Triangles triply in perspective.

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The first part of the present paper reproduces an article contributed by me to the July issue of *Mathesis*, 1900, entitled "Sur les triangles trihomologiques"; the second part contains fresh developments.

I.

1. If two triangles be triply in perspective, and if lines be drawn from the vertices of either to the points where the opposite sides are met by any one of the three axes of perspective, the conic circumscribed to the triangle and touching these lines passes through the two non-corresponding centres of perspective; also if lines be drawn from the vertices of either triangle through any one of the three centres of perspective to meet the opposite sides in three points, the conic touching the sides at these points touches the two non-corresponding axes of perspective.

Demonstration. Let ABC, A′B′C′ be the two triangles; S_1, S_2, S_3 the three centres of perspective, S_1 lying on AA′, S_2 on AB′, and S_3 on AC′; and u_1, u_2, u_3 the three axes of perspective corresponding to S_1, S_2, S_3 respectively. Let Σ_1 be the conic circumscribed to ABC and touching the lines joining the vertices to the points where u_1 meets the opposite sides, and let Σ_2, Σ_3 be the corresponding conics for u_2, u_3. Let σ_1 be the conic inscribed in ABC and touching its sides at the points where these are met by the lines joining S_1 to the opposite vertices, and let σ_2, σ_3 be the corresponding conics for S_2, S_3.

Take ABC as the triangle of reference, and let the coordinates, trilinear or barycentric, of A′, B′, C′ be

$$A'(\alpha_1, \beta_1, \gamma_1),\ B'(\alpha_2, \beta_2, \gamma_2),\ C'(\alpha_3, \beta_3, \gamma_3).$$

The necessary and sufficient condition that the two triangles shall be triply in perspective is

$$\alpha_1\beta_2\gamma_3 = \alpha_2\beta_3\gamma_1 = \alpha_3\beta_1\gamma_2.$$

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Hence $\alpha_3 : \beta_3 : \gamma_3 = 1/\beta_1\gamma_2 : 1/\gamma_1\alpha_2 : 1/\alpha_1\beta$.

The coordinates of S_1, S_2, S_3 may, therefore, be expressed in the form

$$S_1(\gamma_1\alpha_2,\ \beta_1\gamma_2,\ \gamma_1\gamma_2),\quad S_2(\alpha_1\beta_2,\ \beta_1\beta_2,\ \beta_1\gamma_2),\quad S_3(\alpha_1\alpha_2,\ \alpha_1\beta_2,\ \gamma_1\alpha_2).$$

We have also

$$u_1 \equiv \frac{\alpha}{\alpha_1\beta_2(\gamma_1\alpha_2 - \gamma_2\alpha_1)} + \frac{\beta}{\alpha_1\beta_2(\beta_1\gamma_2 - \beta_2\gamma_1)} + \frac{\gamma}{\gamma_1\gamma_2(\alpha_1\beta_2 - \alpha_2\beta_1)} = 0,$$

with similar equations for u_2 and u_3.

Hence

$$\Sigma_1 \equiv \alpha_1\beta_2(\gamma_1\alpha_2 - \gamma_2\alpha_1)/\alpha + \alpha_1\beta_2(\beta_1\gamma_2 - \beta_2\gamma_1)/\beta + \gamma_1\gamma_2(\alpha_1\beta_2 - \alpha_2\beta_1)/\gamma = 0,$$

with similar equations for Σ_2 and Σ_3.

The equation of Σ_1 is obviously satisfied by the coordinates of S_2 and S_3; therefore S_2 and S_3 lie on Σ_1. Similarly it might be shown that S_3 and S_1 lie on Σ_2, and S_1 and S_2 on Σ_3.

By reciprocating, S_1, S_2, S_3 become u_1, u_2, u_3 and *vice versa,* and Σ_1, Σ_2, Σ_3 become σ_1, σ_2, σ_3 and *vice versa.* Hence u_2 and u_3 touch σ_1, u_3 and u_1 touch σ_2, and u_1 and u_2 touch σ_3. Thus the proposition is completely established.

2. *Corollaries and Particular Cases.*

(i) The following theorem is obviously involved in the general one. If a variable conic σ touch the sides BC, CA, AB of a triangle at X, Y, Z, and also touch a fixed line u which meets BC, CA, AB at D, E, F, the locus of the point of intersection S of AX, BY, CZ is the conic Σ circumscribed to ABC and touching AD, BE, CF.

