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THE REGULAR PART OF A SEMIGROUP OF LINEAR TRANSFORMATIONS WITH RESTRICTED RANGE

WORACHEAD SOMMANEE and KRITSADA SANGKHANAN✉

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Abstract

Let V be a vector space and let $T(V)$ denote the semigroup (under composition) of all linear transformations from V into V. For a fixed subspace W of V, let $T(V, W)$ be the semigroup consisting of all linear transformations from V into W. In 2008, Sullivan ['Semigroups of linear transformations with restricted range', *Bull. Aust. Math. Soc.* **77**(3) (2008), 441–453] proved that

$$Q = \{\alpha \in T(V, W) : V\alpha \subseteq W\alpha\}$$

is the largest regular subsemigroup of $T(V, W)$ and characterized Green's relations on $T(V, W)$. In this paper, we determine all the maximal regular subsemigroups of Q when W is a finite-dimensional subspace of V over a finite field. Moreover, we compute the rank and idempotent rank of Q when W is an n-dimensional subspace of an m-dimensional vector space V over a finite field F.

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1. Introduction

Let $T(X)$ be the set of all full transformations from a nonempty set X into itself. If $X = \{1, 2, \ldots, n\}$, with $n \in \mathbb{N}$, we write $T_n = T(X)$. It is well known that $T(X)$ is a regular semigroup under composition of functions. The properties of $T(X)$ have been widely studied. In particular, in 2002, You [15] determined all the maximal regular subsemigroups of $T(X)$ when X is finite.

The *rank* of a semigroup S is the smallest number of elements required to generate S and is denoted by $\operatorname{rank}(S)$, that is,

$$\operatorname{rank}(S) = \min\{|A| : A \subseteq S \text{ and } \langle A\rangle = S\}.$$

If S is generated by its set of idempotents $E(S)$, then the idempotent rank of S is defined by

$$\operatorname{idrank}(S) = \min\{|A| : A \subseteq E(S) \text{ and } \langle A\rangle = S\}.$$

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It is known that $\operatorname{rank}(T_n) = 3$ when $n \geq 3$ and that T_n has no idempotent rank for $n \geq 2$.

For a fixed nonempty subset Y of X, let

$$T(X, Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\},$$

where $X\alpha$ denotes the range of α. Then $T(X, Y)$ is a subsemigroup of $T(X)$. In 1975, Symons [12] described all the automorphisms of $T(X, Y)$. He also determined when $T(X_1, Y_1)$ is isomorphic to $T(X_2, Y_2)$. In 2005, Nenthein, Youngkhong and Kemprasit [6] characterized the regular elements of $T(X, Y)$.

In 2008, Sanwong and Sommanee [9] defined

$$F(X, Y) = \{\alpha \in T(X, Y) : X\alpha \subseteq Y\alpha\}$$

and showed that $F(X, Y)$ is the largest regular subsemigroup of $T(X, Y)$. They also determined a class of maximal inverse subsemigroups of $T(X, Y)$ and characterized its Green's relations.

In 2011, Mendes-Gonçalves and Sullivan [4] determined all the ideals of $T(X, Y)$. In the same year, Sanwong [8] described Green's relations, ideals and all the maximal regular subsemigroups of $F(X, Y)$. Also, the author proved that every regular semigroup S can be embedded in $F(S^1, S)$. Later, in 2013, Sommanee and Sanwong [10] computed the rank of $F(X, Y)$ when X is a finite set. Moreover, they obtained the rank and idempotent rank of its ideals. In 2014, Fernandes and Sanwong [2] computed the rank of $T(X, Y)$.

For a vector space V over a field F, let $T(V)$ be the set of all linear transformations from V into V. It is known that $T(V)$ is a regular semigroup under composition of functions (see [3, page 63]).

For a fixed subspace W of V, let

$$T(V, W) = \{\alpha \in T(V) : V\alpha \subseteq W\}.$$

Then $T(V, W)$ is a subsemigroup of $T(V)$. In 2007, Nenthein and Kemprasit [5] proved that $\alpha \in T(V, W)$ is a regular element of $T(V, W)$ if and only if $V\alpha = W\alpha$. As a consequence, they showed that $T(V, W)$ is regular if and only if either $V = W$ or $W = \{0\}$. Later, in 2008, Sullivan [11] proved that the set

$$Q = \{\alpha \in T(V, W) : V\alpha \subseteq W\alpha\}$$

consisting of all regular elements in $T(V, W)$ is the largest regular subsemigroup of $T(V, W)$ (see [11, Lemma 1]). This semigroup plays a crucial role in the characterization of Green's relations on $T(V, W)$. The author also showed that Q is always a right ideal of $T(V, W)$ and described all the ideals of Q and $T(V, W)$.

Here, we determine all the maximal regular subsemigroups of Q when W is a finite-dimensional subspace of V over a finite field. Moreover, we compute the rank of Q when W is an n-dimensional subspace of an m-dimensional vector space V over a finite field F with q elements.

2. Preliminaries and notation

For convenience, we adopt the convention introduced in [1, page 241]: namely, for $\alpha \in T(X)$, we write

$$\alpha = \begin{pmatrix} A_i \\ a_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$, that $X\alpha = \{a_i\}$ and that $A_i = a_i\alpha^{-1}$, the set of all inverse images of a_i under α.

Similarly, we can use the above notation for a linear transformation in $T(V)$, where V is a vector space. To construct a map $\alpha \in T(V)$, we first choose a basis $\{e_i\}$ for V and a subset $\{u_i\}$ of V and then let $e_i\alpha = u_i$ for each $i \in I$; we then extend this action by linearity to the whole of V. To shorten this process, we simply say, given $\{e_i\}$ and $\{a_i\}$ within context, that $\alpha \in T(V)$ is defined by letting

$$\alpha = \begin{pmatrix} e_i \\ u_i \end{pmatrix}.$$

In this paper, a subspace U of a vector space V generated by a linearly independent subset $\{e_i\}$ of V is denoted by $\langle e_i \rangle$ and, when we write $U = \langle e_i \rangle$, we mean that the set $\{e_i\}$ is a basis of U with $\dim U = |I|$. For $\alpha \in T(V)$, if we write $U\alpha = \langle u_i\alpha \rangle$, it means that the set $\{u_i\alpha\}$ is a basis of the subspace $U\alpha$ of V and that $u_i \in U$ for all i.

Let $\{u_i\}$ be a subset of a vector space V. The expression $\sum a_iu_i$ denotes a finite linear combination

$$a_{i_1}u_{i_1} + a_{i_2}u_{i_2} + \cdots + a_{i_n}u_{i_n}$$

for some $n \in \mathbb{N}$, $u_{i_1}, u_{i_2}, \ldots, u_{i_n} \in \{u_i\}$ and scalars $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$.

A vector space V is the internal direct sum of a family $\{S_1, S_2, \ldots, S_n\}$ of subspaces of V, written as

$$V = S_1 \oplus S_2 \oplus \cdots \oplus S_n$$

if $V = \{s_1 + \cdots + s_n : s_i \in S_i\}$ and $S_i \cap (\sum_{j \neq i} S_j) = \{0\}$. Notice that if $V = S \oplus T$ for some subspaces S and T of V, then T is called a *complement* of S in V.

Lemma 2.1 [13, Theorem 6]. *Let V be an n-dimensional vector space over a finite field F. Let U be any k-dimensional subspace of V. Then there are $|F|^{k(n-k)}$ distinct complements of U in V.*

Lemma 2.2. *Let U, V and W be subspaces of a vector space S such that $S = U \oplus V$. If $V \subseteq W$, then $W = (U \cap W) \oplus V$.*

Proof. It is clear that $(U \cap W) \cap V \subseteq U \cap V = \{0\}$. Let $w \in W \subseteq S = U \oplus V$. Then we can write $w = u + v$ for some $u \in U$ and $v \in V$. Since $w \in W$ and $v \in V \subseteq W$, we obtain $u = w - v \in W$, which implies that $w = u + v \in (U \cap W) + V$. Hence $W = (U \cap W) \oplus V$. □

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Throughout the rest of the paper, we let V be a vector space and W be a fixed subspace of V. For $\alpha \in T(V)$, the *kernel* of α is $\ker\alpha = \{v \in V : v\alpha = 0\}$. Since $W \subseteq V$, we obtain $W\alpha \subseteq V\alpha$. Hence

$$Q = \{\alpha \in T(V, W) : V\alpha \subseteq W\alpha\} = \{\alpha \in T(V, W) : V\alpha = W\alpha\}.$$

The following characterization of Green's relations on $T(V)$ is well known (see [3, page 63]). For $\alpha, \beta \in T(V)$:

(1) $\alpha\mathcal{L}\beta$ if and only if $V\alpha = V\beta$;
(2) $\alpha\mathcal{R}\beta$ if and only if $\ker\alpha = \ker\beta$;
(3) $\alpha\mathcal{D}\beta$ if and only if $\dim(V\alpha) = \dim(V\beta)$; and
(4) $\mathcal{D} = \mathcal{J}$.

