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ON THE STANLEY DEPTH AND SIZE OF MONOMIAL IDEALS

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Abstract. Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over the field $\mathbb{K}$. For every monomial ideal $I \subset S$, we provide a recursive formula to determine a lower bound for the Stanley depth of S/I. We use this formula to prove the inequality $\mathrm{sdepth}(S/I) \geq \mathrm{size}(I)$ for a particular class of monomial ideals.

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1. Introduction. Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over the field $\mathbb{K}$. Let M be a non-zero finitely generated $\mathbb{Z}^n$-graded S-module. Let $u \in M$ be a homogeneous element and $Z \subseteq \{x_1, \ldots, x_n\}$. The $\mathbb{K}$-subspace $u\mathbb{K}[Z]$ generated by all elements uv with $v \in \mathbb{K}[Z]$ is called a *Stanley space* of dimension $|Z|$, if it is a free $\mathbb{K}[\mathbb{Z}]$-module. Here, as usual, $|Z|$ denotes the number of elements of Z. A decomposition $\mathcal{D}$ of M as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of M. The minimum dimension of a Stanley space in $\mathcal{D}$ is called the *Stanley depth* of $\mathcal{D}$ and is denoted by $\mathrm{sdepth}(\mathcal{D})$. The quantity

$$\mathrm{sdepth}(M) := \max\left\{\mathrm{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\right\}$$

is called the *Stanley depth* of M. Stanley **[11]** conjectured that

$$\mathrm{depth}(M) \leq \mathrm{sdepth}(M)$$

for all $\mathbb{Z}^n$-graded S-modules M. This conjecture has been recently disproved in **[1]**. However, the study of the properties of the Stanley depth of $\mathbb{Z}^n$-graded modules is still interesting. For a reader friendly introduction to Stanley decomposition, we refer to **[9]** and for a nice survey on this topic we refer to **[2]**.

Let I be a monomial ideal of S. In **[6]**, Lyubeznik associated a numerical invariant to I which is called size and is defined as follows.

Definition 1.1. Assume that I is a monomial ideal of S. Let $I = \bigcap_{j=1}^{s} Q_j$ be an irredundant primary decomposition of I, where Q_j $(1 \leq j \leq s)$ is a monomial primary ideal of S. Let h be the height of $\sum_{j=1}^{s} Q_j$, and denote by v the minimum number t such that there exist $1 \leq j_1, \dots, j_t \leq s$ with

$$\sqrt{\sum_{i=1}^{t} Q_{j_i}} = \sqrt{\sum_{j=1}^{s} Q_j}.$$

Then, the *size* of I is defined to be $v + n - h - 1$.

Lyubeznik [**6**] proved that for every monomial ideal I, the inequality $\mathrm{depth}(I) \geq \mathrm{size}(I) + 1$ holds true. It is natural to ask whether the inequalities $\mathrm{sdepth}(I) \geq \mathrm{size}(I) + 1$ and $\mathrm{sdepth}(S/I) \geq \mathrm{size}(I)$ hold, for a monomial ideal I. The first inequality was proved by Herzog, Popescu and Vladoiu for square-free monomial ideals in [**4**]. In fact, the method, which is used in [**4**], is the generalization of a method, started by A. Popescu [**7**] and continued by D. Popescu [**8**]. Recently, Tang [**12**] proved the second inequality for square-free monomial ideals. The aim of this paper is to extend Tang's method to prove the inequality $\mathrm{sdepth}(S/I) \geq \mathrm{size}(I)$ for a particular class of monomial ideals containing square-free monomial ideals.

By [**3**, Corollary 1.3.2], a monomial ideal is irreducible if and only if it is generated by pure powers of the variables. Also, by [**3**, Theorem 1.3.1], every monomial ideal of S can be written as the intersection of irreducible monomial ideals and every irredundant presentation in this form is unique. Assume that $I = Q_1 \cap \ldots \cap Q_s$ is the irredundant presentation of I as the intersection of irreducible monomial ideals. Using this presentation, we provide a recursive formula for computing a lower bound for the Stanley depth of S/I (see Theorem 2.7). Assume moreover that for every $1 \leq i \leq s$ and every proper non-empty subset $\tau \subset [s]$ with

$$\sqrt{Q_i} \subseteq \sum_{j \in \tau} \sqrt{Q_j},$$

we have

$$Q_i \subseteq \sum_{j \in \tau} Q_j.$$

Then, we prove that $\mathrm{sdepth}(S/I) \geq \mathrm{size}(I)$ (see Theorem 2.8).

