$$
\frac{d^2}{dsdt}E_\Phi(f_{s,t}) = \int_M \langle v \mid Pw\rangle\, d\mathrm{vol}_g \tag{2.4}
$$

with a symmetric second order partial differential operator P acting on vector fields along f, i.e. on sections v, w of $f^*TV \to M$. The restriction $P^{\perp f}$ of P (or of the bilinear form given by (2.4)) to the orthogonal complement of the image of $df : TM \to f^*TV$ will be called second variation perpendicular to f. The leading symbol of P is determined by the highest order term

$$
-\langle v \mid \bar{\partial}_{k,l}\bar{\partial}_{i,j}\Phi(df^*df)\langle\nabla_{e_i}\nabla_{e_k}w \mid df\,e_l\rangle df\,e_j + \bar{\partial}_{i,j}\Phi(df^*df)\nabla_{e_i}\nabla_{e_j}w\rangle
$$

in Proposition 2.3. Hence we get

PROPOSITION 2.5. *The leading symbol of the second variation of the Φ-energy is*

$$
\sigma(\xi) = \bar{\partial}_{k,l}\bar{\partial}_{i,j}\Phi(df^*df)\xi_i\,\xi_k df\,e_l \otimes df\,e_j + \bar{\partial}_{i,j}\Phi(df^*df)\xi_i\xi_J, \tag{2.6}
$$

for $\xi = \sum_i \xi_i e_i$. Thus

$$
\langle\sigma(\xi)w \mid w\rangle = \bar{\partial}_{k,l}\bar{\partial}_{i,j}\Phi(df^*df)\xi_i\xi_k\langle w \mid df\,e_l\rangle\langle w \mid df\,e_j\rangle + \bar{\partial}_{i,j}\Phi(df^*df)\xi_i\xi_j\|w\|^2
$$

for $\xi \in T_pM^*$ and $w \in (f^*TV)_p$.

REMARK. Let $\phi : M(n \times k) \to \mathbb{R}_0^+$ be a polynomial function, invariant under the action of $O(n) \times O(k)$, i.e. such that $\phi(BXA) = \phi(X)$ for all $B \in O(k)$, $A \in O(n)$ and $X \in M(n \times k)$. For any $X \in M(n \times k)$ we can diagonalize X^*X and find othogonal

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matrices B and A as before such that

$$BXA = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_q \\ 0 & \cdots & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda_1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & \lambda_q & 0 \end{pmatrix}$$

as $q := \min\{n, k\} = n$ or $q = k$. Hence $\phi(X) = \phi(\lambda_1, \dots, \lambda_q)$ is a symmetric polynomial and since $\phi(\pm\lambda_1, \dots, \pm\lambda_q) = \phi(\lambda_1, \dots, \lambda_q)$ this does not involve odd powers of the λ_i. Thus we find a symmetric polynomial Φ in n variables such that $\phi(\lambda_1, \dots, \lambda_q) = \Phi(\lambda_1^2, \dots, \lambda_q^2, 0, \dots, 0)$. This extends to a polynomial $\Phi : M(n \times n)^+ \to \mathbb{R}_0^+$ such that $\phi(X) = \Phi(X^*X)$. For analytic ϕ this construction yields an analytic function Φ.

Note that if ϕ is differentiable we do not necessarily get a differentiable function Φ with the above properties. In general Φ is only differentiable on the set of matrices of full rank q. For instance $\phi(X) := \det(X^*X)^{3/4}$ is differentiable but $\Phi(A) := \det(A)^{3/4}$ is not.

For polynomial ϕ there are polynomials Φ^s and Φ^σ such that

$$\phi(X) = \Phi(X^*X) = \Phi^s(s_1, \dots, s_q) = \Phi^\sigma(\sigma_1, \dots, \sigma_q)$$

where σ_l is the lth elementary symmetric polynomial in the eigenvalues $\lambda_1^2, \dots, \lambda_q^2$ of X^*X determined by

$$\sum_{l=0}^{n} \sigma_l(X^*X)t^l = \det(1 + tX^*X)$$

and

$$s_k = \sum_{l=0}^{n} \lambda_l^{2q} = \mathrm{Tr}((X^*X)^l).$$

In the smooth case one can use a theorem of Glaeser, [**8**], to get smooth functions Φ^s and Φ^σ.