Since Q is a regular subsemigroup of $T(V)$, we obtain by Hall's Theorem [3, Proposition 2.4.2] that

$$\alpha\mathcal{L}\beta \text{ in } Q \text{ if and only if } V\alpha = V\beta \quad \text{and} \quad \alpha\mathcal{R}\beta \text{ in } Q \text{ if and only if } \ker\alpha = \ker\beta.$$

From [11, Lemma 5], $\alpha\mathcal{J}\beta$ in Q if and only if $\dim(V\alpha) = \dim(V\beta)$ and if $\mathcal{D} = \mathcal{J}$ on Q. Thus, we get the following description of Green's relations on Q.

Lemma 2.3. *Let $\alpha, \beta \in Q$. Then:*

(1) $\alpha\mathcal{L}\beta$ *if and only if* $V\alpha = V\beta$;
(2) $\alpha\mathcal{R}\beta$ *if and only if* $\ker\alpha = \ker\beta$;
(3) $\alpha\mathcal{D}\beta$ *if and only if* $\dim(V\alpha) = \dim(V\beta)$; *and*
(4) $\mathcal{D} = \mathcal{J}$.

Moreover, the author in [11] described the ideals of Q as given by the following theorem.

Theorem 2.4 [11, Theorem 8]. *The ideals of Q are precisely the sets*

$$Q_k = \{\alpha \in Q : \dim(V\alpha) \leq k\},$$

where $0 \leq k \leq \dim(W)$.

Throughout the rest of the paper, W is a subspace of V over a field F, with $\dim W = n$. If n is finite, define

$$J(k) = \{\alpha \in Q : \dim(V\alpha) = k\},$$

where $0 \leq k \leq n = \dim W$. Then $J(k)$ is a $\mathcal{J}$-class of Q. Moreover, each $J(k)$ contains at least one idempotent. Indeed, let $\{w_1, \ldots, w_k\}$ be a linearly independent subset of a basis of W. Thus, we can write $V = \langle w_1, \ldots, w_k\rangle \oplus \langle v_j\rangle$ for some subspace $\langle v_j\rangle$ of V and define

$$\epsilon = \begin{pmatrix} w_1 & \cdots & w_k & v_j \\ w_1 & \cdots & w_k & 0 \end{pmatrix}.$$

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Then ϵ is an idempotent in Q with $\dim(V\epsilon) = k$: that is, ϵ is an idempotent in $J(k)$. In particular, $J(0) = \{0_V\}$, where 0_V denotes the linear transformation of V with range $\{0\}$.

We note that $V\alpha = W$ for all $\alpha \in J(n)$ since $V\alpha \subseteq W$ and $\dim(V\alpha) = n = \dim W$ is finite. It follows that $J(n)$ consists of a single $\mathcal{L}$-class.

For every k such that $0 \leq k \leq n$, we let Q_k be defined as in Theorem 2.4. Then

$$Q_k = J(0) \cup J(1) \cup \cdots \cup J(k)$$

and, clearly, $Q_n = Q$.

Since Q is regular and Q_k is an ideal of Q by Theorem 2.4, we obtain that Q_k is a regular subsemigroup of Q. Hence the following lemma.

Lemma 2.5. *Q_k is a regular subsemigroup of Q.*

We state and prove the following lemmas which will be used in this paper.

Lemma 2.6. *If $\epsilon \in T(V)$ is an idempotent, then* $\ker \epsilon \cap V\epsilon = \{0\}$.

Proof. Assume that ϵ is an idempotent in $T(V)$. In general, $0 \in \ker \epsilon \cap V\epsilon$. Let $u \in \ker \epsilon \cap V\epsilon$. Then $u\epsilon = 0$ and $u = v\epsilon$ for some $v \in V$. We obtain $u = v\epsilon = (v\epsilon)\epsilon = u\epsilon = 0$, which implies that $\ker \epsilon \cap V\epsilon = \{0\}$. □

Lemma 2.7. *If $\alpha \in Q$ and $V\alpha = W\alpha = \langle w_i\alpha \rangle$, where $w_i \in W$ for each i, then $\{w_i\}$ is linearly independent and* $V = \ker \alpha \oplus \langle w_i \rangle$.

Proof. See the proof as given in the converse part of [11, Lemma 1]. □

We will denote the set of all automorphisms of V over a field F by $\mathrm{GL}(V)$ and the set of $n \times n$ invertible matrices with coefficients in a field F with q elements by $\mathrm{GL}(n, q)$.

It is well known that $\mathrm{GL}(V)$ is a group under the composition of functions and that $\mathrm{GL}(n, q)$ is a group under matrix multiplication, with the identity matrix as the identity. $\mathrm{GL}(n, q)$ is called the *general linear group of degree n*. If V is an n-dimensional vector space over a finite field F with $|F| = q$, then $\mathrm{GL}(V)$ and $\mathrm{GL}(n, q)$ are isomorphic (see [7, page 219]). Furthermore, $|\mathrm{GL}(n, q)| = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1})$ (see [7, Theorem 8.5]).

3. More results on Q

In this section, we characterize the $\mathcal{H}$-classes of Q and, using this, we show that the $\mathcal{J}$-class consisting of all mappings α in Q whose range has maximum dimension n is a regular subsemigroup of Q. By Lemma 2.3, for $\alpha, \beta \in Q$,

$$\alpha \mathcal{H} \beta \text{ if and only if } V\alpha = V\beta \quad \text{and} \quad \ker \alpha = \ker \beta.$$

Lemma 3.1. *Let $\epsilon \in Q$ be an idempotent. Then* $H_\epsilon = \{\epsilon\sigma : \sigma \in \mathrm{GL}(V\epsilon)\}$.

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Proof. We note that, for each $\sigma \in \mathrm{GL}(V\epsilon)$, $(V\epsilon)\sigma = V\epsilon$ and $\epsilon\sigma\colon V \to V$ is a linear transformation such that $V(\epsilon\sigma) = (V\epsilon)\sigma = V\epsilon \subseteq W$ and $V(\epsilon\sigma) = (V\epsilon)\sigma = (W\epsilon)\sigma = W(\epsilon\sigma)$. Hence $\epsilon\sigma \in Q$ for all $\sigma \in \mathrm{GL}(V\epsilon)$.

Let $\sigma \in \mathrm{GL}(V\epsilon)$. Then, for each $v \in \ker \epsilon\sigma$, $v\epsilon\sigma = 0 = 0\sigma$, which implies that $v\epsilon = 0$ since σ is injective. Thus $v \in \ker\epsilon$ and so $\ker\epsilon\sigma \subseteq \ker\epsilon$. In general, $\ker\epsilon \subseteq \ker\epsilon\sigma$. Hence $\ker\epsilon\sigma = \ker\epsilon$. Since $V\epsilon\sigma = V\epsilon$, it follows that $\epsilon\sigma \in H_\epsilon$.

On the other hand, let $\alpha \in H_\epsilon$. Then $V\alpha = V\epsilon$ and $\ker\alpha = \ker\epsilon$. Since ϵ is the identity in the group H_ϵ, $\epsilon\alpha = \alpha$. Let σ be a restriction of α to $V\epsilon$. Then $\sigma = \alpha|_{V\epsilon}\colon V\epsilon \to V$ is linear and $(V\epsilon)\sigma = (V\epsilon)\alpha = V(\epsilon\alpha) = V\alpha = V\epsilon$: that is, $\sigma\colon V\epsilon \to V\epsilon$ is surjective. To show that σ is injective, let $u, v \in V\epsilon$ be such that $u\sigma = v\sigma$. Thus $u\alpha = v\alpha$, and it follows that $u - v \in \ker\alpha = \ker\epsilon$. So $u - v \in \ker\epsilon \cap V\epsilon = \{0\}$ by Lemma 2.6, which implies that $u = v$. Hence $\sigma \in \mathrm{GL}(V\epsilon)$. For each $t \in V$, $(t\epsilon)\sigma = (t\epsilon)\alpha = t(\epsilon\alpha) = t\alpha$. Hence $\alpha = \epsilon\sigma$. □

For each idempotent $\epsilon \in Q$, it is easy to verify that $H_\epsilon \cong \mathrm{GL}(V\epsilon)$ by mapping $\epsilon\sigma \mapsto \sigma$ for all $\sigma \in \mathrm{GL}(V\epsilon)$. This gives the following lemma.

Lemma 3.2. *Let $\epsilon \in Q$ be an idempotent. Then $H_\epsilon \cong \mathrm{GL}(V\epsilon)$.*

Now suppose that $\dim W = n < \aleph_0$. Then the results for the $\mathcal{J}$-class $J(n)$ are as follows.