If u be projected to infinity, σ becomes a parabola, and AD, BE, CF become parallel to BC, CA, AB, so that Σ becomes the Steiner, or minimum circumscribed, ellipse of ABC.*

Another particular case may be noticed. If u be the fourth common tangent of two circles which touch the sides of ABC, the circumconic which touches AD, BE, CF passes through the two corresponding Gergonne points. It might easily be shown that the centre of this conic is the mid-point of one of the sides of ABC.

(ii) If the triangle A′B′C′ in the general theorem degenerate

* This case is given in Steiner's *Theorie der Kegelschnitte* (ed. Schröter, 1867), p. 287.

into three collinear points lying on a line u, the axes u_1, u_2, u_3 coincide with u, and the conics Σ_1, Σ_2, Σ_3 coalesce. Hence we have the theorem that if Σ be a conic circumscribed to a triangle ABC, and if u be the line containing the points where the tangents to Σ at the vertices meet the opposite sides, and if through any point S_1 on Σ, lines AS_1, BS_1, CS_1 be drawn to meet u in A′, B′, C′ respectively, then the connectors AB′, BC′, CA′ are concurrent in a point S_2, and the connectors AC′, BA′, CB′ are concurrent in a point S_3, and S_2 and S_3 lie on Σ.

By projecting u to infinity, Σ becomes the Steiner ellipse, and we obtain the following theorem. If S_1 be any point on the Steiner ellipse of ABC, and one set of parallels be drawn to AS_1, BS_1, CS_1 through C, A, B, and another set through B, C, A, the first set are concurrent in a point S_2, and the second in a point S_3, and S_2 and S_3 lie on the Steiner ellipse.*

(iii) If the vertices of A′B′C′ in the general theorem coincide in a point S, the centres of perspective S_1, S_2, S_3 obviously coincide with S, and the conics σ_1, σ_2, σ_3 coalesce. Hence we have the theorem, the reciprocal of that in (ii), that if Σ be a conic circumscribed to a triangle ABC, and if D, E, F, lying on a line u, be the points where BC, CA, AB are met by the tangents to Σ at the opposite vertices, and S be any point on Σ, and the connector SD meet CA, AB in M and N′, the connector SE meet AB, BC in N and L′, and the connector SF meet BC, CA in L and M′, then L, M, N and L′, M′, N′ lie on two lines touching the conic σ which touches the sides of ABC at the points where these are met by the connectors of S with the opposite vertices.

By projecting u to infinity, Σ becomes the Steiner ellipse and σ a parabola, and we obtain the following theorem. If through any point S on the Steiner ellipse of a triangle ABC, parallels be drawn to the sides, the parallel to BC meeting CA, AB in M and N′, the parallel to CA meeting AB, BC in N and L′, and the parallel to AB meeting BC, CA in L and M′, then L, M, N and L′, M′, N′ lie on two lines † touching the parabola which touches the sides of ABC

* Stated and proved by means of orthogonal projection by Prof. Neuberg in *Mathesis*, 2nd series, Vol. V., p. 199.

† The proposition up to this point is given in *Mathesis*, 1st series, Vol. II., p. 42, by Prof. Neuberg.

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at the points where these are met by the connectors of S with the opposite vertices.

(iv) Of several applications which might be made to the Brocardian geometry of the triangle, the following, an obvious particular case of the fundamental theorem, may be stated. If A′B′C′ be the first Brocard triangle of ABC, and D, E, F be the points (collinear) where B′C′, C′A′, A′B′ meet BC, CA, AB respectively, then (1) DA′, EB′, FC′ are tangents to the Brocard circle of ABC, and (2) DA, EB, FC are tangents to the circumconic of ABC which passes through the Brocard points.

In a note in the October issue of *Mathesis*, 1900, referring to my article,* M. Ripert points out that if the two triangles ABC, A′B′C′ of the fundamental theorem are inscribed in the same conic Ω,† the triangles determined by the fourth points of intersection of Ω with the conics Σ_1, Σ_2, Σ_3, and with the corresponding conics circumscribed to A′B′C′, Σ_1', Σ_2', Σ_3', are in triple perspective (1) with ABC, A′B′C′, and (2) with each other. M. Ripert supplies a proof of the first part of the proposition, and says that the second part can be established by "un calcul laborieux mais sans difficulté." This note suggested to me the developments contained in the next section.

