GENERAL TRANSFORMATIONS OF BILATERAL COGNATE TRIGONOMETRICAL SERIES OF ORDINARY HYPERGEOMETRIC TYPE

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1. Introduction. Whipple [6] was the first to consider transformations connecting well-poised hypergeometric series as particular cases of relations between cognate trigonometrical series. He used contour integrals of the Barnes type to deduce such transformations. Later Sears [3] gave a systematic theory of general and well-poised transformations of trigonometrical series of any order which included Whipple's results as particular cases.

The object of this paper is to study general transformations of bilateral cognate trigonometrical series in analogy with the ordinary bilateral series introduced by Bailey [1].

Thus, in §2 bilateral trigonometrical series have been defined and later, with the help of Sear's known transformations of ordinary trigonometrical series, a number of transformations connecting general and well-poised series have been deduced in §§4–9.

In §10 particular cases of these transformations have been considered which yield the known transformations of Bailey [1], Slater [4; 5], Jackson [2], and Sears [3].

Lastly, in §11 and onwards I have given direct proofs of the bilateral transformations deduced in §§4–9 by employing contour integrals of the Barnes type similar to those employed by Slater [5] for deducing Sears' general transformations of generalized hypergeometric series.

2. Definitions. If u_n denotes the $(n+1)$th term of the series

$$F(a_1, a_2, \dots, a_{M+1};\ b_1, b_2, \dots, b_M;)$$

then, following Sears' notation, the series

$$\sum_n (-)^n u_n \sin(\lambda + 2n)\theta, \qquad \sum_n u_n \sin(\lambda+2n)\theta,$$

$$\sum_n (-)^n u_n \cos(\lambda + 2n)\theta, \qquad \sum_n u_n \cos(\lambda+2n)\theta$$

will be denoted by the symbols

$$_{M+1}S_M\begin{bmatrix} a_1, \dots, a_{M+1}; & \theta \\ b_1, \dots, b_M & ; \lambda \end{bmatrix}, \qquad _{M+1}S'_M\begin{bmatrix} a_1, \dots, a_{M+1}; & \theta \\ b_1, \dots, b_M & ; \lambda \end{bmatrix}$$

$$_{M+1}C_M\begin{bmatrix} a_1, \dots, a_{M+1}; & \theta \\ b_1, \dots, b_M & ; \lambda \end{bmatrix}, \qquad _{M+1}C'_M\begin{bmatrix} a_1, \dots, a_{M+1}; & \theta \\ b_1, \dots, b_M & ; \lambda \end{bmatrix}$$

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respectively. When $\lambda = a_1$, the first numerator parameter, it will be omitted from each symbol, and, following Whipple, series of the type

$$S\left[\begin{matrix} a_1, \ldots, a_{M+1}; \theta \\ 1 + a_1 - a_2, \ldots, 1 + a_1 - a_{M+1}, \end{matrix}\right]$$

in which $\lambda = a_1$, necessarily, will be called well-poised and will be denoted by ${}_{M+1}S_M(a_1)$. We will also denote series of the type

$$S\left[\begin{matrix} 2a_r - a_1, a_r, a_r + a_2 - a_1, \ldots, a_r + a_{M+1} - a_1; \theta \\ 1 + a_r - a_1, \ldots, 1 + a_r - a_{M+1}, \end{matrix}\right]$$

by the symbol ${}_{M+1}S_M(a_r)$ for $r > 1$; where the numerator parameter $(a_r + a_r - a_1)$ is the first to occur, and the denominator parameter $(1 + a_r - a_r)$ is omitted. Similar notations are used for the series C, S', and C'.

Now we define the bilateral series as follows:

$$\begin{matrix}(2.1)\\(2.2)\end{matrix} \qquad \begin{matrix}{}_pX_p\\{}_pZ_p\end{matrix}\left[\begin{matrix} a_1, \ldots, a_p; \theta \\ b_1, \ldots, b_p; \lambda \end{matrix}\right] = \sum_{-\infty}^{\infty} \frac{(-)^n (a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_p)_n} \begin{matrix}\sin\\\cos\end{matrix}\left\{(\lambda + 2n)\theta\right\}$$

$$\begin{matrix}(2.3)\\(2.4)\end{matrix} \qquad \begin{matrix}{}_pX'_p\\{}_pZ'_p\end{matrix}\left[\begin{matrix} a_1, \ldots, a_p; \theta \\ b_1, \ldots, b_p; \lambda \end{matrix}\right] = \sum_{-\infty}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_p)_n} \begin{matrix}\sin\\\cos\end{matrix}\left\{(\lambda + 2n)\theta\right\}$$

where $(\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha)$.

