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THE PARTIAL-ISOMETRIC CROSSED PRODUCTS BY SEMIGROUPS OF ENDOMORPHISMS AS FULL CORNERS

SRIWULAN ADJI ✉ and SAEID ZAHMATKESH

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Abstract

Suppose that Γ^+ is the positive cone of a totally ordered abelian group Γ, and (A, Γ^+, α) is a system consisting of a C^*-algebra A, an action α of Γ^+ by extendible endomorphisms of A. We prove that the partial-isometric crossed product $A \times_\alpha^{\text{piso}} \Gamma^+$ is a full corner in the subalgebra of $\mathcal{L}(\ell^2(\Gamma^+, A))$, and that if α is an action by automorphisms of A, then it is the isometric crossed product $(B_{\Gamma^+} \otimes A) \times^{\text{iso}} \Gamma^+$, which is therefore a full corner in the usual crossed product of system by a group of automorphisms. We use these realizations to identify the ideal of $A \times_\alpha^{\text{piso}} \Gamma^+$ such that the quotient is the isometric crossed product $A \times_\alpha^{\text{iso}} \Gamma^+$.

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1. Introduction

Let Γ be a totally ordered abelian group, and $\Gamma^+ := \{x \in \Gamma : x \geq 0\}$ the positive cone of Γ. A dynamical system (A, Γ^+, α) is a system consisting of a C^*-algebra A, an action $\alpha : \Gamma^+ \to \text{EndA}$ of Γ^+ by endomorphisms α_x of A such that $\alpha_0 = \text{id}_A$. Since we do not require the algebra A to have an identity element, we need to assume that every endomorphism α_x extends to a strictly continuous endomorphism $\overline{\alpha}_x$ of the multiplier algebra $M(A)$ as it is used in [1, 9], and note that extendibility of α_x may imply $\alpha_x(1_{M(A)}) \neq 1_{M(A)}$.

A partial-isometric covariant representation, the analogue of isometric covariant representation, of the system (A, Γ^+, α) is defined in [10] where the endomorphisms α_s are represented by partial isometries instead of isometries. The partial-isometric crossed product $A \times_\alpha^{\text{piso}} \Gamma^+$ is defined there as the Toeplitz algebra studied in [6] associated to a product system of Hilbert bimodules arising from the underlying

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dynamical system (A, Γ^+, α). This algebra is universal for covariant partial-isometric representations of the system.

The success of the theory of isometric crossed products [2–4, 11–13] has led the authors of [10] to study the structure of the partial-isometric crossed product of the distinguished system $(B_{\Gamma^+}, \Gamma^+, \tau)$, where τ_x acts on the subalgebra B_{Γ^+} of $\ell^\infty(\Gamma^+)$ as the right translation. However, the analogous view of isometric crossed products as full corners in crossed products by groups [1, 8, 16] for partial-isometric crossed products remains unavailable. This is the main task undertaken in the present work.

We construct a covariant partial-isometric representation of (A, Γ^+, α) in the C^*-algebra $\mathcal{L}(\ell^2(\Gamma^+, A))$ of adjointable operators on the Hilbert A-module $\ell^2(\Gamma^+, A)$, and we show that the corresponding representation of the crossed product is an isomorphism of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ onto a full corner in the subalgebra of $\mathcal{L}(\ell^2(\Gamma^+, A))$. We use the idea from [7] for the construction: the embedding π_α of A into $\mathcal{L}(\ell^2(\Gamma^+, A))$, together with the isometric representation $S : \Gamma^+ \to \mathcal{L}(\ell^2(\Gamma^+, A))$, satisfies the equation $\pi_\alpha(a)S_x = S_x\pi(\alpha_x(a))$ for all $a \in A$ and $x \in \Gamma^+$, and then the algebra $\mathcal{T}_{(A,\Gamma^+,\alpha)}$ generated by $\pi(A)$ and $S(\Gamma^+)$ contains $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ as a full corner. However, since the results in [7] are developed to compute and to show that KK-groups of $\mathcal{T}_{(A,\Gamma^+,\alpha)}$ and A are equivalent, the theory is set for unital C^*-algebras and unital endomorphisms: if the algebra is not unital, they use the smallest unitization algebra $\tilde{A}$ and then the extension of endomorphism on $\tilde{A}$ is unital.