II. ‡

Let A′B′C′ be a triangle, the coordinates of whose vertices, barycentric or trilinear, with reference to the triangle ABC, are

$$A'(p\lambda,\ q\mu,\ r\nu),\quad B'(p\mu,\ q\nu,\ r\lambda),\quad C'(p\nu,\ q\lambda,\ r\mu).$$

By causing λ, μ, ν to vary, while p, q, r remain constant, we obtain a group of triangles, T, every one of which is obviously in triple perspective with the triangle of reference ABC. It can readily be

* and to a paper of his own which I have so far been unable to procure, read before the French Association, 1900, entitled, "Sur des groupes de triangles trihomologiques inscrits ou circonscrits à une même conique ou à une famille de coniques."

† This is the case in the theorem of § I., (2), (ii) if we replace Σ by Ω, and interchange A′, B′, C′ with S_1, S_2, S_3; for ABC and $S_1S_2S_3$ are obviously in triple perspective, A′, B′, C′ being the centres.

‡ One or two alterations in the proofs given in this part of the paper as read to the Society have been made on the suggestion of Mr R. F. Muirhead.

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verified that the triangles of the group T are inscribed in the conics of the group

$$\Omega \equiv p\beta\gamma + q\gamma\alpha + r\alpha\beta + \delta\left(\frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r}\right)^2 = 0,$$

where δ is a variable parameter equal to

$$-\frac{pqr(\mu\nu + \nu\lambda + \lambda\mu)}{(\lambda + \mu + \nu)^2}.$$

The triangle of reference is obviously inscribed in the conic of the Ω group for which $\delta = 0$. The conics Ω have double contact with each other on the line

$$k \equiv \frac{\alpha}{p} + \frac{\beta}{q} + \frac{\gamma}{r} = 0\,;$$

k is the line on which the tangents to the conic

$$p\beta\gamma + q\gamma\alpha + r\alpha\beta = 0$$

at the vertices of ABC meet the opposite sides; it is also the trilinear polar with respect to ABC of the point P (p, q, r). Further it could easily be shown that P is the pole of k with respect to every one of the conics Ω.

This being premised, we have the following propositions.

(1) The triangles T are in triple perspective with each other two and two.

(2) The tangents at the vertices of every triangle of the group T to the conic of the group Ω circumscribed to the triangle, meet the opposite sides on the line k.

(3) All the triangles of the group T which are inscribed in the same conic of the group Ω are circumscribed to another conic of the group Ω.

(4) The lines which join the vertices of every triangle of the group T to the points where the opposite sides are touched by the inscribed conic of the group Ω, cointersect in the point P.

(5) P is the trilinear pole of k with respect to every triangle of the group T.

(6) The triangles determined by the three centres of perspective and by the three axes of perspective of any pair of triangles belonging to the group T, also belong to the group.

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(7) The three centres of perspective of two triangles of the group T which are inscribed in, or circumscribed to, the same conic of Ω, lie on k, and the three axes of perspective pass through P.

(8) The lines joining the vertices, two and two, of two triangles of the group T, when not concurrent, form a triangle belonging to the group; and the points of intersection, when not collinear, of the sides taken two and two, also determine a triangle belonging to the group.

(9) If $A'B'C'$, $A''B''C''$ be two triangles of the group T, and $B''C''$, $C''A''$, $A''B''$ meet k in D, E, F respectively, the following triads of lines are concurrent:

$$\begin{aligned} &A'D,\ C'E,\ B'F \text{ in the point U,}\\ &A'E,\ C'F,\ B'D \text{ in the point V,}\\ &A'F,\ C'D,\ B'E \text{ in the point W;}\end{aligned}$$

the triangle UVW belongs to the group T and is inscribed in the same conic of the group Ω as $A'B'C'$;
and $A''.\,A'UPD$, $B''.\,C'UPE$, $C''.\,B'UPF$, etc., are harmonic pencils.

(10) The fourth points of intersection of a conic of the group Ω, circumscribed to a triangle T_1 of the group T, with the three conics circumscribed to T_1 and passing through the vertices, two and two, of any other triangle T_2 of the group T, determine a triangle belonging to the group T.

