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Unitary Equivalence and Similarity to Jordan Models for Weak Contractions of Class C_0

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Abstract. We obtain results on the unitary equivalence of weak contractions of class C_0 to their Jordan models under an assumption on their commutants. In particular, our work addresses the case of arbitrary finite multiplicity. The main tool in this paper is the theory of boundary representations due to Arveson. We also generalize and improve previously known results concerning unitary equivalence and similarity to Jordan models when the minimal function is a Blaschke product.

1 Introduction

We start with some background concerning operators of class C_0 (greater detail can be found in [4] or [20]). Let H^∞ be the algebra of bounded holomorphic functions on the open unit disc $\mathbb{D}$. Let $\mathcal{H}$ be a Hilbert space and T a bounded linear operator on $\mathcal{H}$, which we indicate by $T \in B(\mathcal{H})$. If $T \in B(\mathcal{H})$ is a completely non-unitary contraction, then its associated Sz.-Nagy–Foias H^∞ functional calculus is an algebra homomorphism $\Phi\colon H^\infty \to B(\mathcal{H})$ with the following properties:

(i) $\|\Phi(u)\| \leq u$ for every $u \in H^\infty$;
(ii) $\Phi(p) = p(T)$ for every polynomial p;
(iii) Φ is continuous when H^∞ and $B(\mathcal{H})$ are equipped with their respective weak-star topologies.

We use the notation $\Phi(u) = u(T)$ for $u \in H^\infty$. The contraction T is said to belong to the class C_0 whenever Φ has a nontrivial kernel. It is known in that case that $\ker \Phi = \theta H^\infty$ for some inner function θ, called the minimal function of T, which is uniquely determined up to a scalar factor of absolute value one.

We denote by H^2 the Hilbert space of functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

holomorphic on the open unit disc, equipped with the norm

$$\|f\|_{H^2} = \Big(\sum_{n=0}^{\infty} |a_n|^2\Big)^{1/2}.$$

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For any inner function $\theta \in H^\infty$, the space $H(\theta) = H^2 \ominus \theta H^2$ is closed and invariant for S^*, the adjoint of the shift operator S on H^2. The operator $S(\theta)$ defined by $S(\theta)^* = S^* | (H^2 \ominus \theta H^2)$ is called a Jordan block; it is of class C_0 with minimal function θ.

A more general family of operators consists of the so-called Jordan operators. Start with a collection of inner functions $\Theta = \{\theta_\alpha\}_\alpha$ indexed by the ordinal numbers, such that $\theta_\alpha = 1$ for α large enough and that θ_β divides θ_α whenever $\operatorname{card}(\beta) \geq \operatorname{card}(\alpha)$ (recall that a function $u \in H^\infty$ divides another function $v \in H^\infty$ if $v = uf$ for some $f \in H^\infty$). Let γ be the first ordinal such that $\theta_\gamma = 1$. Then the associated Jordan operator is $J_\Theta = \bigoplus_{\alpha < \gamma} S(\theta_\alpha)$.

The Jordan operators are of fundamental importance in the study of operators of class C_0, as the following theorem from [5] illustrates. Recall first that a bounded injective linear operator with dense range is called a *quasiaffinity*. Two operators $T \in B(\mathcal{H})$ and $T' \in B(\mathcal{H}')$ are said to be *quasisimilar* if there exist quasiaffinities $X \colon \mathcal{H} \to \mathcal{H}'$ and $Y \colon \mathcal{H}' \to \mathcal{H}$ such that $XT = T'X$ and $TY = YT'$.

Theorem 1.1 *For any operator T of class C_0 there exists a unique Jordan operator J that is quasisimilar to T.*

This theorem is one of the main features of the class C_0. Recent investigations have identified special situations in which the relation of quasisimilarity between a multiplicity-free operator T of class C_0 and its Jordan model can be improved to similarity. For instance, the work done in [8] was inspired in part by early results of Apostol [1] (discovered independently in [22]). A link was found between the possibility of achieving similarity between T and $S(\theta)$ and the fact that $\varphi(T)$ has closed range for every inner divisor φ of θ (here θ denotes the minimal function of T). The same problem was studied in [9], albeit from another point of view. Drawing inspiration from the seminal work of Arveson [2], the main question addressed in that paper was whether similarity between T and $S(\theta)$ could be detected via properties of the associated algebras

$$H^\infty(T) = \{u(T) : u \in H^\infty\} \quad \text{and} \quad H^\infty(S(\theta)).$$

More precisely, assuming that these algebras are boundedly isomorphic, does it follow that T and $S(\theta)$ are similar? Partial results along with estimates on the size of the similarity were obtained in [9] in the case where the minimal function is a finite Blaschke product. In both [8] and [9] the considerations also took advantage of (and perhaps reinforced) a well-known connection with the theory of interpolation by bounded holomorphic functions on the unit disc and the so-called (generalized) Carleson condition (see [21] or [15]).

Our work here offers several improvements and generalizations of various results from [2, 3, 8, 9]. As mentioned above, our focus is to describe a relation between an operator of class C_0 and its Jordan model, and we do so in two different settings: up to similarity and up to unitary equivalence. We now present the plan of the paper, state our main results, and explain to what extent those improve upon previous ones.

Section 2 is based on the following result, which is a consequence of the proof of Theorem 3.6.12 in [2] and of Corollary 1 in [3], both due to Arveson. Recall that a vector $x \in \mathcal{H}$ is said to be *cyclic* for $T \in B(\mathcal{H})$ if the smallest closed subspace of $\mathcal{H}$

containing $T^n x$ for every integer $n \geq 0$ is the entire space $\mathcal{H}$. An operator having a cyclic vector is said to be *multiplicity-free*. We denote by $P(T)$ the smallest norm-closed algebra containing T and the identity operator.

Theorem 1.2 *Let $T \in B(\mathcal{H})$ be an irreducible multiplicity-free operator of class C_0 with minimal function θ and with the property that its spectrum does not contain the unit circle. Consider the homomorphism $\Psi\colon P(S(\theta)) \to P(T)$ defined by $\Psi(p(S(\theta))) = p(T)$ for every polynomial p. Assume that Ψ is completely isometric. Then T is unitarily equivalent to $S(\theta)$.*

Our first main result (Theorem 1.3) addresses the case of higher multiplicities and removes the condition on the spectrum of T. We denote by $\{T\}'$ the commutant of the operator T.

Theorem 1.3 *Let $T_1 \in B(\mathcal{H}_1)$ be an operator of class C_0 with the property that $I - T_1^* T_1$ is of trace class and that $\{T_1\}'$ is irreducible. Let $T_2 \in B(\mathcal{H}_2)$ be another operator of class C_0 that is quasisimilar to T_1 and with the property that $\{T_2\}'$ is irreducible. Assume that there exists a completely isometric isomorphism $\varphi\colon \{T_1\}' \to \{T_2\}'$ such that $\varphi(T_1) = T_2$. Then T_1 and T_2 are unitarily equivalent.*

In Section 3, we explore the case where the minimal function is a Blaschke product and show that in this setting, unitary equivalence between a multiplicity-free operator of class C_0 and its Jordan model can be obtained from assumptions weaker than those appearing in the statement above. Throughout, we use the notation

$$b_\lambda(z) = \frac{z-\lambda}{1-\overline{\lambda}z}$$

for the Blaschke factor with root at $\lambda \in \mathbb{D}$ and

$$\widetilde{b}_\lambda(z) = -\frac{\overline{\lambda}}{|\lambda|} b_\lambda(z) = -\frac{\overline{\lambda}}{|\lambda|}\frac{z-\lambda}{1-\overline{\lambda}z}.$$

Given a Blaschke product θ, an inner divisor ψ of θ is said to be *big* if the ratio θ/ψ is a Blaschke factor. Also, a multiplicity-free contraction of class C_0 whose minimal function is a Blaschke product is said to be *maximal* if there exists a big divisor ψ of θ and a unit cyclic vector ξ with the property that $\|\psi(T)\xi\| = 1$. The motivation for Section 3 is the following result due to Arveson (see [2, Lemma 3.2.6]).