Before beginning the proof, we mention that although, the behaviour of Stanley depth with polarization is known [**5**], the following example shows that one can not use the polarization and Tang's result to deduce Theorem 2.8.

EXAMPLE 1.2. Let $I = (x_1^2, x_2x_3)$ be a monomial ideal of $S = \mathbb{K}[x_1, x_2, x_3]$. Then, I satisfies the assumptions of Theorem 2.8 and one can easily check that $\mathrm{size}(I) = 1$. Thus, Theorem 2.8 implies that $\mathrm{sdepth}(S/I) \geq 1$. On the other hand, by applying polarization on I, we obtain the ideal $I^p = (x_1x_4, x_2x_3)$ as a monomial ideal in the polynomial ring $T = \mathbb{K}[x_1, x_2, x_3, x_4]$. One can check that $\mathrm{size}(I^p) = 1$. Now, [**5**, Corollary 4.4] and [**12**, Theorem 3.2] imply that $\mathrm{sdepth}(S/I) = \mathrm{sdepth}(T/I^p) - 1 \geq 1 - 1 = 0$. Note that this inequality is weaker than one obtained by Theorem 2.8.

2. Stanley depth and size. In this section, we prove the main results of this paper. Using the irredundant primary decomposition of a monomial ideal I, we first provide a decomposition for S/I in Corollary 2.5. Then, we use this decomposition to obtain a lower bound for the Stanley depth of S/I (see Theorem 2.7). This lower bound and an inductive argument help us to prove the inequality $\mathrm{sdepth}(S/I) \geq \mathrm{size}(I)$ for a particular class of monomial ideals (see Theorem 2.8).

REMARK 2.1. We emphasize that every decomposition in this paper is valid only in the category of $\mathbb{K}$-vector spaces and not in the category of S-modules.

To obtain a decomposition for S/I, we first need to have decompositions for S and I. The following proposition, provides the required decomposition for S. Before beginning the proof, we remind that for every subset S' of S, the set of monomials belonging to S' is denoted by $\mathrm{Mon}(S')$. Also, for every monomial $u \in S$, the *support* of u, denoted by $\mathrm{Supp}(u)$ is the set of variables which divide u.

PROPOSITION 2.2. *Let $S' = \mathbb{K}[x_1, \ldots, x_r]$, $S'' = \mathbb{K}[x_{r+1}, \ldots, x_n]$, $S = \mathbb{K}[x_1, \ldots, x_n]$ and I be a monomial ideal of S. Assume that*

$$I = Q_1 \cap \ldots \cap Q_s, \ s \geq 2 \tag{†}$$

is the unique irredundant presentation of I as the intersection of irreducible monomial ideals. Suppose that $Q = \sum_{i=1}^{s} Q_i$. For every proper subset $\tau \subset [s]$, set

$$S_\tau = \mathbb{K}\left[x_i \,\middle|\, 1 \leq i \leq r,\ x_i \notin \sum_{j\in\tau} \sqrt{Q_j}\right]$$

and

$$\mathcal{M}_\tau = \left\{u \,\middle|\, u \in \mathrm{Mon}(S') \setminus \sum_{j\in\tau} Q_j\right\} \bigcap \mathbb{K}\left[x_i \,\middle|\, x_i \in \sum_{j\in\tau} \sqrt{Q_j}\right].$$

Then,

$$(*) \qquad S = \Big(\bigoplus_{u\in \mathrm{Mon}(S'\setminus Q)} uS''\Big) \oplus \left(\bigoplus_{\tau\subset[s]} \bigoplus_{w\in\mathcal{M}_\tau} \left(\Big(\bigcap_{j\in[s]\setminus\tau} Q_j \cap wS_\tau\Big) S_\tau[x_{r+1}, \ldots, x_n]\right)\right).$$

Proof. We first prove that every monomial of S belongs to the right-hand side of $(*)$. Let $\alpha \in S$ be a monomial. Then there exist monomials $u \in S'$ and $v \in S''$ such that $\alpha = uv$. If $u \notin Q$, then since $\alpha \in uS''$, it belongs to the first summand. Thus, assume that $u \in Q$.