Here we use the (largest unitization) multiplier algebra $M(A)$ of A, and every endomorphism is extendible to $M(A)$. So we generalize the arguments in [7] to the context of multiplier algebra. When endomorphisms in a given system are unital, then we are in the context of [7], so that the C^*-algebra $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ enjoys all properties of the algebra $\mathcal{T}_{(A,\Gamma^+,\alpha)}$ described in [7]. Moreover, if the action is automorphic action then we show that $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ is a full corner in the crossed product by group action.

Using the corner realization of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$, we identify the kernel of the natural surjective homomorphism $i_A \times i_{\Gamma^+} : A \times_\alpha^{\mathrm{piso}} \Gamma^+ \to A \times_\alpha^{\mathrm{iso}} \Gamma^+$ induced by the canonical isometric covariant pair (i_A, i_{Γ^+}) of (A, Γ^+, α), to get the exact sequence of [7] and the Pimsner–Voiculescu exact sequence in [14].

We begin the paper with a preliminary section containing background material about partial-isometric and isometric crossed products, and then identify the spanning elements of the kernel of the natural homomorphism from the partial-isometric crossed product onto the isometric crossed product of a system (A, Γ^+, α). In Section 3 we construct a covariant partial-isometric representation of (A, Γ^+, α) in $\mathcal{L}(\ell^2(\Gamma^+, A))$ for which it gives an isomorphism of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ onto a full corner of the subalgebra of $\mathcal{L}(\ell^2(\Gamma^+, A))$. In Section 4 we show that when the semigroup Γ^+ is $\mathbb{N}$ the kernel of that natural homomorphism is a full corner in the algebra of compact operators on $\ell^2(\mathbb{N}, A)$. We discuss in Section 5 the theory of partial-isometric crossed products for systems by automorphic actions of the semigroups Γ^+. We show that $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ is a full corner in the classical crossed product $(B_\Gamma \otimes A) \times \Gamma$ of a dynamical system by a group of automorphisms.

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2. Preliminaries

A *partial isometry* V on a Hilbert space H is an operator which satisfies $\|Vh\| = \|h\|$ for all $h \in (\ker V)^{\perp}$. A bounded operator V is a partial isometry if and only if $VV^*V = V$, and then the adjoint V^* is a partial isometry too. Furthermore, the two operators V^*V and VV^* are the orthogonal projections on the initial space $(\ker V)^{\perp}$ and the range VH, respectively. So an element v of a C^*-algebra A is called a partial isometry if $vv^*v = v$.

A *partial-isometric representation* of Γ^+ on a Hilbert space H is a map $V : \Gamma^+ \to B(H)$ such that $V_s := V(s)$ is a partial isometry and $V_sV_t = V_{s+t}$ for every $s, t \in \Gamma^+$. The product ST of two partial isometries S and T is not always a partial isometry, unless S^*S commutes with TT^* [10, Proposition 2.1]. A partial isometry S is called a power partial isometry if S^n is a partial isometry for every $n \in \mathbb{N}$. So a partial isometric representation of $\mathbb{N}$ is determined by a single power partial isometry V_1 because $V_n = V_1^n$. [10, Proposition 3.2] says that if V is a partial-isometric representation of Γ^+, then every V_s is a power partial isometry, and $V_sV_s^*$ commutes with $V_tV_t^*$, $V_s^*V_s$ commutes with $V_t^*V_t$.

A *covariant partial-isometric representation* of (A, Γ^+, α) on a Hilbert space H is a pair (π, V) consisting of a nondegenerate representation $\pi : A \to B(H)$ and a partial-isometric representation $V : \Gamma^+ \to B(H)$ which satisfies

$$\pi(\alpha_s(a)) = V_s\pi(a)V_s^* \quad \text{and} \quad V_s^*V_s\pi(a) = \pi(a)V_s^*V_s \quad \text{for } s \in \Gamma^+, a \in A.$$

Every covariant representation (π, V) of (A, Γ^+, α) extends to a covariant representation $(\overline{\pi}, V)$ of $(M(A), \Gamma^+, \overline{\alpha})$. [10, Lemma 4.3] shows that (π, V) is a covariant representation of (A, Γ^+, α) if and only if

$$\pi(\alpha_s(a))V_s = V_s\pi(a) \quad \text{and} \quad V_sV_s^* = \overline{\pi}(\overline{\alpha}_s(1)) \quad \text{for } s \in \Gamma^+, a \in A.$$

Every system (A, Γ^+, α) admits a nontrivial covariant partial-isometric representation [10, Example 4.6].

