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SEMILINEAR SECOND ORDER ELLIPTIC OSCILLATION

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1. **Introduction.** These pages summarize recent progress on the oscillation problem for semilinear elliptic partial differential equations of the form

$$L(u;x) \equiv \sum_{i,j=1}^{n} D_i[A_{ij}(x)D_ju] + B(x,u) = 0 \tag{1}$$

$$x \in \Omega, \qquad D_i = \partial/\partial x_i, \qquad i = 1, \ldots, n$$

in unbounded domains Ω in n-dimensional Euclidean space R^n. Our attention is restricted to the second order symmetric equation (1), and completeness is not attempted; the emphasis is on results obtained in the last five years.

Points in R^n are denoted by $x = (x_1, \ldots, x_n)$ and the Euclidean length of x is written $|x|$. The following notation will be used throughout:

$$S_a = \{x \in R^n : |x| = a\}, \qquad a > 0;$$

$$G_a = \{x \in R^n : |x| > a\};$$

$$\Omega_a = \Omega \cap G_a;$$

$$G(a,b) = \{x \in R^n : a < |x| < b\}.$$

The measure on S_r and S_1 will be denoted by s and ω, respectively: $ds = r^{n-1}\, d\omega$. Our assumptions below on the functions B and A_{ij}, $i, j = 1, \ldots, n$ in (1) could be relaxed somewhat, but are typical of those in [28–31, 38–41].

(A) Each A_{ij} in (1) is a real-valued function of class $C^1(\Omega)$, and the matrix $A = (A_{ij})$ is symmetric and positive definite in Ω (ellipticity condition);

(B_1) $B(x,t)$ is continuous in $\Omega \times R^1$ with $B(x,-t) = -B(x,t)$ for all $x \in \Omega$, $t > 0$;

(B_2) $B(x,t) \geq p(x)\phi(t)$ for all $x \in \Omega$ and for all $t \geq 0$, where p is continuous in Ω and ϕ is a continuously differentiable function in $[0,\infty)$ with $\phi(t) > 0$ in $(0,\infty)$.

Assumptions (B_1) and (B_2) need to be strengthened or modified for many of the theorems stated in §§3–7.

A function $f : \Omega \to R^1$ is called *oscillatory* in Ω whenever $f(x)$ has a zero in

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Ω_a for all $a > 0$. An elliptic equation (1) (or inequality $uL(u; x) \le 0$) is called oscillatory in Ω whenever every solution of the equation (or inequality, respectively) is oscillatory in Ω. The concept of *nodal oscillation* (or strong oscillation) has frequently been considered [2, 14–16, 22, 33, 34, 38, 40] in the special case that (1) has the linear form (2) below, i.e. $B(x, t)$ has the form $p(x)t$. A nonempty bounded domain $M \subset \Omega$ is called a *nodal domain* of a nontrivial solution u of (2) if and only if $u(x) = 0$ for all $x \in \partial M$. Equation (2) is defined to be *nodally oscillatory* in Ω if and only if there exists a nontrivial solution u_a of (2) with a nodal domain contained in Ω_a for all $a > 0$. Otherwise stated, (2) is nodally oscillatory in Ω if and only if the Dirichlet problem $L(u; x) = 0$ in M, $u = 0$ on ∂M is not uniquely solvable in some nonempty bounded domain M in Ω_a for all $a > 0$. If $M \subset \Omega_a$ is a nodal domain of a nontrivial solution u_a of (2), then *every* solution u of (2) has a zero at some point in $\bar{M}$ by the n-dimensional version of Sturm's separation theorem [37]. Incidentally, the stronger conclusion that every solution of (2) except a constant multiple of u_a has a zero at some point in M, without any boundary regularity hypotheses whatsoever, was proved in 1978 by Heywood, Noussair, and the writer [17]. It follows that a linear equation (2) is oscillatory in an unbounded domain Ω if it is nodally oscillatory in Ω.

The converse of the last statement also is true provided the coefficients in (2) are sufficiently regular; Allegretto [2] proved it in 1974 under the following hypotheses: (i) $p \in C^{3m}(\bar{M})$; (ii) Each $A_{ij} \in C^{3m+1}(\bar{M})$; and (iii) L is uniformly elliptic in M for every bounded subdomain $M \subset \Omega$, where

$$m = \left[\frac{1}{2}\left[\frac{n+6}{2}\right]\right]$$

and $[\beta]$ denotes the largest integer not exceeding β. Allegretto showed how to construct a positive solution of (2) in $\Omega_{a+\varepsilon}$ for any $\varepsilon > 0$ if the Dirichlet problem for (2) is uniquely solvable in every bounded domain M in Ω_a for some $a > 0$. This construction implies Allegretto's equivalence theorem: *Nodal oscillation and* (*weak*) *oscillation of sufficiently regular symmetric linear elliptic equations are equivalent.* We note that the regularity requirements are quite severe, but can be weakened according to Allegretto [2] and Piepenbrink [34]; the latter states that hypotheses (i) and (ii) can be replaced by $p \in C^{2m}$ and each $A_{ij} \in C^{2m+1}$, respectively, e.g. $p \in C^4$ if $n = 2$, 3, 4, or 5. A related theorem was stated earlier by Kuks [22, Theorem 3] without proof and with incomplete hypotheses. Allegretto [2], Glazman [14], and Piepenbrink [33, 34] have all emphasized the relationship between nonoscillation of L and the nature of the lower part of the spectrum of the selfadjoint realization of $-L$. Allegretto [2] gives the interesting example

$$L(u; x) = \sum_{i=1}^{2} [D_i(x_i^{-2} D_i u) + 2x_i^{-4} u]$$

in the unit square $\Omega=\{x\in R^2: 0<x_i<1;\ i=1,2\}$ to illustrate that some kind of global regularity assumptions are needed, even for a regular bounded domain, in order that the existence of a positive solution of (2) in Ω implies the unique solvability of the Dirichlet problem in every subdomain of Ω. In fact, $u(x)=x_1x_2$ is a positive solution in Ω while $u_n(x)=(x_1^2-x_1/n)(x_2^2-x_2/n)$ is a nontrivial solution of $L(u_n; x)=0$ in M, $u_n=0$ on ∂M for the subdomain

$$M=\left\{x\in\Omega: 0<x_i<\frac{1}{n};\ i=1,2\right\},\qquad n=2,3,\ldots,$$

and consequently the Dirichlet problem does not have a unique solution in all subdomains of Ω.

