NON-STATIONARY PROCESSES AND SPECTRUM

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1. In 1964, L. J. Herbst (**3**) introduced the generalized spectral density function

$$\gamma_0 \left| \sum_{j=0}^{p} a_j e^{ij\lambda} \right|^2$$

for a non-stationary process $\{X(t)\}$ defined by

$$(1) \qquad X(t) = \sum_{j=0}^{p} a_j \sigma_{t-j} \eta_{t-j}, \qquad t = 0, \pm 1, \pm 2, \ldots,$$

where $\{\eta(t)\}$ is a real Gaussian stationary process of discrete parameter and independent variates, the (a_j)'s and (σ_j)'s being constants, the latter, which are ordered in time, having their moduli less than a positive number M. γ_0 in the above expression stands for

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \sigma_t^2,$$

where it is supposed that the limit exists and is finite.

2. In this paper we start from a general non-stationary process $\{X(t)\}$ of discrete parameter with finite first moments $\{E(X(t))\}$ and second moments $\{\rho(t, u)\} = \{E(X(t)\overline{X(u)})\}$ (the bar denoting the complex conjugate), and build up the concept of spectrum in terms of $\{\rho(t, u)\}$. The idea is to construct from the double sequence $\{\rho(t, u)\}$ a sequence $\{\rho(k)\}$ such that

$$\rho(k) = \int_{-\pi}^{\pi} e^{ik\lambda}\, d\sigma(\lambda), \qquad k = 0, \pm 1, \pm 2, \ldots,$$

the function $\sigma(\lambda)$ having the properties of a spectral distribution function. If this can be achieved, $\sigma(\lambda)$ may be called the spectral function, and what corresponds to the derivative of its absolutely continuous part as the spectral density at a continuity point.

3. Let us define

$$(2) \quad \sigma_N(\lambda) = \frac{1}{4\pi N}\left[(\pi + \lambda) \sum_{\substack{t,u=-N;\\ u-t=0}}^{N} \rho(t, u) + \sum_k{}' \left(\sum_{\substack{t,u=-N;\\ u-t=k}}^{N} \rho(t, u) \frac{e^{-ik\lambda} - e^{ik\pi}}{-ik} \right) \right],$$

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where the prime in the sum over k indicates that 0 is omitted. As N varies, we get a sequence of functions $\{\sigma_N(\lambda)\}$. We next define, for each positive integer N,

$$\rho_N(k) = \frac{1}{2N} \sum_{\substack{t,u=-N;\\ u-t=k}}^{N} \rho(t, u), \qquad k = 0, \pm 1, \pm 2, \dots,$$

and assume that the limit as N tends to infinity of $\rho_N(k)$ exists and is finite for each integer k. Under this assumption, we prove the following lemma.

LEMMA. *A sequence of positive integers $\{N_j\}$ exists such that $\lim_{N_j\to\infty}\sigma_{N_j}(\lambda)$ exists, and is a bounded, non-decreasing and non-negative function in $(-\pi, \pi)$, vanishing at $-\pi$.*

Proof. Firstly, whatever N may be, $\sigma_N(-\pi) = 0$. Secondly, calculations show that for $-\pi \leqq \lambda_1 < \lambda_2 \leqq \pi$, we have that

$$\sigma_N(\lambda_2) - \sigma_N(\lambda_1) = \frac{1}{4\pi N} \int_{\lambda_1}^{\lambda_2} E \left| \sum_{t=-N}^{N} e^{it\lambda} X(t) \right|^2. \tag{3}$$

Hence, every $\sigma_N(\lambda)$ is zero at $-\pi$, and is a non-decreasing function of λ in $(-\pi, \pi)$, so that $\sigma_N(\lambda)$ is non-negative. We have that

$$\sigma_N(\pi) = \frac{1}{2N} \sum_{\substack{t,u=-N;\\ u-t=0}}^{N} \rho(t, u) + 0 = \rho_N(0).$$

By our assumption that $\lim_{N\to\infty}\rho_N(0)$ exists and is finite, it follows that $\sigma_N(\pi)$ is less than a constant L whatever N may be. Thus, the sequence of functions $\{\sigma_N(\lambda)\}$ satisfies the conditions of Helly's first theorem (see B. V. Gnedenko (**2**)) for yielding a subsequence $\{\sigma_{N_j}(\lambda)\}$ which converges to a function $\sigma(\lambda)$ at all continuity points of the latter. This function will also possess the same properties, viz., it is bounded and non-negative, non-decreasing, and vanishes at $-\pi$. Thus $\sigma(\lambda)$ is a spectral distribution function.

4. We can now establish the following.

THEOREM 1. *If $-\pi$ and π are continuity points of $\sigma(\lambda)$, then*

$$\rho(k) = \int_{-\pi}^{\pi} e^{ik\lambda}\, d\sigma(\lambda), \qquad k = 0, \pm 1, \pm 2, \dots,$$

where $\rho(k)$ is the limit as N tends to infinity of $\rho_N(k)$ which is assumed to exist and be finite for each k.

