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THE CHOWLA-SELBERG METHOD FOR FOURIER EXPANSION OF HIGHER RANK EISENSTEIN SERIES

BY

AUDREY TERRAS[1]

Dedicated to the memory of Robert Arnold Smith

ABSTRACT. The terms of maximal rank in Fourier expansions of Eisenstein series for $\mathrm{GL}(n, \mathbb{Z})$ are obtained by an analogue of a method of Chowla and Selberg. The coefficients involve matrix analogues of divisor functions as well as K-Bessel functions for $\mathrm{GL}(n)$. The discussion involves a few properties of Hecke operators.

1. **Introduction**. Our goal is to give a simple discussion of Fourier expansions of Eisenstein series for the *general linear group* $\Gamma_n = \mathrm{GL}(n, \mathbb{Z})$ of $n \times n$ integral matrices of determinant ± 1. We concentrate here on the terms of maximal rank in these expansions. When $n = 2$, the method is that of Chowla and Selberg [5].

There are many number-theoretic applications of these Fourier expansions and the analogues for Siegel modular forms (see Chowla and Selberg [loc. cit.], Bump and Goldfeld [4], Siegel ([27], Vol. II, pp. 97–137), Maass [18], [19], Terras [30]).

These Eisenstein series are of interest because they form the continuous spectrum of the Laplacian on the fundamental domain for Γ_n in the symmetric space of $\mathrm{GL}(n, \mathbb{R})$. Thus these series are basic ingredients in analogues of the Poisson summation formula such as (5) below, or Selberg's trace formula (see Arthur [1], Langlands [17], Selberg [24], and Terras [30]). There are many number-theoretic and geometric applications of these noneuclidean Poisson summation formulas in studies of the statistical properties of Γ_n (see for example Hejhal [8], Mennicke [20], Terras [30], Wallace [34], [35]).

Fourier expansions similar to those described here are obtained by Bump [3], Imai and Terras [12], Proskurin [23], and Takhtadzyan and Vinogradov [28] — all for $\mathrm{GL}(3, \mathbb{Z})$. A very special Eisenstein series for $\mathrm{GL}(n, \mathbb{Z})$ was considered in [29] using very different methods (the Bruhat decomposition).

In order to describe our results, we need some basic definitions. More details of the fundamental theory of symmetric spaces and discrete isometry groups can be found in Helgason [9], Maass [18], Selberg [24], and Terras [30].

Let $\mathcal{P}_n$ denote the space of *positive definite symmetric* $n \times n$ *real matrices* $Y = (y_{ij})_{1 \leq i,j \leq n}$. Here "$Y$ positive definite" means that if

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$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n - 0, \quad \text{then} \quad Y[x] = {}^t x Y x = \sum_{i,j=1}^{n} x_i y_{ij} x_j > 0.$$

We shall use ${}^t x$ to denote the transpose of x. The general linear group $G = \mathrm{GL}(n, \mathbb{R})$ of nonsingular $n \times n$ real matrices g acts on $Y \in \mathcal{P}_n$ via $Y[g] = {}^t g Y g$. We can then identify $\mathcal{P}_n$ with the symmetric space $K\backslash G$, for $G = \mathrm{GL}(n, \mathbb{R})$, $K = O(n)$, the maximal compact subgroup of orthogonal matrices. The identification comes from the mapping of $K\backslash G$ onto $\mathcal{P}_n$ given by $Kg \longmapsto I[g]$.

The space $\mathcal{P}_n$ is turned into a Riemannian manifold by defining the G-invariant *arc length* element ds by:

$$ds^2 = \mathrm{Tr}((Y^{-1}\,dY)^2), \qquad dY = (dy_{ij})_{1 \le i,j \le n}.$$

The corresponding *G-invariant measure* on $\mathcal{P}_n$ is:

$$d\mu_n = d\mu_n(Y) = |Y|^{-(n+1)/2} \prod_{i \le j} dy_{ij}, \quad dy_{ij} = \text{Lebesgue measure},$$

$$|Y| = \det(Y).$$

And the *Laplace operator* is:

$$\Delta = \mathrm{Tr}((Y\partial/\partial Y)^2), \quad \partial/\partial Y = (\tfrac{1}{2}(1 + \delta_{ij})\partial/\partial y_{i,j})_{1 \le i,j \le n}.$$

There are also G-invariant differential operators L_j defined by:

$$L_j = \mathrm{Tr}((Y\partial/\partial Y)^j), \quad j = 1, 2, \ldots, n,$$

forming an algebraically independent basis for the algebra $D(\mathcal{P}_n)$ of all G-invariant differential operators on $\mathcal{P}_n$.

It will be useful to introduce the *determinant one surfaces*:

$$(1) \quad \begin{cases} \mathrm{SL}(n, D) = \{g \in \mathrm{GL}(n, D) \mid |g| = \det(g) = 1\}, \\ \mathrm{SO}(n) = \{g \in O(n) \mid |g| = \det(g) = 1\}, \\ \mathcal{SP}_n = \{Y \in \mathcal{P}_n \mid |Y| = \det(Y) = 1\} \cong \mathrm{SO}(n)\backslash \mathrm{SL}(n, \mathbb{R}). \end{cases}$$

We will use the notation:

$$(2) \quad \begin{cases} Y = t^{1/n} Y^0, \quad \text{for} \quad Y \in \mathcal{P}_n,\ Y^0 \in \mathcal{SP}_n,\ t = |Y| = \det(Y), \\ d\mu_n(Y) = t^{-1}\,dt\,dY^0, \quad \text{where} \quad dY^0 \text{ is an } \mathrm{SL}(n.\ \mathbb{R})\text{-invariant measure on } \mathcal{SP}_n. \end{cases}$$

Fourier analysis on $\mathcal{P}_n$ has been developed by many mathematicians, especially Harish-Chandra and Helgason (see Helgason [9] and Terras [30]). To describe it, one must define elementary eigenfunctions of the algebra $D(\mathcal{P}_n)$ which we call *power functions*, for $Y \in \mathcal{P}_n$, $s \in \mathbb{C}^n$:

$$(3) \qquad p_s(Y) = \prod_{j=1}^{n} |Y_j|^{s_j}, \quad \text{if} \quad Y = \begin{pmatrix} Y_j & * \\ * & * \end{pmatrix}, \quad Y_j \in \mathcal{P}_j, \quad |Y_j| = \det(Y_j).$$

See Maass ([18], p. 69) for the proof that these power functions are indeed eigenfunctions of the G-invariant differential operators on $\mathcal{P}_n$.