Theorem 1.4 *Let $T \in B(\mathcal{H})$ be a multiplicity-free operator of class C_0 whose minimal function is a finite Blaschke product θ. Assume that T is maximal. Then T is unitarily equivalent to $S(\theta)$.*

The following is our second main result. One improvement that it offers over the previous theorem is the possibility for θ to be an infinite Blaschke product.

Theorem 1.5 *Let $T \in B(\mathcal{H})$ be a multiplicity-free operator of class C_0 whose minimal function is a Blaschke product θ. Assume that $\|\psi(T)\| = 1$ for some big inner divisor ψ of θ. Then T is unitarily equivalent to $S(\theta)$.*

Let us also emphasize here that the condition $\|\psi(T)\| = 1$ is formally weaker than maximality: although cyclic vectors are known to be plentiful (see Theorem 4.7), it is not immediately clear that an operator must achieve its norm on one of them. That this is indeed the case follows from the proof of Theorem 1.4, and to the best of our knowledge it has not been observed before.

Finally, in Section 4 we are concerned with similarity rather than unitary equivalence. The basic idea is to weaken the condition appearing in Theorem 1.5 while still obtaining similarity between T and its Jordan model. Our results improve upon the work that was done in [9].

2 Unitary Equivalence and Boundary Representations

In this section, we investigate $*$-representations of C^*-algebras related to C_0 operators and their connection with unitary equivalence of such operators to their Jordan models. The first result we need is inspired by the discussion found in [2, p. 201]. We denote by $\mathcal{K}(\mathcal{H})$ the ideal of compact operators on a Hilbert space $\mathcal{H}$.

Lemma 2.1 *Let $T \in B(\mathcal{H})$ be an operator that is not unitary and with the property that $I - TT^*$ and $I - T^*T$ are compact and that $\{T\}'$ is irreducible.*

(i) *If we denote by $\mathfrak{J}$ the closed ideal of $C^*(\{T\}')$ generated by $I - T^*T$ and $I - TT^*$, then $\mathfrak{J} = \mathcal{K}(\mathcal{H})$.*

(ii) *Assume that π is a $*$-representation of $C^*(\{T\}')$. Then $\pi(T)$ is unitarily equivalent to $(T \otimes I_{\mathcal{H}'}) \oplus U$, where U is a unitary operator with spectrum contained in the essential spectrum of T and $\mathcal{H}'$ is another Hilbert space.*

Proof The fact that $C^*(\{T\}')$ is irreducible immediately implies that the ideal $\mathfrak{J}$ is irreducible (see [10, Lemma I.9.15]). By assumption, $\mathfrak{J}$ is a nonzero C^*-subalgebra of $\mathcal{K}(\mathcal{H})$. Thus $\mathcal{K}(\mathcal{H}) = \mathfrak{J}$ by [10, Corollary I.10.4], which proves (i). Moreover, [2, Lemma 3.4.4] shows that the representation π can be decomposed as

$$\pi(x) = \pi_1(x) \oplus \pi_2(x + \mathfrak{J})$$

for every $x \in C^*(\{T\}')$, where the $*$-representation π_1 is the unique extension to $C^*(\{T\}')$ of a $*$-representation of $\mathfrak{J}$ and π_2 is a $*$-representation of $C^*(\{T\}')/\mathfrak{J}$. Since $\mathfrak{J} = \mathcal{K}(\mathcal{H})$, it is well known (see [10, Corollary I.10.7]) that $\pi_1 \restriction \mathfrak{J}$ must be unitarily equivalent to a multiple of the identity representation, and by uniqueness so must be π_1. On the other hand, $\pi_2(T)$ is a unitary operator with spectrum contained in the essential spectrum of T, which shows (ii) and finishes the proof. ■

We note that in the statement above we allow for both $\mathcal{H}'$ and the space on which U acts to be zero. In other words, one of the pieces U or $T \otimes I_{\mathcal{H}'}$ can be absent. This is the case if we specialize Lemma 2.1 to contractions of class C_0.

Lemma 2.2 *Let $T_1 \in B(\mathcal{H}_1)$ be an operator of class C_0 with the property that $\{T_1\}'$ is irreducible and $I - T_1^*T_1, I - T_1T_1^*$ are compact. Let $T_2 \in B(\mathcal{H}_2)$ be another operator of class C_0. Assume that there exists a $*$-homomorphism*

$$\pi \colon C^*\big(\{T_1\}'\big) \longrightarrow B(\mathcal{H}_2)$$

such that $\pi(T_1) = T_2$. Then T_2 is unitarily equivalent to $T_1 \otimes I_{\mathcal{H}'}$ for some Hilbert space $\mathcal{H}'$.

Proof By virtue of Lemma 2.1, we see that $T_2 = \pi(T_1)$ is unitarily equivalent to $(T_1 \otimes I_{\mathcal{H}'}) \oplus U$ for some unitary U. Since T_2 is of class C_0, it must be completely nonunitary, and thus U acts on the zero space so that T_2 is in fact unitarily equivalent to $T_1 \otimes I_{\mathcal{H}'}$. ■

Next, we need some results of Bercovici and Voiculescu (see [6]). Recall that a contraction T is said to be *weak* if $I - T^*T$ belongs to the ideal of trace class operators.

Theorem 2.3 *Let T be an operator of class C_0 with Jordan model J.*

(i) *T is a weak contraction if and only if T^* is a weak contraction.*
(ii) *T is a weak contraction if and only if J is a weak contraction.*

Lemma 2.4 *Let T_1 and T_2 be quasisimilar weak contractions of class C_0. If T_1 is unitarily equivalent to $T_2 \otimes I_{\mathcal{H}'}$, then $\mathcal{H}'$ is one dimensional and T_1 is unitarily equivalent to T_2.*

Proof This follows immediately from a consideration of the determinant functions of T_1 and T_2, which must be equal (see [4, section 6.3] for more details). ■

The next corollary is the link between $*$-representations and unitary equivalence.

Corollary 2.5 *Let $T_1 \in B(\mathcal{H}_1)$ be a weak contraction of class C_0 with the property that $\{T_1\}'$ is irreducible. Let $T_2 \in B(\mathcal{H}_2)$ be another operator of class C_0 that is quasisimilar to T_1. Assume that there exists a $*$-homomorphism*

$$\pi\colon C^*(\{T_1\}') \longrightarrow B(\mathcal{H}_2)$$

such that $\pi(T_1) = T_2$. Then T_2 is unitarily equivalent to T_1.

Proof Since T_2 and T_1 are quasisimilar, they share the same Jordan model. By Theorem 2.3, we see that $I - T_i^*T_i$ and $I - T_iT_i^*$ are of trace class for $i = 1, 2$. In light of Lemma 2.2, we know that T_2 is unitarily equivalent to $T_1 \otimes I_{\mathcal{H}'}$, and an application of Lemma 2.4 shows that T_1 and T_2 are unitarily equivalent. ■

It is typically quite difficult to construct $*$ -representations of $C^*(\{T\}')$ in order to apply this result. We follow here one method to obtain such representations that is originally due to Arveson [2]. Recall that given a unital (nonself-adjoint) subalgebra $\mathcal{A} \subset B(\mathcal{H}_1)$, we say that an irreducible $*$-representation $\pi\colon C^*(\mathcal{A}) \to B(\mathcal{H})$ is a *boundary representation* for $\mathcal{A}$ if the only unital completely positive extension of $\pi|\mathcal{A}$ to $C^*(\mathcal{A})$ is π itself (we refer the reader to [2] or [16] for further details and definitions). Our next goal is to establish that for weak contractions of class C_0 with irreducible commutant, the identity representation of $C^*(\{T\}')$ is a boundary representation for $\{T\}'$. The main tool is the following result, known as Arveson's boundary theorem (see [3, Theorem 2.1.1]).