Theorem 3.3. *Suppose that $\dim W = n < \aleph_0$ and let $\{\epsilon_p : p \in P\}$ be the set of all idempotents in $J(n)$. Then*

$$J(n) = \bigcup_{p\in P} H_{\epsilon_p}$$

is a disjoint union of groups, all of which are isomorphic.

Proof. Since $\{\epsilon_p : p \in P\} \neq \emptyset$ and $\mathcal{H} \subseteq \mathcal{J}$, $\bigcup_{p\in P} H_{\epsilon_p} \subseteq J(n)$. Now, let $\alpha \in J(n)$. Then $\dim(V\alpha) = n$. We suppose that $\ker\alpha = \langle u_i\rangle$ and $V\alpha = W\alpha = \langle w_j\alpha\rangle$, where $w_j \in W$. Then $|J| = \dim(\langle w_j\alpha\rangle) = \dim(V\alpha) = n$ and $V = \ker\alpha \oplus \langle w_j\rangle = \langle u_i\rangle \oplus \langle w_j\rangle$, by Lemma 2.7. We can write

$$\alpha = \begin{pmatrix} u_i & w_j \\ 0 & w_j\alpha \end{pmatrix}$$

and define

$$\epsilon = \begin{pmatrix} u_i & w_j \\ 0 & w_j \end{pmatrix}.$$

Then ϵ is an idempotent in Q such that $V\epsilon = \langle w_j\rangle$: that is, $\dim(V\epsilon) = |J| = n$. Hence ϵ is an idempotent in $J(n)$ and, as observed before, $V\epsilon = W$. Since $\ker\alpha = \langle u_i\rangle = \ker\epsilon$ and $V\alpha = W = V\epsilon$, this implies that $\alpha \in H_\epsilon \subseteq \bigcup_{p\in P} H_{\epsilon_p}$: that is, $J(n) \subseteq \bigcup_{p\in P} H_{\epsilon_p}$. Therefore $J(n) = \bigcup_{p\in P} H_{\epsilon_p}$ and $H_{\epsilon_p} \cong \mathrm{GL}(V\epsilon_p) = \mathrm{GL}(W)$ for all $p \in P$, by Lemma 3.2. □

Lemma 3.4. *Let ϵ_i and ϵ_j be idempotents in $J(n)$ and let $\sigma \in \mathrm{GL}(W)$. Then $\epsilon_i\sigma\epsilon_j = \epsilon_i\sigma$ and $\epsilon_i\epsilon_j = \epsilon_i$.*

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Proof. Let $v \in V$. Then $v\epsilon_i \in V\epsilon_i = W$, and so $(v\epsilon_i)\sigma \in W = V\epsilon_j$. Thus $v\epsilon_i\sigma = u\epsilon_j$ for some $u \in V$, and it follows that $v\epsilon_i\sigma\epsilon_j = (u\epsilon_j)\epsilon_j = u\epsilon_j = v\epsilon_i\sigma$ (since ϵ_j is an idempotent). Thus $\epsilon_i\sigma\epsilon_j = \epsilon_i\sigma$. In particular, if $\sigma = 1_W \in \mathrm{GL}(W)$, then $\epsilon_i 1_W\epsilon_j = \epsilon_i 1_W$: that is, $\epsilon_i\epsilon_j = \epsilon_i$. □

Lemma 3.5. *Suppose* $\dim W = n < \aleph_0$ *and let* $\{\epsilon_p : p \in P\}$ *be the set of all idempotents in* $J(n)$. *Then* $\{\epsilon_p : p \in P\}$ *is a left zero semigroup and* $H_{\epsilon_i}H_{\epsilon_j} = H_{\epsilon_i}$ *for all* $i, j \in P$.

Proof. Assume that ϵ_i and ϵ_j are idempotents in $J(n)$. Then $\epsilon_i\epsilon_j = \epsilon_i$ (by Lemma 3.4), and so $\{\epsilon_p : p \in P\}$ is a left zero subsemigroup. Let $\alpha \in H_{\epsilon_i}$ and $\beta \in H_{\epsilon_j}$. Then $\alpha = \epsilon_i\sigma$ and $\beta = \epsilon_j\delta$ for some $\sigma, \delta \in \mathrm{GL}(V\epsilon_i) = \mathrm{GL}(V\epsilon_j) = \mathrm{GL}(W)$ by Lemma 3.1. From Lemma 3.4, it follows that $\alpha\beta = (\epsilon_i\sigma)(\epsilon_j\delta) = (\epsilon_i\sigma)\delta = \epsilon_i(\sigma\delta)$, where $\sigma\delta \in \mathrm{GL}(W) = \mathrm{GL}(V\epsilon_i)$, so $\alpha\beta \in H_{\epsilon_i}$ and $H_{\epsilon_i}H_{\epsilon_j} \subseteq H_{\epsilon_i}$. On the other hand, let $\gamma \in H_{\epsilon_i}$. By Lemma 3.1, there is $\rho \in \mathrm{GL}(V\epsilon_i) = \mathrm{GL}(W)$ such that $\gamma = \epsilon_i\rho$. Lemma 3.4 implies that $\gamma = \epsilon_i\rho\epsilon_j = (\epsilon_i\rho)\epsilon_j = \gamma\epsilon_j \in H_{\epsilon_i}H_{\epsilon_j}$. □

Since $J(n) = \bigcup_{p\in P} H_{\epsilon_p}$, where $\{\epsilon_p : p \in P\}$ is the set of all idempotents in $J(n)$, and $H_{\epsilon_i}H_{\epsilon_j} = H_{\epsilon_i}$ for all idempotents ϵ_i, ϵ_j in $J(n)$, it follows that $J(n)$ is a subsemigroup of Q. Moreover, it is easy to see that $J(n)$ is regular since $J(n)$ is a union of groups. This gives the following theorem.

Theorem 3.6. *$J(n)$ is a regular subsemigroup of Q.*

4. Maximal regular subsemigroups

In this section, let $n \geq 0$ be an integer and let W be an n-dimensional subspace of V over a finite field F. We will describe all maximal regular subsemigroups of Q. To do this, we need the following preliminary Lemma.

Lemma 4.1. *$J(n-1) \subseteq J(n)\alpha J(n)$ for all $\alpha \in J(n-1)$.*

Proof. Let $\alpha \in J(n-1)$ and write $V\alpha = W\alpha = \langle w_i\alpha\rangle$, where $\{w_i\}$ is a linearly independent subset of W, with $|I| = n-1$. Then, by Lemma 2.7, $V = \ker\alpha \oplus \langle w_i\rangle$. By Lemma 2.2, $W = (\ker\alpha \cap W) \oplus \langle w_i\rangle$ and $\dim(\ker\alpha \cap W) = 1$ since $\dim(W) = n$ and $\dim(\langle w_i\rangle) = n-1$. We let $\ker\alpha \cap W = \langle u\rangle$ for some $u \in \ker\alpha \cap W \subseteq \ker\alpha$ and write $\ker\alpha = \langle u\rangle \oplus \langle u_j\rangle$ for some subspace $\langle u_j\rangle$ of $\ker\alpha$. Thus $W = \langle u\rangle \oplus \langle w_i\rangle$ and $V = \langle u\rangle \oplus \langle u_j\rangle \oplus \langle w_i\rangle$. So we can write

$$\alpha = \begin{pmatrix} u & u_j & w_i \\ 0 & 0 & w_i\alpha \end{pmatrix}.$$

Let β be any element in $J(n-1)$. As above, $V\beta = W\beta = \langle w'_i\beta\rangle$ for some $w'_i \in W$, $V = \ker\beta \oplus \langle w'_i\rangle$, $\ker\beta \cap W = \langle u'\rangle$ for some $u' \in \ker\beta \cap W$, $\ker\beta = \langle u'\rangle \oplus \langle u'_j\rangle$ for some subspace $\langle u'_j\rangle$ of $\ker\beta$, and $V = \langle u'\rangle \oplus \langle u'_j\rangle \oplus \langle w'_i\rangle$. Hence, we can write

$$\beta = \begin{pmatrix} u' & u'_j & w'_i \\ 0 & 0 & w'_i\beta \end{pmatrix}.$$

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Since $|I| = n-1$, there are w and w' in W such that $W = \langle w_i\alpha\rangle \oplus \langle w\rangle$ and $W = \langle w'_i\beta\rangle \oplus \langle w'\rangle$. Let $V = W \oplus \langle v_j\rangle = \langle w_i\alpha\rangle \oplus \langle w\rangle \oplus \langle v_j\rangle$ for some subspace $\langle v_j\rangle$ of V. Define $\gamma, \delta \in T(V, W)$ by

$$\gamma = \begin{pmatrix} u'_j & u' & w'_i \\ 0 & u & w_i \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} v_j & w & w_i\alpha \\ 0 & w' & w'_i\beta \end{pmatrix}.$$