Definition 2.1. A partial-isometric crossed product of (A, Γ^+, α) is a triple (B, i_A, i_{Γ^+}) consisting of a C^*-algebra B, a nondegenerate homomorphism $i_A : A \to B$, and a partial-isometric representation $i_{\Gamma^+} : \Gamma^+ \to M(B)$ such that:

(i) the pair (i_A, i_{Γ^+}) is a covariant representation of (A, Γ^+, α) in B;
(ii) for every covariant partial-isometric representation (π, V) of (A, Γ^+, α) on a Hilbert space H there is a nondegenerate representation $\pi \times V$ of B on H which satisfies $(\pi \times V) \circ i_A = \pi$ and $\overline{(\pi \times V)} \circ i_{\Gamma^+} = V$; and
(iii) the C^*-algebra B is spanned by $\{i_{\Gamma^+}(s)^* i_A(a) i_{\Gamma^+}(t) : a \in A, s, t \in \Gamma^+\}$.

Remark 2.2. Proposition 4.7 of [10] shows that such (B, i_A, i_{Γ^+}) always exists, and it is unique up to isomorphism: if (C, j_A, j_{Γ^+}) is a triple that satisfies properties (i)–(iii) then there is an isomorphism of B onto C which carries (i_A, i_{Γ^+}) into (j_A, j_{Γ^+}).

We use the standard notation $A \times_\alpha \Gamma^+$ for the crossed product of (A, Γ^+, α), and we write $A \times_\alpha^{\text{piso}} \Gamma^+$ if we want to distinguish it from the other kind of crossed product.

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[10, Theorem 4.8] asserts that a covariant representation (π, V) of (A, Γ^+, α) on H induces a faithful representation $\pi \times V$ of $A \times_\alpha \Gamma^+$ if and only if π is faithful on $(V_s^* H)^\perp$ for all $s > 0$, and this condition is equivalent to saying that π is faithful on the range of $(1 - V_s^* V_s)$ for all $s > 0$.

2.1. Isometric crossed products. The above definition of partial-isometric crossed products is analogous to that for isometric crossed products: the endomorphisms α_s are implemented by partial isometries instead of isometries.

We recall that an *isometric representation* V of Γ^+ on a Hilbert space H is a homomorphism $V : \Gamma^+ \to B(H)$ such that each V_s is an isometry and $V_{s+t} = V_s V_t$ for all $s, t \in \Gamma^+$. A pair (π, V), consisting of a nondegenerate representation π of A and an isometric representation V of Γ^+ on H, is a *covariant isometric representation* of (A, Γ^+, α) if $\pi(\alpha_s(a)) = V_s \pi(a) V_s^*$ for all $a \in A$ and $s \in \Gamma^+$. The *isometric crossed product* $A \times_\alpha^{\mathrm{iso}} \Gamma^+$ is generated by a universal isometric covariant representation (i_A, i_{Γ^+}), such that there is a bijection $(\pi, V) \mapsto \pi \times V$ between covariant isometric representations of (A, Γ^+, α) and nondegenerate representations of $A \times_\alpha^{\mathrm{iso}} \Gamma^+$. We note that some systems (A, Γ^+, α) may not have a nontrivial covariant isometric representation, in which case their isometric crossed products give no information about the systems.

When $\alpha : \Gamma^+ \to \mathrm{End(A)}$ is an action of Γ^+ such that every α_x is an automorphism of A, then every isometry V_s in a covariant isometric representation (π, V) is a unitary. Thus $A \times_\alpha^{\mathrm{iso}} \Gamma^+$ is isomorphic to the classical group crossed product $A \times_\alpha \Gamma$. For more general situations, [1, 8] show that we get, by dilating the system (A, Γ^+, α), a C^*-algebra B and an action β of the group Γ by automorphisms of B such that $A \times_\alpha^{\mathrm{iso}} \Gamma^+$ is isomorphic to the full corner $p(B \times_\alpha \Gamma)p$ where p is the unit $1_{M(A)}$ in B.