It is easily seen that

$$\begin{matrix}(2.5)\\(2.6)\end{matrix} \qquad \begin{matrix}{}_pX_p\\{}_pZ_p\end{matrix}\left[\begin{matrix} a_1, \ldots, a_p; \theta \\ b_1, \ldots, b_p; \lambda \end{matrix}\right] = \begin{matrix}{}_{p+1}S_p\\{}_{p+1}C_p\end{matrix}\left[\begin{matrix} 1, a_1, \ldots, a_p; \theta \\ b_1, \ldots, b_p; \lambda \end{matrix}\right]$$

$$\pm \frac{(1 - b_1) \ldots (1 - b_p)}{(1 - a_1) \ldots (1 - a_p)} \begin{matrix}{}_{p+1}S_p\\{}_{p+1}C_p\end{matrix}\left[\begin{matrix} 1, 2 - b_1, \ldots, 2 - b_p; \theta \\ 2 - a_1, \ldots, 2 - a_p;\ 2 - \lambda \end{matrix}\right],$$

where on the right-hand side the positive or negative sign is to be taken according as the series is of the type ${}_pX_p$ or ${}_pZ_p$.

Similarly,

$$\begin{matrix}(2.7)\\(2.8)\end{matrix} \qquad \begin{matrix}{}_pX'_p\\{}_pZ'_p\end{matrix}\left[\begin{matrix} a_1, \ldots, a_p; \theta \\ b_1, \ldots, b_p; \lambda \end{matrix}\right] = \begin{matrix}{}_{p+1}S'_p\\{}_{p+1}C'_p\end{matrix}\left[\begin{matrix} 1, a_1, \ldots, a_p; \theta \\ b_1, \ldots, b_p; \lambda \end{matrix}\right]$$

$$\mp \frac{(1 - b_1) \ldots (1 - b_p)}{(1 - a_1) \ldots (1 - a_p)} \begin{matrix}{}_{p+1}S'_p\\{}_{p+1}C'_p\end{matrix}\left[\begin{matrix} 1, 2 - b_1, \ldots, 2 - b_p; \theta \\ 2 - a_1, \ldots, 2 - a_p; 2 - \lambda \end{matrix}\right],$$

where on the right-hand side the negative or positive sign is to be taken according as the series is of the type ${}_pX'_p$ or ${}_pZ'_p$.

The convergence factor $\Re(\Sigma b_p - \Sigma a_p - 1)$ will be henceforth denoted by y.

The series (2.1) and (2.2) converge when either

$$y > 0, \quad -\pi \leqslant 2\theta \leqslant \pi$$

or

$$-1 < y \leqslant 0, \quad -\pi < 2\theta < \pi.$$

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The series (2.3) and (2.4) converge when either

$$y > 0, \quad 0 \leqslant \theta \leqslant \pi$$

or

$$-1 < y \leqslant 0, \quad 0 < \theta < \pi.$$

When $y > 0$ all the series converge uniformly and absolutely in the variable θ or in λ but when $-1 < y \leqslant 0$ the convergence is, in general, conditional.

3. Notation. Let

$$G(a_1, \ldots, a_M; b_1, \ldots, b_N) = \left\{\prod_{r=1}^{M} \Gamma(a_r)\right\}\left\{\prod_{1}^{N} \Gamma(b_r)\right\}^{-1}$$

$$A = \left\{\prod_{2}^{M} G(a_r; b_r)\right\}\left\{\prod_{M+1}^{M+N} G(1 - b_r; 1 - a_r)\right\}$$

$$A(a_1) = \left\{\prod_{2}^{M} G(a_r - a_1; b_r - a_1)\right\}\left\{\prod_{M+1}^{M+N} G(1 + a_1 - b_r; 1 + a_1 - a_r)\right\} \times \left\{\Gamma(b_1 - a_1)\right\}^{-1}$$

$$A(b_{M+1}) = \left\{\prod_{1}^{M} G(1 + a_r - b_{M+1}; 1 + b_r - b_{M+1})\right\} \times \left\{\prod_{M+2}^{M+N} G(b_{M+1} - b_r; b_{M+1} - a_r)\right\}^{-1} \left\{\Gamma(b_{M+1} - a_{M+1})\right\}^{-1}$$

$$B = A \operatorname{cosec} \pi a_{M+1}/\Gamma(1 - a_{M+N+1})$$

$$B(a_1) = A(a_1)\, G(a_1; 1 + a_1 - a_{M+N+1}) \operatorname{cosec} \pi(a_{M+1} - a_1)$$