It is easy to verify that $V\gamma = W\gamma = \langle u\rangle \oplus \langle w_i\rangle = W$ and $V\delta = W\delta = \langle w'\rangle \oplus \langle w'_i\beta\rangle = W$. It follows that $\gamma, \delta \in J(n)$. Moreover, $\beta = \gamma\alpha\delta \in J(n)\alpha J(n)$. Therefore $J(n-1) \subseteq J(n)\alpha J(n)$. □

Theorem 4.2. *The set $Q_{n-2} \cup J(n)$ is a maximal regular subsemigroup of Q.*

Proof. Since Q_{n-2} is an ideal of Q and $J(n)$ is a regular subsemigroup of Q, Q_{n-2} is a regular subsemigroup of Q and so $Q_{n-2} \cup J(n)$ is a regular subsemigroup of Q. To show that $Q_{n-2} \cup J(n)$ is a maximal regular subsemigroup of Q, suppose that there is a regular subsemigroup S of Q such that $Q_{n-2} \cup J(n) \subsetneq S \subseteq Q$. Thus there exists $\alpha \in J(n-1) \cap S$. Let β be any element in $J(n-1)$. Then, by Lemma 4.1, there are $\gamma, \delta \in J(n) \subseteq S$ such that $\beta = \gamma\alpha\delta \in S$. Hence $J(n-1) \subseteq S$, and it follows that $S = Q$. □

In what follows, let $E(J(n)) = \{\epsilon_p : p \in P\}$ be the set of all idempotents in $J(n)$. For each $\epsilon_p \in E(J(n))$, $H_{\epsilon_p} \cong \mathrm{GL}(V\epsilon_p) = \mathrm{GL}(W)$ by Lemma 3.2. Since $\mathrm{GL}(W) \cong \mathrm{GL}(n, q)$ is a finite group, where $q = |F|$, it follows that H_{ϵ_p} is a finite group for all $p \in P$. Let U be a maximal subgroup of $\mathrm{GL}(W)$ and let $\Phi_p : H_{\epsilon_p} \to \mathrm{GL}(W)$ be an isomorphism defined by $\epsilon_p\delta \mapsto \delta$ for all $\delta \in \mathrm{GL}(W)$. Let

$$M_p = U\Phi_p^{-1} = \{\epsilon_p\delta : \delta \in U\}.$$

Then M_p is a maximal subgroup of H_{ϵ_p} for all $p \in P$.

Lemma 4.3. *Let M_p be defined as above and let $M = \bigcup_{p\in P} M_p$. Then M is a maximal regular subsemigroup of $J(n)$.*

Proof. Clearly, $M \subseteq \bigcup_{p\in P} H_{\epsilon_p} = J(n)$. Let $\alpha, \beta \in M$. Then $\alpha \in M_p$ and $\beta \in M_q$ for some $p, q \in P$. We can write $\alpha = \epsilon_p\sigma$ and $\beta = \epsilon_q\delta$ for some $\sigma, \delta \in U$. It follows, from Lemma 3.4, that $\alpha\beta = (\epsilon_p\sigma)(\epsilon_q\delta) = \epsilon_p\sigma\delta = \epsilon_p(\sigma\delta)$, where $\sigma\delta \in U$. Thus $\alpha\beta \in M_p \subseteq M$ and so M is a subsemigroup of $J(n)$. In addition, M is regular since each M_p is a group.

We prove the maximality of M. Let T be a regular subsemigroup of $J(n)$ such that $M \subsetneq T \subseteq J(n)$. Let $\alpha \in T \setminus M$. Then $\alpha \in H_{\epsilon_p} \setminus M_p$ for some $p \in P$. It is clear that the semigroup $\langle M_p \cup \{\alpha\}\rangle$ of $J(n)$ generated by $M_p \cup \{\alpha\}$ is contained in T. It is well known that every finite subsemigroup of a group is a subgroup. Therefore, since H_{ϵ_p} is a finite group, its subsemigroup $\langle M_p \cup \{\alpha\}\rangle$ is a subgroup of H_{ϵ_p} such that $M_p \subsetneq \langle M_p \cup \{\alpha\}\rangle \subseteq H_{\epsilon_p}$. Thus $H_{\epsilon_p} = \langle M_p \cup \{\alpha\}\rangle$ by the maximality of M_p. Let β be any element in $J(n)$. Thus $\beta \in H_{\epsilon_q}$ for some $q \in P$, and we write $\beta = \epsilon_q\rho$ for some $\rho \in \mathrm{GL}(W)$. Since $\epsilon_q\epsilon_p = \epsilon_q$ by Lemma 3.4, we get that $\beta = \epsilon_q\rho = (\epsilon_q\epsilon_p)\rho = \epsilon_q(\epsilon_p\rho)$ such that $\epsilon_q \in M_q \subseteq T$ and $\epsilon_p\rho \in H_{\epsilon_p} = \langle M_p \cup \{\alpha\}\rangle \subseteq T$. Hence $\beta \in T$ and $J(n) \subseteq T$. Therefore $T = J(n)$. □

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Theorem 4.4. *Let M be as in Lemma 4.3. Then $Q_{n-1} \cup M$ is a maximal regular subsemigroup of Q.*

Proof. Since the ideals Q_{n-1} and M are regular subsemigroups of Q, we obtain that $Q_{n-1} \cup M$ is a regular subsemigroup of Q. We prove that $Q_{n-1} \cup M$ is maximal. Let S be a regular subsemigroup of Q such that $Q_{n-1} \cup M \subsetneq S \subseteq Q$. Then $M \subsetneq S \cap J(n) \subseteq J(n)$ and $S \cap J(n)$ is a subsemigroup of $J(n)$. Clearly, $S \cap J(n) = S \cap (\bigcup_{p \in P} H_{\epsilon_p}) = \bigcup_{p \in P}(S \cap H_{\epsilon_p})$. We show that $S \cap H_{\epsilon_p} = M_p$ or $S \cap H_{\epsilon_p} = H_{\epsilon_p}$ for all $p \in P$. Let $p \in P$ and $S \cap H_{\epsilon_p} \neq M_p$. Thus $M_p \subsetneq S \cap H_{\epsilon_p}$ and there exists $\alpha \in (S \cap H_{\epsilon_p}) \setminus M_p$. Since M_p is a maximal subgroup of H_{ϵ_p} and $M_p \subsetneq \langle M_p \cup \{\alpha\} \rangle \subseteq H_{\epsilon_p}$, we obtain $H_{\epsilon_p} = \langle M_p \cup \{\alpha\} \rangle \subseteq S$ and so $S \cap H_{\epsilon_p} = H_{\epsilon_p}$. Thus $S \cap J(n) = \bigcup_{p \in P}(S \cap H_{\epsilon_p})$ is a disjoint union of groups, which implies that $S \cap J(n)$ is a regular subsemigroup of $J(n)$. Since $M \subsetneq S \cap J(n) \subseteq J(n)$ and M is a maximal regular subsemigroup of $J(n)$ by Lemma 4.3, we get $S \cap J(n) = J(n)$. Therefore $S = Q$. □

Next, we prove that there are only two types of maximal regular subsemigroups of Q, as given in Theorems 4.2 and 4.4. To do this, we need the following four lemmas.

Lemma 4.5. *Let T be a maximal regular subsemigroup of $J(n) = \bigcup_{p \in P} H_{\epsilon_p}$. Let $R = \{r \in P : T \cap H_{\epsilon_r} \neq \emptyset\}$ and $T_r = T \cap H_{\epsilon_r}$ for all $r \in R$. Let Φ_p be defined as in Lemma 4.3 ($p \in P$) and let $T_r\Phi_r = V_r$ in $\mathrm{GL}(W)$ for all $r \in R$. Then:*

(1) *$V_r = V_s$ for all $r, s \in R$; and*

(2) *T_r are maximal subgroups of H_{ϵ_r} for all $r \in R$.*

Proof. We note that T_r is a subgroup of H_{ϵ_r} for all $r \in R$ since $T \cap H_{\epsilon_r}$ is a finite subsemigroup of H_{ϵ_r}. In addition, $T_r = \{\epsilon_r\sigma : \sigma \in V_r\}$, where ϵ_r is an idempotent of H_{ϵ_r}.

(1) Assume that $r, s \in R$ and ϵ_r, ϵ_s are idempotents of T_r and T_s, respectively. Let $\sigma \in V_r$. Then $\epsilon_r\sigma \in T_r$ and $\epsilon_s\sigma = \epsilon_s(\epsilon_r\sigma) \in T_sT_r \subseteq T$. Hence $\epsilon_s\sigma \in T \cap H_{\epsilon_s} = T_s$, and this implies that $\sigma \in V_s$ and hence $V_r \subseteq V_s$. Similarly, we can show that $V_s \subseteq V_r$. Therefore $V_r = V_s$.