3. Applications.

3.1. Isometric immersions. For isometric immersions the preceding formulae simplify substantially. By invariance $d\Phi(id)$ must be some multiple $\lambda\mathrm{Tr}$ of the trace. We have the following.

THEOREM 1. *Let $f : M \to V$ be an isometric immersion and assume that $d\Phi(id) \neq 0$. Then*

1. *f is Φ-harmonic if and only if it is harmonic.*
2. *If $\lambda > 0$ then the leading symbol of $P^{\perp f}$ is positive definite, hence the second variation perpendicular to f has finite index.*

Proof. (1) For an isometric immersion or a Riemannian submersion the first term in (2.2) vanishes. Since an isometric immersion f has $df^*df = id$ we get

$$\tau_\Phi(f) = \bar{\partial}_{i,j}\Phi(id)\nabla_{e_i} df\, e_j = 2\lambda\mathrm{Tr}\nabla df = 2\lambda\tau(f).$$

(2) On vector fields w normal to f, i.e perpendicular to the $df e_l$ in (2.6), the first summand in (2.6) vanishes. As before the second summand is some multiple of the trace which shows that

$$\sigma(\xi) = \bar{\partial}_{i,j}\Phi(df^* df)\xi_i\xi_j = 2\lambda\|\xi\|^2 > 0$$

for $\xi \neq 0$. Thus the restriction of P to $(\text{Image }(df))^\perp \subset f^*TV$ is elliptic with positive definite leading symbol and therefore has only finitely many negative eigenvalues. □

3.2. Stability of isometries. By invariance, the second derivative $d^2\Phi(id)$ is a homogeneous polynomial of degree 2. Therefore there are $\mu, \nu \in \mathbb{R}$ such that

$$d^2\Phi(id)(H) = \mu \operatorname{Tr}(H^2) + \nu(\operatorname{Tr} H)^2$$

The second variation formula in Proposition 2.3 simplifies to

$$\begin{aligned}
I_\Phi(f)(v, v) &= -\langle v \mid R_{v,e_i}(\bar{\partial}_{i,j}\Phi(id)e_j)\rangle \\
&\quad + \bar{\partial}_{k,l}\bar{\partial}_{i,j}\Phi(id)\langle \nabla_{e_i} v \mid e_j\rangle\langle \nabla_{e_k} v \mid e_l\rangle + \bar{\partial}_{i,j}\Phi(id)\langle \nabla_{e_i} v \mid \nabla_{e_j} v\rangle \\
&= -2\lambda \operatorname{Ric}(v) \\
&\quad + \mu\big(\langle \nabla_{e_i} v \mid e_j\rangle + \langle \nabla_{e_j} v \mid e_i\rangle\big)^2 \\
&\quad + 4\nu\langle \nabla_{e_i} v \mid e_i\rangle\langle \nabla_{e_k} v \mid e_k\rangle + 2\lambda\langle \nabla_{e_i} v \mid \nabla_{e_i} v\rangle \\
&= -2\lambda \operatorname{Ric}(v) \\
&\quad + 2\mu(\|\nabla v\|^2 + \operatorname{Tr}((\nabla v)^2)) + 4\nu(\operatorname{div}(v))^2 + 2\lambda\|\nabla v\|^2 \\
&= -2\lambda \operatorname{Ric}(v) + 2(\mu + \lambda)\|\nabla v\|^2 + 2\mu \operatorname{Tr}((\nabla v)^2) + 4\nu(\operatorname{div}(v))^2 \\
&= -2\lambda \operatorname{Ric}(v) + \mu\|L_v g\|^2 + 4\nu(\operatorname{div}(v))^2 + 2\lambda\|\nabla v\|^2
\end{aligned}$$