Let $\tau = \{i \in [s] \mid u \notin Q_i\}$. Since $u \in Q$, it follows that τ is a proper subset of $[s]$. Now, there exist monomials

$$w \in \mathbb{K}\left[x_i \,\middle|\, 1 \leq i \leq r,\ x_i \in \sum_{j\in\tau} \sqrt{Q_j}\right] \quad \text{and} \quad w' \in S_\tau$$

such that $u = ww'$. Since for every $j \in \tau$, we have $u \notin Q_j$, it follows that $w \notin Q_j$, for every $j \in \tau$. This shows that $w \in \mathcal{M}_\tau$. On the other hand, $u \in \bigcap_{j\in[s]\setminus\tau} Q_j$ and hence $u \in \bigcap_{j\in[s]\setminus\tau} Q_j \cap wS_\tau$. Therefore,

$$\alpha = uv \in \Big(\bigcap_{j\in[s]\setminus\tau} Q_j \cap wS_\tau\Big) S_\tau[x_{r+1}, \ldots, x_n].$$

It turns out that

$$S = \sum_{u \in \mathrm{Mon}(S' \setminus Q)} uS'' + \sum_{\tau \subset [s]} \sum_{w \in \mathcal{M}_\tau} \left(\left(\bigcap_{j \in [s] \setminus \tau} Q_j \cap wS_\tau \right) S_\tau[x_{r+1}, \ldots, x_n] \right).$$

We now show that the sum is direct. We consider the following cases.

CASE 1. For every pair of monomials $u_1, u_2 \in S' \setminus Q$, we have $u_1 S'' \cap u_2 S'' = 0$, since

$$S'' \cap \mathrm{Supp}(u_1) = S'' \cap \mathrm{Supp}(u_2) = \emptyset.$$

CASE 2. We prove that for every subset τ of $[s]$ and every pair of monomials $u \in S' \setminus Q$ and $w \in \mathcal{M}_\tau$, we have

$$uS'' \cap \left(\left(\bigcap_{j \in [s] \setminus \tau} Q_j \cap wS_\tau \right) S_\tau[x_{r+1}, \ldots, x_n] \right) = 0.$$

Indeed, assume by the contrary that there exists a monomial

$$v \in uS'' \cap \left(\left(\bigcap_{j \in [s] \setminus \tau} Q_j \cap wS_\tau \right) S_\tau[x_{r+1}, \ldots, x_n] \right).$$

Let v' be the monomial obtained from v by applying the map $x_i \mapsto 1$, for every $r + 1 \leq i \leq n$. Then, $v' = u$ and on the other hand,

$$v' \in \bigcap_{j \in [s] \setminus \tau} Q_j \cap wS_\tau.$$

Therefore, $u \in \bigcap_{j \in [s] \setminus \tau} Q_j$, which is a contradiction by $u \notin Q$.

CASE 3. We prove that for every subset τ of $[s]$ and every pair of distinct monomials $w_1, w_2 \in \mathcal{M}_\tau$,

$$\left(\left(\bigcap_{j \in [s] \setminus \tau} Q_j \cap w_1 S_\tau \right) S_\tau[x_{r+1}, \ldots, x_n] \right) \cap \left(\left(\bigcap_{j \in [s] \setminus \tau} Q_j \cap w_2 S_\tau \right) S_\tau[x_{r+1}, \ldots, x_n] \right) = 0.$$

Indeed, assume by the contrary that there exists a monomial

$$v \in \left(\left(\bigcap_{j \in [s] \setminus \tau} Q_j \cap w_1 S_\tau \right) S_\tau[x_{r+1}, \ldots, x_n] \right) \cap \left(\left(\bigcap_{j \in [s] \setminus \tau} Q_j \cap w_2 S_\tau \right) S_\tau[x_{r+1}, \ldots, x_n] \right).$$

Let v' be the monomial obtained from v by applying the map $x_i \mapsto 1$, for every i with $x_i \in S_\tau[x_{r+1}, \ldots, x_n]$. Since $v \in w_1 S_\tau[x_{r+1}, \ldots, x_n]$ and

$$w_1 \in \mathbb{K}\left[x_i \;\middle|\; 1 \leq i \leq r,\ x_i \in \sum_{j \in \tau} \sqrt{Q_j} \right],$$

we conclude that $v' = w_1$. Similarly, $v' = w_2$, which implies that $w_1 = w_2$ and this is a contradiction.