(2) Suppose that there exists $s \in R$ such that T_s is not a maximal subgroup of H_{ϵ_s}. Then T_s is properly contained in a maximal subgroup N_s of H_{ϵ_s}. Let $U = N_s\Phi_s \subseteq \mathrm{GL}(W)$. Thus U is a maximal subgroup of $\mathrm{GL}(W)$ such that $V_s = T_s\Phi_s \subsetneq N_s\Phi_s = U$. Now, let $M = \bigcup_{p \in P} M_p$, where $M_p = U\Phi_p^{-1}$ for all $p \in P$. Thus M is a maximal regular subsemigroup of $J(n)$ by Lemma 4.3. Since $T_r\Phi_r = V_r = V_s \subsetneq U$ for all $r \in R$ by (1), it follows that $T_r \subsetneq U\Phi_r^{-1} = M_r$ for all $r \in R$. So $T = \bigcup_{r \in R} T_r \subsetneq \bigcup_{r \in R} M_r \subseteq \bigcup_{p \in P} M_p = M \subsetneq J(n)$, which contradicts the maximality of T. Hence T_r is a maximal subgroup of H_{ϵ_r} for every $r \in R$. □

Lemma 4.6. *T is a maximal regular subsemigroup of $J(n)$ if and only if there is a maximal subgroup U of $\mathrm{GL}(W)$ such that $T = \bigcup_{p \in P} M_p$, where $M_p = U\Phi_p^{-1}$ for all $p \in P$.*

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Proof. One direction is clear by Lemma 4.3. Now, let T be a maximal regular subsemigroup of $J(n)$. Let $R = \{r \in P : T \cap H_{\epsilon_r} \neq \emptyset\}$. If $R = \emptyset$, it is clear that $T = \emptyset$, which is a contradiction. Thus $R \neq \emptyset$ and there is $r \in R$ such that $T_r = T \cap H_{\epsilon_r} \neq \emptyset$. By Lemma 4.5(2), T_r is a maximal subgroup of H_{ϵ_r}. It follows that $T_r\Phi_r = V_r$ is a maximal subgroup of $\mathrm{GL}(W)$. Let $U = V_r$ and $M_p = U\Phi_p^{-1}$ for all $p \in P$. We claim that $T = \bigcup_{p\in P} M_p$. Let $\alpha \in T$. Then $\alpha \in T \cap H_{\epsilon_s}$ for some $s \in P$. Since $T \cap H_{\epsilon_s} \neq \emptyset$, $s \in R$ and $T_s = T \cap H_{\epsilon_s}$ is a maximal subgroup of H_{ϵ_p} by Lemma 4.5(2). Let $T_s\Phi_s = V_s$. Then, by Lemma 4.5(1), $V_r = V_s$ and so $\alpha\Phi_s \in T_s\Phi_s = V_s = V_r = U$. This implies that $\alpha \in U\Phi_s^{-1} = M_s \subseteq \bigcup_{p\in P} M_p$ and hence $T \subseteq \bigcup_{p\in P} M_p$. Since $\bigcup_{p\in P} M_p$ is a maximal regular subsemigroup of $J(n)$ containing T by Lemma 4.3, we obtain $T = \bigcup_{p\in P} M_p$ by the maximality of T. □

Lemma 4.7. *For* $0 \leq k \leq n-1$, $Q_k = \langle J(k)\rangle$.

Proof. Let $0 \leq k \leq n-1$ and $\alpha \in Q_k$. If $\alpha \in J(k)$, then $\alpha \in \langle J(k)\rangle$. Now, let $\alpha \in Q_{k-1}$. Then $\alpha \in J(t)$ for some $0 \leq t \leq k-1$, that is, $\dim(V\alpha) = t$. Suppose that $V\alpha = W\alpha = \langle w_i\alpha\rangle$, where $w_i \in W$ and $|I| = t$. Then, by Lemma 2.7, $V = \ker\alpha \oplus \langle w_i\rangle$. Since $\langle w_i\rangle \subseteq W$, $W = (\ker\alpha \cap W) \oplus \langle w_i\rangle$ by Lemma 2.2. If $\dim(\ker\alpha \cap W) \leq 1$, then

$$t + 2 \leq k + 1 \leq n = \dim(W) = \dim(\ker\alpha \cap W) + |I| \leq 1 + t,$$

which is a contradiction. Hence $\dim(\ker\alpha \cap W) \geq 2$ and so there are distinct u, v in a basis of $\ker\alpha \cap W$. Thus $\{u, v\}$ is linearly independent and we write $\ker\alpha = \langle u, v\rangle \oplus \langle v_s\rangle$ for some subspace $\langle v_s\rangle$ of $\ker\alpha$. It follows that $V = \langle u, v\rangle \oplus \langle v_s\rangle \oplus \langle w_i\rangle$ and we can write

$$\alpha = \begin{pmatrix} u & v & v_s & w_i \\ 0 & 0 & 0 & w_i\alpha \end{pmatrix}.$$

We let $W = \langle w_i\alpha\rangle \oplus \langle w_j\rangle$ for some subspace $\langle w_j\rangle$ of W. Since $|I| = t < n$, $\{w_j\} \neq \emptyset$ and there exists $w \in \{w_j\} \setminus \{w_i\alpha\}$ such that $\{w, w_i\alpha\}$ is linearly independent. Define $\beta, \gamma \in Q$ by

$$\beta = \begin{pmatrix} u & v & v_s & w_i \\ 0 & u & 0 & w_i \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} u & v & v_s & w_i \\ 0 & w & 0 & w_i\alpha \end{pmatrix}.$$

Then $\alpha = \beta\gamma$ such that $V\beta = \langle u, w_i\rangle$ and $V\gamma = \langle w, w_i\alpha\rangle$: that is, $\dim(V\beta) = \dim(V\gamma) = |I| + 1 = t + 1$. By the principle of induction on t, we conclude that $Q_k = \langle J(k)\rangle$ for all $0 \leq k \leq n-1$. □

Lemma 4.8. *Let S be a maximal regular subsemigroup of Q. Then the following statements hold.*

(1) *If $S \cap J(n) = J(n)$, then $S \cap J(n-1) = \emptyset$.*
(2) *If $S \cap J(n) \subsetneq J(n)$, then $S \cap J(n)$ is a maximal regular subsemigroup of $J(n)$.*

Proof. We note that $S \cap J(n) \neq \emptyset$. In fact, if $S \cap J(n) = \emptyset$, then $S \subseteq Q_{n-1} \subsetneq Q_{n-1} \cup M \subsetneq Q$, where M is a maximal regular subsemigroup of $J(n)$ as defined in Lemma 4.3. By Theorem 4.4, $Q_{n-1} \cup M$ is a regular subsemigroup of Q, which contradicts the maximality of S.

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(1) Assume that $S \cap J(n) = J(n)$. Thus $J(n) \subseteq S$. We suppose that there is $\alpha \in S \cap J(n-1) \subseteq J(n-1)$. Then, by Lemma 4.1, $J(n-1) \subseteq J(n)\alpha J(n) \subseteq S\alpha S \subseteq S$ and it follows from Lemma 4.7 that

$$Q_{n-1} = \langle J(n-1) \rangle \subseteq S.$$

Then $Q = Q_{n-1} \cup J(n) \subseteq S \cup J(n) = S$ since $J(n) \subseteq S$, and hence $S = Q$, which contradicts the maximality of S. Therefore $S \cap J(n-1) = \emptyset$.

(2) Assume that $S \cap J(n) \subsetneq J(n)$. Since $S \cap J(n) \neq \emptyset$, $S \cap J(n)$ is a subsemigroup of $J(n)$ and $S \cap H_{\epsilon_r} \neq \emptyset$ for some $r \in P$. Let $R = \{r \in P : S \cap H_{\epsilon_r} \neq \emptyset\}$. Thus $S \cap J(n) = \bigcup_{r \in R}(S \cap H_{\epsilon_r})$. Since $S \cap H_{\epsilon_r}$ is a finite subsemigroup of H_{ϵ_r} for all $r \in R$, it follows that $S \cap H_{\epsilon_r}$ is a subgroup of H_{ϵ_r} for all $r \in R$ and hence $S \cap J(n) = \bigcup_{r \in R}(S \cap H_{\epsilon_r})$ is a disjoint union of groups. Thus $S \cap J(n)$ is a regular subsemigroup of $J(n)$. If $S \cap J(n)$ is not maximal under these conditions, then there exists a maximal regular subsemigroup T of $J(n)$ such that $S \cap J(n) \subsetneq T \subsetneq J(n)$. It is easy to see that $Q_{n-1} \cup T$ is a regular subsemigroup of Q with $S \subsetneq Q_{n-1} \cup T \subsetneq Q$, which contradicts the maximality of S. Therefore $S \cap J(n)$ is a maximal regular subsemigroup of $J(n)$, as required. □

Theorem 4.9. *Let S be a maximal regular subsemigroup of Q. Then S is either of the form:*

(1) $Q_{n-2} \cup J(n)$*; or*
(2) $Q_{n-1} \cup M$*, where M is a maximal regular subsemigroup of $J(n)$ as defined in Lemma 4.3.*

Proof. By Theorems 4.2 and 4.4, $Q_{n-2} \cup J(n)$ and $Q_{n-1} \cup M$ are maximal subsemigroups of Q. Conversely, we consider two cases.