$$B(b_{M+1}) = -A(b_{M+1})\, G(b_{M+1} - 1; b_{M+1} - a_{M+N+1}) \operatorname{cosec} \pi(b_{M+1} - a_{M+1})$$

$$P = \Gamma(a_1)\left\{\prod_{2}^{M+1} G(a_r, a_r - a_1)\right\}\left\{\prod_{M+2}^{2M} G(1 + a_1 - a_r, 1 - a_r)\right\}^{-1}$$

$$Q(a_2) = \left\{\prod_{1}^{M+1}{}' G(a_r - a_2, a_2 + a_r - a_1)\right\} \times \left\{\prod_{M+2}^{2M} G(1 + a_2 - a_r, 1 + a_1 - a_2 - a_r)\right\}^{-1}$$

$$R = P/G(1 + a_1 - a_{2M+1}, 1 - a_{2M+1})$$

$$T(a_2) = Q(a_2)/G(1 + a_2 - a_{2M+1}, 1 + a_1 - a_2 - a_{2M+1})$$

$$U = \Gamma(a_1)\left\{\prod_{2}^{M+2} G(a_r, a_r - a_1)\right\}\left\{\prod_{M+3}^{2M+1} G(1 + a_1 - a_r, 1 - a_r)\right\}^{-1}$$

$$V(a_2) = \left\{\prod_{1}^{M+2}{}' G(a_r - a_2, a_r + a_2 - a_1)\right\} \times \left\{\prod_{M+3}^{2M+1} G(1 + a_2 - a_r, 1 + a_1 - a_2 - a_r)\right\}^{-1}$$

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$$X = UG(1 + a_1 - a_{2M+1}, 1 - a_{2M+1})$$

$$Y(a_2) = V(a_2)\,G(1 + a_2 - a_{2M+1}, 1 + a_1 - a_2 - a_{2M+1}).$$

The primes in the product symbols denote the omission of the gamma functions with zero arguments.

4. The general bilateral transformations. Sears [**3**] has proved the following general theorem for ordinary cognate trigonometrical series:

(4.1)
(4.2)

$$B\begin{matrix}S\\C\end{matrix}\begin{bmatrix}a_1, \ldots, a_{M+N+1}; & \theta\\ b_1, \ldots, b_{M+N} & ; \lambda\end{bmatrix}$$

$$= \mp B(a_1)\begin{matrix}S\\C\end{matrix}\begin{bmatrix}a_1, 1 + a_1 - b_1, \ldots, 1 + a_1 - b_{M+N} & ; & \theta\\ 1 + a_1 - a_2, \ldots, 1 + a_1 - a_{M+N+1}; & - & \lambda + 2a_1(1 - m\pi/\theta)\end{bmatrix}$$

$$\mp \text{ idem } (a_1; a_2, \ldots, a_{M+1})$$

$$+ B(b_{M+1})\begin{matrix}S\\C\end{matrix}\begin{bmatrix}1 + a_1 - b_{M+1}, \ldots, 1 + a_{M+N+1} - b_{M+1} & ; \theta\\ 2 - b_{M+1}, 1 + b_1 - b_{M+1}, \ldots, 1 + b_{M+N} - b_{M+1}; & \lambda + 2 - 2b_{M+1}(1 - m\pi/\theta)\end{bmatrix}$$

$$+ \text{ idem } (b_{M+1}; b_{M+2}, \ldots, b_{M+N}),$$

valid when either

$$y > 0, \quad (2m - 1)\pi \leqslant 2\theta \leqslant (2m + 1)\pi$$

or

$$-1 < y \leqslant 0, \quad (2m - 1)\pi < 2\theta < (2m + 1)\pi,$$

where m is a positive integer.

In (4.1) and (4.2) let us put $b_{M+1} = a_1, \ldots, b_{M+N} = a_M$ with $N = M$ and also let $a_{M+1} \to 1$. We find that all the series in (4.1) and (4.2) reduce to series of the type ${}_{M+1}S_M$ (or ${}_{M+1}C_M$) and the series corresponding to the parameter a_{M+1} on the right combines with the series on the left-hand side to give a bilateral series of the type ${}_MX_M$ (or ${}_MZ_M$). Also, the series corresponding to a_r combines with the series corresponding to b_{M+r} on the right for $r = 1, 2, \ldots, M$, to give corresponding bilateral series.