The *Helgason-Fourier transform* of K-invariant functions $f: \mathcal{P}_n/K \to \mathbb{C}$ has the form:

$$\hat{f}(s) = \int_{\mathcal{P}_n} f(Y)\,\overline{p_s(Y)}\,d\mu_n(Y), \qquad s \in \mathbb{C}^n. \tag{4}$$

The transform has an inversion formula due to Harish-Chandra. It can also be identified with the Selberg transform appearing in the Selberg trace formula.

For number theory, one needs to connect analysis on $\mathcal{P}_n$ and analysis on $\mathcal{P}_n/\Gamma_n$, $\Gamma_n = GL(n, \mathbb{Z})$. One such result is the Poisson summation formula, which is a preliminary to Selberg's trace formula. Given a smooth compactly supported function $f: \mathcal{P}_n/K \to \mathbb{C}$, the *Poisson summation formula* says (cf. Arthur [1], Langlands [17]):

$$\sum_{\gamma \in \Gamma_n/\pm I} f(I[\gamma]) = \sum_{\pi} c_\pi^{-1} \sum_{\varphi \in \mathcal{B}_\pi} \int_{r \in \mathbb{C}^{|\pi|},\ \mathrm{Re}\, r = \text{constant}} \hat{f}(s(\varphi, r))|E(\varphi, r|I)|^2\, dr. \tag{5}$$

Here π runs through all inequivalent partitions of n; i.e., decompositions of $n = n_1 + \cdots + n_q$, $n_i \in \mathbb{Z}^+$, with $|\pi| = q$, c_π = a positive constant, $\mathcal{B}_\pi$ is a complete orthonormal set of automorphic forms φ defined from $\varphi_i \in L^2(\mathcal{SP}_{n_i}/\Gamma_{n_i})$, which are eigenfunctions of $D(\mathcal{SP}_{n_i})$, as follows:

$$\varphi(Y) = \prod_{i=1}^{q} \varphi_i(a_i^0), \quad Y = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_q \end{pmatrix} \begin{bmatrix} I_{n_1} & & * \\ & \ddots & \\ 0 & & I_{n_q} \end{bmatrix}, \quad a_i \in \mathcal{P}_{n_i},$$

$$a_i = a_i^0 |a_i|^{1/n_i}, \quad |a_i^0| = 1.$$

If $r \in \mathbb{C}^q$, we can define the power function

$$\psi(Y) = \prod_{i=1}^{q} |a_i|^{-r_i}.$$

When Re r_i is sufficiently large, we define the Eisenstein series by:

$$E(\varphi, r|Y) = \sum_{\gamma \in \Gamma_n/P_\pi} \varphi(Y[\gamma])\psi(Y[\gamma]),$$

where P_π is the *parabolic subgroup* of matrices with block form:

$$\gamma = \begin{pmatrix} \gamma_1 & & * \\ & \ddots & \\ 0 & & \gamma_q \end{pmatrix}, \qquad \gamma_i \in \Gamma_{n_i}. \tag{6}$$

Finally $\hat{f}(s(\varphi, r))$ is the Fourier transform (4) evaluated at $s(\varphi, r) \in \mathbb{C}^n$ determined by a basic lemma of Selberg [24] which says that eigenfunctions of G-invariant differential operators are eigenfunctions of G-invariant integral operators (see Terras [31]). We do not use (5) here. It is stated to motivate our study of the special Eisenstein series associated to partitions with $q = 2$ or n.

Minkowski [21] found that, up to boundary identifications, a *fundamental domain* for $\mathcal{P}_n/\Gamma_n$, $\Gamma_n = GL(n, \mathbb{Z})$, has the form:

$$\mathcal{M}_n = \{Y \in \mathcal{P}_n | Y[a] \geq y_{ii}, \quad \forall a \in \mathbb{Z}^n \cdot \ni \cdot g.c.d.(a_i, \ldots, a_n) = 1, \quad y_{i,i+1} \geq 0, \quad 1 \leq i \leq n\}.$$

He also found that the Euclidean volume of $\{Y \in \mathcal{M}_n| \ |Y| \leq 1\}$ is:

$$2(n+1)^{-1} \prod_{K=2}^{n} \Lambda(k/2), \quad \text{where} \quad \Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s).$$

This fact can be proved from properties of Eisenstein series (8) (see Siegel [27], Vol. I, pp. 459–468; Vol. III, pp. 328–333; Maass [18], pp. 231ff; and Terras [30]).

2. **Eisenstein Series**. We consider the following space of *automorphic forms for* $\Gamma_n = \mathrm{GL}(n, \mathbb{Z})$ corresponding to a system $\lambda = \lambda_L \in \mathbb{C}$ of eigenvalues for the differential operators $L \in D(\mathcal{P}_n)$:

$$(7) \quad \begin{cases} \mathcal{A}(\Gamma_n, \lambda) = \{f: \mathcal{P}_n/\Gamma_n \to \mathbb{C} | Lf = \lambda f, \ L \in D(\mathcal{P}_n), \ |f(Y)| \leq C|p_s(Y)|\}, \\ \mathcal{A}^0(\Gamma_n, \lambda) = \{f: \mathcal{SP}_n/\Gamma_n \to \mathbb{C} | Lf = \lambda f, \ L \in D(\mathcal{SP}_n), \ |f(Y)| \leq C|p_s(Y)|\}. \end{cases}$$

These forms generalize the Maass wave forms when $n = 2$ (see Terras [30] for the history of these matters). There are, in fact, many parallels with the classical theory of holomorphic modular forms on the Poincaré upper half plane (c.f. Hecke [7]) and Siegel modular forms (c.f. Siegel [27], Vol. II, pp. 97–137).