CASE 4. We prove that for every pair of proper subsets τ_1, τ_2 of $[s]$ with $\tau_1 \neq \tau_2$ and every pair of monomials $w_1 \in \mathcal{M}_{\tau_1}$ and $w_2 \in \mathcal{M}_{\tau_2}$,

$$\left(\Big(\bigcap_{j\in[s]\setminus\tau_1} Q_j \cap w_1 S_{\tau_1}\Big) S_{\tau_1}[x_{r+1}, \dots, x_n]\right) \cap \left(\Big(\bigcap_{j\in[s]\setminus\tau_2} Q_j \cap w_2 S_{\tau_2}\Big) S_{\tau_2}[x_{r+1}, \dots, x_n]\right) = 0.$$

Indeed, assume by the contrary that there exists a monomial

$$v \in \left(\Big(\bigcap_{j\in[s]\setminus\tau_1} Q_j \cap w_1 S_{\tau_1}\Big) S_{\tau_1}[x_{r+1}, \dots, x_n]\right) \cap \left(\Big(\bigcap_{j\in[s]\setminus\tau_2} Q_j \cap w_2 S_{\tau_2}\Big) S_{\tau_2}[x_{r+1}, \dots, x_n]\right).$$

Since $\tau_1 \neq \tau_2$, without loss of generality, we may assume that $\tau_1 \nsubseteq \tau_2$. Thus, there exists an integer $j_0 \in \tau_1 \setminus \tau_2$. Let v' be the monomial obtained from v by applying the map $x_i \mapsto 1$, for every $r + 1 \leq i \leq n$. Then,

$$v' \in \Big(\bigcap_{j\in[s]\setminus\tau_1} Q_j \cap w_1 S_{\tau_1}\Big) \cap \Big(\bigcap_{j\in[s]\setminus\tau_2} Q_j \cap w_2 S_{\tau_2}\Big),$$

in particular $v' \in Q_{j_0}$. On the other hand, by $v' \in w_1 S_{\tau_1}$, we conclude that there exists a monomial $w_0 \in S_{\tau_1}$, such that $v' = w_0 w_1$. Since $w_1 \in \mathcal{M}_{\tau_1}$, we see that $w_1 \notin Q_{j_0}$. Also, by the definition of S_{τ_1}, we conclude that $w_0 \notin \sqrt{Q_{j_0}}$. Since Q_{j_0} is a primary ideal, $v' = w_0 w_1 \notin Q_{j_0}$, which is a contradiction. This completes the proof of the proposition. $\square$

REMARK 2.3. Notice that in the decomposition of Proposition 2.2, the summand corresponding to $\tau = \emptyset$ is equal to $(I \cap S')S$, because $\mathcal{M}_\emptyset = \{1\}$ and $S_\emptyset = S'$.

In the following proposition, we provide a decomposition for I.

PROPOSITION 2.4. *Under the assumptions as in Proposition 2.2, suppose further that one of the irreducible monomial ideals in the decomposition* (†) *of I is $(x_1^{a_1}, \dots, x_r^{a_r})$, where $a_1, \dots, a_r$ are positive integers. Then, there is a decomposition of I:*

$$\begin{aligned} I = &\Big((I \cap S')S\Big) \oplus \\ &\bigoplus_{\emptyset\neq\tau\subset[s]} \bigoplus_{w\in\mathcal{M}_\tau} \left(\left(\Big(\bigcap_{j\in[s]\setminus\tau} Q_j \cap w S_\tau\Big) S_\tau[x_{r+1}, \dots, x_n]\right)\right. \\ &\left.\cap \left(\Big(\bigcap_{j\in\tau} Q_j \cap w S''\Big) S_\tau[x_{r+1}, \dots, x_n]\right)\right), \end{aligned}$$

where τ runs over all non-empty proper subsets of $[s]$.