Theorem 2.6 *Let $\mathcal{A} \subset B(\mathcal{H})$ be an irreducible unital subalgebra with the property that $C^*(\mathcal{A})$ contains $\mathcal{K}(\mathcal{H})$ and that the quotient map*

$$q\colon B(\mathcal{H}) \longrightarrow B(\mathcal{H})/\mathcal{K}(\mathcal{H})$$

is not completely isometric on $\mathcal{A}$. Then the identity representation of $C^*(\mathcal{A})$ is a boundary representation for $\mathcal{A}$.

In order to apply this theorem, we also require the following fact from [13].

Theorem 2.7 *If $T \in B(\mathcal{H})$ is an operator of class C_0 with the property that $I - T^*T$ and $I - TT^*$ are compact, then there exists a function $u \in H^\infty$ with the property that $u(T)$ is a nonzero compact operator.*

We now achieve the desired result. In fact, we only require $I - T^*T$ and $I - TT^*$ to be compact, as opposed to trace class.

Corollary 2.8 *If $T \in B(\mathcal{H})$ is an operator of class C_0 such that $I - T^*T$ and $I - TT^*$ are compact and $\{T\}'$ is irreducible, then the identity representation of $C^*(\{T\}')$ is a boundary representation for $\{T\}'$.*

Proof First, we see that $C^*(\{T\}')$ contains $\mathcal{K}(\mathcal{H})$ by virtue of Lemma 2.1 (i). Moreover, by Theorem 2.7 there exists a nonzero compact operator of the form $u(T)$ for some $u \in H^\infty$. This operator necessarily commutes with T and $q(u(T)) = 0$, and thus Theorem 2.6 completes the proof. ■

The following result is Theorem 1.2 of [2]. It is the key to obtaining $*$-representations of $C^*(\{T\}')$.

Theorem 2.9 *Let $\mathcal{A} \subset B(\mathcal{H}_1), \mathcal{B} \subset B(\mathcal{H}_2)$ be unital subalgebras and let $\varphi\colon \mathcal{A} \to \mathcal{B}$ be a unital completely isometric algebra isomorphism. Let $\pi_{\mathcal{B}}$ be a $*$-representation of $C^*(\mathcal{B})$ that is a boundary representation for $\mathcal{B}$. Then there exists a $*$-representation $\pi_{\mathcal{A}}$ of $C^*(\mathcal{A})$ that is a boundary representation for $\mathcal{A}$ and such that $\pi_{\mathcal{B}} \circ \varphi = \pi_{\mathcal{A}}$ on $\mathcal{A}$.*

Finally, we come to the main result of this section.

Theorem 2.10 *Let $T_1 \in B(\mathcal{H}_1)$ be a weak contraction of class C_0 with the property that $\{T_1\}'$ is irreducible. Let $T_2 \in B(\mathcal{H}_2)$ be another operator of class C_0 that is quasisimilar to T_1 and with the property that $\{T_2\}'$ is irreducible. Assume that there exists a completely isometric isomorphism $\varphi\colon \{T_1\}' \to \{T_2\}'$ such that $\varphi(T_1) = T_2$. Then T_1 and T_2 are unitarily equivalent.*

Proof By Theorem 2.3, we see that $I - T_i^*T_i$ and $I - T_iT_i^*$ are of trace class for each $i = 1, 2$. In light of Corollary 2.8, the identity representation of $C^*(\{T_2\}')$ is a boundary representation for $\{T_2\}'$. Therefore, we may apply Theorem 2.9 to obtain a $*$-representation $\pi\colon C^*(\{T_1\}') \to B(\mathcal{H}_2)$ that satisfies $\pi|\{T_1\}' = \varphi$. An application of Corollary 2.5 finishes the proof. ■

We close this section by examining more closely the irreducibility assumption appearing above. Obviously, the main interest of Theorem 2.10 lies in the case where T_2 is the Jordan model of T_1. In that case, the irreducibility assumption on $\{T_1\}'$ is necessary to obtain unitary equivalence in view of the following fact.

Proposition 2.11 *If $J = \bigoplus_\alpha S(\theta_\alpha)$ is a Jordan operator, then $\{J\}'$ is irreducible.*

Proof Set $\mathcal{H} = \bigoplus_\alpha H(\theta_\alpha)$. Let $M \subset \mathcal{H}$ be a proper reducing subspace for $\{J\}'$. Let P_α denote the orthogonal projection of $\mathcal{H}$ onto the $H(\theta_\alpha)$ component. Since P_α commutes with J for every α, we see that M is reducing for each P_α and hence it can be written as $M = \bigoplus_\alpha M_\alpha$. Now, the operator $P_\alpha J P_\alpha$ also commutes with J, whence each M_α is reducing for $P_\alpha J P_\alpha$. Since this operator is unitarily equivalent to $S(\theta_\alpha)$ and Jordan blocks are known to be irreducible, we must have either $M_\alpha = P_\alpha \mathcal{H}$ or $M_\alpha = 0$. We proceed to show that each M_α must be equal to 0. For the rest of the proof, for each α we identify $P_\alpha \mathcal{H}$ with $H(\theta_\alpha)$.

Since we assume that $M \neq \mathcal{H}$, we must have $M_{\alpha_0} = 0$ for some α_0. Now, any operator X acting on $\mathcal{H}$ may be written as $X = (X_{\alpha\beta})_{\alpha,\beta}$, where

$$X_{\alpha\beta} \colon H(\theta_\beta) \longrightarrow H(\theta_\alpha).$$

If $\gamma < \alpha_0$, consider the operator $Y(\gamma)$ defined by $Y(\gamma)_{\alpha_0\gamma} = P_{H(\theta_{\alpha_0})}|H(\theta_\gamma)$ and $Y(\gamma)_{\alpha\beta} = 0$ otherwise. It is easily verified that $Y(\gamma)$ commutes with J and thus $Y(\gamma)M \subset M$, which in turn implies that

$$P_{H(\theta_{\alpha_0})} M_\gamma \subset M_{\alpha_0} = 0.$$

This forces M_γ to be equal to 0, since the other case, $M_\gamma = H(\theta_\gamma)$, is impossible:

$$P_{H(\theta_{\alpha_0})} H(\theta_\gamma) = H(\theta_{\alpha_0}) \neq 0.$$

Therefore, $M_\gamma = 0$ whenever $\gamma < \alpha_0$. Assume now that $\gamma > \alpha_0$ and consider the operator $Z(\gamma)$ defined by

$$Z(\gamma)_{\alpha_0\gamma} = P_{H(\theta_{\alpha_0})}(\theta_{\alpha_0}/\theta_\gamma)(S)|H(\theta_\gamma)$$

and $Z(\gamma)_{\alpha\beta} = 0$ otherwise. It is easily verified that $Z(\gamma)$ commutes with J and thus $Z(\gamma)M \subset M$, which in turn implies that

$$P_{H(\theta_{\alpha_0})}(\theta_{\alpha_0}/\theta_\gamma)(S) M_\gamma \subset M_{\alpha_0} = 0.$$

This forces M_γ to be equal to 0, since the other possibility $M_\gamma = H(\theta_\gamma)$ is impossible:

$$P_{H(\theta_{\alpha_0})}(\theta_{\alpha_0}/\theta_\gamma)(S) H(\theta_\gamma) = (\theta_{\alpha_0}/\theta_\gamma)(S) H(\theta_\gamma) \neq 0.$$

Thus, $M_\gamma = 0$ for every $\gamma > \alpha_0$, and the proof is complete. ■

We obtain a simpler version of Theorem 2.10 that applies to Jordan operators.