There are examples of cusp forms belonging to congruence subgroups of $\mathrm{GL}(3, \mathbb{Z})$ corresponding via generalizations of Hecke theory to Hecke L-functions of cubic number fields (see Jacquet, Piatetskii-Shapiro and Shalika [14]). There are also cuspidal examples corresponding via Hecke theory to Rankin-Selberg L-functions (see Gelbart and Jacquet [6] and Moreno and Shahidi [22]).

We do not consider such examples here for our subject is Eisenstein series. The main result (Theorem 1) gives an explicit Fourier expansion for such Eisenstein series in a special case.

For $\varphi \in \mathcal{A}^0(\Gamma_m, \lambda)$, $s \in \mathbb{C}$, with $\mathrm{Re}\, s > n/2$, $m < n$, define the *Eisenstein series* by:

$$(8) \qquad E(\varphi, s|Y) = E_{m,n-m}(\varphi, s|Y) = \sum_{\substack{A=(A_1 *)\in\Gamma_n/P(m,n-m) \\ A_1 \in \mathbb{Z}^{m\times n}}} \varphi(Y[A_1]^0)|Y[A_1]|^{-s}.$$

We are using the notation (2), (6) above. The series (8) converges by an integral test coming out of integral formulas of Siegel ([27], Vol. III, pp. 39–96) and Wishart [36]. These integral formulas combine to give (c.f. Terras [32]):

$$\int_{\mathcal{SP}_n/\Gamma_n} \sum_{N\in\mathbb{Z}^{n\times m}, rk(N)=m} f(W[N])\, dW = c\int_{\mathcal{P}_m} f(Y)|Y|^{n/2}\, d\mu_m(Y).$$

for a positive constant c. It is rather easy to get a convergence argument out of this integral formula, assuming that $\varphi \in L^1(\mathcal{SP}_m/\Gamma_m)$.

When $n = 2$, the Eisenstein series (8) is quite well understood (see Chowla and Selberg [5] or Terras [30]). When $\varphi \equiv 1$, the series (8) is closely related to a zeta function (first considered by Koecher [16]) which generalizes the Epstein zeta function. We will soon discuss this (see formulas (12) and (16)). When φ is itself an Eisenstein series built up only out of Eisenstein series (i.e., all the φ's are identically one at each level), then (8) is easily seen to be an *Eisenstein series first considered by Selberg* [24] for $s \in \mathbb{C}^n$, $Y \in \mathcal{P}_n$:

$$(9) \qquad E_{(n)}(s|Y) = \sum_{\Gamma_n/P_{(n)}} p_{-s}(Y[\gamma]), \ \operatorname{Re} s_j > 1, \ j = 1, \ldots, n.$$

Here $P_{(n)}$ denotes the minimal parabolic subgroup associated to the partition with $|\pi| = n$. An integral formula similar to that stated above gives the convergence region indicated (see Terras [32]). Selberg obtained the analytic continuation of (9) to a meromorphic function of $s \in \mathbb{C}^n$ with $n!$ functional equations (see Maass [18], §17; and Terras [30]).

It is fairly easy to see that the Eisenstein series $E(\varphi, s|Y)$ defined by (8) are indeed automorphic forms in $\mathcal{A}(\Gamma_n, \mu)$ for some eigenvalue system μ, since $\varphi(W^0)|W|^{-s}$ is clearly an eigenfunction of the G-invariant integral operators on $W \in \mathcal{P}_m$.

To see that $|E(\varphi, s|Y)| \leq C|p_s(Y)|$, one can argue as follows. Suppose that $Y \in \mathcal{P}_n$ has *partial Iwasawa decomposition:*

$$(10) \qquad Y = \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \begin{bmatrix} I_m & X \\ 0 & I_{n-m} \end{bmatrix}, \ V \in \mathcal{P}_m, \ W \in \mathcal{P}_{n-m}, \ X \in \mathbb{R}^{m \times (n-m)},$$

and

$$A_1 = \begin{pmatrix} B \\ C \end{pmatrix}, \ B \in \mathbb{Z}^{m \times m}, \ C \in \mathbb{Z}^{m \times (n-m)}.$$

Then $Y[A_1] = V[B + XC] + W[C]$. If W approaches infinity in the sense that the geodesic distance between W and I blows up, then the eigenvalues of W must approach infinity. We shall assume that all these eigenvalues blow up. If so, the only way that $|Y[A_1]|$ can avoid approaching infinity is if $C = 0$. So as W approaches infinity,

$$(11) \qquad E\left(\varphi, s \,\middle|\, \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\right) \sim \varphi(V^0)|V|^{-s}.$$

This gives the first term in the Fourier expansion of the Eisenstein series with respect to the X-variable in (10) and shows that the series (8) does not represent a cusp form. Our operation of sending W to infinity gives an analogue of the Siegel Φ-operator in the theory of Siegel modular forms (cf. Maass [18]).

Why do the Eisenstein series require analytic continuation in the s-variable? Consider, for example, the series (9). The negativity of the Laplacian on $L^2(\mathcal{S}\mathcal{P}_n/\Gamma_n)$ implies that s in $E_{(n)}(s|Y)$ corresponds to an element of the spectrum of Δ if $\operatorname{Re} s_i = 1/2$, $i = 1, \ldots, n-1$ (see Terras [30]).