This is a generalisation of M. Ripert's proposition.

(11) If T_1, T_2, T_3 be three triangles of the group T, the three circumconics of T_3 which pass through the vertices of T_1, two and two, cut the three circumconics of T_3 which pass through the vertices of T_2, two and two, in three triads of points, excluding the vertices of T_3, which determine three triangles belonging to the group T.

Propositions correlative to (9), (10), and (11) are obtained by reciprocation.

The proof of the foregoing theorems can be simplified by transforming the coordinates of the vertices of $A'B'C'$ so that $p=q=r$. The transformation is homographic, and consequently the groups T and Ω are projectively related to the groups T′ and Ω' into which they are transformed. Hence to establish the propositions enunciated for T and Ω, which, it will be noticed, involve only projective properties, it is sufficient to prove the corresponding statements for T′ and Ω'.

The coordinates of the vertices of a triangle of the group T′ are of the form

$$(\lambda, \mu, \nu), \quad (\mu, \nu, \lambda), \quad (\nu, \lambda, \mu).$$

If we restrict ourselves, as we may without loss of generality, to *barycentric* coordinates, the vertices of such a triangle form what are called *isobaric* points, and the triangle itself may be said, for convenience, to be *isobaric* with respect to ABC, the triangle of reference.

Let A′B′C′ be an isobaric triangle with respect to ABC, the coordinates of A′, B′, C′, with reference to ABC, being

$$A'(\lambda, \mu, \nu), \quad B'(\mu, \nu, \lambda), \quad C'(\nu, \lambda, \mu).$$

The form of the coordinates shows at once that A′B′C′ is in triple perspective with ABC. The equations of the sides of A′B′C′ are

$$\begin{aligned} B'C' &\equiv l\alpha + m\beta + n\gamma = 0, \\ C'A' &\equiv m\alpha + n\beta + l\gamma = 0, \\ A'B' &\equiv n\alpha + l\beta + m\gamma = 0, \end{aligned}$$

where $\quad l = \mu\nu - \lambda^2, \quad m = \nu\lambda - \mu^2, \quad n = \lambda\mu - \nu^2.$

The perpendiculars from A′, B′, C′ on BC are proportional to λ, μ, ν. Hence the equation of BC with reference to A′B′C′ is

$$\lambda\alpha + \mu\beta + \nu\gamma = 0.$$

Similarly the equations of CA and AB with reference to A′B′C′ are

$$\mu\alpha + \nu\beta + \lambda\gamma = 0,$$

$$\text{and} \quad \nu\alpha + \lambda\beta + \mu\gamma = 0.$$

The form of these equations shows that ABC is isobaric with respect to A′B′C′, the coordinates of the vertices of the former with reference to the latter being

$$A(l, m, n), \quad B(m, n, l), \quad C(n, l, m).$$

Let A″B″C″ be another triangle of the T′ group. Let the coordinates of its vertices with reference to ABC be

$$A''(\lambda', \mu', \nu'), \quad B''(\mu', \nu', \lambda'), \quad C''(\nu', \lambda', \mu');$$

and let $\quad l' = \mu'\nu' - \lambda'^2, \quad m' = \nu'\lambda' - \mu'^2, \quad n' = \lambda'\mu' - \nu'^2.$

Then the perpendiculars from A″, B″, C″ to B′C′ are proportional to L, M, N where

$$L = l\lambda' + m\mu' + n\nu', \quad M = l\mu' + m\nu' + n\lambda', \quad N = l\nu' + m\lambda' + n\mu'.$$

Therefore the equation of B′C′ with reference to A″B″C″ is

$$L\alpha + M\beta + N\gamma = 0.$$

Similarly the equations of C′A′ and A′B′ with reference to A″B″C″ are

$$N\alpha + L\beta + M\gamma = 0,$$

$$\text{and} \quad M\alpha + N\beta + L\gamma = 0.$$

The form of these equations shows that A′B′C′ and A″C″B″ are isobaric with respect to each other. The coordinates of the vertices of A′B′C′ with reference to A″B″C″ are

$$A'(L', M', N'), \quad B'(N', L', M'), \quad C'(M', N', L'),$$

where
$$L' : M' : N' = MN - L^2 : NL - M^2 : LM - N^2$$
$$= \lambda l' + \mu m' + \nu n' : \lambda m' + \mu n' + \nu l' : \lambda n' + \mu l' + \nu m'.$$

The coordinates of the vertices of A″B″C″ with reference to A′B′C′ are readily found to be

$$A''(L, M, N), \quad B''(N, L, M), \quad C''(M, N, L).$$

It thus appears that the triangles of the T′ group are isobaric with respect to each other, and are, consequently, triply in perspective, two and two. Proposition (1) follows from this immediately. We also infer that any triangle of either the T′ or the T group may be taken as the triangle of reference without any alteration in the general form of the coordinates of the vertices of the other triangles of the group.