§2 contains a brief description of oscillation criteria for linear elliptic equations, dating back to a result of Glazman [14], thereby extending to R^n various one-dimensional criteria of Kneser [37], Leighton [24], Moore [25], and others. The existence of positive solutions of (1) in exterior domains is the main topic of §3. In the case of the Emden–Fowler equation, for which $B(x, t)$ in (1) has the form $p(x)t^\gamma$, $\gamma>0$, specific criteria for the existence of a positive solution are used in §§5 and 6 to generate necessary and sufficient conditions for (1) to be oscillatory in both the superlinear case ($\gamma>1$) and the sublinear case ($0<\gamma<1$). These results extend theorems of Atkinson [7] and Belohorec [8, 9, 44] for ordinary differential equations to n dimensions, and constitute the only known characterizations of oscillatory partial differential equations at this date. §3 contains specific upper and lower bounds for positive solutions, and also a criterion for the existence of a *bounded* positive solution as a consequence of Nehari's one-dimensional criterion [26]. Some recent superlinear oscillation criteria [29] arise from a Riccati-type transformation (Lemma 11) together with *a priori* lower bounds for positive solutions of (1) (Lemma 9), as described in §4. The method of spherical means [28] is used in §5 to generate sharp oscillation criteria for Schrödinger equations; in fact, these are characterizations of oscillatory equations in some cases [30, 31]. A brief discussion of perturbed linear equations, without sign restrictions on the coefficient functions, is given in §7, and some typical oscillation theorems in arbitrary unbounded domains are described in §8.

2. **Linear equations.** The first oscillation theorems for elliptic equations were obtained by Glazman [14], Headley [15, 16], Kreith [19, 20], Kreith and Travis [21], and the author [16, 38–41] in the *linear* case

$$L(u; x)=\sum_{i,j=1}^{n} D_i[A_{ij}(x)D_ju]+p(x)u=0, \tag{2}$$

where p and (A_{ij}) are as in assumptions (A) and (B_2). Some of these results will now be described in the case that the unbounded domain Ω is an exterior domain, i.e. there exists a positive number a such that $G_a\subset\Omega$.

Let $\lambda(x)$ denote the (necessarily positive) largest eigenvalue of the matrix $(A_{ij}(x))$ and let f be any piecewise C^1 function in $(0, \infty)$ satisfying $f(r) \geq \max_{|x|=r} \lambda(x)$, $0 < r < \infty$. Also let $g(r) = \min_{|x|=r} p(x)$, $0 < r < \infty$. Then [16] equation (2) is nodally oscillatory in an exterior domain $\Omega \subset R^n$ if either

$$(3) \qquad \int_a^\infty \frac{dr}{r^{n-1}f(r)} = +\infty \quad \text{and} \quad \int_a^\infty r^{n-1}g(r)\,dr = +\infty$$

or

$$(4) \qquad \int_a^\infty \frac{dr}{r^{n-1}f(r)} < \infty \quad \text{and} \quad \int_a^\infty r^{n-1}h_n^\mu(r)g(r)\,dr = +\infty$$

for some $a > 0$ and $\mu > 1$, where

$$h_n(r) = \int_r^\infty \frac{dt}{t^{n-1}f(t)}$$

If $n = 1$, the conditions (3) reduce to the Leighton–Wintner oscillation criterion for an ordinary linear equation [24, 43] while (4) reduce to Moore's criteria [25]. If $\lambda(x)$ is bounded in Ω, additional nodal oscillation criteria [16] are

$$(5) \qquad \int_a^\infty rg(r)\,dr = +\infty \quad \text{if} \quad n = 2,$$

$$(6) \qquad \int_a^\infty r^{1-\delta}g(r)\,dr = +\infty \quad \text{if} \quad n \geq 3,$$

for some positive numbers a and δ; and

$$(7) \qquad \liminf_{r\to\infty} r^2 g(r) > \lambda_0(n-2)^2/4$$

where λ_0 is an upper bound on $\lambda(x)$ in Ω. Condition (7) reduces to Glazman's criterion [14] if (A_{ij}) in (2) is the identity matrix, and becomes Kneser's classical criterion in one dimension. The results (3)–(7) are proved in [16], and in fact are extended to unbounded domains which are not necessarily exterior domains: it is merely required that Ω be large enough at ∞ to contain a cone $C_\alpha = \{x \in R^n : x^i \geq |x| \cos \alpha\}$ for some i, $1 \leq i \leq n$, and some α in $(0, \pi]$. Recently Noussair and the author [29] have sharpened and extended (5) and (6) as indicated below. For these results, $p_M(r)$ denotes the spherical mean of p over the sphere S_r:

$$(8) \qquad p_M(r) = \frac{1}{\omega(S_1)} \int_{S_1} p(x)\,d\omega,$$

where ω denotes the measure on S_1. The boundedness of the largest eigenvalue $\lambda(x)$ of $(A_{ij}(x))$ is weakened to

$$\max_{|x|=r} \lambda(x) \le C[\log(\log r)]^{\delta}, \qquad r>e, \quad \text{if} \quad n=2; \tag{9}$$

$$\max_{|x|=r} \lambda(x) \le C(\log r)^{\delta}, \qquad r>1, \quad \text{if} \quad n\ge 3, \tag{10}$$

for some numbers $C>0$ and δ, $0<\delta\le 1$.