How many ways can the Eisenstein series be continued? There are at least three methods (c.f. Hejhal [8], Vol. II, Appendix F; Langlands [17]; Maass [19]; Selberg [24]; and Terras [30]). Our aim here is to make use of a method that is somewhat special to arithmetic groups. We shall see that this method has interesting implications for the Fourier coefficients. The idea goes back to Riemann when $n = 1$ and was developed by Siegel, Selberg, Maass, etc. for $n > 1$. We call the technique "*Riemann's method of theta functions*". To use it, one must connect the Eisenstein series (8), which is a sum over Γ_n, with a zeta function, which is a sum over as big a subset of $\mathbb{Z}^{n\times m}$ as possible. We define this *zeta function* as:

$$(12) \qquad Z(\varphi, s|Y) = Z_{m,n-m}(\varphi, s|Y) = \sum_{A\in\mathbb{Z}^{n\times m} rk\ m/\Gamma_m} \varphi(Y[A]^0)|Y[A]|^{-s},$$

for $\varphi \in \mathcal{A}^0(\Gamma_m, \lambda)$, $s \in \mathbb{C}$, Re $s > n/2$, $m \leq n$. The sum is over a complete set of representatives for the $n \times m$ integral matrices A of rank m, modulo $\mathrm{GL}(m, \mathbb{Z})$. The special case $\varphi \equiv 1$ is Koecher's generalization of Epstein's zeta function (see [16]). When $\varphi \equiv 1$ and $n = m$, this zeta function is the analogue of the Dedekind zeta function for the simple algebra $\mathbb{Q}^{n\times m}$; i.e.

$$Z_{n,0}(1, s|I) = \sum_{A\in\mathbb{Z}^{n\times n} rk\ n/\Gamma_n} |A|^{-2s} = \prod_{j=0}^{n-1} \zeta(2s - j),$$

a result which follows from (14) below. Such zeta functions were studied by Käte Hey [11]. But the analytic continuation problem was not resolved until Siegel ([27], Vol. 1, pp. 459–468, Vol. III, pp. 328–333) and Selberg (see Maass [18], pp. 231 ff.) and Terras [30]).

In order to relate (8) and (12), we need a few facts about Hecke operators for Γ_n.

3. **Hecke Operators**. If $k \in \mathbb{Z}^+$ and $f: \mathcal{SP}_n/\Gamma_n \to \mathbb{C}$, we define the *Hecke operator* $T_k^{(n)} = T_k$ by:

$$(13) \qquad T_k f(Y) = \sum_{A\in M_n(k)/\Gamma_n} f(Y[A]^0),$$

where $M_n(k) = \{A \in \mathbb{Z}^{n\times n} \,|\, |A| = k\}$ and we use the notation (2). A complete set of representatives for $M_n(k)/\Gamma_n$ is easily seen to be given by:

$$(14) \qquad \begin{pmatrix} d_1 & & d_{ij} \\ & \ddots & \\ 0 & & d_n \end{pmatrix}, \quad \prod_{j=1}^{n} d_j = k, \ d_j > 0, \ 0 \leq d_{ij} < d_i.$$

These Hecke operators are quite analogous to those considered by Hecke [7] and Maass [19]. A detailed discussion can be found in Terras [30, 33]. More general operators of this sort have been studied by Satake, Tamagawa and others (see Shimura [26] and Bump [3]).

It is shown in [33] that the linear maps $T_k: \mathcal{A}^0(\Gamma_n, \lambda) \to \mathcal{A}^0(\Gamma_n, \lambda)$ are self-adjoint operators with respect to the usual inner product on $L^2(\mathcal{SP}_n/\Gamma_n, dW)$, normalizing measures as in (2). Moreover, the T_k form a commutative ring of operators and are simultaneously diagonalizable. One can give explicit formulas for Euler factors:

$$\sum_{r\geq 0} T_{p^r} X^r, \quad p = \text{prime}, \quad X = \text{indeterminate}.$$

In fact, there is an analogue of Hecke's theory of L-functions corresponding to eigenfunctions of Hecke operators. Suppose $\varphi \in \mathcal{A}^0(\Gamma_n, \lambda)$ is such an eigenfunction with $T_k\varphi = u_k\varphi$, $u_k \in \mathbb{C}$. Then define the *L-function associated to* φ by:

$$L_\varphi(s) = \sum_{k \geq 1} u_k k^{-s}, \quad \operatorname{Re} s > n/4. \tag{15}$$

It is shown in [33] that $L_\varphi(s)$ has analytic continuation as a meromorphic function of $s \in \mathbb{C}$ with a functional equation (the method being the same as that sketched below for Eisenstein series).

We have introduced the Hecke operators because we have the following *relation between the Eisenstein series* (8) *and the zeta function* (12) when $\varphi \in \mathcal{A}^0(\Gamma_m, \lambda)$ is an eigenfunction for all the Hecke operators $T_k^{(m)}$ with $T_k^{(m)}\varphi = u_k^{(m)}\varphi$,

$$Z(\varphi, s|Y) = L_\varphi(2s)E(\varphi, s|Y). \tag{16}$$

See [33] for the simple proof. Note, however, that one must understand the zeros of $L_\varphi(2s)$ in order to divide by it (see Shahidi [25]).

The analytic continuation needed here requires a *theta function* defined for $Y \in \mathcal{P}_n$, $X \in \mathcal{P}_m$, $m \leq n$, by:

$$\theta_r(Y, X) = \sum_{A \in \mathbb{Z}^{n \times m},\, rk A = r} \exp\{-\pi \operatorname{Tr}(Y[A]X)\}, \qquad \theta = \sum_{r=0}^{m} \theta_r. \tag{17}$$

We will also require the *gamma function* for $\mathcal{P}_m$, defined for suitably restricted $s \in \mathbb{C}^m$ by:

$$\begin{aligned} \Gamma_m(s) &= \int_{\mathcal{P}_m} p_s(X) \exp\{-\operatorname{Tr}(X)\}\, d\mu_m \\ &= \pi^{m(m-1)/4} \prod_{j=1}^{m} \Gamma\left(s_j + \ldots + s_m - \frac{j-1}{2}\right). \end{aligned} \tag{18}$$

Hopefully the use of the same letter for a special function and a discrete group will not cause too big a disaster. Formula (18) was first proved by Ingham and Siegel in a special case (see Terras [30]).