Corollary 2.12 *Let $T \in B(\mathcal{H})$ be a weak contraction of class C_0 with the property that $\{T\}'$ is irreducible. Let J be the Jordan model of T. Assume that there exists a completely isometric isomorphism $\varphi \colon \{T\}' \to \{J\}'$ such that $\varphi(T) = J$. Then T and J are unitarily equivalent.*

Proof Simply combine Theorem 2.10 and Proposition 2.11. ∎

Finally, we show that for certain minimal functions, the irreducibility of $\{T\}'$ is automatic. We first need two preliminary facts. The first one is from [5]. We recall that the double commutant of T, denoted by $\{T\}''$, is defined as the algebra of operators that commute with the commutant,

$$\{T\}'' = (\{T\}')' = \{A \in B(\mathcal{H}) : AX = XA \text{ for every } X \in \{T\}'\}.$$

Theorem 2.13 *Let T be an operator of class C_0 with minimal function θ and let $X \in \{T\}''$. Then there exists a function $v \in H^\infty$ with the property that v has no nonconstant common inner divisor with θ and that $Xv(T) = u(T)$ for some function $u \in H^\infty$.*

The following lemma is [4, Proposition 2.4.9].

Lemma 2.14 *Let $u \in H^\infty$ and $T \in B(\mathcal{H})$ be an operator of class C_0 with minimal function θ. Then $u(T)$ is a quasiaffinity if and only if u and θ have no nonconstant common inner divisor.*

We now show that if the inner divisors of the minimal function θ satisfy a certain property, then the commutant $\{T\}'$ is always irreducible.

Proposition 2.15 *Let $\theta \in H^\infty$ be an inner function with the property that for every inner divisor φ of θ, we have that φ and θ/φ have a nonconstant common inner divisor, unless $\varphi = 1$ or $\varphi = \theta$. Let T be an operator of class C_0 with minimal function θ. Then $\{T\}''$ contains no idempotents besides 0 and I, and $\{T\}'$ is irreducible.*

Proof The second statement clearly follows from the first, so it suffices to show that if $E \in \{T\}''$ satisfies $E^2 = E$, then $E = I$ or $E = 0$. By Theorem 2.13, we see that there $Ev(T) = u(T)$ for some functions $u, v \in H^\infty$, where v and θ have no nonconstant common inner divisor. We compute

$$u(T)^2 = E^2 v(T)^2 = Ev(T)^2 = u(T)v(T),$$

whence $u^2 - uv = \theta f$ for some $f \in H^\infty$. If we define $\varphi \in H^\infty$ to be the greatest common inner divisor of u and θ, we can write $u = \varphi g$ where g and θ have no nonconstant common inner divisor. Now, we see that

$$\varphi g^2 - gv = \frac{\theta}{\varphi} f,$$

which implies that the greatest common inner divisor of φ and θ/φ divides gv. By choice of g and v and by assumption on θ, we see that $\varphi = 1$ or $\varphi = \theta$. If $\varphi = \theta$, then $u(T) = 0$ and the equation

$$0 = u(T) = Ev(T)$$

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along with Lemma 2.14 implies that $E = 0$. If $\varphi = 1$, then by Lemma 2.14, $u(T)$ is a quasiaffinity, which forces E to be a quasiaffinity as well, by virtue of the equation

$$Ev(T) = v(T)E = u(T).$$

But E has closed range (being idempotent), and thus it must be invertible. The equation $E^2 = E$ then yields $E = I$, and the proof is complete. ■

A moment's thought reveals that an inner function θ satisfying the condition of the previous proposition must be of one of two types: either a power of a Blaschke factor $\theta(z) = (b_\lambda(z))^n$ or a singular inner function associated with a point mass measure on the unit circle

$$\theta(z) = \exp\left(t\frac{z+\zeta}{z-\zeta}\right),$$

where $t > 0$ and $|\zeta| = 1$. In fact, these are the inner functions whose inner divisors are completely ordered by divisibility (see [4, Proposition 4.2.6]). Moreover, Proposition 2.15 extends a recent result of Jiang and Yang (see [12]) that deals with the case of T being a Jordan block $S(\theta)$. In this special case, the result holds under the weaker condition that the function θ does not admit a so-called corona decomposition. We now formulate another corollary of the main result of this section.

Corollary 2.16 *Let $T_1 \in B(\mathcal{H}_1)$ be a weak contraction of class C_0 with minimal function θ. Assume that the inner divisors of θ are completely ordered by divisibility. Let $T_2 \in B(\mathcal{H}_2)$ be another operator of class C_0 which is quasisimilar to T_1. Assume that there exists a completely isometric isomorphism $\varphi\colon \{T_1\}' \to \{T_2\}'$ such that $\varphi(T_1) = T_2$. Then T_1 and T_2 are unitarily equivalent.*

Proof This is a mere restatement of Theorem 2.10 using Proposition 2.15 and the discussion that follows its proof. ■

3 Unitary Equivalence and Maximality for Blaschke Products

In this section, we show that in the case where the minimal function θ of a multiplicity-free operator is a Blaschke product, we may replace the assumption on the existence of a completely isometric isomorphism appearing in the previous section by a condition on the norm of a single operator. In fact, we investigate the maximality condition appearing in Theorem 1.4 and set out to prove Theorem 1.5, which improves it significantly. The first step is an estimate which, although completely elementary, is very useful (the reader might want to compare it with [9, Lemma 3.7]).

Lemma 3.1 *Let $T \in B(\mathcal{H})$ be a contraction and let $h \in \mathcal{H}$ such that $\|Th\| \geq \delta\|h\|$ for some $\delta > 0$. Then*

$$\left\| b_\mu(T)h\right\| \geq \frac{\delta - |\mu|}{1+|\mu|}\|h\|$$

for every $\mu \in \mathbb{D}$.

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Proof We have that $b_\mu(T)(1-\overline{\mu}T) = T-\mu$ so that

$$\big\| b_\mu(T)(1-\overline{\mu}T)h\big\| \geq \big(\delta - |\mu|\big)\, \|h\|.$$

Thus,

$$\|b_\mu(T)h\| \geq \frac{1}{\|1-\overline{\mu}T\|}\big\| b_\mu(T)(1-\overline{\mu}T)h\big\| \geq \frac{\delta-|\mu|}{1+|\mu|}\|h\|. \qquad \blacksquare$$

We also require the following fact (see [18]).

Theorem 3.2 *Let $T \in B(\mathcal{H})$ be a multiplicity-free operator of class C_0 with minimal function θ. Then every closed invariant subspace $M \subset \mathcal{H}$ of T is of the form*

$$M = \ker \varphi(T) = \overline{(\theta/\varphi)(T)\,\mathcal{H}}$$

for some inner divisor φ of θ. Conversely, if φ is an inner divisor of θ, then

$$M = \ker \varphi(T) = \overline{(\theta/\varphi)(T)\,\mathcal{H}}$$

is an invariant subspace for T, and the minimal function of $T|M$ is equal to φ.

The next lemma is used to prove the main result of this section, but it is also of independent interest. The very basic Lemma 3.1 first comes into play here.