THEOREM 1. *The linear inequality* $uL(u;x)\le 0$ *is oscillatory in an exterior domain* Ω *in* R^2, R^n $(n\ge 3)$ *if* (9), (10) *hold, respectively, and there exists a positive number a such that*

$$\int_a^{\infty}\left[r\log r p_M(r)-\frac{f(r)}{4r\log r}\right]dr=+\infty \quad \text{if} \quad n=2; \tag{11}$$

$$\int_a^{\infty}\left[rp_M(r)-\frac{(n-2)^2f(r)}{r}\right]dr=+\infty \quad \text{if} \quad n\ge 3. \tag{12}$$

These criteria are also *nodal* oscillation criteria if p and the A_{ij} in (2) are sufficiently regular because of Allegretto's equivalence theorem [2], as described in §1. In the case of a constant matrix (A_{ij}) in (2), conditions (9) and (10) hold automatically, and hence (11) and (12) are oscillation criteria if $f(r)$ is replaced by the (constant) largest eigenvalue λ of (A_{ij}). Another oscillation criterion obtained in [29] is

$$\int_{\Omega} p(x)\,dx=+\infty \quad \text{and} \quad \int_a^{\infty}\frac{r^{1-n}\,dr}{f(r)}=\infty$$

for some $a>0$. Positivity of $p(x)$ throughout Ω is not required for any of these results.

The sharpness of (11) and (12) can be seen by considering the special case that (2) is a radial Schrödinger equation $\Delta u+p_M(r)u=0$; then (11) and (12) are known [37] to be sharp one-dimensional criteria for the damped ordinary equation arising from separation of variables. Of course (11) and (12) do not characterize oscillatory equations (2): No characterizations are known even for *ordinary* linear equations. It is interesting to compare (11) with the characterization (32) in Theorem 19 (§5) of oscillatory *superlinear* equations (2) of the special Emden–Fowler type (14).

3. **Existence of positive solutions.** Although the existence theorems below are true for the general semilinear elliptic equation (1), by essentially the same proofs, we shall state them for the Schrödinger equation

$$L(u;x)=\Delta u+B(x,u)=0, \qquad x\in\Omega \tag{13}$$

and in particular the Emden–Fowler equation

$$\Delta u+p(x)u^{\gamma}=0, \qquad x\in\Omega, \qquad \gamma>0 \tag{14}$$

since the interesting applications are in these special cases. In particular, necessary and sufficient conditions for (14) to be oscillatory are generated in both the superlinear case ($\gamma > 1$) and the sublinear case ($0 < \gamma < 1$). These are the only known characterizations of oscillatory partial differential equations, extending well-known theorems of Atkinson [7] and Belohorec [8, 44] to n dimensions.

For a bounded domain $M \subset R^n$, the Hölder norms of functions $u: \bar{M} \to R^1$ are defined (as usual) by [23]

$$\|u\|_{\alpha,\bar{M}} = \sup_{\substack{x,y \in M \\ x \neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

$$\|u\|_{m+\alpha,\bar{M}} = \sum_{|i|=m} \|D^i u\|_{\alpha,\bar{M}} + \sum_{|i| \le m} \sup_{x \in \bar{M}} |D^i u(x)|,$$

$$0 < \alpha < 1, \qquad m = 1, 2, \ldots,$$

where i denotes a multi-index of length $|i|$. The Hölder space $C^{m+\alpha}(\bar{M})$ is the set of all continuous real-valued functions on $\bar{M}$ such that $\|u\|_{m+\alpha,\bar{M}}$ is finite, $0 < \alpha < 1$, $m = 0, 1, 2, \ldots$.

Equation (13) is to be considered in an exterior domain Ω in R^n under the hypotheses below.

(R) Regularity hypothesis. $B \in C^\alpha(\bar{M} \times J)$, for some α in $0 < \alpha < 1$, for every bounded domain $M \subset \Omega$ and for every interval $J = [0, b]$, $b > 0$.

(MN) Monotony and nonnegativity hypotheses. $0 \le B(x, t_1) \le B(x, t_2)$ for all t_1, t_2 satisfying $0 \le t_1 \le t_2$ and for all $x \in \Omega$.

Definition. A *solution* of $L(u; x) = 0$ in an exterior domain Ω is a function $u \in C^{2+\alpha}(\bar{M})$ for every bounded subdomain $M \subset \Omega$, with α as in hypothesis (*R*), such that $L(u; x) = 0$ for all $x \in \Omega$. A solution of $L(u; x) \le 0$ or $L(u; x) \ge 0$ is defined similarly.

Theorem 2 [30, 31]. *Let L, Ω and α be as above and suppose a is a positive number such that $G_a \subset \Omega$. If there exists a positive* (*uniformly positive, respectively*) *solution $v(x)$ of $L(v; x) \le 0$ in G_a, then equation* (13) *has a positive* (*uniformly positive*) *solution $u(x)$ in G_a. If in addition there exists a positive solution of $L(w; x) \ge 0$ in G_a such that $w(x) \le v(x)$ throughout $G_a \cup S_a$, then* (13) *has a solution $u(x)$ satisfying $w(x) \le u(x) \le v(x)$ throughout $G_a \cup S_a$.*

An essential part of the proof is the inductive construction of a nonincreasing sequence of functions u_j in $G_a \cup S_a$ which are solutions of (13) in annular subdomains $\{x \in R^n : a < |x| < a + j\}$, $j = 0, 1, 2, \ldots$. This can be done by means of the maximum principle [35] and a theorem of Amann [6, p. 142], providing conditions for the existence of a solution u_j of (12) in a bounded domain which is squeezed between a subsolution and a supersolution. Then the sequence is shown [30] to converge to a solution of (13) in G_a by means of Sobolev embedding [23], classical Schauder estimates [12, p. 335], and L^p estimates developed by Agmon, Douglis, and Nirenberg [1, p. 704].

COROLLARY 3. *A necessary and sufficient condition for the existence of a positive solution of* (13) *in an exterior domain* G_a *is the existence of a positive supersolution of* (13) *in* G_a.