Finally, writing c_1 for a_{M+2}, c_2 for $a_{M+3}, \ldots, c_M$ for a_{2M+1}, we get the transformations

(4.3)
(4.4)

$$\prod_1^M G(a_r, 1 - a_r; b_r, 1 - c_r)\,\begin{matrix}{}_MX_M\\{}_MZ_M\end{matrix}\begin{bmatrix}c_1, \ldots, c_M; \theta\\ b_1, \ldots, b_M; \lambda\end{bmatrix}$$

$$= \mp \Gamma(a_1)\Gamma(1 - a_1)\prod_1^M{}' G(a_r - a_1, 1 + a_1 - a_r; b_r - a_1, 1 + a_1 - c_r)$$

$$\times \begin{matrix}{}_MX_M\\{}_MZ_M\end{matrix}\begin{bmatrix}1 + a_1 - b_1, \ldots, 1 + a_1 - b_M; & \theta\\ 1 + a_1 - c_1, \ldots, 1 + a_1 - c_M; & - \lambda + 2a_1(1 - m\pi/\theta)\end{bmatrix}$$

$$\mp \text{ idem } (a_1; a_2, \ldots, a_M)$$

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valid when either

$$y > 0, \quad (2m-1)\pi \leqslant 2\theta \leqslant (2m+1)\pi$$

or

$$-1 < y \leqslant 0, \quad (2m-1)\pi < 2\theta < (2m+1)\pi.$$

5. Sears has also proved that

(5.1)
(5.2)
$$B\,\begin{matrix} S' \\ C' \end{matrix}\begin{bmatrix} a_1, \ldots, a_{M+N+1};\ \theta \\ b_1, \ldots, b_{M+N}\ \ ;\ \lambda \end{bmatrix}$$

$$= \mp B(a_1)\,\begin{matrix} S' \\ C' \end{matrix}\begin{bmatrix} a_1, 1+a_1-b_1, \ldots, 1+a_1-b_{M+N}; & \theta \\ 1+a_1-a_2, \ldots, 1+a_1-a_{M+N+1}\ ; & -\lambda+2a_1-(2m+1)\pi a_1/\theta \end{bmatrix}$$

$$\mp \text{ idem } (a_1; a_2, \ldots, a_{M+1})$$

$$- B(b_{M+1})\,\begin{matrix} S' \\ C' \end{matrix}\begin{bmatrix} 1+a_1-b_{M+1}, \ldots, 1+a_{M+N+1}-b_{M+1} & ;\theta \\ 2-b_{M+1}, 1+b_1-b_{M+1}, \ldots, 1+b_{M+N}-b_{M+1}; & \lambda+2(1-b_{M+1})+(2m+1)\pi b_{M+1}/\theta \end{bmatrix}$$

$$- \text{ idem } (b_{M+1}; b_{M+2}, \ldots, b_{M+N}).$$

Using the same substitutions as in §4 we arrive at the following transformations

(5.3)
(5.4)
$$\prod_1^M G(a_r, 1-a_r; b_r, 1-c_r)\ \begin{matrix} {}_MX'_M \\ {}_MZ'_M \end{matrix}\begin{bmatrix} c_1, \ldots, c_M; \theta \\ b_1, \ldots, b_M; \lambda \end{bmatrix}$$

$$= \mp \Gamma(a_1)\Gamma(1-a_1)\prod_1^M G(a_r-a_1, 1+a_1-a_r; b_r-a_1, 1+a_1-c_r)$$

$$\times \begin{matrix} {}_MX'_M \\ {}_MZ'_M \end{matrix}\begin{bmatrix} 1+a_1-b_1, \ldots, 1+a_1-b_M; & \theta \\ 1+a_1-c_1, \ldots, 1+a_1-c_M; & -\lambda+2a_1-(2m+1)\pi a_1/\theta \end{bmatrix}$$

$$\mp \text{ idem } (a_1; a_2, \ldots, a_M)$$

valid under the conditions

$$y > 0, \quad m\pi \leqslant \theta \leqslant (m+1)\pi$$

or

$$-1 < y \leqslant 0, \quad m\pi < \theta < (m+1)\pi.$$

6. Well-poised trigonometrical transformations. Sears [**3**] has also proved the following nine transformation theorems for well-poised trigonometrical series.