The analytic continuation of the Eisenstein series $E(\varphi, s|Y)$ can now be based on the formula:

$$2\pi^{-ms}\Gamma_m(r(\varphi, s))Z(\varphi, s|Y) = \int_{\mathcal{P}_m/\Gamma_m} \theta_m(Y, X)\varphi((X^0)^{-1})\,|X|^s\, d\mu_m(X). \tag{19}$$

Here the variables $r = r(\varphi, s) \in \mathbb{C}^m$ are determined by:

$$\frac{L\varphi((X^0)^{-1})|X|^s}{\varphi((X^0)^{-1}|X|^s} = \frac{Lp_r(X)}{p_r(X)}.$$

This is derived from Selberg's basic principle, mentioned earlier, that eigenfunctions of G-invariant differential operators are eigenfunctions of G-invariant integral operators (see [30, 31, 33] or Maass [18], §16).

Riemann's method of theta functions produces an analytic continuation of (19) by splitting the fundamental domain into the regions with $|Y| \leq 1$ and $|Y| \geq 1$, then sending Y to Y^{-1} in the first region and making use of the transformation formula:

$$\theta(Y^{-1}, X^{-1}) = |Y|^{m/2}|X|^{n/2}\theta(Y, X).$$

The lower rank terms of theta in (17) produce divergent integrals. Selberg found G-invariant differential operators which annihilate the θ_r for $r < m$. This allows one to obtain the analytic continuation. The details are in the references mentioned at the end of the preceding paragraph.

The Eisenstein series (8) are eigenfunctions of the Hecke operators (13). The following proposition gives an explicit formula for the eigenvalue.

PROPOSITION 1. *Eigenvalues for the action of Hecke operators on Eisenstein series. Using the definitions* (8) *of* $E(\varphi, s|Y)$ *and* (13) *of* $T_k^{(n)}$, *we have*

$$T_k^{(n)}E(\varphi, s|Y) = u_k^{(n)}E(\varphi, s|Y),$$

where

$$u_k^{(n)} = k^{2ms/n} \sum_{t|k} d_{n-m}(k/t)\, t^{n-m-2s} u_t^{(m)}.$$

Here $T_t^{(m)}\varphi = u_t^{(m)}\varphi$ *and*

$$d_r(v) = \sum_{\prod_{j=1}^r d_j = v,\, d_j \in \mathbb{Z}^+} d_1^{r-1} d_2^{r-2} \dots d_{r-1}.$$

PROOF. Clearly we need representatives of $M_n(k)/P(m, n-m)$. One can write $A \in M_n(k)$ as $A = BC$, with

$$B = (B_1 *) \in \Gamma_n/P(m, n-m),\ C = \begin{pmatrix} F & H \\ 0 & G \end{pmatrix} \in M_n(k),$$
$$F \in \mathbb{Z}^{m\times m},\ G \in \mathbb{Z}^{(n-m)\times(n-m)}.$$

It follows that the sum $A \in M_n(k)/P(m, n-m)$ is equivalent to summing over $B = (B_1 *) \in \Gamma_n/P(m, n-m)$, and $F \in M_m(t)/\Gamma_m$, $G \in M_{n-m}(k/t)/\Gamma_{n-m}$, H mod F, for all divisors t of k. The notation "H mod F" means that we want a complete set of representatives for the equivalence relation $H \sim H'$ if and only if $H' = FU + H$, for some $U \in \mathbb{Z}^{m\times(n-m)}$. The number of $G \in M_{n-m}(k/t)/\Gamma_{n-m}$ is found from (14) and we count $d_{n-m}(k/t)$ such G's. The number of H mod F is easily seen to be $|F|^{n-m} = t^{n-m}$.

Thus, setting $P = P(m, n-m)$, we have:

$$T_k^{(n)}E_{m,n-m}(\varphi, s|Y) = \sum_{B\in M_n(k)/\Gamma_n} \sum_{A=(A_1 *)\in\Gamma_n/P,\, A_1\in\mathbb{Z}^{n\times m}} \varphi(Y[BA_1]^0)|Y[k^{-1/n}BA_1]|^{-s}$$

$$= k^{2ms/n} \sum_{(A_1 *)=A\in M_n(k)/P,\, A_1\in\mathbb{Z}^{n\times m}} \varphi(Y[A_1]^0)\,|Y[A_1]|^{-s}$$

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$$= k^{2ms/n} \sum_{(B_1{}^*)\in\Gamma_n/P} \Big(\sum_{t|k} d_{n-m}(k/t)t^{n-m-2s} u_t\Big)\, \varphi(Y[B_1]^0)|Y[B_1]|^{-s}$$

$$= u_k^{(n)} E_{m,n-m}(\varphi, s|Y). \qquad \square$$

One would expect the eigenvalues computed in Proposition 1 to be related to the Fourier coefficients computed in Theorem 1, but we will not pursue that question here.

4. **Fourier Expansions**. We know that $Z(\varphi, s|Y)$ defined in (12) is a periodic function of each entry of the matrix X when Y has the partial Iwasawa decomposition (10) and thus it has a Fourier expansion:

$$Z\left(\varphi, s \,\middle|\, \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\right) = \sum_{N\in\mathbb{Z}^{n\times m}} c_N(V, W) \exp\{2\pi i \operatorname{Tr}({}^tNX)\}.$$

There is quite a long history of the derivation of such expansions. Here we are mainly motivated by the work of Siegel and Maass for $\mathrm{Sp}(n, \mathbb{Z})$ (see [18] and [27], Vol. II, pp. 97–137).