The triangles of the T′ group are inscribed in the group of conics

$$\Omega' \equiv \beta\gamma + \gamma\alpha + \alpha\beta + \delta(\alpha + \beta + \gamma)^2 = 0.$$

When $\delta = 0$, this is the Steiner, or minimum circumscribed, ellipse of the triangle of reference. Since the line

$$k' \equiv \alpha + \beta + \gamma = 0$$

on which the conics have double contact is now the line at infinity, the other conics are concentric with this ellipse, and similar and similarly situated to it, the common centre P′ being the centroid of the triangle of reference.

The polar of a point (λ, μ, ν) with respect to Ω' is

$$\Sigma a\{(2\delta+1)(\lambda+\mu+\nu)-\lambda\}=0.$$

This is obviously parallel to the line

$$\lambda a+\mu\beta+\nu\gamma=0,$$

which, in its turn, can be readily proved parallel to the line

$$a(\mu\nu-\lambda^2)+\beta(\nu\lambda-\mu^2)+\gamma(\lambda\mu-\nu^2)=0,$$

the connector of the points (μ, ν, λ) and (ν, λ, μ). Hence, as a particular case, the tangent to a conic of the Ω' group at a vertex of a triangle of the T′ group inscribed in it, is parallel to the opposite side of the triangle. From this we infer the truth of Proposition (2).

Another important inference from the same result is that every triangle of the T′ group is an inscribed triangle of maximum area in the ellipse of the Ω' group in which it is inscribed. Otherwise: Every conic of the Ω' group is the Steiner ellipse of every triangle of the T′ group inscribed in it. This form of statement makes it clear that any triangle of either the T′ or the T group may be taken as the triangle of reference without any alteration in the general form of the equation for the associated group of conics.

It is well known * that the group of triangles of maximum area inscribed in an ellipse are circumscribed to a concentric, similar and similarly situated ellipse; that the lines joining the points where the latter ellipse touches their sides, to the opposite vertices, cointersect in the centre of the ellipses; and that this point is the centroid of all the triangles. Applying these statements to the triangles of the T′ group and extending them to those of the T group, we obtain Propositions (3), (4) and (5).†

The coordinates of the vertices of any triangle of the T′ group, with reference to any other triangle of the group, are of the form

$$(\lambda, \mu, \nu), \quad (\mu, \nu, \lambda), \quad (\nu, \lambda, \mu).$$

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* See "*Résumé des propriétés concernant les triangles d'aire maximum inscrits dans l'ellipse*," by M. E. N. Barisien, *Mathesis*, 2nd series, Vol. V., p. 42.

† Of the many properties that may be deduced from the fact that the T′ group consists of triangles of maximum area inscribed in concentric, similar and similarly situated ellipses, one of the most interesting is that all the triangles of the group have the same Brocard angle.

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Hence the three centres of perspective of the two triangles are

$$\left(\frac{1}{\lambda}, \frac{1}{\nu}, \frac{1}{\mu}\right), \quad \left(\frac{1}{\nu}, \frac{1}{\mu}, \frac{1}{\lambda}\right), \quad \left(\frac{1}{\mu}, \frac{1}{\lambda}, \frac{1}{\nu}\right),$$

and the three axes of perspective

$$\alpha/l + \beta/n + \gamma/m = 0,$$
$$\alpha/n + \beta/m + \gamma/l = 0,$$
$$\alpha/m + \beta/l + \gamma/n = 0,$$

where $l = \mu\nu - \lambda^2$, $m = \nu\lambda - \mu^2$, $n = \lambda\mu - \nu^2$.

Thus the triangles determined by the three centres and the three axes of perspective are isobaric with respect to the triangle of reference, and therefore belong to the group T′. Proposition (6) follows immediately.