(6.1) $P_{2M}\, S'_{2M-1}(a_1) + Q(a_2)\ {}_{2M}S'_{2M-1}(a_2) + \text{idem } (a_2; a_3, \ldots, a_{M+1}) = 0$

(6.2) $P \cos \tfrac{1}{2}\pi a_1\ {}_{2M}S_{2M-1}(a_1) + Q(a_2) \cos \tfrac{1}{2}\pi(2a_2 - a_1)\ {}_{2M}S_{2M-1}(a_2)$
$+ \text{ idem } (a_2; a_3, \ldots, a_{M+1}) = 0$

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$$(6.3)\quad P\sin\tfrac{1}{2}\pi a_1\ {}_{2M}C_{2M-1}(a_1)+Q(a_2)\sin\tfrac{1}{2}\pi(2a_2-a_1)\ {}_{2M}C_{2M-1}(a_2) + \text{idem }(a_2;a_3,\ldots,a_{M+1})=0$$

$$(6.4)\quad U\sin\pi a_1\ {}_{2M+1}C_{2M}(a_1)+V(a_2)\sin\pi(2a_2-a_1)\ {}_{2M+1}C_{2M}(a_2) + \text{idem }(a_2;a_3,\ldots,a_{M+2})=0$$

$$(6.5)\quad U\cos\tfrac{1}{2}\pi a_1\ {}_{2M+1}S'_{2M}(a_1)+V(a_2)\cos\tfrac{1}{2}\pi(2a_2-a_1)\ {}_{2M+1}S'_{2M}(a_2) + \text{idem }(a_2;a_3,\ldots,a_{M+2})=0$$

$$(6.6)\quad U\sin\tfrac{1}{2}\pi a_1\ {}_{2M+1}C'_{2M}(a_1)+V(a_2)\sin\tfrac{1}{2}\pi(2a_2-a_1)\ {}_{2M+1}C'_{2M}(a_2) + \text{idem }(a_2;a_3,\ldots,a_{M+2})=0$$

$$(6.7)\quad U\ {}_{2M+1}S_{2M}(a_1)+V(a_2)\ {}_{2M+1}S_{2M}(a_2)+\text{idem }(a_2;a_3,\ldots,a_{M+2})=0$$

$$(6.8)\quad X\sin\pi a_1\ {}_{2M}C'_{2M-1}(a_1)+Y(a_2)\sin\pi(2a_2-a_1)\ {}_{2M}C'_{2M-1}(a_2) + \text{idem }(a_2;a_3,\ldots,a_{M+2})=0$$

$$(6.9)\quad R\ {}_{2M+1}S_{2M}(a_1)+T(a_2)\ {}_{2M+1}S_{2M}(a_2)+\text{idem }(a_2;a_3,\ldots,a_{M+1})=0$$

The conditions of validity for (6.1), (6.5), (6.6), (6.8) are either $y > 0$, $0 \leqslant \theta \leqslant \pi$ or $-1 < y \leqslant 0$ and $0 < \theta < \pi$.

The results (6.2), (6.3), (6.4), and (6.9) are true for either $y > 0$, $-\pi \leqslant 2\theta \leqslant \pi$ or for $-1 < y \leqslant 0$ and $-\pi < 2\theta < \pi$.

The result (6.7), however, is true for either $y > 0$, $-3\pi \leqslant 2\theta \leqslant 3\pi$ or $-1 < y \leqslant 0$ and $-3\pi < 2\theta < 3\pi$ $(2\theta \neq \pm\pi)$.

7. Transformations of well-poised bilateral series. Consider first the transformations (6.1), (6.2), and (6.3). In each one of them put $M = 2N + 1$ (an odd integer) and then let $a_{2N+2} = 1$, $a_2 = 1 + a_1 - a_3$, $a_4 = 1 + a_1 - a_5$, etc., and, in general, $a_{2N} = 1 + a_1 - a_{2N+1}$. The series then reduce to one of the type ${}_{2N+1}S'_{2N}$, ${}_{2N+1}S_{2N}$, and ${}_{2N+1}C_{2N}$ respectively.

Simplifying the coefficients of the series on the left and, finally, writing a for a_1, b_1 for a_{2N+3}, b_2 for a_{2N+4}, etc. and, in general, b_{2N} for a_{4N+2}; a_1 for a_3, a_2 for a_5, etc. and in general, a_N for a_{2N+1}, we have, on combining the series in pairs as in §4, the following three transformations for ${}_{2N}X'_{2N}$, ${}_{2N}X_{2N}$, and ${}_{2N}Z_{2N}$ series, respectively:

$$(7.1)\qquad P'\ {}_{2N}X'_{2N}(a) = Q'(a_1)\ {}_{2N}X'_{2N}(a_1)+\text{idem }(a_1;a_2,\ldots,a_N),$$