Terms in such Fourier expansions always have two parts. One part is *arithmetic*-either a singular series (for functions defined as in (8)) or a divisor function (for functions defined as in (12)). When $n = 2$, the arithmetic part of the kth Fourier coefficient for $k \geqslant 1$ is essentially the eigenvalue in Proposition 1; i.e., $k^s \sum_{0<t|k} t^{1-2s} = k^s \sigma_{1-2s}(k)$. The singular series version of this is:

$$\sum_{\substack{c>0,\, d \bmod c \\ (d,c)=1}} c^{-2s} \exp(2\pi i k d/c) = \sigma_{1-2s}(k)/\zeta(2s).$$

When $\varphi \equiv 1$ and $n = 2m$, we found in [29] that the arithmetic part of the term corresponding to $N \in \mathbb{Z}^{m\times m}$ of rank m in the Fourier expansion of $E(1, s|Y)$ is essentially the singular series:

$$\sum_{R\in(\mathbb{Q}/\mathbb{Z})^{m\times m}} \nu(R)^{-2s} \exp\{2\pi i \operatorname{Tr}({}^tRN)\}.$$

Here $\nu(R)$ is the product of the reduced denominators of the elementary divisors of R. Siegel [loc. cit.] obtains an analogous result for holomorphic Eisenstein series for $\mathrm{Sp}(n, \mathbb{Z})$. Lower rank terms are more complicated to describe. We will find here that the arithmetic part of our Fourier expansions cannot so easily be separated out.

The terms of the Fourier expansion have a second part which is *analytic*-a matrix argument confluent hypergeometric function. For $\mathrm{GL}(n, \mathbb{Z})$ one obtains analogues either of K-Bessel or Whittaker functions. We mostly work with K-Bessel type functions, as we are attempting to stay close to the Siegel-type Fourier expansions. The Whittaker-type functions appear in the expansions discussed by Bump [3] (see also [13] and [14], for examples of the Whittaker model theory).

We define two kinds of K-Bessel functions, using the upper case letter for the first type of function and the lower case letter for the second. Formula (23) of Bengtson [2] relates the two functions. The K-Bessel function is useful because it has good convergence properties and the k-Bessel function is useful because the differential equations it satisfies are quite obvious. The theory of these functions is described in detail by Bengtson [2] and Terras [30]. Special cases were considered by Herz [10].

The *K-Bessel function* for $\mathcal{P}_n$ is defined for $s \in \mathbb{C}^n$, $A, B \in \mathcal{P}_n$ by:

$$(20) \qquad K_n(s|A,B) = K(s|A,B) = \int_{Y \in \mathcal{P}_n} p_s(Y) \exp\{-\mathrm{Tr}\,(AY + BY^{-1})\}\, d\mu_n(Y).$$

Note that when $n = 1$, this is essentially the *classical K-Bessel function* for $a > 0$:

$$(21) \qquad K_s(a) = \frac{1}{2} \int_{y>0} y^{s-1} \exp\left\{-\frac{a}{2}(y + 1/y)\right\} dy.$$

We find easily that $K_1(s|a,b) = 2(b/a)^{s/2} K_s(2\sqrt{ab})$. For our purposes it is useful to separate the arguments A and B, though one can change variables and reduce one argument to the identity. If A or B is singular, one must restrict s to a suitable half plane for convergence. For example, $K_n(s|A,0) = p_s(A^{-1})\Gamma_n(s)$, with the notation (18).

The *k-Bessel function* for $\mathcal{P}_n$ is defined for $s \in \mathbb{C}^n$, $Y \in \mathcal{P}_n$, $A \in \mathbb{R}^{n\times m}$ by:

$$(22) \qquad k_{m,n-m}(s|Y,A) = k(s|Y,A)$$
$$= \int_{X \in \mathbb{R}^{n\times m}} p_{-s}\left(Y^{-1}\begin{bmatrix} I_m & 0 \\ {}^tX & I_{n-m}\end{bmatrix}\right) \exp\{2\pi i\, \mathrm{Tr}\,({}^tAX)\}\, dX.$$

Here s must be suitably restricted for convergence.

As a function of Y, $k(s|Y,A)$ is an eigenfunction of all the G-invariant differential operators on $\mathcal{P}_n$, and it has the transformation property:

$$k\left(s\,\middle|\, Y\begin{bmatrix} I_m & X \\ 0 & I_{n-m}\end{bmatrix}, A\right) = \exp\{2\pi i\, \mathrm{Tr}\,({}^tAX)\} k(s|Y,A).$$

Moreover, $|k(s|Y,A)| \leq C|p_r(Y)|$ as Y approaches infinity.

Bengtson [2] proves the following *formula relating the two Bessel functions*:

$$(23) \quad \Gamma_m(-s^*)\, k_{m,n-m}\left(s, 0\,\middle|\, \begin{pmatrix} V & 0 \\ 0 & W\end{pmatrix}, A\right) = \pi^{m(n-m)/2}\, |W|^{m/4} K_m(s\#\,|W[\pi\,{}^tA], V^{-1}),$$

where $V \in \mathcal{P}_m$, $W \in \mathcal{P}_{n-m}$, $s \in \mathbb{C}^m$, $s\# = (0,\ldots,0,(n-m)/2) - s$, and $s^* = (s_{m-1}, \ldots, s_1, -\Sigma_1^m s_j)$.

The preceding formulas say that our Bessel functions come from *characters* χ_A *of the abelian group*:

$$N(m, n-m) = \left\{\begin{pmatrix} I_m & X \\ 0 & I_{n-m}\end{pmatrix} \,\middle|\, X \in \mathbb{R}^{m\times(n-m)}\right\},$$

defined by

$$(24) \qquad \chi_A\begin{pmatrix} I_m & x \\ 0 & I_{n-m}\end{pmatrix} = \exp\{2\pi i\, \mathrm{Tr}\,({}^tAX)\} \text{ for fixed } A \in \mathbb{R}^{m\times(n-m)}.$$

All of these functions are connected with the representation theory of the *nilpotent group*

$$N = \left\{ \begin{pmatrix} 1 & & x_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \,\middle|\, x_{ij} \in \mathbb{R} \right\}. \tag{25}$$

Kirillov [15] describes the irreducible unitary representations of N as being of two types — one dimensional or infinite dimensional. The one dimensional representations χ_a of N are parameterized by vectors $a \in \mathbb{R}^{n-1}$ and defined by:

$$\chi_a \begin{pmatrix} 1 & & x_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \exp\left\{ 2\pi i \sum_{1}^{n-1} a_i x_{i,i+1} \right\}.$$

These representations lead to the *Whittaker functions* of Jacquet [13] and Bump [3]. The infinite dimensional irreducible unitary representations of N are induced from a character χ_A as in (24) with $m = [(n+1)/2]$. These latter representations lead to our *k-Bessel functions* (and they also contribute to the Plancherel formula for N).