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Proof. It is clear that every monomial of the sum

$$\Big((I \cap S')S\Big) + \sum_{\emptyset \neq \tau \subset [s]} \sum_{w \in \mathcal{M}_\tau} \Bigg(\bigg(\Big(\bigcap_{j \in [s]\setminus\tau} Q_j \cap wS_\tau\Big) S_\tau[x_{r+1}, \dots, x_n]\bigg) \cap \bigg(\Big(\bigcap_{j \in \tau} Q_j \cap wS''\Big) S_\tau[x_{r+1}, \dots, x_n]\bigg)\Bigg),$$

belongs to I. Thus, we prove that every monomial of I belongs to the above sum. Assume that $\alpha \in I$ is a monomial. Then there exist monomials $u_1 \in S'$ and $u_2 \in S''$ such that $\alpha = u_1u_2$. Since $I \subseteq (x_1^{a_1}, \dots, x_r^{a_r})$, we conclude that $u_1 \in (x_1^{a_1}, \dots, x_r^{a_r}) \subseteq Q$ and hence

$$\alpha \notin \bigoplus_{u \in \mathrm{Mon}(S' \setminus Q)} uS''.$$

Therefore, Proposition 2.2 shows that there exists a proper subset τ of $[s]$ and a monomial $w \in \mathcal{M}_\tau$ such that

$$\alpha \in \Big(\bigcap_{j \in [s]\setminus\tau} Q_j \cap wS_\tau\Big) S_\tau[x_{r+1}, \dots, x_n].$$

If $\tau = \emptyset$, then Remark 2.3 implies that $\alpha \in (I \cap S')S$.

Thus, assume that $\tau \neq \emptyset$. It is sufficient to prove that

$$\alpha \in (\Big(\bigcap_{j \in \tau} Q_j \cap wS''\Big) S_\tau[x_{r+1}, \dots, x_n]\bigg).$$

Remind that $\alpha = u_1u_2$, where $u_1 \in S'$ and $u_2 \in S''$. It is clear that $u_1 \in wS_\tau$. Therefore, there exists a monomial $u' \in S_\tau$ such that $u_1 = wu'$. Hence, $\alpha = wu'u_2$. It follows from the definition of S_τ that for every $j \in \tau$, we have $u' \notin \sqrt{Q_j}$. Since for every $j \in \tau$, we have $\alpha \in I \subseteq Q_j$ and Q_j is a primary ideal, we conclude that $wu_2 \in \bigcap_{j\in\tau} Q_j$. This shows that $wu_2 \in \bigcap_{j\in\tau} Q_j \cap wS''$. Hence,

$$\alpha = wu'u_2 \in (\Big(\bigcap_{j \in \tau} Q_j \cap wS''\Big) S_\tau[x_{r+1}, \dots, x_n]\bigg),$$

and it implies that

$$I = \Big((I \cap S')S\Big) + \sum_{\emptyset \neq \tau \subset [s]} \sum_{w \in \mathcal{M}_\tau} \Bigg(\bigg(\Big(\bigcap_{j \in [s]\setminus\tau} Q_j \cap wS_\tau\Big) S_\tau[x_{r+1}, \dots, x_n]\bigg) \cap \bigg(\Big(\bigcap_{j \in \tau} Q_j \cap wS''\Big) S_\tau[x_{r+1}, \dots, x_n]\bigg)\Bigg).$$

It now follows from Proposition 2.2 that the sum is in fact a direct sum. □

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The following corollary is an immediate consequence of Propositions 2.2, 2.4 and Remark 2.3. It provides a decomposition for S/I and helps us to determine a lower bound for the Stanley depth of S/I.