COROLLARY 4. *For arbitrary* ε *in* $(0, n-2)$, $n \geq 3$, (13) *has a solution* $u(x)$ *in* $G_a \subset R^n$, *for some* $a > 0$, *satisfying the inequalities*

$$|x|^{2-n} \leq u(x) \leq |x|^{2-n+\varepsilon}, \qquad x \in G_a$$

if

$$\limsup_{r\to\infty} r^{n-\varepsilon} \max_{|x|=r} B(x, |x|^{2-n+\varepsilon}) \leq \varepsilon(n-2-\varepsilon). \tag{15}$$

For the Emden–Fowler equation (14), condition (15) reduces to

$$\limsup_{r\to\infty} r^b P(r) \leq \varepsilon(n-2-\varepsilon), \qquad n \geq 3,$$

where

$$P(r) = \max_{|x|=r} p(x), \qquad b = n-(n-2)\gamma + (\gamma-1)\varepsilon.$$

COROLLARY 5. *For arbitrary* $\varepsilon > 0$ *and* K_1, K_2 *with* $0 < K_1 < K_2$, *if*

$$\limsup_{r\to\infty} r^{2+\varepsilon} \max_{|x|=r} B(x, K_2) \leq \varepsilon^2, \tag{16}$$

then there exists $a > 0$ *such that* (13) *has a solution* $u(x)$ *in* $G_a \subset R^2$ *satisfying*

$$K_1 \leq u(x) \leq K_2 - r^{-\varepsilon}.$$

In particular (13) has a positive *bounded* solution in G_a if (15) or (16) is satisfied. A stronger result is obtained for (14) on the basis of a one-dimensional criterion of Nehari [26, p. 103]; this is stated as the next Corollary.

COROLLARY 6. *For* $\gamma > 0$, *the Emden–Fowler equation* (14) *has a bounded positive solution in an exterior domain in* R^n *if p is a nonnegative function in* $C^\alpha(\bar{M})$, $0 < \alpha < 1$, *for every bounded subdomain* $M \subset R^n$, *and if*

$$\int_c^\infty r \log rP(r)\, dr < \infty, \qquad n = 2 \tag{17}$$

$$\int_c^\infty r^\sigma P(r)\, dr < \infty, \qquad n \geq 3, \tag{18}$$

for some $c > 0$, *where* $\sigma = (n-1) - \gamma(n-2)$.

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It is shown in [30] that nonoscillation criteria for (13) or (14) follow easily from Theorem 2. We illustrate the procedure below by proving that (17), (18) are nonoscillation criteria for (14) in the superlinear case $\gamma > 1$.

LEMMA 7. *In* (14) *suppose that* $\gamma > 0$, *that* $p(x)$ *is nonnegative in an exterior domain* $\Omega \subset R^n$ *with* $p \in C^\alpha(\bar{M})$ *for every bounded domain* $M \subset \Omega$, *and that* $P \in C^\alpha[a, b]$ *for all* $b > a$, $0 < \alpha < 1$, *where* $P(r) = \max p(x)$ *on* $|x| = r$, *as before. Then a sufficient condition for* (14) *to be nonoscillatory in* Ω *is the existence of a positive solution* $\rho \in C^{2+\alpha}[a, b]$ *of the ordinary differential equation* (19) *below for some number* $a > 0$ *and for all* $b > a$:

$$\frac{d}{dr}\left(r^{n-1}\frac{d\rho}{dr}\right) + r^{n-1}P(r)\rho^\gamma(r) = 0, \qquad r > 0. \tag{19}$$

Proof. The function v in G_a defined by $v(x) = \rho(r)$, $r = |x| \geq a$ satisfies (in the case that (13) reduces to (14))

$$\begin{aligned} r^{n-1}L(v; x) &= \frac{d}{dr}\left(r^{n-1}\frac{d\rho}{dr}\right) + r^{n-1}p(x)\rho^\gamma(r) \\ &\leq \frac{d}{dr}\left(r^{n-1}\frac{d\rho}{dr}\right) + r^{n-1}P(r)\rho^\gamma(r), \end{aligned}$$

and hence $L(v; x) \leq 0$ for all $x \in G_a$ by (19). Then Theorem 2 shows that (14) has a positive solution $u(x)$ in G_a.

THEOREM 8. *Under the hypotheses on p and P in Lemma* 7, (17) *and* (18) *are sufficient conditions for the superlinear* (*i.e.* $\gamma > 1$) *Emden–Fowler equation* (14) *to be nonoscillatory in an exterior domain* Ω *in* R^2 *and* R^n, $n \geq 3$, *respectively.*

Proof. For $n = 2$, Liouville's change of variables $r = e^s$, $h(s) = \rho(e^s)$ transforms (19) into

$$h''(s) + e^{2s}P(e^s)h^\gamma(s) = 0, \qquad \gamma > 1,$$

which has a positive solution $h(s)$ for sufficiently large s if there is a positive number s_0 such that

$$\int_{s_0}^{\infty} se^{2s}P(e^s)\, ds < \infty \tag{20}$$

by Atkinson's theorem [7]. Since (20) is equivalent to (17), it follows that (17) is sufficient for (19) (when $\gamma > 1$, $n = 2$) to have a positive solution $\rho(r)$ in $[a, \infty)$ for some $a > 0$. Since $P \in C^\alpha[a, b]$ for all $b > a$, $0 < \alpha < 1$, standard regularity theorems [23] show that $\rho \in C^{2+\alpha}[a, b]$ for all $b > a$. The conclusion in the case $n = 2$ then follows from Lemma 7. The case $n \geq 3$ is quite similar and is described in [30].

4. **Superlinear equations.** We now return our attention to the general semilinear elliptic equation (1) under the assumptions stated in §1. The following additional notation will be required:

$$L_0(u; x) = \sum_{i,j=1}^{n} D_i[A_{ij}(x)D_ju];$$

$$\rho(r) = \sup_{|x|=r} |L_0(|x|; x)\Big[\sum_{i,j} A_{ij}(x)D_i\, |x|\, D_j\, |x|\Big]^{-1};$$

$$S(r; b) = \int_r^b \exp\left[\int_a^t -\rho(s)\, ds\right] dt, \qquad a \le r \le b;$$

$$R(r; b) = S(r; b)/S(a; b);$$

$$R(r) = \lim_{b\to\infty} R(r; b);$$

$$U(a) = \inf_{|x|=a} u(x).$$

LEMMA 9. *Every positive solution u of $L_0(u; x) \le 0$ for $|x| \ge a$ satisfies the inequality $u(x) \ge U(a)R(|x|)$ for $|x| \ge a$.*

If $B(x, u)$ in (1) is nonnegative for all $x \in \Omega$ and for all $u > 0$, Lemma 9 gives an a priori lower bound $U(a)R(|x|)$ on any positive solution $u(x)$ of $L(u; x) \le 0$ in Ω, and enables us to deduce superlinear oscillation criteria from linear criteria. Lemma 9 is proved in [29] as a consequence of the Hopf maximum principle [35]. Related results have been derived and exploited by Allegretto [3]. In the superlinear case, assumptions (A), (B_1), and (B_2) will be augmented by the following additional ones:

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(SL_1) p in assumption (B_2) is everywhere nonnegative in Ω;
(SL_2) $\psi(t) = \phi(t)/t$ is nondecreasing for all $t > 0$.