Lemma 3.3 *Let $T \in B(\mathcal{H})$ be a multiplicity-free operator of class C_0 whose minimal function is a Blaschke product θ. Let $\xi \in \mathcal{H}$ be a unit vector satisfying $\|\psi(T)\xi\| = 1$ for some big inner divisor ψ of θ. Then ξ is cyclic.*

Proof Let $M \subset \mathcal{H}$ be the smallest closed invariant subspace for T that contains ξ. By Theorem 3.2, there must exist an inner divisor φ of θ with the property that $M = \ker\varphi(T)$. The desired conclusion will follow if we show that $\varphi = \theta$, because then $M = \mathcal{H}$. Assume on the contrary that φ is a proper divisor of θ. Then there exists a big divisor ω of θ with the property that $\omega(T)\xi = 0$. Note that $\psi(T)\xi \neq 0$ by assumption so that $\psi \neq \omega$. Now, there exists $\lambda, \mu \in \mathbb{D}$ distinct zeros of θ such that $\psi = \theta/b_\lambda$ and $\omega = \theta/b_\mu$. Choose $z \in \mathbb{D}$ with the property that $b_z \circ b_\mu = b_\lambda$. Using Lemma 3.1 and the fact that

$$1 = \|\psi(T)\xi\| = \left\| b_\mu(T)\Big(\frac{\theta}{b_\lambda b_\mu}\Big)(T)\xi\right\|,$$

we find that

$$\begin{aligned} 0 = \|\omega(T)\xi\| &= \left\| b_\lambda(T)\Big(\frac{\theta}{b_\lambda b_\mu}\Big)(T)\xi\right\| \\ &= \left\| (b_z\circ b_\mu)(T)\Big(\frac{\theta}{b_\lambda b_\mu}\Big)(T)\xi\right\| \\ &\geq \frac{1-|z|}{1+|z|}\left\| \Big(\frac{\theta}{b_\lambda b_\mu}\Big)(T)\xi\right\| \geq \frac{1-|z|}{1+|z|}\|\psi(T)\xi\|, \end{aligned}$$

which is a contradiction, since $\psi(T)\xi \neq 0$. ■

Before moving on to the next step towards the main result of this section, we recall an elementary fact (see [9]).

Lemma 3.4 *Let $T \in B(\mathcal{H})$ be a multiplicity-free operator of class C_0 with minimal function $\theta = b_{\lambda_1} \cdots b_{\lambda_N}$. Let $\xi \in \mathcal{H}$ be a cyclic vector for T. Then the vectors*

$$\xi, b_{\lambda_1}(T)\xi, (b_{\lambda_1} b_{\lambda_2})(T)\xi, \dots, (b_{\lambda_1} \cdots b_{\lambda_{N-1}})(T)\xi$$

form a basis for $\mathcal{H}$.

We require an infinite-dimensional version of Lemma 3.4. First, we need another basic fact (see [4, Theorem 2.4.6]).

Lemma 3.5 *Let $T \in B(\mathcal{H})$ be an operator of class C_0 with minimal function θ. Given a family $\{\theta_n\}_n$ of inner divisors of θ with least common inner multiple φ, we have that* $\ker \varphi(T)$ *is the smallest closed subspace containing* $\ker \theta_n(T)$ *for every n.*

We now proceed to establish a more general version of Lemma 3.4.

Lemma 3.6 *Let $T \in B(\mathcal{H})$ be a multiplicity-free operator of class C_0 whose minimal function is a Blaschke product θ. Let $\xi \in \mathcal{H}$ be a unit vector which is also cyclic for T. Then for every big divisor ψ of θ, we have that $\mathcal{H}$ is the smallest closed subspace containing $\varphi(T)\xi$ for every inner divisor φ of ψ.*

Proof We can write $\theta = \prod_{n=0}^{\infty} \theta_n$, where each θ_n is a power of a Blaschke factor with the property that θ_n and θ_m have no nonconstant common inner divisor if $n \neq m$. For each $n \geq 0$, put $\theta_n = \widetilde{b}_{\lambda_n}^{d_n}$ for some $\lambda_n \in \mathbb{D}$ and some positive integer d_n. Without loss of generality, we may assume that $\psi = \theta / b_{\lambda_0}$.

By Lemma 3.4, we see that for each $n \geq 1$ the set

$$\left\{ (b_{\lambda_n}^k)(T)h \mid 0 \leq k \leq d_n \right\}$$

is a basis for $\ker(b_{\lambda_0}\theta_n)(T)$ whenever h is a cyclic vector for $T|\ker(b_{\lambda_0}\theta_n)(T)$. Since ξ is cyclic for T, we get from Theorem 3.2 that $(\theta/(b_{\lambda_0}\theta_n))(T)\xi$ is cyclic for $T|\ker(b_{\lambda_0}\theta_n)(T)$ and thus

$$\left\{ \left(\frac{\theta}{b_{\lambda_0}\theta_n} b_{\lambda_n}^k \right)(T)\xi : 0 \leq k \leq d_n \right\}$$

is a basis for $\ker(b_{\lambda_0}\theta_n)(T)$ for every $n \geq 1$. Note in addition that $\frac{\theta}{b_{\lambda_0}\theta_n} b_{\lambda_n}^k$ divides $\psi = \theta / b_{\lambda_0}$ for every k. A similar argument shows that

$$\left\{ \left(\frac{\theta}{\theta_0} b_{\lambda_0}^k \right)(T)\xi : 0 \leq k \leq d_0 - 1 \right\}$$

is a basis for $\ker \theta_0(T)$. Note once again that $\frac{\theta}{\theta_0} b_{\lambda_0}^k$ divides $\psi = \theta / b_{\lambda_0}$ for every $0 \leq k \leq d_0 - 1$.

Therefore, the smallest closed subspace containing all the vectors of the form $\varphi(T)\xi$ where φ is an inner divisor of ψ contains

$$\ker \theta_0(T) \cup \bigcup_{n=1}^{\infty} \ker(b_{\lambda_0}\theta_n)(T).$$

Since the least common inner multiple of θ_0 and the family $\{b_{\lambda_0}\theta_n\}_n$ is θ, the conclusion follows from Lemma 3.5. ■

We now come to the main result of this section. In light of Lemmas 3.6 and 3.3, it is a consequence of [2, Theorem 3.2.9] and the surrounding circle of ideas. However, we feel that the following proof (which is an adaptation of that of [2, Lemma 3.2.6]) is instructive and more direct, so we provide it nonetheless.

Theorem 3.7 *Let $T \in B(\mathcal{H})$ be a multiplicity-free operator of class C_0 whose minimal function is a Blaschke product θ. Assume that $\|\psi(T)\| = 1$ for some big inner divisor ψ of θ. Then T is unitarily equivalent to $S(\theta)$.*

Proof Set $\psi = \theta/b_\lambda$. By Theorem 3.2, we know that $N = \overline{\psi(T)\,\mathcal{H}}$ is an invariant subspace for T with the property that $T|N$ has minimal function equal to b_λ. Thus, $T|N$ must be quasisimilar to $S(b_\lambda)$ by Theorem 1.1, and we conclude that N is one-dimensional. In other words, $\psi(T)$ has rank 1 and there exists a unit vector $\xi \in \mathcal{H}$ with the property that $\|\psi(T)\xi\| = 1$. It is easily verified that this implies that $\|\varphi(T)\xi\| = 1$ for every inner divisor φ of ψ. Note also that the vector ξ is cyclic for T by Lemma 3.3.