$$(7.2)\quad P'\cos\tfrac{1}{2}\pi a\ {}_{2N}X_{2N}(a)= Q'(a_1)\cos\tfrac{1}{2}\pi(2a_1-a)\ {}_{2N}X_{2N}(a_1) + \text{idem }(a_1;a_2,\ldots,a_N),$$

$$(7.3)\quad P'\sin\tfrac{1}{2}\pi a\ {}_{2N}Z_{2N}(a) = Q'(a_1)\sin\tfrac{1}{2}\pi(2a_1-a)\ {}_{2N}Z_{2N}(a_1) + \text{idem }(a_1;a_2,\ldots,a_N),$$

where

$$P' = \Gamma(a)\Gamma(1-a)\left\{\prod_1^N G(a_r, 1-a_r, 1+a-a_r, a_r-a)\right\} \times\left\{\prod_1^{2N} G(1+a-b_r, 1-b_r)\right\}^{-1},$$

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$$Q'(a_1) = \Gamma(a_1 - a)\Gamma(1 + a - a_1)\Gamma(a_1)\Gamma(1 - a_1)$$
$$\times \left\{ \prod_1^{N}{}' G(1 + a - a_1 - a_r, a_1 + a_r - a, a_r - a_1, 1 + a_1 - a_r) \right\}$$
$$\times \left\{ \prod_1^{2N} G(1 + a_1 - b_r, 1 + a - a_1 - b_r) \right\}^{-1},$$

and ${}_MX_M(a)$, ${}_MX_M(a_r)$ denote the series

$${}_MX_M\left[\begin{matrix} b_1, b_2, \ldots, b_M & ; \theta \\ 1 + a - b_1, \ldots, 1 + a - b_M; a \end{matrix}\right],$$

and

$${}_MX_M\left[\begin{matrix} a_r + b_1 - a, \ldots, a_r + b_M - a; \ \theta \\ 1 + a_r - b_1, \ldots, 1 + a_r - b_M; 2a_r - a \end{matrix}\right]$$

respectively, with similar notations for X', Z, and Z' series also.

The transformations (7.1), (7.2), and (7.3) are valid under the same conditions as those required for (6.1), (6.2), and (6.3) respectively.

8. We now pass on to the consideration of (6.4), (6.5), (6.6), and (6.7). Taking $M = 2N$ (an even integer) and treating them by a method similar to that of §7, we get the transformations:

(8.1) $$P' \, \Gamma(1 + a - b_{2N}) \, \Gamma(1 - b_{2N}) \sin \pi a \, {}_{2N-1}Z_{2N-1}(a)$$
$$= Q'(a_1) \, \Gamma(1 + a_1 - b_{2N}) \, \Gamma(1 + a - a_1 - b_{2N}) \sin \pi(2a_1 - a) \, {}_{2N-1}Z_{2N-1}(a_1)$$
$$+ \text{ idem } (a_1; a_2, \ldots, a_N),$$

(8.2) $$P' \, \Gamma(1 + a - b_{2N}) \, \Gamma(1 - b_{2N}) \cos \tfrac{1}{2}\pi a \, {}_{2N-1}X'_{2N-1}(a)$$
$$= Q'(a_1) \, \Gamma(1 + a_1 - b_{2N}) \, \Gamma(1 + a - a_1 - b_{2N}) \cos \tfrac{1}{2}\pi(2a_1 - a) {}_{2N-1}X'_{2N-1}(a_1)$$
$$+ \text{ idem } (a_1; a_2, \ldots, a_N),$$

(8.3) $$P' \, \Gamma(1 + a - b_{2N}) \, \Gamma(1 - b_{2N}) \sin \tfrac{1}{2}\pi a \, {}_{2N-1}Z'_{2N-1}(a)$$
$$= Q'(a_1) \, \Gamma(1 + a_1 - b_{2N}) \, \Gamma(1 + a - a_1 - b_{2N}) \sin \tfrac{1}{2}\pi(2a_1 - a) \, {}_{2N-1}Z'_{2N-1}(a_1)$$
$$+ \text{ idem } (a_1; a_2, \ldots, a_N),$$

(8.4) $$P' \, \Gamma(1 + a - b_{2N}) \, \Gamma(1 - b_{2N}) \, {}_{2N-1}X_{2N-1}(a)$$
$$= Q'(a_1) \, \Gamma(1 + a_1 - b_{2N}) \, \Gamma(1 + a - a_1 - b_{2N}) \, {}_{2N-1}X_{2N-1}(a_1)$$
$$+ \text{ idem } (a_1; a_2, \ldots, a_N),$$

respectively. These transformations are valid under the same conditions as those necessary for (6.4), (6.5), (6.6), and (6.7) respectively.