COMPARISON THEOREM 10. *Under assumptions* (A), (B_1), (B_2), (SL_1), *and* (SL_2), *the superlinear inequality $uL(u; x) \le 0$ is oscillatory in an exterior domain Ω in R^n if the linear inequality*

$$L_0(u; x) + p(x)\psi(\varepsilon R(|x|))u \le 0 \tag{21}$$

has no eventually positive solution in Ω for any positive number ε.

Proof. If $u(x)$ were a positive solution of $uL(u; x) \le 0$ throughout G_a for some $a > 0$ (chosen large enough so that $G_a \subset \Omega$, without loss of generality), then

$$\begin{aligned} 0 &\ge L_0(u(x); x) + B(x, u(x)) \\ &\ge L_0(u(x); x) + p(x)\psi(u(x))u(x) \\ &\ge L_0(u(x); x) + p(x)\psi(U(a)R(|x|))u(x) \end{aligned}$$

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by Lemma 9, (B_2), (SL_1), and the nondecreasing hypothesis (SL_2) on ψ. Thus $u(x)$ satisfies (21) with $\varepsilon = U(a) > 0$, and cannot be eventually positive by hypothesis. Similarly a solution of $L(u; x) \geq 0$ cannot be everywhere negative in G_a.

In [29] Noussair and the writer utilized a generalized Riccati-type transformation [36] defined in terms of an arbitrary positive absolutely continuous function α in $[0, \infty)$. This transformation maps positive C^1 scalar functions u in Ω into n-vector functions w defined by

$$w(x) = -\alpha(|x|)[\phi(u(x))]^{-1}(A\,\nabla u)(x) \tag{22}$$

Matrix notation is used in (22) and below: In particular A^{-1} denotes the inverse of the matrix $A = (A_{ij})$ and $*$ denotes the transpose.

LEMMA 11. *If u is a positive-valued solution of $L(u; x) \leq 0$ in Ω, under assumptions* (A), (B_1), *and* (B_2), *then the n-vector function w given by* (22) *satisfies the Riccati inequality*

$$\operatorname{div} w(x) \geq \alpha(r)p(x) + \frac{\phi'(y(x))}{\alpha(r)}(w^*A^{-1}w)(x) + \frac{\alpha'(r)}{\alpha(r)}(w^*v)(x), \tag{23}$$

where $\nu(x) = x/r$, is the outward unit normal to S_r, $r = |x|$.

The proof is given in [29]. In the case of the linear inequality (21), the Riccati inequality (23) reduces to

$$\operatorname{div} w(x) \geq \alpha(r)p(x)\psi(\varepsilon R(|x|)) + \frac{(w^*A^{-1}w)(x)}{\alpha(r)} + \frac{\alpha'(r)}{\alpha(r)}(w^*\nu)(x). \tag{24}$$

By the choice $\alpha(r) = \log r$ $(r > 1)$ for $n = 2$ and $\alpha(r) = r^{2-n}$ $(r > 0)$ for $n \geq 3$ in (24), the following results are obtained in [29] by a modification of Coles' one-dimensional procedure [10].

THEOREM 12. *Under the assumptions of Theorem* 10, *the superlinear inequaltiy $uL(u; x) \leq 0$ is oscillatory in an exterior domain Ω of R^2 if* (9) *holds and there exists a positive number a such that*

$$\int_a^\infty \left[r \log r\, \psi(\varepsilon R(r))p_M(r) - \frac{f(r)}{4r \log r}\right] dr = +\infty \tag{25}$$

for all $\varepsilon > 0$, where $p_M(r)$ is given by (8) *and $f(r)$ is defined in* §2.

THEOREM 13. *Under the assumptions of Theorem* 10, *the superlinear inequality $uL(u; x) \leq 0$ is oscillatory in an exterior domain Ω of R^n, $n \gtrsim 3$, if* (10) *holds and there exists a positive number a such that*

$$\int_a^\infty \left[r\psi(\varepsilon R(r))p_M(r) - \frac{(n-2)^2 f(r)}{r}\right] dr = +\infty \tag{26}$$

for all $\varepsilon > 0$.

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In the special case of the Emden–Fowler equation (14), $L_0(u; x) = (\Delta u)(x)$, and one can check that $R(r)$ in Lemma 9 is given by $R(r) = (a/r)^{n-2}$, $n \geq 2$. In the superlinear case, $\gamma > 1$ in (14) and we assume as usual that γ is a quotient of odd positive integers in order to have $\phi(-t) = (-t)^\gamma = -\phi(t)$ for all $t > 0$, as required in assumption (B_1). Then $\psi(t) = \phi(t)/t = t^{\gamma-1}$, and for $f(r) = 1$, $n = 2$, and $\psi(\varepsilon_1 R(r)) = \varepsilon_1^{\gamma-1} = \varepsilon$, (25) specializes to

$$\int_a^\infty \left[\varepsilon r \log r p_M(r) - \frac{1}{4r \log r}\right] dr = +\infty \tag{27}$$

for all $\varepsilon > 0$. Similarly for $f(r) = 1$, $n \geq 3$, and $\psi(\varepsilon_1 R(r)) = \varepsilon r^{(2-n)(\gamma-1)}$, where $\varepsilon = (\varepsilon_1 a^{n-2})^{\gamma-1}$, (26) specializes to

$$\int_a^\infty \left[\varepsilon r^\sigma p_M(r) - \frac{(n-2)^2}{r}\right] dr = +\infty \tag{28}$$

for all $\varepsilon > 0$, where $\sigma = (n-1) - \gamma(n-2)$, $n \geq 3$. It can be seen that (27) and (28) are quite sharp oscillation criteria by comparison with the necessary and sufficient conditions for oscillation in the next section.