Proof. Since $e^{ik\lambda}$ is continuous, and $\{\sigma_{N_j}(\lambda)\}$ are uniformly bounded, we have from Helly's second theorem (see B. V. Gnedenko (**2**)) that

$$\int_{-\pi}^{\pi} e^{ik\lambda}\,d\sigma(\lambda) = \lim_{N_j\to\infty} \int_{-\pi}^{\pi} e^{ik\lambda}\,d\sigma_{N_j}(\lambda)$$

$$= \lim_{N_j\to\infty} \int_{-\pi}^{\pi} e^{ik\lambda}\frac{1}{4\pi N_j}\Bigg[\sum_{\substack{t,u=-N;\\ u-t=0}}^{N} \rho(t,u)$$

$$+ \sum_k{}' \Bigg(\sum_{\substack{t,u=-N_j;\\ u-t=k}}^{N_j} \rho(t,u)\Bigg)e^{-ik\lambda}\Bigg]d\lambda$$

$$= \lim_{N_j\to\infty} \frac{1}{2N_j} \sum_{\substack{t,u=-N_j;\\ u-t=k}}^{N_j} \rho(t,u) = \rho(k).$$

Thus

$$\int_{-\pi}^{\pi} e^{ik\lambda}\,d\sigma(\lambda) = \rho(k)$$

for each given integer k.

5. In view of (3) we may refer to

$$D(\lambda) = \lim_{N_j\to\infty} \frac{1}{4\pi N_j} E\left|\sum_{t=-N_j}^{N_j} e^{it\lambda}X(t)\right|^2 \tag{4}$$

as the spectral density of $\{X(t)\}$ at λ.

Next, we prove the following theorem.

THEOREM 2. *For the process defined by* (1), *the generalized spectral density of Herbst is the same as that given by* (4).

Proof. We shall mention two details regarding the use of symbols and then give the proof. Since the symbol j is used as a suffix for N in the subsequence, we shall replace j in (1) by f. Again, the time-dependent constants (σ_t) will be replaced by (c_t). With these changes,

$$D(\lambda) = \lim_{N_j\to\infty} \frac{1}{4\pi N_j} E\left|\sum_{t=-N_j}^{N_j} e^{it\lambda}\left(\sum_{f=0}^{p} a_f c_{t-f}\eta_{t-f}\right)\right|^2$$

which, if written as

$$\frac{1}{2\pi}\lim_{N_j\to\infty}\frac{1}{2N_j}\left(\sum_{t=-N_j+p}^{N_j-p} |c_t|^2 + R_{N_j}\right)\left|\sum_{f=0}^{p} a_f e^{if\lambda}\right|^2,$$

yields $|R_{N_j}|$ is bounded by a constant which can be chosen in terms of $p, a_0, a_1, a_2, \ldots, a_p$ and M. Then it will follow that

$$D(\lambda) = \frac{1}{2\pi}\lim_{N_j\to\infty}\frac{2(N_j-p)}{2N_j}\frac{1}{2(N_j-p)}\left(\sum_{t=-N_j+p}^{N_j-p}|c_t|^2 + R_{N_j}\right)\left|\sum_{f=0}^{p} a_f e^{if\lambda}\right|^2$$

$$= \frac{1}{2\pi}\gamma_0\left|\sum_{f=0}^{p} a_f e^{if\lambda}\right|^2.$$

Here, γ_0 is the limit of the average overtime instants starting from a negative integral value and going up to the corresponding positive integral value, instead of being the average overtime, the initial instant being taken as zero.

Again, an extra constant factor $1/2\pi$ occurs in our expression. This is because our spectral range is $(-\pi, \pi)$.

Taking these into account, we find that $D(\lambda)$, given by (4), is the generalized spectral density of Herbst for the process $\{X(t)\}$ defined by (1).

6. Remarks. 1. We had to use only the stationary and the non-autocorrelation properties of the η-process to get the spectral density. The Gaussian nature of the process variates was not required, but it would be helpful in forming consistent estimates of it.

2. When

$$\lim_{N\to\infty} \frac{d}{d\lambda}\sigma_N(\lambda) = \frac{d}{d\lambda}\left(\lim_{N\to\infty}\sigma_N(\lambda)\right)$$

we have that $D(\lambda) = d\sigma(\lambda)/d\lambda$. This will hold, for instance, if the series obtained by term-by-term differentiation of the right side of (2) is uniformly convergent.

3. The relationship of the present concept of spectrum to that of the evolutionary spectrum of M. B. Priestley (3) will be taken up in a later study.

4. Spectra of processes obtained by replacing η's by variates of more general stationary processes can also be obtained by the method given in this paper.

5. Some of the arguments employed in this paper have similarity with those of U. Grenander and M. Rosenblatt (**4**) in their proof of the Herglotz theorem.

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