Corollary 2.5. *Under the assumptions as in Proposition 2.2, suppose further that one of the irreducible monomial ideals in the decomposition (†) of I is $(x_1^{a_1}, \dots, x_r^{a_r})$, where $a_1, \dots, a_r$ are positive integers. Then, there is a decomposition of S/I:*

$$S/I = \Big(\bigoplus_{u \in \mathrm{Mon}(S' \setminus Q)} uS''\Big) \oplus$$
$$\bigoplus_{\tau \subset [s]} \bigoplus_{w \in \mathcal{M}_\tau} \frac{\Big(\Big(\bigcap_{j \in [s]\setminus\tau} Q_j \cap wS_\tau\Big) S_\tau[x_{r+1}, \dots, x_n]\Big)}{\Big(\Big(\bigcap_{j \in [s]\setminus\tau} Q_j \cap wS_\tau\Big) S_\tau[x_{r+1}, \dots, x_n]\Big) \cap \Big(\Big(\bigcap_{j \in \tau} Q_j \cap wS''\Big) S_\tau[x_{r+1}, \dots, x_n]\Big)},$$

where τ runs over all non-empty proper subsets of $[s]$.

The following lemma is a modification of [**12**, Lemma 2.3]. In fact, for $w = 1$, it implies [**12**, Lemma 2.3]. Using this lemma, we are able to find a lower bound for the Stanley depth of summands appearing in Corollary 2.5.

Lemma 2.6. *Let $S_1 = \mathbb{K}[x_1, \dots, x_n]$ and $S_2 = \mathbb{K}[y_1, \dots, y_m]$ be polynomial rings with disjoint set of variables and assume that $S_3 = \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_m]$. Assume also that $S = \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_t]$ is a polynomial ring containing S_3. Suppose that $I, J \subset S$ are monomial ideals and $w \in S \setminus J$ is a monomial. Set $I_1 = I \cap wS_1$ and $J_1 = J \cap wS_2$. Then,*

$$\mathrm{sdepth}_{S_3}\Big(\frac{I_1S_3}{I_1S_3 \cap J_1S_3}\Big) \geq \mathrm{sdepth}_{S_1}\Big((I : w) \cap S_1\Big) + \mathrm{sdepth}_{S_2}\Big(\frac{S_2}{(J : w) \cap S_2}\Big).$$

Proof. We note that every monomial in I_1S_3 is divisible by w. Thus, the S_3-modules $I_1S_3/(I_1S_3 \cap J_1S_3)$ and $(I_1S_3 : w)/((I_1S_3 : w) \cap (J_1S_3 : w))$ are isomorphic. Hence,

$$\mathrm{sdepth}_{S_3}\Big(\frac{I_1S_3}{I_1S_3 \cap J_1S_3}\Big) = \mathrm{sdepth}_{S_3}\Big(\frac{(I_1S_3 : w)}{(I_1S_3 : w) \cap (J_1S_3 : w)}\Big).$$

Moreover, by the definition of I_1 and J_1, we have $(I_1S_3 : w) = ((I_1S_3 : w) \cap S_1)S_3$ and $(J_1S_3 : w) = ((J_1S_3 : w) \cap S_2)S_3$. Therefore, it follows from [**12**, Lemma 2.3] and the above equality that

$$\mathrm{sdepth}_{S_3}\Big(\frac{I_1S_3}{I_1S_3 \cap J_1S_3}\Big) \geq \mathrm{sdepth}_{S_1}\Big((I_1S_3 : w) \cap S_1\Big) + \mathrm{sdepth}_{S_2}\Big(\frac{S_2}{(J_1S_3 : w) \cap S_2}\Big).$$

Since $(I_1S_3 : w) \cap S_1 = (I : w) \cap S_1$ and $(J_1S_3 : w) \cap S_2 = (J : w) \cap S_2$, the assertion follows. □

In the following theorem, we determine a lower bound for the Stanley depth of S/I. It is a generalization of [**12**, Theorem 2.4].

THEOREM 2.7. *Under the assumptions as in Corollary 2.5, there is an inequality*

$$\mathrm{sdepth}_S(S/I) \geq \min\Bigg\{n-r, \mathrm{sdepth}_{S_\tau}\Big(\bigcap_{j\in[s]\setminus\tau}(Q_j : w)\cap S_\tau\Big) + \mathrm{sdepth}_{S''}\Bigg(S''\Big/\Big(\bigcap_{j\in\tau}Q_j\cap S''\Big)\Bigg)\Bigg\},$$

where the minimum is taking over all non-empty proper subset $\tau \subset [s]$ and all $w \in \mathcal{M}_\tau$ such that $(\bigcap_{j\in[s]\setminus\tau}Q_j \cap wS_\tau) \neq 0$.