5. **Oscillation criteria by spherical means.** Sharper criteria than (27) and (28) were obtained in [28], under the slightly stronger hypotheses (SL) below:

$$B(x, t) \geq P_1(|x|)\phi(t) \quad \text{for all} \quad x \in \Omega, \qquad t \geq 0, \tag{SL}$$

where P_1 is continuous and nonnegative in $[0, \infty)$, and ϕ is a continuously differentiable convex function in $[0, \infty)$ with $\phi(t) > 0$ and $\phi'(t) \geq 0$ in $(0, \infty)$. This requires slightly more than the union of (B_2), (SL_1), and (SL_2), and actually could be weakened somewhat [28]. The method in [28] associates every solution $u(x)$ of (13) (or (14)) with its spherical mean $u_M(r)$ over the sphere S_r of radius r, as defined by (8):

$$u_M(r) = \frac{1}{s(S_r)} \int_{S_r} u(x)\, ds = \frac{1}{\omega(S_1)} \int_{S_1} u(x)\, d\omega.$$

Lemma 14. *If $f: G_a \to R^1$ is a $C^1(G_a)$ function for some number $a > 0$, then*

$$\int_{S_r} (D_i f)(x)\, ds = \frac{d}{dr} \int_{S_r} f(x) \frac{x_i}{r}\, ds$$

for $i = 1, 2, \ldots, n$, $a < r < \infty$.

Lemma 15. *Under assumptions (B_1) and (SL), the spherical mean $u_M(r)$ of a positive-valued solution $u \in C^2(G_a)$ of (13) satisfies the ordinary differential inequality*

$$[r^{n-1} u_M'(r)]' + r^{n-1} P_1(r) \phi[u_M(r)] \leq 0 \tag{29}$$

for $a < r < \infty$.

The idea of the proof of Lemma 15 [28] is to use Lemma 14 twice to verify by direct calculation that $u_M(r)$ satisfies

$$(30) \qquad [r^{n-1}u_M'(r)]' = -\frac{1}{\omega(S_1)}\int_{S_r} B(x, u(x))\, ds.$$

Then (29) follows upon application of Jensen's inequality for convex functions [32, p. 160] to $\phi[u(x)]$ on S_r:

$$\phi[u_M(r)] = \phi\left[\frac{1}{s(S_r)}\int_{S_r} u(x)\, ds\right] \le \frac{1}{s(S_r)}\int_{S_r} \phi[u(x)]\, ds;$$

$$\begin{aligned}\int_{S_r} B(x, u(x))\, ds &\ge P_1(r)\int_{S_r} \phi[u(x)]\, ds\\ &\ge P_1(r)s(S_r)\phi[u_M(r)]\\ &= \omega(S_1)r^{n-1}P_1(r)\phi[u_M(r)]\end{aligned}$$

where (SL) has been used, and hence (30) implies (29).

THEOREM 16. *Under assumptions* (B_1) *and* (SL) *the Schrödinger equation* (13) *is oscillatory in an exterior domain* Ω *in* R^n *if the ordinary differential inequality* (29) *has no solution* $u_M(r)$ *which is positive throughout* $[r_0, \infty)$ *for any positive number* r_0.

This is an easy consequence of Lemma 15, and reduces the oscillation problem for (13) to an analogous problem for the ordinary differential inequaltiy (29). This constitutes considerable progress since (29) is much easier to analyse directly than (13), and in fact special cases of (29) have been studied in detail for some time, see e.g. [13, 42, 44] for extensive bibliographies. Various sufficient conditions for (29) to have no eventually positive solution as $r \to \infty$ also are developed in [28], and applied via Theorem 16 to yield the following oscillation theorems for (13) and (14).

THEOREM 17. *Under hypotheses* (B_1) *and* (SL), *the Schrödinger equation* (13) *is oscillatory in an exterior domain* Ω *in* R^2 *if there exists a positive number* c *such that both*

$$(31) \qquad \int^{\infty} \frac{du}{\phi(u)} < \infty,$$

and

$$(32) \qquad \int_c^{\infty} r \log r P_1(r)\, dr = +\infty.$$

THEOREM 18. *In the Emden–Fowler equation* (14) *suppose that* $\gamma > 1$, *that* γ *is a quotient of odd positive integers, and that* $P_1(r)$ *is continuous and nonnegative*

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in $[0, \infty)$, *where* $P_1(r) = \min p(x)$ *on the sphere* $|x| = r$. *Then a sufficient condition for* (14) *to be oscillatory in an exterior domain* Ω *in* R^n, $n \geq 3$, *is*

$$\int_c^\infty r^\sigma P_1(r)\, dr = +\infty \tag{33}$$

for some $c > 0$, *where* $\sigma = n - 1 - \gamma(n-2)$, $\gamma > 1$.

Modifications of Theorem 18 given in [28] apply to the Schrödinger equation (13).

Examination of the sufficient conditions (17) and (18) (of Theorem 8) for (14) to be nonoscillatory shows that (32) and (33) are very close to characterizations of oscillatory equations (14) in R^2 and R^n, $n \geq 3$, respectively. In fact, (32) and (33) *are* characterizations of oscillatory equations (14) under the additional hypothesis

$$\liminf_{r \to \infty} P_1(r)/P(r) > 0. \tag{34}$$

This condition merely restricts $p(x)$ from oscillating too severely on the sphere S_r of radius r for all sufficiently large r.

THEOREM 19. *Suppose that* $p(x)$ *in* (14) *is nonnegative in an exterior domain* $\Omega \subset R^n$, $p \in C^\alpha(\bar{M})$ *for every bounded domain* $M \subset \Omega$, $P \in C^\alpha[a, b]$ *for all* $b > a$, $a > 0$, $0 < \alpha < 1$, *and* (34) *is satisfied. Then* (32) *or* (33) *is necessary and sufficient for a superlinear equation* (14) *to be oscillatory in* Ω *according as* $n = 2$ *or* $n \geq 3$, *respectively.*

Since the hypotheses of Theorem 19 certainly imply those of Theorem 17 (specialized to the Emden–Fowler case (14)) and Theorem 18, the sufficiency of (32) and (33) follow from those theorems.