Let us now denote by $U\colon \mathcal{K} \to \mathcal{K}$ the minimal unitary dilation of T. The operator $\varphi(U)$ is unitary for every inner divisor φ of ψ, whence

$$\|\varphi(U)\xi\| = 1 = \|\varphi(T)\xi\|.$$

These equalities coupled with the relation $\varphi(T) = P_{\mathcal{H}}\varphi(U)|\mathcal{H}$ force $\varphi(T)\xi = \varphi(U)\xi$ for every inner divisor φ of ψ. Consequently, for integers n, m such that $n > m$, we have

$$\begin{aligned}\left\langle b_\lambda^n(U)\xi, b_\lambda^m(U)\xi\right\rangle &= \left\langle b_\lambda^{n-m}(U)\xi, \xi\right\rangle = \left\langle \psi(U)b_\lambda^{n-m}(U)\xi, \psi(U)\xi\right\rangle \\ &= \left\langle \psi(U)b_\lambda^{n-m}(U)\xi, \psi(T)\xi\right\rangle = \left\langle (\psi b_\lambda^{n-m})(T)\xi, \psi(T)\xi\right\rangle = 0,\end{aligned}$$

whence $\{b_\lambda^n(U)\xi\}_{n\in\mathbb{Z}}$ is an orthonormal set that generates a Hilbert space $\mathcal{K}_0 \subset \mathcal{K}$. Now define an operator $\Lambda\colon \mathcal{K}_0 \to L^2$ such that $\Lambda b_\lambda^n(U)\xi = b_\lambda^n k_\lambda$ for every integer n, where $k_\lambda \in H^\infty$ is defined as

$$k_\lambda(z) = \frac{1 - |\lambda|^2}{1 - \overline{\lambda}z}.$$

It is easily verified that Λ is unitary and $\Lambda b_\lambda(U) = b_\lambda(M_z)\Lambda$, where M_z denotes the unitary operator of multiplication by z on L^2. Since Blaschke factors can be uniformly approximated on $\overline{\mathbb{D}}$ by polynomials and

$$b_{-\lambda} \circ b_\lambda(z) = z = b_\lambda \circ b_{-\lambda}(z),$$

we see that

$$\Lambda U = M_z \Lambda, \quad \bigvee_{n=0}^{\infty} b_\lambda^n(U)\xi = \bigvee_{n=0}^{\infty} U^n \xi \quad \text{and} \quad \bigvee_{n=0}^{\infty} b_\lambda^n k_\lambda = \bigvee_{n=0}^{\infty} z^n k_\lambda.$$

If we put $\mathcal{H}_0 = \bigvee_{n=0}^{\infty} b_\lambda^n(U)\xi$, then clearly $\mathcal{H}_0$ is invariant under U and

$$\Lambda \mathcal{H}_0 = \bigvee_{n=0}^{\infty} z^n k_\lambda = H^2,$$

since a straightforward calculation shows that k_λ is an outer function. Moreover, we find

$$\Lambda\varphi(T)\xi = \Lambda\varphi(U)\xi = \varphi(M_z)k_\lambda = \varphi k_\lambda = \varphi(S(\theta))k_\lambda$$

for every inner divisor φ of ψ, and thus $\Lambda \mathcal{H} = H(\theta)$ by Lemma 3.6 (here we used the well-known fact that k_λ is cyclic for $S(\theta)$). In particular, we see that

$$\mathcal{H} = \Lambda^* H(\theta) \subset \Lambda^* H^2 = \mathcal{H}_0 \,.$$

If we set $W = \Lambda | \mathcal{H}$, then we obtain another unitary operator $W \colon \mathcal{H} \to H(\theta)$. Using $\Lambda \mathcal{H} = H(\theta)$ along with $\Lambda(\mathcal{H}_0 \ominus \mathcal{H}) = \theta H^2$, we conclude that

$$P_{H(\theta)} \Lambda \restriction \mathcal{H}_0 = \Lambda P_{\mathcal{H}} \restriction \mathcal{H}_0 \,.$$

Since $U \mathcal{H}_0 \subset \mathcal{H}_0$, we have $U \mathcal{H} \subset \mathcal{H}_0$ and

$$\begin{aligned} WT &= (\Lambda \restriction \mathcal{H})T = \Lambda P_{\mathcal{H}} U \restriction \mathcal{H} = (\Lambda P_{\mathcal{H}} \restriction \mathcal{H}_0) U \restriction \mathcal{H} \\ &= (P_{H(\theta)} \Lambda \restriction \mathcal{H}_0) U \restriction \mathcal{H} = P_{H(\theta)} \Lambda U \restriction \mathcal{H} = P_{H(\theta)} M_z \Lambda \restriction \mathcal{H} \\ &= (P_{H(\theta)} M_z \restriction H(\theta)) \Lambda \restriction \mathcal{H} = S(\theta)(\Lambda \restriction \mathcal{H}) = S(\theta) W, \end{aligned}$$

so that T is unitarily equivalent to $S(\theta)$. ∎

Recall now the following well-known consequence of the commutant lifting theorem (see [17]).

Theorem 3.8 *The map $u + \theta H^\infty \mapsto u(S(\theta))$ establishes an isometric algebra isomorphism between $H^\infty/\theta H^\infty$ and $\{S(\theta)\}'$. In particular,*

$$\|u(S(\theta))\| = \inf\{\|u + \theta f\|_{H^\infty} : f \in H^\infty\}$$

for every $u \in H^\infty$.

We close this section by stating a simpler version of Theorem 3.7.

Corollary 3.9 *Let $T \in B(\mathcal{H})$ be a multiplicity-free operator of class C_0 whose minimal function is a Blaschke product θ. Assume that the map $\Psi \colon H^\infty(T) \to H^\infty(S(\theta))$ defined by $\Psi(u(T)) = u(S(\theta))$ is contractive. Then T is unitarily equivalent to $S(\theta)$.*

Proof This follows directly from Theorem 3.7. Indeed, if λ is a zero of θ, then

$$\Big\|\Big(\frac{\theta}{b_\lambda}\Big)(T)\Big\| \geq \Big\|\Psi\Big(\Big(\frac{\theta}{b_\lambda}\Big)(T)\Big)\Big\| = \Big\|\Big(\frac{\theta}{b_\lambda}\Big)(S(\theta))\Big\|$$
$$= \inf\Big\{\Big\|\frac{\theta}{b_\lambda}+\theta f\Big\| : f \in H^\infty\Big\} = \inf\Big\{\|1+b_\lambda f\| : f \in H^\infty\Big\} = 1,$$

where we used Theorem 3.8. ■

Note that in the setting of that corollary, we do not need to assume the irreducibility of the commutant (compare with Theorem 2.12).

4 Similarity and Lower Bounds for Big Divisors of Finite Blaschke Products

The focus of this section shifts from unitary equivalence to similarity. Let T be a multiplicity-free operator of class C_0 whose minimal function θ is a Blaschke product. We saw in Section 3 (Theorem 3.7) that under, the assumption that $\|\psi(T)\| = 1$ for some big divisor ψ of θ, T and $S(\theta)$ must be unitarily equivalent. In this final section, we investigate the possibility of obtaining a weaker conclusion, namely similarity, from a weaker assumption on the norm of $\psi(T)$. This problem was studied in [9] where the following partial result was obtained.

Theorem 4.1 *Let $T_1 \in B(\mathcal{H}_1)$, $T_2 \in B(\mathcal{H}_2)$ be multiplicity-free operators of class C_0 with minimal function $\theta = b_{\lambda_1}\cdots b_{\lambda_N}$. Define*

$$\eta = \sup_{1\leq j,k\leq N} \frac{|b_{\lambda_j}(\lambda_k)|^{1/2}}{(1-\max\{|\lambda_j|,|\lambda_k|\}^2)^{1/2}}.$$

Assume that

$$\big\|(\theta/b_{\lambda_N})(T_1)\big\| > \beta + 5\sqrt{2}\eta. \quad \textit{and} \quad \big\|(\theta/b_{\lambda_N})(T_2)\big\| > \beta + 5\sqrt{2}\eta$$

for some constant β satisfying

$$\Big(1-\frac{1}{(N-1)^2}\Big)^{1/2} < \beta < 1.$$

Then there exists an invertible operator $X\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that $XT_1 = T_2X$ and

$$\max\big\{\|X\|, \|X^{-1}\|\big\} \leq C(\beta, N),$$

where $C(\beta,N) > 0$ is a constant depending only on β and N.

We should remark at this point that in the above setting the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional, and thus Theorem 1.1 implies that T_1 and T_2 must be similar. Thus, the relevance of Theorem 4.1 lies in the control over the norm of the similarity rather than in the existence of the similarity. This control allows one to obtain similarity results for infinite Blaschke products having certain nice properties. We refer the curious reader to [9] for such applications related to interpolation by bounded holomorphic functions on the unit disc.