If the two triangles are inscribed in the same conic of the group Ω', taking as before one of them as the triangle of reference, we have the condition

$$\frac{1}{\lambda} + \frac{1}{\mu} + \frac{1}{\nu} = 0$$

Hence the three centres of perspective lie on

$$\alpha + \beta + \gamma = 0,$$

the line at infinity.

Again, from the condition $\Sigma 1/\lambda = 0$, we deduce

$$l : m : n = \lambda : \mu : \nu.$$

Hence the equations of the three axes of perspective are

$$\alpha/\lambda + \beta/\nu + \gamma/\mu = 0,$$
$$\alpha/\nu + \beta/\mu + \gamma/\lambda = 0,$$
$$\alpha/\mu + \beta/\lambda + \gamma/\nu = 0.$$

These lines obviously cointersect in the point (1, 1, 1), the common centre of the conics. This establishes Proposition (7).

The coordinates of the vertices of any triangle A′B′C′ of the T′ group, with reference to any other triangle ABC of the group, being

$$A'(\lambda, \mu, \nu), \quad B'(\mu, \nu, \lambda), \quad C'(\nu, \lambda, \mu),$$

we have

$$A'A \equiv 0.\alpha + \nu\beta - \mu\gamma = 0,$$
$$B'C \equiv \nu\alpha - \mu\beta + 0.\gamma = 0,$$
$$C'B \equiv -\mu\alpha + 0.\beta + \nu\gamma = 0.$$

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The form of these equations shows that the triangle formed by A′A, B′C, C′B belongs to the isobaric group. The same is true for the triangles formed by B′B, C′A, A′C and C′C, A′B, B′A. Thus the first part of Proposition (8) is established.

Again, from the equations of the sides of A′B′C′ it is evident that the coordinates of the points of intersection of the pairs

$$(B'C',\ BC),\quad (C'A',\ AB),\quad (A'B',\ CA)$$

are $(0,\ n,\ -m),\quad (n,\ -m,\ 0),\quad (-m,\ 0,\ n),$

and consequently that the triangle formed by these points belongs to the isobaric group. The same may be proved true of the triangles formed by the points of intersection of the pairs

$$(C'A',\ BC),\quad (A'B',\ AB),\quad (B'C',\ CA);$$

and $(A'B',\ BC),\quad (B'C',\ AB),\quad (C'A',\ CA).$

Thus Proposition (8) is completely established.

Parallels through A′, B′, C′

(1) to BC, CA, AB, (2) to CA, AB, BC, and (3) to AB, BC, CA obviously cointersect in the three points

$$U(\lambda,\ \nu,\ \mu,),\quad V(\nu,\ \mu,\ \lambda),\quad W(\mu,\ \lambda,\ \nu).$$

From the form of these coordinates, UVW is seen to belong to the isobaric group, and to be inscribed in the same conic of the group Ω' as A′B′C′. Obviously also the pairs (AA′, AU), (BB′, BU), (CC′, CU), etc., are isotomic conjugates with respect to ABC.* We thus obtain Proposition (9).

The coordinates of the vertices of any triangle A′B′C′ of the T′ group, with reference to any other triangle ABC of the group, being $A'(\lambda,\ \mu,\ \nu),\quad B'(\mu,\ \nu,\ \lambda),\quad C'(\nu,\ \lambda,\ \mu),$

we have as equations of the three conics Σ_1, Σ_2, Σ_3, circumscribed to ABC and passing through the pairs of points (B′, C′), (C′, A′), (A′, B′),

$$\Sigma_1 \equiv \frac{l}{\lambda\alpha} + \frac{m}{\mu\beta} + \frac{n}{\nu\gamma} = 0,$$

$$\Sigma_2 \equiv \frac{m}{\mu\alpha} + \frac{n}{\nu\beta} + \frac{l}{\lambda\gamma} = 0,$$

$$\Sigma_3 \equiv \frac{n}{\nu\alpha} + \frac{l}{\lambda\beta} + \frac{m}{\mu\gamma} = 0,$$

where $l = \mu\nu - \lambda^2$, etc.

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* This theorem is given by M. G. Rogier, in a "Note sur les points isobariques," *Journal de Mathématiques Spéciales*, 1887, p. 103.