If (A, Γ^+, α) is the distinguished system $(B_{\Gamma^+}, \Gamma^+, \tau)$ of the unital C^*-algebra $B_{\Gamma^+} := \overline{\mathrm{span}}\{1_s \in \ell^\infty(\Gamma^+) : s \in \Gamma^+\}$ spanned by the characteristic function

$$1_s(x) = \begin{cases} 1 & \text{if } x \geq s, \\ 0 & \text{if } x < s, \end{cases}$$

and the action $\tau : \Gamma^+ \to \mathrm{End(B_{\Gamma^+})}$ is given by the translation on $\ell^\infty(\Gamma^+)$ which satisfies $\tau_t(1_s) = 1_{s+t}$. Then [4] shows that any isometric representation V of Γ^+ induces a unital representation $\pi_V : 1_s \mapsto V_s V_s^*$ of B_{Γ^+} such that (π_V, V) is a covariant isometric representation of $(B_{\Gamma^+}, \Gamma^+, \tau)$, and the representation $\pi_V \times V$ of $B_{\Gamma^+} \times_\tau^{\mathrm{iso}} \Gamma^+$ is faithful provided all V_s are nonunitary. Since the isometric representation given by the Toeplitz representation $T : s \mapsto T_s$ of Γ^+ on $\ell^2(\Gamma^+)$ is nonunitary, then $\pi_T \times T$ is an isomorphism of $B_{\Gamma^+} \times_\tau^{\mathrm{iso}} \Gamma^+$ onto the Toeplitz algebra $\mathcal{T}(\Gamma)$.

We consider the two kinds of crossed products $(A \times_\alpha^{\mathrm{iso}} \Gamma^+, i_A, i_{\Gamma^+})$ and $(A \times_\alpha^{\mathrm{piso}} \Gamma^+, j_A, j_{\Gamma^+})$ of a dynamical system (A, Γ^+, α). The equation

$$i_{\Gamma^+}(s)^* i_{\Gamma^+}(s) i_A(a) = i_A(a) i_{\Gamma^+}(s)^* i_{\Gamma^+}(s)$$

is automatic because i_{Γ^+} is an isometric representation of Γ^+.

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Therefore we have a covariant partial-isometric representation (i_A, i_{Γ^+}) of (A, Γ^+, α) in the C^*-algebra $A \times_\alpha^{\mathrm{iso}} \Gamma^+$, and the universal property of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ gives a nondegenerate homomorphism

$$\phi := i_A \times i_{\Gamma^+} : (A \times_\alpha^{\mathrm{piso}} \Gamma^+, j_A, j_{\Gamma^+}) \longrightarrow (A \times_\alpha^{\mathrm{iso}} \Gamma^+, i_A, i_{\Gamma^+}),$$

which satisfies $\phi(j_{\Gamma^+}(x)^* j_A(a) j_{\Gamma^+}(y)) = i_{\Gamma^+}(x)^* i_A(a) i_{\Gamma^+}(y)$ for all $a \in A$ and $x, y \in \Gamma^+$. Consequently ϕ is surjective, and then we have a short exact sequence

$$0 \longrightarrow \ker \phi \longrightarrow A \times_\alpha^{\mathrm{piso}} \Gamma^+ \xrightarrow{\phi} A \times_\alpha^{\mathrm{iso}} \Gamma^+ \longrightarrow 0.$$

In the next proposition, we identify spanning elements for the ideal $\ker \phi$.

Proposition 2.3. *Suppose that (A, Γ^+, α) is a dynamical system. Then*

$$\ker \phi = \overline{\mathrm{span}}\{j_{\Gamma^+}(x)^* j_A(a)(1 - j_{\Gamma^+}(t)^* j_{\Gamma^+}(t)) j_{\Gamma^+}(y) : a \in A, x, y, t \in \Gamma^+\}. \tag{2.1}$$

Before we prove this proposition, we first want to show the following lemma.

Lemma 2.4. *For $t \in \Gamma^+$, let P_t be the projection $1 - j_{\Gamma^+}(t)^* j_{\Gamma^+}(t)$. Then the set $\{P_t : t \in \Gamma^+\}$ is a family of increasing projections in the multiplier algebra $M(A \times_\alpha^{\mathrm{piso}} \Gamma^+)$, which satisfy the following equations: $j_A(a)P_t = P_t j_A(a)$ for $a \in A$ and $t \in \Gamma^+$,*

$$P_x j_{\Gamma^+}(y)^* = \begin{cases} 0 & \text{if } x \le y \\ j_{\Gamma^+}(y)^* P_{x-y} & \text{if } x > y \end{cases} \quad \text{and} \quad P_x P_y = \begin{cases} P_x & \text{if } x \le y \\ P_y & \text{if } x > y. \end{cases}$$