The following result connects our K-Bessel functions with those in [28] for $n = 3$.

PROPOSITION 2. *Inductive formula for K-Bessel functions. For $s \in \mathbb{C}^n$, there are parameters $r_1 \in \mathbb{C}^m$, $r_2 \in \mathbb{C}^{n-m}$ such that*

$$p_s \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = p_{r_1}(A)|A|^{(m-n)/2} p_{r_2}(B)|B|^{m/2},$$

for every $A \in \mathcal{P}_m$, $B \in \mathcal{P}_{n-m}$. Then

$$K_n\left(s \,\middle|\, \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{bmatrix} I_m & 0 \\ {}^tQ & I_{n-m} \end{bmatrix}, I\right)$$

$$= \int_{X \in \mathbb{R}^{m \times (n-m)}} K_m(r_1 | A + B[{}^tX + {}^tQ], I) K_{n-m}(r_2 | B, I + I[X])\, dX.$$

PROOF. Write

$$Y = \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \begin{bmatrix} I_m & X \\ 0 & I_{n-m} \end{bmatrix}, \; d\mu_n(Y) = |V|^{(m-n)/2}\, d\mu_m(V) |W|^{-m/2} d\mu_{n-m}(W) dX$$

In formula (20). Note that

$$\operatorname{Tr}\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{bmatrix} I_m & 0 \\ {}^tQ & I_{n-m} \end{bmatrix} Y + Y^{-1} \right)$$

$$= \operatorname{Tr}(AV + BV[X + Q] + BW + V^{-1} + W^{-1}[{}^tX] + W^{-1}). \quad \square$$

Now we can consider the Fourier expansions of Eisenstein series $E(\varphi, s|Y)$ defined in (8). *It will be convenient to assume that $n \geq 2m$.* Our method is analogous to that of Chowla and Selberg [5] for the case $n = 2$, $m = 1$. We consider only Fourier expansions with respect to the parabolic subgroup involved in the definition of the Eisenstein series. And we shall restrict ourselves to consideration of the terms of maximal rank.

We begin with formula (19). From the partial Iwasawa decomposition (10)

$$Y = \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \begin{bmatrix} I_m & Q \\ 0 & I_{n-m} \end{bmatrix},$$

we find that the partial theta function of maximal rank given by (17) is:

$$\theta_m(Y,X) = \sum_{\substack{\binom{B}{C} \in \mathbb{Z}^{n\times m} rk\ m \\ B \in \mathbb{Z}^{m\times m}}} \exp\left\{-\pi \operatorname{Tr}\left(Y\begin{bmatrix} B \\ C \end{bmatrix} X\right)\right\}$$

$$= \sum_{\substack{\binom{B}{C} \in \mathbb{Z}^{n\times m} rk\ m \\ B \in \mathbb{Z}^{m\times m}}} \exp\{-\pi \operatorname{Tr}(V[B + QC]X + W[C]X)\}.$$

The terms of maximal rank in the Fourier expansion correspond to the matrices C of rank m in $\mathbb{Z}^{(n-m)\times m}$. We shall consider the other terms elsewhere. It follows that, for such terms, B is summed over the full lattice $\mathbb{Z}^{m\times m}$. The classical Poisson summation formula can then be applied to the sum over B and leads to the following formula for the terms with C of rank m in $\theta_m(Y,X)$:

$$|X|^{-m/2}|V|^{-m/2} \sum_{B\in\mathbb{Z}^{m\times m}} \exp\{2\pi i \operatorname{Tr}({}^tBQC) - \pi \operatorname{Tr}(V^{-1}[B]X^{-1})\}.$$

Substitution of this result in (19) shows that the terms of rank m come from $B \in \mathbb{Z}^{mxm}$, $C \in \mathbb{Z}^{(n-m)xm}$ with B and C both of rank m. These terms are:

$$(26) \qquad |V|^{-m/2} \sum_{\substack{B\in\mathbb{Z}^{m\times m} rk\ m/\Gamma_m \\ C\in\mathbb{Z}^{(m-m)\times m} rk\ m}} \exp\{2\pi i \operatorname{Tr}(C^tBQ)\} I(\varphi, s - m/2 | \pi W[C], \pi V^{-1}[B]),$$

where

$$(27) \qquad I(\varphi, s|A,B) = \int_{X\in\mathcal{P}_m} |X|^s \varphi(X^{0-1}) \exp\{-\operatorname{Tr}(AX + BX^{-1})\} d\mu_m(X),$$

for $A, B \in \mathcal{P}_m$, $s \in \mathbb{C}$.

Thus it is useful to relate $I(\varphi, s|A,B)$ with $K_m(r|A,B)$. We do this in a special case. Of course, it is clear that $I(1,s|A,B) = K(1,s|A,B)$.

PROPOSITION 3. *Suppose $A, B \in \mathcal{P}_m$, $r \in \mathbb{C}^m$ with* Re $r_i > 1$. *If $E_{(m)}(r|Y)$ is the Eisenstein series (9), we have the following expression for the integral (27) in terms of K-Bessel functions:*

$$I(E_{(m)}(r|*)) = \int_{\mathcal{P}_m} E_{(m)}(r|Y) \exp\{-\operatorname{Tr}(AY + BY^{-1})\} d\mu_m(Y)$$

$$= 2 \sum_{u\in P_{(m)}\backslash\Gamma_m} K_m(-r|A[{}^tu], B[u^{-1}]).$$

Here $P_{(m)}$ stands for the minimal parabolic subgroup of Γ_m, as in (9).