The equation of the conic of the Ω' group circumscribed to ABC is

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=0.$$

Hence the fourth points of intersection of this conic with Σ_1, Σ_2, Σ_3 are $\quad (\lambda', \mu', \nu'), \quad (\mu', \nu', \lambda'), \quad (\nu', \lambda', \mu'),$

where $$\lambda'=\frac{\mu\nu}{\mu-\nu}, \quad \mu'=\frac{\nu\lambda}{\nu-\lambda}, \quad \nu'=\frac{\lambda\mu}{\lambda-\mu}.$$

These points are isobaric. Hence we deduce Proposition (10).

The coordinates of the vertices of two triangles T_1 and T_2 of the T′ group, with reference to a third triangle T_3 of the group, being

$$(\lambda_1, \mu_1, \nu_1), \quad (\mu_1, \nu_1, \lambda_1), \quad (\nu_1, \lambda_1, \mu_1)$$

and $$(\lambda_2, \mu_2, \nu_2), \quad (\mu_2, \nu_2, \lambda_2), \quad (\nu_2, \lambda_2, \mu_2),$$

the equations of the conics $\Sigma_{ik}(i=1, 2\,;\ k=1, 2, 3)$ circumscribed to T_3 and passing through the vertices of the triangles T_i, two and two, are

$$\Sigma_{i1}\equiv\frac{l_i}{\lambda_i\alpha}+\frac{m_i}{\mu_i\beta}+\frac{n_i}{\nu_i\gamma}=0,$$

$$\Sigma_{i2}\equiv\frac{m_i}{\mu_i\alpha}+\frac{n_i}{\nu_i\beta}+\frac{l_i}{\lambda_i\gamma}=0,$$

$$\Sigma_{i3}\equiv\frac{n_i}{\nu_i\alpha}+\frac{l_i}{\lambda_i\beta}+\frac{m_i}{\mu_i\gamma}=0,$$

where $$l_i=\mu_i\nu_i-\lambda_i^2, \quad m_i=\nu_i\lambda_i-\mu_i^2, \quad n_i=\lambda_i\mu_i-\nu_i^2.$$

Therefore the coordinates of the fourth points of intersection of the pairs $(\Sigma_{11}, \Sigma_{21})$, $(\Sigma_{12}, \Sigma_{22})$, $(\Sigma_{13}, \Sigma_{23})$, are

$$(\lambda', \mu', \nu'), \quad (\mu', \nu', \lambda'), \quad (\nu', \lambda', \mu'),$$

where $$\lambda'\left(\frac{m_1n_2}{\mu_1\nu_2}-\frac{m_2n_1}{\mu_2\nu_1}\right)=\mu'\left(\frac{n_1l_2}{\nu_1\lambda_2}-\frac{n_2l_1}{\nu_2\lambda_1}\right)=\nu'\left(\frac{l_1m_2}{\lambda_1\mu_2}-\frac{l_2m_1}{\lambda_2\mu_1}\right).$$

These points, therefore, are isobaric. The same may be proved for the fourth points of intersection of the pairs

$$(\Sigma_{11}, \Sigma_{22}), \quad (\Sigma_{12}, \Sigma_{23}), \quad (\Sigma_{13}, \Sigma_{21}),$$

and of the pairs $\quad (\Sigma_{11}, \Sigma_{23}), \quad (\Sigma_{12}, \Sigma_{21}), \quad (\Sigma_{13}, \Sigma_{22}).$

Thus Proposition (11) is established.

Of theorems the enunciation of which is identical for both the T and the T′ groups the following may be stated. (i) The vertices of the triangle formed by the axes of perspective of two triangles of the group are the trilinear poles, with respect to either triangle, of the sides of the triangle determined by the centres of perspective; (ii) the centres of perspective are the trilinear poles, with respect to either triangle, of the lines which join the trilinear poles of the axes of perspective.*

In conclusion it may be noticed that the triangles of the T group may be projected into a group of equilateral triangles inscribed in a group of concentric circles. In this way a number of the properties of the group may be very simply obtained, as, for example, that every triangle of the group is in quadruple perspective with an infinity of other triangles of the group, P and k being the fourth centre and axis of perspective.

* These theorems are proved for the case of three isobaric points and the triangle of reference by M. Rogier, *loc. cit.*