Proof. For $s \ge t$ in Γ^+,

$$\begin{aligned} P_s - P_t &= (1 - j_{\Gamma^+}(s)^* j_{\Gamma^+}(s)) - (1 - j_{\Gamma^+}(t)^* j_{\Gamma^+}(t)) \\ &= j_{\Gamma^+}(t)^* j_{\Gamma^+}(t) - j_{\Gamma^+}(s)^* j_{\Gamma^+}(s) \\ &= j_{\Gamma^+}(t)^* j_{\Gamma^+}(t) - j_{\Gamma^+}(t)^* j_{\Gamma^+}(s-t)^* j_{\Gamma^+}(s-t) j_{\Gamma^+}(t) \\ &= j_{\Gamma^+}(t)^* P_{s-t} j_{\Gamma^+}(t) = j_{\Gamma^+}(t)^* P_{s-t} P_{s-t} j_{\Gamma^+}(t) \\ &= [P_{s-t} j_{\Gamma^+}(t)]^* [P_{s-t} j_{\Gamma^+}(t)]. \end{aligned}$$

So $P_s - P_t \ge 0$, and hence $P_s \ge P_t$.

If $x \le y$, then

$$\begin{aligned} P_x j_{\Gamma^+}(y)^* &= (1 - j_{\Gamma^+}(x)^* j_{\Gamma^+}(x)) j_{\Gamma^+}(x)^* j_{\Gamma^+}(y-x)^* \\ &= [j_{\Gamma^+}(x)^* - j_{\Gamma^+}(x)^* j_{\Gamma^+}(x) j_{\Gamma^+}(x)^*] j_{\Gamma^+}(y-x)^* = 0, \end{aligned}$$

and if $x > y$,

$$\begin{aligned} P_x j_{\Gamma^+}(y)^* &= j_{\Gamma^+}(y)^* - j_{\Gamma^+}(x)^* j_{\Gamma^+}(x) j_{\Gamma^+}(y)^* \\ &= j_{\Gamma^+}(y)^* - j_{\Gamma^+}(y)^* j_{\Gamma^+}(x-y)^* j_{\Gamma^+}(x-y) j_{\Gamma^+}(y) j_{\Gamma^+}(y)^* \\ &= j_{\Gamma^+}(y)^* - j_{\Gamma^+}(y)^* j_{\Gamma^+}(x-y)^* j_{\Gamma^+}(x-y) \overline{j}_A(\overline{\alpha}_y(1)) \\ &= j_{\Gamma^+}(y)^* - j_{\Gamma^+}(y)^* \overline{j}_A(\overline{\alpha}_y(1)) j_{\Gamma^+}(x-y)^* j_{\Gamma^+}(x-y) \\ &= j_{\Gamma^+}(y)^* - [j_{\Gamma^+}(y)^* j_{\Gamma^+}(y) j_{\Gamma^+}(y)^*] j_{\Gamma^+}(x-y)^* j_{\Gamma^+}(x-y) \\ &= j_{\Gamma^+}(y)^* P_{x-y}. \end{aligned}$$

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Next we use the equation

$$j_{\Gamma^+}(x)^* j_{\Gamma^+}(x) j_{\Gamma^+}(y)^* j_{\Gamma^+}(y) = j_{\Gamma^+}(\max\{x, y\})^* j_{\Gamma^+}(\max\{x, y\}) \text{ for any } x, y \in \Gamma^+,$$

to compute

$$\begin{aligned} P_x P_y &= (1 - j_{\Gamma^+}(x)^* j_{\Gamma^+}(x))(1 - j_{\Gamma^+}(y)^* j_{\Gamma^+}(y)) \\ &= 1 - j_{\Gamma^+}(x)^* j_{\Gamma^+}(x) - j_{\Gamma^+}(y)^* j_{\Gamma^+}(y) + j_{\Gamma^+}(x)^* j_{\Gamma^+}(x) j_{\Gamma^+}(y)^* j_{\Gamma^+}(y) \\ &= 1 - j_{\Gamma^+}(x)^* j_{\Gamma^+}(x) - j_{\Gamma^+}(y)^* j_{\Gamma^+}(y) + j_{\Gamma^+}(\max\{x, y\})^* j_{\Gamma^+}(\max\{x, y\}) \\ &= \begin{cases} P_x & \text{if } x \le y, \\ P_y & \text{if } x > y. \end{cases} \end{aligned}$$