It may be noted that the transformations (8.1), (8.2), and (8.4) can also be obtained by putting $b_{2N} = \frac{1}{2}(1 + a)$ in (7.3), (7.1), and (7.2) respectively.

9. Next, in (6.8) take $M = 2N$ and, after reduction as in §7 and §9, change N to $(N + 1)$. This gives the transformation

(9.1) $$P'\Gamma(1 + a - a_{N+1})\Gamma(a_{N+1} - a)\Gamma(1 - a_{N+1})\Gamma(a_{N+1}) \sin \pi a \, {}_{2N}Z'_{2N}(a)$$
$$= Q'(a_1) \, \Gamma(a_1 - a + a_{N+1}) \, \Gamma(1 + a - a_1 - a_{N+1}) \, \Gamma(a_{N+1} - a_1)$$
$$\times \Gamma(1 + a_1 - a_{N+1}) \sin \pi(2a_1 - a) \, {}_{2N}Z'_{2N}(a_1) + \text{ idem } (a_1; a_2, \ldots, a_{N+1}).$$

Lastly, in (6.9), take $M = 2N + 1$ and as before we obtain the transformation

$$(9.2)\quad \{P'/\Gamma(1+a-b_{2N+1})\,\Gamma(1-b_{2N+1})\}\ {}_{2N+1}X_{2N+1}(a)$$
$$= \{Q'(a_1)/\Gamma(1+a_1-b_{2N+1})\,\Gamma(1+a-a_1-b_{2N+1})\}\ {}_{2N+1}X_{2N+1}(a_1)$$
$$+ \text{ idem } (a_1; a_2, \ldots, a_N).$$

10. Special cases. In this section we shall briefly mention some of the interesting particular cases of the transformations deduced in §§4–9.

If we take $\theta = 0$ or $\frac{1}{2}\pi$ in all the above transformations we get the transformations of ordinary bilateral series with arguments $+1$ or -1, as the case may be.

In particular, $\theta = \frac{1}{2}\pi$ in (7.1) gives Slater [**4**, (11)]. Also, $\theta = 0$ in (7.3), (8.1), (8.3), (9.1) gives Slater [**4**, (11), (14), (13), (12)].

As indicated by Slater [**4**] her trasnformations include all those of Bailey [**1**].

We can also by suitable substitutions rediscover Sears' original theorems on trigonometric series.

11. Bilateral trigonometrical integrals. We shall now give a direct proof of the general transformations deduced in §§4–9. Let us first consider the general transformation (4.3). For simplicity we shall prove it for the case when $m = 0$.

Consider the integral

$$\frac{1}{2\pi i}\int_C G\begin{bmatrix} a_1+s, \ldots, a_M+s, 1-a_1-s, \ldots, 1-a_M-s, -s, 1+s; \\ b_1+s, \ldots, b_M+s, 1-c_1-s, \ldots, 1-c_M-s, \end{bmatrix} e^{2is\theta} ds$$

where the contour of integration C is a circle of radius R with origin as centre, and R is so chosen that the circle does not pass through any of the poles of the integrand. The parameters a, b, and c are supposed real for simplicity's sake. θ is necessarily a real quantity. The parameters are such that none of the members of the two sequences

$$-1-n, -n-a_1, \ldots, -n-a_M$$

and

$$n, 1-a_1+n, \ldots, 1-a_M+n$$

coincide.

Now the integrand for $\Re s > 0$ can be written as

$$\frac{\Gamma(c_1+s)\ldots\Gamma(c_M+s)}{\Gamma(b_1+s)\ldots\Gamma(b_M+s)}\ \frac{\pi\sin\pi(c_1+s)\ldots\sin\pi(c_M+s)}{\sin\pi s\sin\pi(a_1+s)\ldots\sin\pi(a_M+s)}\, e^{2is\theta}.$$

Writing $s = Re^{i\phi}$, $-\frac{1}{2}\pi \leqslant \phi \leqslant \frac{1}{2}\pi$, we find that the first fraction in the above product is

$$O(R^{\Sigma c_r - \Sigma b_r})$$

and the second fraction is

(i) bounded for $R \to \infty$ if $-\pi \leqslant 2\theta \leqslant \pi$, and

(ii) $O[\exp(-2\theta R\sin\phi - R\pi|\sin\phi|)]$ when $R \to \infty$.