Conversely, if (14) is oscillatory in Ω, then Theorem 8 shows that the integrals on the left sides of (17) and (18) diverge to $+\infty$. By (34) there exist positive numbers r_0 and ε such that $P_1(r) \geq \varepsilon P(r)$ whenever $r \geq r_0$, and consequently

$$\int_{r_0}^\infty r \log r P_1(r)\, dr \geq \varepsilon \int_{r_0}^\infty r \log r P(r)\, dr = +\infty,$$

proving the necessity of (32), and similarly the necessity of (33).

6. **Sublinear equations.** Oscillation and nonoscillation criteria for (13) and (14) will be described under assumptions (B_1), (B_2), and the sublinear hypothesis below:

(SB) $\phi(t)$ in (B_2) satisfies $\phi'(t) > 0$ and $\Phi(t) < \infty$ for all $t > 0$, where

$$\Phi(t) = \int_0^t \frac{ds}{\phi(s)}.$$

In particular the Emden–Fowler equation (14) satisfies (SB) in the case $0<\gamma<1$.

THEOREM 20. *If* (B_1), (B_2), *and* (SB) *are satisfied, a sufficient condition for the sublinear inequality* $uL(u;x)\le 0$ *(given by* (13)) *to be oscillatory in an exterior domain* $\Omega\subset R^n$ $(n\ge 2)$ *is*

$$\int_c^\infty r p_M(r)\,dr=+\infty \tag{35}$$

for some $c>0$, *where the spherical mean* $p_M(r)$ *of* $p(x)$ *over* S_r *is defined by* (8).

The proof given by Kitamura and Kusano [18] for the special case $L(u;x)=\Delta u+p(x)\phi(u)$ can be extended to (13). A different proof for $n=2$ was given in [29], based on the Riccati inequality (23). Neither $B(x,t)$ nor $p(x)$ is required to be everywhere positive for Theorem 20 to be true.

Sharp nonoscillation criteria for (14) in the sublinear case $0<\gamma<1$ are obtained on the basis of Lemma 7; these are stated in the theorem below.

THEOREM 21 [31]. *Under the hypotheses on p and P in Lemma* 7, *a sufficient condition for the sublinear (i.e.* $0<\gamma<1$) *Emden–Fowler equation* (14) *to be nonoscillatory in an exterior domain* Ω *in* R^n *is*

$$\int_c^\infty r(\log r)^\gamma P(r)\,dr<\infty \quad \text{if} \quad n=2, \tag{36}$$

$$\int_c^\infty rP(r)\,dr<\infty \quad \text{if} \quad n\ge 3 \tag{37}$$

for some $c>0$.

It is interesting to compare (36) and (37) with the corresponding criteria (17) and (18) in the superlinear case $(\gamma>1)$, as stated in Theorem 8. The proof is very similar to that of Theorem 8, except that Atkinson's criterion (20) is replaced by Belohorec's criterion [8, 44]

$$\int_{s_0}^\infty s^\gamma e^{2s}P(e^s)\,ds<\infty, \qquad s=\log r \tag{38}$$

(when $n=2$), which is equivalent to (36). The argument (for $n=2$) in Theorem 8 is then essentially unchanged.

Combination of Theorems 20 and 21 yields a necessary and sufficient condition for the sublinear equation (14) to be oscillatory in $\Omega\subset R^n$, $n\ge 3$ provided hypothesis (34) is added, thereby extending Belohorec's one-dimensional characterization of sublinear oscillation to dimensions $n\ge 3$. However, the case $n=2$ is still unresolved.

THEOREM 22. *If p and P satisfy the hypotheses of Theorem* 19 *and if* (34) *holds, then a necessary and sufficient condition for the sublinear (i.e.* $0<\gamma<1$) *Emden–Fowler equation* (14) *to be oscillatory in an exterior domain* $\Omega\subset R^n$, $n\ge 3$ *is*

$$\int_c^\infty rP(r)\,dr=+\infty \tag{39}$$

for some positive number c.

The oscillation criteria (32), (33), (39), etc. for (14) have the form (where γ is a quotient of odd positive integers)

$$\int_c^\infty m(r)P_1(r)\,dr=+\infty$$

for some radial function m and some number $c>0$, where $P_1(r)=\min p(x)$ on the sphere $|x|=r$. Under the hypotheses associated with Theorems 19, 22, etc., some of these criteria are characterizations of oscillatory equations (14). A summary of the known results is tabulated below according to dimension n and semilinear exponent γ. All the *nonlinear* entries correspond to necessary and sufficient conditions for oscillation, except the entry for $n=2$, $0<\gamma<1$.

TABLE OF MULTIPLIERS $m(r)$ FOR OSCILLATION OF (14)

Dimension	Sublinear ($0<\gamma<1$)	Linear ($\gamma=1$)	Superlinear ($\gamma>1$)
1	r^γ NASC, 1961 Belohorec [8]	$r^b, b<1$ 1955 Moore [25]	r NASC, 1955 Atkinson [7]
2	r (35), 1979 Kitamura and Kusano [18]	$r\,(\log r)^b, b<1$ (11), 1975 Noussair and I [28]	$r\log r$ NASC (32), 1979 Noussair and I [30]
$n\ge 3$	r NASC, (39), 1979 Noussair and I [31]	$r^b, b<1$ (6), 1968 Headley and I [16]	$r^\sigma, \sigma=n-1-\gamma(n-2)$ NASC (33), 1979 Noussair and I [30]

7. **Perturbed linear equations.** The elliptic equation (1) is called a *perturbed linear equation* when $B(x,u)$ in (1) has the form

$$B(x,u)=q_1(x)u+\sum_{j=2}^{J} q_j(x)\psi_j(u), \qquad x\in\Omega \tag{40}$$

under the assumptions (PL_1) and (PL_2) below:

(PL_1) Each q_j is a continuous real-valued function in Ω, $j=1,2,\ldots,J$;

(PL_2) Each ψ_j is an odd C^1 function in $(-\infty,\infty)$ with $\psi_j(t)>0$ for all $t>0$. The functions q_j are *not* required to be everywhere positive, precluding the

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deduction of oscillation criteria via comparison theorems for (1) [37, 41]. We define functions p and ϕ by

$$p(x)=\min[q_1(x),\ldots,q_J(x)], \qquad x\in\Omega \tag{41}$$

$$\phi(u)=u+\sum_{j=2}^{J}\psi_j(u), \qquad -\infty<u<\infty.$$

Then $B(x,u)\geq p(x)\phi(u)$ for all $x\in\Omega$ and for all $u\geq 0$, and hence (B_1) and (B_2) in §1 are implied by (PL_1) and (PL_2). The following oscillation criteria for a perturbed linear equation (1) or inequality $uL(u;x)\leq 0$ are proved in [29] by methods similar to those described in §4 above.