Proof. Note that for every non-empty proper subset $\tau \subset [s]$ and every $w \in \mathcal{M}_\tau$, we have $w \notin Q_j$, for all $j \in \tau$. Also, $\mathrm{Supp}(w) \cap S'' = \emptyset$. This shows that for every $j \in \tau$, we have $(Q_j : w) \cap S'' = Q_j \cap S''$. Now, the assertion follows from Corollary 2.5 and Lemma 2.6. To apply Lemma 2.6, for every summand appearing in Corollary 2.5, set $I = \bigcap_{j\in[s]\setminus\tau}Q_j$, $J = \bigcap_{j\in\tau}Q_j$, $S_1 = S_\tau$, $S_2 = S''$ and $S_3 = S_\tau[x_{r+1}, \ldots, x_n] \subseteq S$. □

We are now ready to prove the main result of this paper. In the proof of the following theorem, we use the first statement of [**4**, Lemma 3.2]. Notice that a counterexample by H. Shen shows that the second statement of this Lemma is not true for non-square-free monomial ideals.

THEOREM 2.8. *Let I be a monomial ideal of S. Assume that*

$$I = Q_1 \cap \ldots \cap Q_s$$

is the unique irredundant presentation of I as the intersection of irreducible monomial ideals. Suppose that for every $1 \leq i \leq s$ and every proper non-empty subset $\tau \subset [s]$ with

$$\sqrt{Q_i} \subseteq \sum_{j\in\tau}\sqrt{Q_j},$$

we have

$$Q_i \subseteq \sum_{j\in\tau}Q_j.$$

Then, $\mathrm{sdepth}(S/I) \geq \mathrm{size}_S(I)$.

Proof. We prove the assertion by induction on s. Without loss of generality, assume that $Q_1 = (x_1^{a_1}, \ldots, x_r^{a_r})$, for some integer r with $1 \leq r \leq n$. If $s = 1$, then $I = Q_1$ and it is clear that $\mathrm{size}_S(I) = n - r$. On the other hand, it follows from [**10**, Theorem 1.1] that $\mathrm{sdepth}(S/I) = n - r$. Thus, there is nothing to prove in this case. Hence, assume that $s \geq 2$.

Set $S' = \mathbb{K}[x_1, \ldots, x_r]$ and $S'' = \mathbb{K}[x_{r+1}, \ldots, x_n]$. It is obvious from the definition of size that $\mathrm{size}_S(I) \leq n - r$. Therefore, using Theorem 2.7, it is enough to prove that for every non-empty proper subset $\tau \subset [s]$ and every $w \in \mathcal{M}_\tau$ with $(\bigcap_{j\in[s]\setminus\tau}Q_j \cap wS_\tau) \neq 0$, we have

$$\mathrm{sdepth}_{S_\tau}\Big(\bigcap_{j\in[s]\setminus\tau}(Q_j : w)\cap S_\tau\Big) + \mathrm{sdepth}_{S''}\Bigg(S''\Big/\Big(\bigcap_{j\in\tau}Q_j\cap S''\Big)\Bigg) \geq \mathrm{size}_S(I).$$

Hence, we fix a non-empty proper subset $\tau \subset [s]$ and a monomial $w \in \mathcal{M}_\tau$ such that $(\bigcap_{j\in[s]\setminus\tau} Q_j \cap wS_\tau) \neq 0$. If $\bigcap_{j\in\tau} Q_j \cap S'' = 0$, then

$$\begin{aligned}&\operatorname{sdepth}_{S_\tau}\Big(\bigcap_{j\in[s]\setminus\tau}(Q_j : w)\cap S_\tau\Big) + \operatorname{sdepth}_{S''}\left(S''\Big/\Big(\bigcap_{j\in\tau} Q_j \cap S''\Big)\right)\\ &\geq n - r \geq \operatorname{size}_S(I).\end{aligned}$$