Case 1. $S \cap J(n) = J(n)$. Then $S \cap J(n-1) = \emptyset$ by Lemma 4.8(1). Thus $S \subseteq Q_{n-2} \cup J(n)$, where $Q_{n-2} \cup J(n)$ is a maximal regular subsemigroup of Q by Theorem 4.2. Hence $S = Q_{n-2} \cup J(n)$.

Case 2. $S \cap J(n) \subsetneq J(n)$. Then, by Lemma 4.8(2), $S \cap J(n)$ is a maximal regular subsemigroup of $J(n)$. It follows from Lemma 4.6 that $S \cap J(n) = \bigcup_{p \in P} M_p$, where $M_p = U\Phi_p^{-1}$ for all $p \in P$ with a maximal subgroup U of $\mathrm{GL}(W)$. We let $M = \bigcup_{p \in P} M_p$. Then $M = S \cap J(n)$ and $S \subseteq Q_{n-1} \cup M$ such that $Q_{n-1} \cup M$ is a maximal regular subsemigroup of Q by Theorem 4.4. Hence $S = Q_{n-1} \cup M$. □

Next, we consider the case when $V = W$ and V is a finite-dimensional vector space with $\dim V = n$. Clearly, $Q = T(V)$ and $J(n) = \{\alpha \in T(V) : \dim(V\alpha) = n\}$. For each $\alpha \in J(n)$, $V\alpha = V$, which implies that $\alpha : V \to V$ is a bijective linear transformation. Hence $J(n) = \mathrm{GL}(V)$. The following corollary comes directly from Theorem 4.9.

Corollary 4.10. *Let V be an n-dimensional vector space over a finite field F and let S be a maximal regular subsemigroup of $T(V)$. Then S is either of the form:*

(1) $Q_{n-2} \cup \mathrm{GL}(V)$*; or*
(2) $Q_{n-1} \cup M$*, where M is a maximal subgroup of* $\mathrm{GL}(V)$.

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5. Rank and idempotent rank of Q

In this section, we aim to find the rank and idempotent rank of Q. Suppose that W is an n-dimensional subspace of an m-dimensional vector space V over a finite field F with q elements.

Lemma 5.1. *Let $\alpha \in J(n-1)$. Then $Q = \langle J(n) \cup \{\alpha\}\rangle$. Hence* $\operatorname{rank}(Q) \leq \operatorname{rank}(J(n)) + 1$.

Proof. By Lemmas 4.1 and 4.7, $J(n-1) \subseteq J(n)\alpha J(n) \subseteq \langle J(n) \cup \{\alpha\}\rangle$ and $Q_{n-1} = \langle J(n-1)\rangle \subseteq \langle J(n) \cup \{\alpha\}\rangle$. Thus

$$Q = Q_{n-1} \cup J(n) \subseteq \langle J(n) \cup \{\alpha\}\rangle \cup J(n) = \langle J(n) \cup \{\alpha\}\rangle,$$

since $J(n) \subseteq \langle J(n) \cup \{\alpha\}\rangle$. Hence $Q = \langle J(n) \cup \{\alpha\}\rangle$, and it follows that $\operatorname{rank}(Q) \leq \operatorname{rank}(J(n)) + 1$. □

For $n \geq 1$, $\langle J(n)\rangle = J(n) \neq Q$ and an element in $J(n)$ cannot be written as a product of some elements in Q_{n-1} since Q_{n-1} is an ideal. Thus

$$\operatorname{rank}(Q) \geq \operatorname{rank}(J(n)) + 1.$$

Therefore, by Lemma 5.1, we obtain the following lemma.

Lemma 5.2. *For $n \geq 1$,* $\operatorname{rank}(Q) = \operatorname{rank}(J(n)) + 1$.

To determine the rank of Q, we need to find the rank of $J(n)$. We know that, for each $\alpha \in J(n)$, $V\alpha = W\alpha = W$. Let $W = W\alpha = \langle w_i\alpha\rangle$, where $w_i \in W$ for all i. Then, by Lemma 2.7, $\{w_i\}$ is linearly independent and $V = \ker\alpha \oplus \langle w_i\rangle$. Since $\langle w_i\rangle \subseteq W$ and $\dim(\langle w_i\rangle) = |I| = \dim(\langle w_i\alpha\rangle) = \dim(W) = n$ is finite, $W = \langle w_i\rangle$. Hence $V = \ker\alpha \oplus W$ and $\dim(\ker\alpha) = m - n$ for all $\alpha \in J(n)$.

Lemma 5.3. *Let $\epsilon_i, \epsilon_j \in E(J(n))$. Then $\epsilon_i = \epsilon_j$ if and only if* $\ker\epsilon_i = \ker\epsilon_j$.

Proof. Assume that $\ker\epsilon_i = \ker\epsilon_j$. Thus $\epsilon_i \mathcal{R} \epsilon_j$. Since $V\epsilon_i = W = V\epsilon_j$, $\epsilon_i \mathcal{L} \epsilon_j$, which implies that $\epsilon_i \mathcal{H} \epsilon_j$. Hence $\epsilon_i \in H_{\epsilon_j}$ and so $\epsilon_i = \epsilon_j$ since the group $\mathcal{H}$-class H_{ϵ_j} contains only one idempotent ϵ_j. The converse is clear. □

Theorem 5.4. *$J(n)$ has $q^{n(m-n)}$ distinct $\mathcal{H}$-classes.*

Proof. It is clear that the number of distinct $\mathcal{H}$-classes of $J(n)$ equals $|E(J(n))|$. Let C be the set of all complements of W in V. Since $V = W \oplus \ker\epsilon_p$ for all $\epsilon_p \in E(J(n))$ by the above note, it follows that $\ker\epsilon_p \in C$ for all $\epsilon_p \in E(J(n))$. We define $\phi : E(J(n)) \to C$ by $\epsilon_p\phi = \ker\epsilon_p$ for all $\epsilon_p \in E(J(n))$. By using Lemma 5.3, it is easy to see that ϕ is injective. We prove that ϕ is surjective. Let $T \in C$. Thus $V = W \oplus T$, and we can write $W = \langle w_1, \ldots, w_n\rangle$ and $T = \langle v_1, \ldots, v_{m-n}\rangle$. Define

$$\epsilon = \begin{pmatrix} w_1 & \cdots & w_n & v_1 & \cdots & v_{m-n} \\ w_1 & \cdots & w_n & 0 & \cdots & 0 \end{pmatrix}.$$

Then $\epsilon \in E(J(n))$ and $\ker\epsilon = T$. Hence $\epsilon\phi = \ker\epsilon = T$ and so ϕ is bijective. Therefore $|E(J(n))| = |C| = q^{n(m-n)}$ by Lemma 2.1. □

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By Theorem 5.4, $|E(J(n))| = |P| = q^{n(m-n)}$. Then we can write $P = \{1, 2, \ldots, q^{n(m-n)}\}$. Hence

$$J(n) = \bigcup_{i=1}^{q^{n(m-n)}} H_{\epsilon_i}.$$

Since $H_{\epsilon_i} \cong \mathrm{GL}(V\epsilon_i) = \mathrm{GL}(W) \cong \mathrm{GL}(n, q)$ for all $1 \le i \le q^{n(m-n)}$, we obtain $|H_{\epsilon_i}| = |\mathrm{GL}(n, q)| = (q^n - 1)(q^n - q)(q^n - q^2)\cdots(q^n - q^{n-1})$ for all $1 \le i \le q^{n(m-n)}$.