PROOF. Suppose that the nilpotent group $N \subset \mathrm{GL}(m, \mathbb{R})$ is as defined in (25). It follows that

$$2\int_{\mathcal{P}_m} E_{(m)}(r|Y) \exp\{-\mathrm{Tr}(AY + BY^{-1})\} d\mu_m(Y)$$

$$= \int_{\mathcal{P}_m/\Gamma_m} \sum_{\tau \in \Gamma_m/\Gamma_m \cap N} p_{-r}(Y[\tau]) \sum_{\gamma \in \Gamma_m} \exp\{-\mathrm{Tr}(AY[\gamma] + B(Y[\gamma])^{-1})\} d\mu_m(Y)$$

$$= \int_{\mathcal{P}_m/N} p_{-r}(Y) \int_{N/N\cap\Gamma_m} \sum_{\gamma \in \Gamma_m} \exp\{-\mathrm{Tr}(AY[n\gamma] + B(Y[n\gamma])^{-1})\} dn\, d\mu_m(Y).$$

Here the "2" comes from the order of the center of Γ_m. On the other hand,

$$K_m(-r|A, B) = \int_{\mathcal{P}_m/N} p_{-r}(Y) \int_N \exp\{-\mathrm{Tr}(AY[n] + B(Y[n])^{-1})\} dn\, d\mu_m(Y).$$

Thus, it suffices to show the following easily verified equality:

$$\int_{N/N\cap\Gamma_m} \sum_{\gamma \in \Gamma_m} \exp\{-\mathrm{Tr}(AY[n\gamma] + B(Y[n\gamma])^{-1}\} dn$$

$$= \sum_{u \in \Gamma_m \cap N \backslash \Gamma_m} \int_{n \in N} \exp\{-\mathrm{Tr}(A[{}^t u]Y[n] + B[u^{-1}](Y[n])^{-1})\} dn. \quad \square$$

We collect our results in the following theorem.

THEOREM 1. *The nonsingular terms in Fourier expansions of Eisenstein series. Suppose that* $\varphi(Y^{0-1}) = E_{(m)}(r|Y)$ *as in (9) and* $n \geqslant 2m$. *Set* $\Lambda(\varphi, s|Y) = 2\pi^{-ms}\Gamma_m(r(\varphi, x))Z(\varphi, s|Y)$ *which is the left-hand side of formula (19). Then* $\Lambda(\varphi, s|Y)$ *is a periodic function of X in the partial Iwasawa decomposition:*

$$Y = \begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} \begin{bmatrix} I_m & X \\ 0 & I_{n-m} \end{bmatrix}, \quad X \in \mathbb{R}^{m\times(n-m)}, \quad V \in \mathcal{P}_m, \quad W \in \mathcal{P}_{n-m}.$$

The corresponding Fourier series for $\Lambda(\varphi, s|Y)$ *is given by:*

$$\Lambda(\varphi, s|Y) = \sum_{N \in \mathbb{Z}^{m\times(n-m)}} c_N(V, W) \exp\{2\pi i \mathrm{Tr}({}^t NX)\}.$$

The terms with N of maximal rank m have the form:

$$\frac{1}{2}|V|^{-m/2} C_N(V, W) =$$

$$\sum_{\substack{B \in \mathbb{Z}^{m\times m}/\Gamma_m \\ N = B^t C,\ \text{some } C \in \mathbb{Z}^{(n-m)\times m}}} \sum_{a \in P_{(m)}\backslash\Gamma_m} K_m((-r, s - m/2)|\pi W[C^t a], \pi V^{-1}[Ba^{-1}]).$$

Questions:

(1) (a) Are there similar results when $\varphi(Y)$ is a cusp form?

(b) What are the lower rank Fourier coefficients?

We will publish the answers to these questions elsewhere.

(2) Can one make use of Hecke-type operators to move the B around in $I(\varphi, s|\pi W[C], \pi V^{-1}[B])$? These operators would be associated to a matrix $N \in \mathbb{Z}^{m x(n-m)}$ and defined for $f: \mathcal{P}_m \to \mathbb{C}$ by:

$$T_N f(Y) = \sum_{\substack{B \in \mathbb{Z}^{m\times m}/\Gamma_m \\ N = B'C, \text{ some } C \in \mathbb{Z}^{(n-m)\times m}}} f(Y[B])$$

For example, we find that $T_N E_{(m)}(r|Y) = a_N(r) E_{(m)}(r|Y)$, where

$$(28) \qquad a_N(r) = \sum_{D|N} p_{-r}(I[D]), \quad D = \begin{pmatrix} d_1 & & d_{ij} \\ & \ddots & \\ 0 & & d_m \end{pmatrix}, d_{ij} \bmod d_i, d_i > 0$$

where "$D|N$" means there is a $C \in \mathbb{Z}^{(n-m)\times m}$ such that $N = D'C$. This means that each elementary divisor of D divides the corresponding one in N. Thus, if $E_{(m)}(r|W) = \varphi(W^{0-1})|W|^s$, the coefficient c_N in the Fourier series of $\Lambda(\varphi, s|Y)$ from Theorem 1 is:

$$(29) \quad c_N(V, W) =$$
$$2|V|^{-m/2} a_N(r) \sum_{a \in P_{(m)}\backslash\Gamma_m} K_m((-r, s - m/2)|\pi W[{}^tN^ta], \pi V^{-1}[a^{-1}]).$$

Finally, we state *an integral formula*. Normalizing integrals over $\mathcal{S}\mathcal{P}_m$ as in (3), we have for $\varphi \in \mathcal{A}^0(\Gamma_m, \lambda)$, $s \in \mathbb{C}$, $A, B \in \mathcal{P}_m$, and $I(\varphi, s|A, B)$ as in (27):

$$(30) \quad I(\varphi, s|A, B) =$$
$$\frac{2}{m} \int_{W \in \mathcal{S}\mathcal{P}_m} \varphi(W^{-1}) K_{ms}\left(2\sqrt{\operatorname{Tr}(AW)\operatorname{Tr}(BW^{-1})}\right) \left(\frac{\operatorname{Tr}(BW^{-1})}{\operatorname{Tr}(AW)}\right)^{ms/2} dW,$$

where $K_s(y)$ is the classical K-Bessel function (21). There is a similar formula for $K(r|A, B)$.

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Department of Mathematics
C-012, U.C.S.D.
La Jolla, CA 92093