since $\operatorname{Tr}(\nabla v) = \operatorname{div}(v)$. Comparing this with the Bochner formula (see e.g. [**10**]):

$$\int_M -\operatorname{Ric}(v) - \frac{1}{2}\|L_v g\|^2 + (\operatorname{div}(v))^2 + \|\nabla v\|^2 = 0 \tag{3.1}$$

we obtain

THEOREM 2. *Assume that $\mu \geq -\lambda$ and that $2\nu \geq \lambda$. Then any isometry of M is Φ-stable.*

We now derive a sufficient criterion for the identity map on a sphere to be unstable. To that end let v be the gradient vectorfield on $S^n \subset \mathbb{R}^{n+1}$ of the restriction of a linear map $p : \mathbb{R}^{n+1} \to \mathbb{R}$, $p(x) = \langle p, x\rangle$ for a unit vector $p \in \mathbb{R}^{n+1}$ as in [**9**]. Then $\|v(x)\|^2 + p(x)^2 = 1$ and $\nabla_x v = -px$ for all $x \in TS^n$, hence $\langle \nabla_{e_i} v, e_j\rangle = -p\delta_{i,j}$. Since the Ricci curvature of S^n is $\operatorname{Ric}(v) = (n-1)\|v\|^2$, the formula for the index form yields

$$I_\Phi(v, v) = -2\lambda(n-1)\|v\|^2 + (4\mu n + 4\nu n^2 + 2\lambda n)p^2. \tag{3.2}$$

Denoting by ω_{n-1} the volume of the standard $(n-1)$-sphere we compute

$$\begin{aligned}\int_{S^n} \|v\|^2 &= \omega_{n-1} \int_{-\pi/2}^{\pi/2} \cos(\theta)^{n+1} d\theta \\ &= \omega_{n-1}\left([\sin(\theta)\cos(\theta)^n]_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \sin(\theta)^2 \cos(\theta)^{n-1} d\theta \right) \\ &= n \int_{S^n} p^2.\end{aligned}$$

Inserting this into (3.2) shows the following

THEOREM 3. *If*

$$\lambda(n-2) > 2\mu + 2\nu n$$

then $id : S^n \to S^n$ *is* Φ*-unstable.*

3.3. Examples. For some of the functionals mentioned in the introduction theorems 2 and 3 give:

1. For the p-energy, $\Phi(A) = (\mathrm{Tr}(A))^p$ we compute $\lambda = pn^{p-1}$, $\mu = 0$ and $\nu = p(p-1)n^{p-2}$. Thus id_{S^n} is unstable if $n > 2p$. Isometries are generally stable if $n \leq 2(p-1)$.
2. The exponential energy, $\Phi(A) = e^{\mathrm{Tr}A}$, has $\lambda = e^n, \mu = 0, \nu = e^n$. Thus isometries are always stable for E_Φ. This is the proof of [**2**].
3. For $\Phi(A) = \mathrm{Tr}(A^p)$ we get $\lambda = p$, $\mu = p(p-1)$ and $\nu = 0$. Thus id_{S^n} is unstable if $n > 2p$.
4. For $\Phi(A) = \mathrm{Tr}\exp(A)$ we get $\lambda = e$, $\mu = e$, $\nu = 0$. Therefore id_{S^n} is unstable if $n > 4$.
5. For $\Phi(A) = \det(A)$ we get $\lambda = 1$, $\mu = -1$, $\nu = 1$. Thus any isometry is stable for $E_{\det}$.
6. Let $\alpha_1, \ldots, \alpha_n$ be the eigenvalues of A and if $n \geq 2$ define the discriminant $\Phi(A) := \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2$. Then E_Φ has $\lambda = \mu = \nu = 0$ and the second variation at an isometry vanishes.

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