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From (i) it follows that the integral round the semicircle on the right of the imaginary axis tends to zero if

$$\left(\sum_1^M b_r - \sum_1^M c_r\right) > 1, \qquad -\pi \leqslant 2\theta \leqslant \pi.$$

Again from (ii) it follows with the help of Jordan's Lemma [**7**, p. 115, §6.222] that the integral round the same semicircle tends to zero if

$$\left(\sum_1^M b_r - \sum_1^M c_r\right) > 0, \qquad -\pi < 2\theta < \pi.$$

Similarly, when $\Re s < 0$ we write the integrand as

$$\frac{\Gamma(1-b_1-s)\ldots\Gamma(1-b_M-s)}{\Gamma(1-c_1-s)\ldots\Gamma(1-c_M-s)}\;\frac{\pi\sin\pi(b_1+s)\ldots\sin\pi(b_M+s)}{\sin\pi s\sin\pi(a_1+s)\ldots\sin\pi(a_M+s)}\,e^{2is\theta}$$

and similar remarks follow under the same conditions. Thus we have shown that the integral round the circle $|z| = R$ tends to zero as $R \to \infty$ under the above two sets of conditions.

For the sake of brevity let us suppose that

$$r_{n,m} \qquad (m = 0, 1, \ldots, M)$$

are the residues of the integrand at the poles n, $1 - a_1 + n, \ldots, 1 - a_M + n$, and

$$R_{n,m} \qquad (m = 0, 1, \ldots, M)$$

are the residues of the integrand at the poles $-1-n, -n-a_1, \ldots, -n-a_M$.

Then, if $\sum b_r - \sum c_r > 1$ and $-\pi \leqslant 2\theta \leqslant \pi$, the series of residues

$$\sum_{n=0}^{\infty} r_{n,m}, \quad \sum_{n=0}^{\infty} R_{n,m} \qquad (m = 0, 1, \ldots, M),$$

converge absolutely and there is no difficulty in letting $R \to \infty$. Hence, by Cauchy's Theorem,

$$\sum_{m=0}^{M}\sum_{n=0}^{\infty} r_{n,m} + \sum_{m=0}^{M}\sum_{n=0}^{\infty} R_{n,m} = 0.$$

Introducing a parameter λ by multiplying the above equation by $e^{i\lambda\theta}$ throughout and equating the real and imaginary parts we obtain, after combining the series $\sum r_{n,s}$ with $\sum R_{n,s}$, the transformations (4.4) and (4.3) respectively.

In the case when $\sum b_r - \sum c_r > 0$ and $-\pi < 2\theta < \pi$, the series of residues are only conditionally convergent, but, with a little labour we can justify the limiting process of letting $R \to \infty$. The details are similar to those given by Whipple [**6**] and hence have been omitted.

In the case when the parameters a, b, and c are complex, we have to replace the conditions of convergence by their real parts and instead of equating the real and imaginary parts in the final series of residues we have to add or subtract the conjugate series.

In order to obtain the transformations (5.3) and (5.4) we consider the above integral with $e^{i(\pi+2\theta)s}$ instead of the exponential factor $e^{2i\theta s}$ in the integrand, and proceed in exactly the same manner.

12. Well-poised bilateral trigonometrical integrals. If

$$P_N(s) = \Gamma(a+s)\,\Gamma(a_1+s)\ldots\Gamma(a_N+s)\,\Gamma(1+a-a_1+s)\ldots\Gamma(1+a-a_N+s) \times \Gamma(a_1-a-s)\ldots\Gamma(a_N-a-s)\,\Gamma(-s)\,\Gamma(1+s)\,\Gamma(1-a-s) \times \Gamma(1-a_1-s)\ldots\Gamma(1-a_N-s)$$

and

$$Q_{2N}(s) = \left\{\prod_{r=1}^{2N} G(1+a-b_r+s,\, 1-b_r-s)\right\}^{-1}$$

then

$$\frac{1}{2\pi i}\int_C P_N(s)\,Q_{2N}(s)\cos(\tfrac{1}{2}a+s)\pi\, e^{i\theta(2s+a)}ds, \tag{12.1}$$

$$\frac{1}{2\pi i}\int_C P_N(s)\,Q_{2N}(s)\, e^{i\theta(2s+a)}ds, \tag{12.2}$$

$$\frac{1}{2\pi i}\int_C P_N(s)\,Q_{2N}(s)\sin(\tfrac{1}{2}a+s)\pi\, e^{i\theta(2s+a)}ds, \tag{12.3}$$

where C is the same contour as before, give the transformations (7.2), (7.1), (7.3), respectively. The rest of the transformations can easily be obtained by obvious modifications of the above integrals.

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