THEOREM 23. *The perturbed linear inequality $uL(u;x)\leq 0$ (given by* (1) *and* (40)) *is oscillatory in an exterior domain Ω of R^2* (R^n, $n\geq 3$, *respectively*) *under hypotheses* (A), (PL_1) *and* (PL_2) *if* (9) *and* (11) ((10 *and* (12), *respectively*) *are satisfied, where $p(x)$ is given by* (41).

For example, if (A_{ij}) in (1) is a constant matrix with largest eigenvalue λ, oscillation criteria for (1), (40) are

$$\int_a^\infty\left[r\log r p_M(r)-\frac{\lambda}{4r\log r}\right]dr=+\infty, \qquad n=2,$$

$$\int_a^\infty\left[rp_M(r)-\frac{(n-2)^2\lambda}{r}\right]dr=+\infty, \qquad n\geq 3,$$

for some $a>0$, where $p_M(r)$ is given by (8), (41).

8. **Oscillation in general domains.** The foregoing theory applies essentially only to unbounded domains Ω of the *exterior* type. In the case of the linear equation (2), sufficient conditions for (nodal) oscillation were obtained in [40] without special hypotheses on the geometry of Ω. These results are written in terms of the functional

$$F[u;M]=\int_M\left[\sum_{i,j=1}^{n}A_{ij}(x)D_iuD_ju-p(x)u^2\right]dx \tag{42}$$

on a nonempty regular bounded domain $M\subset\Omega$, where the admissible functions u in (42) are supposed to be real-valued piecewise C^1 functions on $\bar{M}$.

THEOREM 24. *Equation* (2) *is nodally oscillatory in an unbounded domain $\Omega\subset R^n$ if Ω contains a sequence of nonempty regular bounded domains M_k, $k=1,2,\ldots,$ with piecewise C^1 boundaries, having the following properties:*

(i) *For arbitrary $r>0$ there exists an integer $k_0(r)$ such that $M_k\subset\Omega_r$ for all $k\geq k_0(r)$; and*
(ii) *There exists a nontrivial piecewise C^1 function u_k on each M_k such that*

$u_k(x)=0$ for all $x \in \partial M_k$ and $F[u_k; M_k] \le 0$ whenever $k \ge K$ for some positive integer K.

This is proved in [40] by Courant's minimum principle for eigenvalues [11, p. 399]. We can prove, furthermore, under the same hypotheses, that the linear *inequality* $vL(v; x) \le 0$ is oscillatory in Ω by appealing to Picone's identity [39, 41]. In fact, if v is a positive or negative solution of the inequality throughout $\bar{M}_k$ for $k \ge \max(k_0(r), K)$, integration of Picone's identity over M_k yields the contradiction

$$\int_{M_k} \frac{u_k^2(x)}{v(x)} L(v; x)\, dx + F[u_k; M_k] > 0$$

unless v is a constant multiple of u_k in $\bar{M}_k$. Therefore every solution v of $vL(v; x) \le 0$ has a zero in Ω_r for all $r > 0$. *Nodal* oscillation of (2) follows from Allegretto's equivalence theorem [2] (see §1) if $p \in C^\alpha$ and each $A_{ij} \in C^{\alpha+1}$, where

$$\alpha = 3\left[\frac{1}{2}\left[\frac{n+6}{2}\right]\right].$$

THEOREM 25. *Equation* (2) *is nodally oscillatory in an unbounded domain $\Omega \subset R^n$ if Ω contains a sequence of disks $M_k(x_k;\ a_k) = \{x : |x - x_k| < a_k\}$ with the following properties:*

(i) $\lim_{k\to\infty}(|x_k| - a_k) = +\infty$;

(ii) $(A_{ij}(x))$ *is bounded in* $\bigcup_k M_k(x_k; a_k)$; *and*

(iii) $\lim_{k\to\infty} a_k^{2-n} \int_{M_k[x_k;(1/2)a_k]} p(x)\, dx = +\infty$.

This is deduced from Theorem 24 if u_k is selected to be the piecewise linear function of $r = |x - x_k|$ which is 1 in $0 \le r \le a_k/2$ and 0 for $r \ge a_k$. Various specializations of Theorem 25 lead to nodal oscillation criteria for (2) in domains Ω which are quasiconical, quasicylindrical, or even of the "spiral" type containing no infinite ray. These results also have been extended by Noussair [27] and Allegretto [5] to certain elliptic operators of arbitrary even order.

Allegretto [4] has obtained results of similar type for semilinear equations (1) under the following geometric assumptions: (i) As in Theorem 24, there exists an integer k_0 such that $M_k \subset \Omega_r$ for all $k \ge k_0$; (ii) As in Theorem 25, M_k is specialized to an n-disk $M_k(x_k; a_k)$ of radius a_k and centre x_k; and (iii) For each integer $k = 1, 2, \ldots$, $x_{k+1} \in M_k$, $M_{k+1} \not\subset M_k$, and $x_k \ne x_j$ if $k \ne j$. Allegretto defines a function g_k in $M_k(x_k; \frac{1}{2}a_k)$, $k = 1, 2, \ldots$, depending on the geometry and the coefficient functions in (1) in a complicated way; then, if $(A_{ij}(x))$ is bounded (as a form) in Ω, a sufficient condition for (1) to be oscillatory in Ω is that

$$\lim_{k\to\infty}\left\{a_k^{2-n} \int_{M_k[x_k;\,(1/2)a_k]} F(x; \varepsilon g_k(x))\, dx\right\} = +\infty$$

for every positive number ε, where $F(x, t) = t^{-1}B(x, t)$. Illustrations of this criterion are given in [4] when (1) reduces to the Emden–Fowler equation (14) in unbounded domains of various specific types.

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