Lemma 5.5. *Let $\epsilon_p \in E(J(n))$ and $\alpha \in J(n)$. Then $H_\alpha = \alpha H_{\epsilon_p}$.*

Proof. Assume that $\alpha \in H_{\epsilon_r}$ for some $r \in P$. Then, by Lemma 3.1, we can write $\alpha = \epsilon_r\sigma$ for some $\sigma \in \mathrm{GL}(W)$. Let $\beta \in H_\alpha = H_{\epsilon_r}$. Then there exists $\delta \in \mathrm{GL}(W)$ such that $\beta = \epsilon_r\delta$. By Lemma 3.4,

$$\alpha\epsilon_p(\sigma^{-1}\delta) = (\epsilon_r\sigma)\epsilon_p(\sigma^{-1}\delta) = (\epsilon_r\sigma\epsilon_p)(\sigma^{-1}\delta) = (\epsilon_r\sigma)\sigma^{-1}\delta = \epsilon_r\delta = \beta.$$

Since $\sigma^{-1}\delta \in \mathrm{GL}(W)$, $\epsilon_p(\sigma^{-1}\delta) \in H_{\epsilon_p}$ and $\beta = \alpha(\epsilon_p(\sigma^{-1}\delta)) \in \alpha H_{\epsilon_p}$. Hence $H_\alpha \subseteq \alpha H_{\epsilon_p}$. For the other containment, we let $\gamma = \alpha\lambda \in \alpha H_{\epsilon_p}$ for some $\lambda \in H_{\epsilon_p}$. Thus $\lambda = \epsilon_p\rho$ for some $\rho \in \mathrm{GL}(W)$. It follows, from Lemma 3.4, that $\gamma = (\epsilon_r\sigma)(\epsilon_p\rho) = (\epsilon_r\sigma\epsilon_p)\rho = (\epsilon_r\sigma)\rho = \epsilon_r(\sigma\rho) \in H_{\epsilon_r} = H_\alpha$ since $\sigma\rho \in \mathrm{GL}(W)$. Hence $\alpha H_{\epsilon_p} \subseteq H_\alpha$. □

To find the rank of $J(n)$, we use the following theorems which appeared in [14].

Since F is a finite field with q elements, the multiplicative group $F \setminus \{0\}$ is a cyclic group and we let a be a generator of $F \setminus \{0\}$. Let E_{ij} denote the $n \times n$ matrix over the field F for which entries in row i and column j equal one and the others are zero.

Theorem 5.6 [14]. *For $n \ge 3$, the group $\mathrm{GL}(n, q)$ is generated by the two elements $A = I + E_{n1} + (a-1)E_{22}$ and $B = E_{12} + E_{23} + \cdots + E_{(n-1)n} + E_{n1}$, where I is the identity matrix.*

Theorem 5.7 [14]. *For $q > 2$, the group $\mathrm{GL}(2, q)$ is generated by the two elements*

$$A = \begin{bmatrix} 0 & r \\ 1 & s \end{bmatrix}, \quad B = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$

for some $r, s \in F$.

Theorem 5.8 [14]. *The group $\mathrm{GL}(2, 2)$ is generated by the two elements*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

In what follows, we let $W = \langle w_1, w_2, \ldots, w_n \rangle$. Since $\mathrm{GL}(W) \cong \mathrm{GL}(n, q)$, we can find an isomorphism in $\mathrm{GL}(W)$ which corresponds to the generators of $\mathrm{GL}(n, q)$, as follows.

By Theorem 5.6, for $n \ge 3$, the group $\mathrm{GL}(W)$ is generated by the two elements

$$\alpha = \begin{pmatrix} w_1 & w_2 & w_3 & \cdots & w_n \\ w_1 & aw_2 & w_3 & \cdots & w_1 + w_n \end{pmatrix}$$

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and

$$\beta = \begin{pmatrix} w_1 & w_2 & w_3 & \cdots & w_n \\ w_2 & w_3 & w_4 & \cdots & w_1 \end{pmatrix}.$$

By Theorem 5.7, for $n = 2$ and $q > 2$, the group $\mathrm{GL}(W)$ is generated by the two elements

$$\alpha = \begin{pmatrix} w_1 & w_2 \\ rw_2 & w_1 + sw_2 \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} w_1 & w_2 \\ aw_1 & w_2 \end{pmatrix}$$

for some $r, s \in F$.

By Theorem 5.8, for $n = 2 = q$, the group $\mathrm{GL}(W)$ is generated by the two elements

$$\alpha = \begin{pmatrix} w_1 & w_2 \\ w_2 & w_1 + w_2 \end{pmatrix}$$

and

$$\beta = \begin{pmatrix} w_1 & w_2 \\ w_1 + w_2 & w_2 \end{pmatrix}.$$

Let

$$\alpha' = \begin{pmatrix} w_1 & w_2 \\ w_2 & w_1 \end{pmatrix}.$$

We see that $\alpha'\beta = \alpha$. Hence, for $n = 2 = q$, $\mathrm{GL}(W)$ is generated by α' and β.

For the case when $n = 1$, the group $\mathrm{GL}(W)$ is generated by a single element. In fact, $W = \langle w_1 \rangle$ and

$$\mathrm{GL}(W) = \left\{ \gamma_c = \begin{pmatrix} w_1 \\ cw_1 \end{pmatrix} : c \in F \setminus \{0\} \right\}.$$

We let a be a generator of $F \setminus \{0\}$ and $\gamma_c \in \mathrm{GL}(W)$ for some $c \in F \setminus \{0\}$. Then $\gamma_a \in \mathrm{GL}(W)$ and $c = a^k$ for some $k \in \mathbb{N}$, so it follows that $\gamma_c = \gamma_{a^k} = (\gamma_a)^k \in \langle \gamma_a \rangle$. Hence $\mathrm{GL}(W) = \langle \gamma_a \rangle$.

From the above results, we conclude the following lemma.

Lemma 5.9.

$$\mathrm{rank}(\mathrm{GL}(W)) = \begin{cases} 1 & \text{if } n = 1, \\ 2 & \text{if } n \geq 2. \end{cases}$$

Lemma 5.10. *For $m > n \geq 1$, $\mathrm{rank}(J(n)) = q^{n(m-n)}$.*

Proof. Assume that A is a generating set of $J(n)$ such that $|A| = \mathrm{rank}(J(n))$. For each $i \in \{1, \ldots, q^{n(m-n)}\}$, we let $\alpha \in H_{\epsilon_i}$. Then $\alpha = \beta_1\beta_2 \cdots \beta_k$ for some $\beta_1, \ldots, \beta_k \in A$. Since $\ker \beta_1 \subseteq \ker \alpha$ and $\dim(\ker \beta_1) = \dim(\ker \alpha) = m - n$ is finite (by the remark before Lemma 5.3), $\ker \beta_1 = \ker \alpha$. It is known that $V\beta_1 = W = V\alpha$, and thus $\beta_1 \in H_\alpha = H_{\epsilon_i}$. Hence $A \cap H_{\epsilon_i} \neq \emptyset$ for all $1 \leq i \leq q^{n(m-n)}$, which implies that $|A| \geq q^{n(m-n)}$ and so $\mathrm{rank}(J(n)) \geq q^{n(m-n)}$.

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Next, we prove that $J(n)$ is generated by $q^{n(m-n)}$ elements. Let $W = \langle w_1, w_2, \ldots, w_n\rangle$ and $V = \langle w_1, w_2, \ldots, w_n\rangle \oplus \langle v_{n+1}, v_{n+2}, \ldots, v_m\rangle$. Define

$$\epsilon_1 = \begin{pmatrix} w_1 & \cdots & w_n & v_{n+1} & \cdots & v_m \\ w_1 & \cdots & w_n & 0 & \cdots & 0 \end{pmatrix}.$$

Then ϵ_1 is an idempotent in $J(n)$ such that $V\epsilon_1 = \langle w_1, \ldots, w_n\rangle = W$ and $\ker \epsilon_1 = \langle v_{n+1}, v_{n+2}, \ldots, v_m\rangle$. We note that $H_\alpha = \alpha H_{\epsilon_1}$ for all $\alpha \in J(n)$ by Lemma 5.5.

Now, we split the proof into four cases.

Case 1. $n = 1$. We can see that

$$H_{\epsilon_1} = \left\{\gamma_c = \begin{pmatrix} w_1 & v_2 & \cdots & v_m \\ cw_1 & 0 & \cdots & 0 \end{pmatrix} : c \in F \setminus \{0\}\right\}$$

and $(\gamma_c)^k = \gamma_{c^k}$ for all $c \in F \setminus \{0\}$ and $k \in \mathbb{N}$. Let γ_c be any element in H_{ϵ_1} for some $c \in F \setminus \{0\}$. Since F is a finite field, there is a generator a of $F \setminus \{0\}$ such that $c = a^k$ for some $k \in \mathbb{N}$. Thus $\gamma_a \in H_{\epsilon_1}$ and $\gamma_c = \gamma_{a^k} = (\gamma_a)^k$. Hence H_{ϵ_1} is generated by γ_a and $J(n) = \langle \gamma_a, \epsilon_2, \ldots, \epsilon_{q^{m-1}}\rangle$.