On the other hand, the presence of the quantity η in the previous statement is unexpected and seems artificial. Moreover, it has the unpleasant consequence of restricting the minimal functions to which the theorem applies, since clearly η must be smaller than $(5\sqrt{2})^{-1}$. The main result of this section removes η completely at the cost of a slightly stronger assumption on the operators (which is automatically satisfied by Jordan blocks, however). In particular, it applies to arbitrary finite Blaschke products.

The main technical tool we require is the following fact, which can be inferred from the work done in [9].

Theorem 4.2 *Let $T_1 \in B(\mathcal{H}_1), T_2 \in B(\mathcal{H}_2)$ be multiplicity-free operators of class C_0 whose minimal function is a finite Blaschke product θ with N roots. Assume that there exist unit cyclic vectors $\xi_1 \in \mathcal{H}_1, \xi_2 \in \mathcal{H}_2$ and a constant β with the property that*

$$\|\varphi(T_1)\xi_1\| \geq \beta > \Big(1 - \frac{1}{(N-1)^2}\Big)^{1/2}$$

and

$$\|\varphi(T_2)\xi_2\| \geq \beta > \Big(1 - \frac{1}{(N-1)^2}\Big)^{1/2}$$

for every inner divisor φ of θ. Then there exists an invertible operator $X\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that $XT_1 = T_2X$ and

$$\max\big\{ \|X\|, \|X^{-1}\| \big\} \leq C(\beta, N),$$

where $C(\beta, N) > 0$ is a constant depending only on β and N.

Before we proceed, we establish some auxiliary results. The first one is well known, but we provide the proof for the reader's convenience.

Lemma 4.3 *Let $\theta_1, \theta_2 \in H^\infty$ be inner functions such that there exist $u_1, u_2 \in H^\infty$ with the property that $\theta_1 u_1 + \theta_2 u_2 = 1$. Let $T \in B(\mathcal{H})$ be an operator of class C_0 with minimal function $\theta_1\theta_2$. Then there exists an invertible operator X such that*

$$XTX^{-1} = T|\ker\theta_1(T) \oplus T|\ker\theta_2(T),$$

$$\|X\| \leq \big(\|u_1\|^2_{H^\infty} + \|u_2\|^2_{H^\infty}\big)^{1/2} \quad \text{and} \quad \|X^{-1}\| \leq \sqrt{2}.$$

Proof Define

$$X\colon \mathcal{H} \longrightarrow \ker\theta_1(T) \oplus \ker\theta_2(T) \quad \text{as} \quad Xf = (\theta_2 u_2)(T)f \oplus (\theta_1 u_1)(T)f$$

for every $f \in \mathcal{H}$. If $Xf = 0$, then

$$f = (\theta_1 u_1 + \theta_2 u_2)(T)f = 0,$$

and thus X is injective. Given $g_1 \in \ker \theta_1(T)$ and $g_2 \in \ker \theta_2(T)$, we see that

$$Xg_1 = (\theta_2 u_2)(T) g_1 \oplus 0 = (1 - \theta_1 u_1)(T) g_1 \oplus 0 = g_1 \oplus 0$$

and

$$Xg_2 = 0 \oplus (\theta_1 u_1)(T) g_2 = 0 \oplus (1 - \theta_2 u_2)(T) g_2 = 0 \oplus g_2,$$

which shows that X is surjective. Notice also that $XT = (T \oplus T)X$. Therefore, we see that T is similar to

$$T|\ker \theta_1(T) \oplus T|\ker \theta_2(T).$$

It remains only to estimate the norm of X and X^{-1}. For $f \in \mathcal{H}$, we have

$$\begin{aligned}
\|Xf\| &= \|(\theta_2 u_2)(T) f \oplus (\theta_1 u_1)(T) f\| \\
&= \big(\|(\theta_1 u_1)(T) f\|^2 + \|(\theta_2 u_2)(T) f\|^2 \big)^{1/2} \\
&\leq \big(\|(\theta_1 u_1)(T)\|^2 + \|(\theta_2 u_2)(T)\|^2 \big)^{1/2} \|f\| \\
&\leq \big(\|u_1\|^2_{H^\infty} + \|u_2\|^2_{H^\infty} \big)^{1/2} \|f\|
\end{aligned}$$

and

$$\begin{aligned}
\|f\| &= \|(\theta_1 u_1 + \theta_2 u_2)(T) f\| \\
&\leq \|(\theta_1 u_1)(T) f\| + \|(\theta_2 u_2)(T) f\| \\
&\leq \sqrt{2} (\|(\theta_1 u_1)(T) f\|^2 + \|(\theta_2 u_2)(T) f\|^2)^{1/2} \\
&= \sqrt{2} \|Xf\|
\end{aligned}$$

by the Cauchy–Schwarz inequality. This completes the proof. ■

Lemma 4.4 *Let $\theta_1, \theta_2 \in H^\infty$ be inner functions such that*

$$\inf_{z \in \mathbb{D}} \big\{ |\theta_1(z)| + |\theta_2(z)| \big\} = \delta > 0.$$

Let $T \in B(\mathcal{H})$ be an operator of class C_0 with minimal function $\theta_1 \theta_2$. Then there exists an invertible operator X such that

$$XTX^{-1} = T{\restriction}\ker \theta_1(T) \oplus T{\restriction}\ker \theta_2(T), \quad \|X^{-1}\| \leq \sqrt{2}, \quad \textit{and} \quad \|X\| \leq C(\delta),$$

where $C(\delta) > 0$ is a constant depending only on δ.

Proof This is an easy consequence of Lemma 4.3 and the estimates associated with Carleson's corona theorem (see [7] or [14, Theorem 3.2.10]). ■

In applying that lemma, the following estimate will prove to be useful.

Lemma 4.5 *Let $E, F \subset \mathbb{D}$ be two finite subsets of cardinality at most N, and let $\theta_E, \theta_F \in H^\infty$ be the associated Blaschke products. Assume that there exists $r > 0$ such that $|e - f| \geq r$ for every $e \in E$, $f \in F$. Then*

$$\inf_{z\in\mathbb{D}}\big\{ |\theta_E(z)| + |\theta_F(z)|\big\} > (r/4)^N.$$

Proof Throughout the proof we put

$$d(A, B) = \inf\big\{ |a - b| : a \in A, b \in B\big\}$$

whenever $A, B \subset \mathbb{C}$.

First note that

$$|b_\lambda(z)| = \Big| \frac{z-\lambda}{1-\overline{\lambda}z}\Big| \geq \frac{|z-\lambda|}{2}$$

for every $z \in \mathbb{D}$. In particular, we see that $|\theta_E(z)| \geq (r/4)^N$ for every $z \in \mathbb{D}$ such that $d(z, E) \geq r/2$. Now if $d(z, E) < r/2$, then $d(z, F) \geq r/2$ in view of the triangle inequality and of our assumption on the sets E and F. Thus, we conclude that

$$|\theta_F(z)| \geq (r/4)^N$$

if $d(z, E) < r/2$. Combining these inequalities yields

$$|\theta_E(z)| + |\theta_F(z)| \geq (r/4)^N$$

for every $z \in \mathbb{D}$. ∎

Next, we need an elementary combinatorial lemma.