Thus, assume that $\bigcap_{j\in\tau} Q_j \cap S'' \neq 0$. In particular, $1 \notin \tau$. If $S_\tau = \mathbb{K}$, then it follows from the definition of S_τ that

$$\sqrt{Q_1} \subseteq \sum_{j\in\tau} \sqrt{Q_j}.$$

Hence, by assumption

$$Q_1 \subseteq \sum_{j\in\tau} Q_j.$$

Since $S_\tau = \mathbb{K}$, it follows from $(\bigcap_{j\in[s]\setminus\tau} Q_j \cap wS_\tau) \neq 0$ and the above inclusion that

$$w \in \cap_{j\in[s]\setminus\tau} Q_j \subseteq Q_1 \subseteq \sum_{j\in\tau} Q_j,$$

which is a contradiction by the definition of $\mathcal{M}_\tau$. Therefore, assume that $S_\tau \neq \mathbb{K}$. In other words, S_τ is a polynomial ring of positive dimension.

Since $(\bigcap_{j\in[s]\setminus\tau} Q_j \cap wS_\tau) \neq 0$, we conclude that $\bigcap_{j\in[s]\setminus\tau}(Q_j : w) \cap S_\tau$ is a non-zero ideal of S_τ. It follows from [**2**, Corollary 2.4] that

$$\operatorname{sdepth}_{S_\tau}\Big(\bigcap_{j\in[s]\setminus\tau}(Q_j : w)\Big) \geq 1.$$

Also, for every $i \in \tau$ and every proper subset $\tau' \subset \tau$, with

$$\sqrt{Q_i \cap S''} \subseteq \sum_{j\in\tau'} \sqrt{Q_j \cap S''},$$

we have

$$\sqrt{Q_i} \subseteq \sum_{j\in\tau'\cup\{1\}} \sqrt{Q_j}$$

and the assumption implies that

$$Q_i \subseteq \sum_{i\in\tau'\cup\{1\}} Q_j.$$

Thus,

$$Q_i \cap S'' \subseteq \sum_{i\in\tau'} Q_j \cap S''.$$

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Thus, the induction hypothesis together with the first statement of [**4**, Lemma 3.2] implies that

$$\begin{aligned}&\mathrm{sdepth}_{S_\tau}\Big(\bigcap_{j\in[s]\setminus\tau}(Q_j : w)\cap S_\tau\Big) + \mathrm{sdepth}_{S''}\left(S''\Big/\Big(\bigcap_{j\in\tau}Q_j\cap S''\Big)\right)\\ &\geq 1 + \mathrm{size}_{S''}\Big(\bigcap_{j\in\tau}Q_j\cap S''\Big) \geq \mathrm{size}_S(I).\end{aligned}$$

□

REMARK 2.9.

(1) Every square-free monomial ideal satisfies the assumption of Theorem 2.8. Indeed, assume that I is a square-free monomial ideal and $I = Q_1 \cap \ldots \cap Q_s$ is the irredundant presentation of I as the intersection of irreducible monomial ideals. Then, for every integer i with $1 \leq i \leq s$, the ideal Q_i is a prime ideal which is generated by a subset of variables. Thus, $Q_i = \sqrt{Q_i}$, for every $1 \leq i \leq s$. This shows that I satisfies the assumption of Theorem 2.8. Therefore, Theorem 2.8 is an extension of Tang's result [**12**, Theorem 3.2].

(2) Note that every monomial ideal satisfying the assumption of Theorem 2.8 has no embedded associated prime. Indeed, assume that $\sqrt{Q_i} \subseteq \sqrt{Q_j}$ for $i \neq j$. Then, the assumption of Theorem 2.8 implies that $Q_i \subseteq Q_j$, which is contradiction, because the intersection $Q_1 \cap \ldots \cap Q_s$ is irredundant.

REMARK 2.10. We have no example of a monomial ideal I such that $\mathrm{sdepth}(S/I) < \mathrm{size}_S(I)$. Thus, it may be true that for every monomial ideal I, the inequality $\mathrm{sdepth}(S/I) \geq \mathrm{size}_S(I)$ holds. However, the method we used in this paper does not look applicable for the general case.

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