Case 2. $n = 2$ and $q = 2$. It is known that

$$H_{\epsilon_1} \cong \mathrm{GL}(W) = \left\langle \begin{pmatrix} w_1 & w_2 \\ w_2 & w_1 \end{pmatrix}, \begin{pmatrix} w_1 & w_2 \\ w_2 & w_1 + w_2 \end{pmatrix} \right\rangle.$$

We define $\gamma_1, \gamma_2 \in H_{\epsilon_1}$ by

$$\gamma_1 = \begin{pmatrix} w_1 & w_2 & v_3 & \cdots & v_m \\ w_2 & w_1 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \gamma_2 = \begin{pmatrix} w_1 & w_2 & v_3 & \cdots & v_m \\ w_2 & w_1 + w_2 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus $H_{\epsilon_1} = \langle \gamma_1, \gamma_2\rangle$. Define $\gamma_2' \in J(n)$ by

$$\gamma_2' = \begin{pmatrix} w_1 & w_2 & v_3 & \cdots & v_m \\ w_2 & w_1 + w_2 & w_2 & \cdots & w_2 \end{pmatrix}.$$

Since $m > n$, $\ker \gamma_2' \neq \ker \epsilon_1$ and so $\gamma_2' \notin H_{\epsilon_1}$. Moreover, $\gamma_2 = \epsilon_1 \gamma_2' = (\gamma_1)^2 \gamma_2'$. Suppose that $\gamma_2' \in H_{\epsilon_p} \neq H_{\epsilon_1}$ for some $2 \leq p \leq 2^{2(m-2)}$. Hence $J(n) = \langle \gamma_1, \epsilon_2, \ldots, \epsilon_{p-1}, \gamma_2', \epsilon_{p+1}, \ldots, \epsilon_{2^{2(m-2)}}\rangle$.

Case 3. $n = 2$ and $q \geq 3$. By the same reason as that given in Case 2, $H_{\epsilon_1} = \langle \lambda_1, \lambda_2\rangle$, where

$$\lambda_1 = \begin{pmatrix} w_1 & w_2 & v_3 & \cdots & v_m \\ aw_1 & w_2 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \lambda_2 = \begin{pmatrix} w_1 & w_2 & v_3 & \cdots & v_m \\ rw_2 & w_1 + sw_2 & 0 & \cdots & 0 \end{pmatrix}$$

for some $r, s \in F$ and a is a generator of the multiplicative group of $F \setminus \{0\}$. We define $\lambda_2' \in J(n)$ by

$$\lambda_2' = \begin{pmatrix} w_1 & w_2 & v_3 & \cdots & v_m \\ rw_2 & w_1 + sw_2 & w_2 & \cdots & w_2 \end{pmatrix}.$$

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Then $\lambda_2' \notin H_{\epsilon_1}$ and $\lambda_2 = \epsilon_1\lambda_2' = (\lambda_1)^k\lambda_2'$, where $a^k = 1 \in F$ for some $k \in \mathbb{N}$. Suppose that $\lambda_2' \in H_{\epsilon_p} \neq H_{\epsilon_1}$ for some $2 \leq p \leq q^{2(m-2)}$. Hence

$$J(n) = \langle \lambda_1, \epsilon_2, \ldots, \epsilon_{p-1}, \lambda_2', \epsilon_{p+1}, \ldots, \epsilon_{q^{2(m-2)}} \rangle.$$

Case 4. $n \geq 3$. By the same reason as that given in Case 2, $H_{\epsilon_1} = \langle \mu_1, \mu_2 \rangle$, where

$$\mu_1 = \begin{pmatrix} w_1 & w_2 & \cdots & w_{n-1} & w_n & v_{n+1} & \cdots & v_m \\ w_2 & w_3 & \cdots & w_n & w_1 & 0 & \cdots & 0 \end{pmatrix}$$

and

$$\mu_2 = \begin{pmatrix} w_1 & w_2 & w_3 & \cdots & w_{n-1} & w_n & v_{n+1} & \cdots & v_m \\ w_1 & aw_2 & w_3 & \cdots & w_{n-1} & w_1 + w_n & 0 & \cdots & 0 \end{pmatrix}$$

when a is a generator of the multiplicative group of $F \setminus \{0\}$. Define

$$\mu_2' = \begin{pmatrix} w_1 & w_2 & w_3 & \cdots & w_{n-1} & w_n & v_{n+1} & \cdots & v_m \\ w_1 & aw_2 & w_3 & \cdots & w_{n-1} & w_1 + w_n & w_1 & \cdots & w_1 \end{pmatrix}.$$

Then $\mu_2' \notin H_{\epsilon_1}$ and $\mu_2 = \epsilon_1\mu_2' = (\mu_1)^n\mu_2'$. Suppose that $\mu_2' \in H_{\epsilon_p} \neq H_{\epsilon_1}$ for some $2 \leq p \leq q^{n(m-n)}$. Hence $J(n) = \langle \mu_1, \epsilon_2, \ldots, \epsilon_{p-1}, \mu_2', \epsilon_{p+1}, \ldots, \epsilon_{q^{n(m-n)}} \rangle$.

From the above four cases, we obtain that $\mathrm{rank}(J(n)) \leq q^{n(m-n)}$. □

The following theorem comes directly from Lemmas 5.2 and 5.10.

THEOREM 5.11. *For $m > n \geq 1$, $\mathrm{rank}(Q) = q^{n(m-n)} + 1$.*

Observe that if $V = W$ with $m = n \geq 1$, then $Q = T(V)$ and $J(n) = \mathrm{GL}(W)$. By Lemma 5.2,

$$\mathrm{rank}(T(V)) = \mathrm{rank}(Q) = \mathrm{rank}(J(n)) + 1 = \mathrm{rank}(\mathrm{GL}(W)) + 1.$$

Then, by Lemma 5.9, we establish the following theorem.

THEOREM 5.12. *Let V be an n-dimensional vector space over a finite field F. Then*

$$\mathrm{rank}(T(V)) = \begin{cases} 2 & \text{if } n = 1, \\ 3 & \text{if } n \geq 2. \end{cases}$$

If $m = n = 0$ or $m > n = 0$, then $W = \{0\}$ and $|Q| = |T(V)| = 1$ and hence

$$\mathrm{rank}(Q) = \mathrm{rank}(T(V)) = 1.$$

We end this section by describing the idempotent rank of Q.

Clearly, if $n = 0$, then $\mathrm{idrank}(Q) = \mathrm{rank}(Q) = 1$. Now, assume that $n \geq 1$. Recall that $|H_{\epsilon_i}| = (q^n - 1)(q^n - q)(q^n - q^2)\cdots(q^n - q^{n-1})$ for all $1 \leq i \leq q^{n(m-n)}$. We consider three cases.

Case 1. $n = 1$ and $q = 2$. Then $Q = J(0) \cup J(1) = \{0_V\} \cup J(1)$ and $|H_{\epsilon_i}| = 2^1 - 1 = 1$: that is, $H_{\epsilon_i} = \{\epsilon_i\}$ for all $1 \leq i \leq q^{n(m-n)} = 2^{m-1}$. Hence $Q = \{0_V\} \cup \{\epsilon_i : 1 \leq i \leq 2^{m-1}\}$ and so $\mathrm{idrank}(Q) = \mathrm{rank}(Q) = 2^{m-1} + 1$.

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Case 2. $n = 1$ and $q \geq 3$. Then $|H_{\epsilon_i}| = q^1 - 1 = q - 1 \geq 2$ for all $1 \leq i \leq q^{m-1}$. It follows that $\langle E(J(n)) \rangle = \{\epsilon_1, \ldots, \epsilon_{q^{m-1}}\} \neq J(n)$ and hence Q cannot be generated by its idempotents.

Case 3. $n \geq 2$. Then $q^n - 1 \geq 4 - 1 = 3$ since $q \geq 2$, which implies that $|H_{\epsilon_i}| = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1}) \geq 3$ for all $1 \leq i \leq q^{n(m-n)}$. Thus $\langle E(J(n)) \rangle = \{\epsilon_1, \ldots, \epsilon_{q^{n(m-n)}}\} \neq J(n)$ and hence Q cannot be generated by its idempotents.

This leads to the following theorem.

Theorem 5.13.

$$\operatorname{idrank}(Q) = \begin{cases} 1 & \text{if } n = 0, \\ 2^{m-1} + 1 & \text{if } n = 1 \text{ and } q = 2. \end{cases}$$

If $(n = 1$ *and* $q \geq 3)$ *or* $n \geq 2$*, then* Q *has no idempotent rank.*

In the case when $V = W$, $Q = T(V)$ and $m = n$. Then, by Theorem 5.13, we obtain the following corollary.

Corollary 5.14. *Let V be an n-dimensional vector space over a finite field F with q elements. Then*

$$\operatorname{idrank}(T(V)) = \begin{cases} 1 & \text{if } n = 0, \\ 2^{n-1} + 1 & \text{if } n = 1 \text{ and } q = 2. \end{cases}$$

If $(n = 1$ *and* $q \geq 3)$ *or* $n \geq 2$*, then* Q *has no idempotent rank.*

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WORACHEAD SOMMANEE, Department of Mathematics and Statistics,
Faculty of Science and Technology, Chiang Mai Rajabhat University,
Chiang Mai 50300, Thailand
e-mail: worachead_som@cmru.ac.th

KRITSADA SANGKHANAN, Department of Mathematics,
Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand
e-mail: kritsada.s@cmu.ac.th