Lemma 4.6 *Let $\varepsilon > 0$ and $\lambda_1, \dots, \lambda_N \in \mathbb{C}$. Then there exists an integer $1 \leq k \leq N$ with the property that the set $\{\lambda_1, \dots, \lambda_N\}$ can be written as the disjoint union of*

$$E_k = \{\lambda_j : |\lambda_j - \lambda_1| < \varepsilon\, 2^{-(N+1-k)}\}$$

and

$$F_k = \{\lambda_j : |\lambda_j - \mu| \geq \varepsilon\, 2^{-(N+1-k)} \text{ for every } \mu \in E_k\}.$$

Proof Put $S_N = \{\lambda_1, \dots, \lambda_N\}$. It is clear that E_k and F_k are disjoint and that $\lambda_1 \in E_k$ for every $1 \leq k \leq N$. Consider the set $G_k = S_N \setminus (E_k \cup F_k)$ for every $1 \leq k \leq N$. An element λ_j lies in G_k if it does not belong to E_k, but there exists $\mu \in E_k$ with the property that

$$|\lambda_j - \mu| < \frac{\varepsilon}{2^{N+1-k}}.$$

By the triangle inequality, we see that $G_k \cup E_k \subset E_{k+1}$. If G_k is nonempty for each $1 \leq k \leq N-1$, this last inclusion implies that E_k contains at least k elements for each $1 \leq k \leq N$, so that $E_N = S_N$ and G_N is empty, and the lemma follows. ∎

One last bit of preparation is necessary. The next fact is found in [19] (it was independently discovered by Herrero; see [11]).

Theorem 4.7 *Let $T \in B(\mathcal{H})$ be a multiplicity-free operator of class C_0. Then the set of cyclic vectors for T is a dense G_δ in $\mathcal{H}$.*

Finally, we come to our similarity result, which improves Theorem 4.1 in the sense that it removes any restriction on the roots of the minimal function θ.

Theorem 4.8 *Let $T_1 \in B(\mathcal{H}_1)$, $T_2 \in B(\mathcal{H}_2)$ be multiplicity-free operators of class C_0 whose minimal function is a finite Blaschke product θ with N roots. Assume that there exist constants β', β such that*

$$\|\varphi(T_1)|\ker\psi(T_1)\| > \beta' > \beta > \Big(1 - \frac{1}{(N-1)^2}\Big)^{1/4},$$

$$\|\varphi(T_2)|\ker\psi(T_2)\| > \beta' > \beta > \Big(1 - \frac{1}{(N-1)^2}\Big)^{1/4},$$

whenever ψ is a nonconstant inner divisor of θ and φ is a proper inner divisor of ψ. Then there exists an invertible operator X with the property that $XT_1 = T_2X$ and

$$\max\big\{\|X\|, \|X^{-1}\|\big\} \le C(N,\beta,\beta'),$$

where $C(N,\beta,\beta') > 0$ is a constant depending only on N, β, and β'.

Proof Put $\theta = b_{\lambda_1}\cdots b_{\lambda_N}$. We proceed by induction on N. The case $N = 1$ is trivial, since the equations

$$b_{\lambda_1}(T_1) = b_{\lambda_1}(T_2) = 0$$

then imply that T_1 and T_2 are equal to the same multiple of the identity operator. Assume that the conclusion holds for Blaschke products with at most $N-1$ roots. For each $1 \le k \le N$ we set $\psi_k = \theta/b_{\lambda_k}$. Since $\|\psi_N(T_i)\| > \beta'$, by Theorem 4.7 we can find a unit cyclic vector $\xi_i \in \mathcal{H}$ such that $\|\psi_N(T_i)\xi_i\| > \beta'$ for $i = 1, 2$. For $1 \le k < N$ we see that

$$\psi_k(T_i)\xi_i = b_{\lambda_N}(T_i)\Big(\frac{\theta}{b_{\lambda_k}b_{\lambda_N}}\Big)(T_i)\xi_i,$$

while

$$\psi_N(T_i)\xi_i = b_{\lambda_k}(T_i)\Big(\frac{\theta}{b_{\lambda_k}b_{\lambda_N}}\Big)(T_i)\xi_i,$$

and thus by Lemma 3.1 we find

$$\begin{aligned}
\|\psi_k(T_i)\xi_i\| &= \Big\| b_{\lambda_N}(T_i)\Big(\frac{\theta}{b_{\lambda_k}b_{\lambda_N}}\Big)(T_i)\xi_i\Big\| \\
&= \Big\|(b_\mu \circ b_{\lambda_k})(T_i)\Big(\frac{\theta}{b_{\lambda_k}b_{\lambda_N}}\Big)(T_i)\xi_i\Big\| \\
&\ge \frac{\beta' - |\mu|}{1+|\mu|}\Big\|\Big(\frac{\theta}{b_{\lambda_k}b_{\lambda_N}}\Big)(T_i)\xi_i\Big\| \\
&\ge \frac{\beta' - |\mu|}{1+|\mu|}\|\psi_N(T_i)\xi_i\| \ge \Big(\frac{\beta' - |\mu|}{1+|\mu|}\Big)\beta',
\end{aligned}$$

where $\mu = -b_{\lambda_N}(\lambda_k)$. Now choose $r > 0$ such that

$$\Big(\frac{\beta' - |\mu|}{1 + |\mu|}\Big)\beta' > \beta^2$$

if $|\lambda_k - \lambda_N| < r$. Clearly, r depends only on β and β', and if $|\lambda_k - \lambda_N| < r$, then $\|\psi_k(T_i)\xi_i\| > \beta^2$. Thus, the desired conclusion follows from Theorem 4.2 in case where $\sup_{1\leq k\leq N} |\lambda_k - \lambda_N| < r$. Assume therefore that this supremum is at least r. In that case, Lemma 4.6 allows us to write

$$\{\lambda_1, \dots, \lambda_N\} = E \cup F$$

where E and F are disjoint and nonempty, $|\lambda - \lambda_N| < r$ for every $\lambda \in E$, and $|\lambda - \mu| > r2^{-N}$ for every $\lambda \in E, \mu \in F$. Let θ_E (resp. θ_F) be the Blaschke product associated with the elements of E (respectively F). By Lemmas 4.4 and 4.5, for each $i = 1, 2$ there exists an invertible operator Y_i with the property that

$$Y_i T Y_i^{-1} = T_i|\ker\theta_E(T_i) \oplus T_i|\ker\theta_F(T_i) \quad \text{and} \quad \max\big\{\|Y_i\|, \|Y_i^{-1}\|\big\} \leq C_1,$$

where $C_1 > 0$ depends only on N, β and β'. Note now that the minimal function of $T_i|\ker\theta_E(T_i)$ (resp. $T_i|\ker\theta_F(T_i)$) is θ_E (resp. θ_F) by virtue of Theorem 3.2. Since E and F have cardinality strictly less than N, we are done by induction. ■

In the case where one of the operators is a Jordan block, we obtain a simpler version of the previous result by making use of another property of Jordan blocks, found in [4, Proposition 3.1.10].

Lemma 4.9 *Let φ be an inner divisor of the inner function θ. Then the operator $S(\theta)|\ker\varphi(S(\theta))$ is unitarily equivalent to $S(\varphi)$.*

Corollary 4.10 *Let $T \in B(\mathcal{H})$ be a multiplicity-free operator of class C_0 whose minimal function is a finite Blaschke product θ with at most N roots. Assume that there exist constants β, β' such that*

$$\|\varphi(T)|\ker\psi(T)\| > \beta' > \beta > \Big(1 - \frac{1}{(N-1)^2}\Big)^{1/4}$$

whenever ψ is an inner divisor of θ and φ is an inner divisor of ψ. Then there exists an invertible operator X with the property that $XT = S(\theta)X$ and

$$\max\big\{\|X\|, \|X^{-1}\|\big\} \leq C(N, \beta, \beta'),$$

where $C(N, \beta, \beta') > 0$ is a constant depending only on N, β, and β'.

Proof This follows directly from Lemma 4.9 and Theorems 3.8 and 4.8. ■

As was done in [9], this corollary can be applied to obtain similarity results for some infinite Blaschke products. Pursuing those applications here would lead us outside of the intended scope of the paper, so let us simply mention that the proofs follow the same lines as those from [9].

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