This concludes the proof. □

Proof of Proposition 2.3. We clarify that the right-hand side of (2.1), that

$$\mathcal{I} := \overline{\operatorname{span}}\{j_{\Gamma^+}(x)^* j_A(a)(1 - j_{\Gamma^+}(t)^* j_{\Gamma^+}(t)) j_{\Gamma^+}(y) : a \in A, \text{ and } x, y, t \in \Gamma^+\}$$

is an ideal of $(A \times_\alpha^{\text{piso}} \Gamma^+, j_A, j_{\Gamma^+})$, by showing that $j_A(b)\mathcal{I}$ and $j_{\Gamma^+}(s)\mathcal{I}$, $j_{\Gamma^+}(s)^*\mathcal{I}$ are contained in $\mathcal{I}$ for all $b \in A$ and $s \in \Gamma^+$. The last containment is trivial. For the first two, we compute using the partial-isometric covariance of (j_A, j_{Γ^+}) to get the following equations for $b \in A$, $s, x \in \Gamma^+$:

$$j_A(b) j_{\Gamma^+}(x)^* = [j_{\Gamma^+}(x) j_A(b^*)]^* = [j_A(\alpha_x(b^*)) j_{\Gamma^+}(x)]^* = j_{\Gamma^+}(x)^* j_A(\alpha_x(b)),$$

and

$$j_{\Gamma^+}(s) j_{\Gamma^+}(x)^* = \begin{cases} j_{\Gamma^+}(x - s)^* j_{\Gamma^+}(x) j_{\Gamma^+}(x)^* = j_{\Gamma^+}(x - s)^* \overline{j}_A(\overline{\alpha}_x(1)) & \text{if } s < x, \\ j_{\Gamma^+}(x) j_{\Gamma^+}(x)^* = \overline{j}_A(\overline{\alpha}_x(1)) & \text{if } s = x, \\ j_{\Gamma^+}(s - x) j_{\Gamma^+}(x) j_{\Gamma^+}(x)^* = \overline{j}_A(\overline{\alpha}_s(1)) j_{\Gamma^+}(s - x) & \text{if } s > x. \end{cases}$$

Consequently,

$$j_A(b) j_{\Gamma^+}(x)^* j_A(a) P_t j_{\Gamma^+}(y) = j_{\Gamma^+}(x)^* j_A(\alpha_x(b)a) P_t j_{\Gamma^+}(y) \in \mathcal{I},$$

and

$$j_{\Gamma^+}(s) j_{\Gamma^+}(x)^* j_A(a) P_t j_{\Gamma^+}(y) = j_{\Gamma^+}(x - s)^* j_A(\overline{\alpha}_x(1)a) P_t j_{\Gamma^+}(y) \in \mathcal{I}$$

whenever $b \in A$ and $t, s \le x$ in Γ^+. If $s > x$, then

$$P_t j_{\Gamma^+}(s - x)^* = \begin{cases} 0 & \text{for } t \le s - x, \\ j_{\Gamma^+}(s - x)^* P_{t-(s-x)} & \text{for } t > s - x. \end{cases}$$

Therefore

$$\begin{aligned} j_{\Gamma^+}(s) j_{\Gamma^+}(x)^* j_A(a) P_t j_{\Gamma^+}(y) &= \overline{j}_A(\overline{\alpha}_s(1)) j_{\Gamma^+}(s - x) j_A(a) P_t j_{\Gamma^+}(y) \\ &= \overline{j}_A(\overline{\alpha}_s(1)) j_A(\alpha_{s-x}(a)) j_{\Gamma^+}(s - x) P_t j_{\Gamma^+}(y) \\ &= j_A(\overline{\alpha}_s(1)\alpha_{s-x}(a))[P_t j_{\Gamma^+}(s - x)^*]^* \, j_{\Gamma^+}(y), \end{aligned}$$

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which is the zero element of $\mathcal{I}$ for $t \le s - x$, and is the element

$$j_A(\overline{\alpha}_s(1)\alpha_{s-x}(a))P_{t-(s-x)}j_{\Gamma^+}(s-x+y) \text{ of } \mathcal{I} \quad \text{for } t > s-x.$$

So $j_{\Gamma^+}(s)j_{\Gamma^+}(x)^* j_A(a)P_t j_{\Gamma^+}(y)$ belongs to $\mathcal{I}$, and $\mathcal{I}$ is an ideal of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$.

We now show the equation $\ker\phi = \mathcal{I}$. The inclusion $\mathcal{I} \subset \ker\phi$ follows from the fact that $\mathcal{I}$ is an ideal of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$, and that $\overline{\phi}(P_t) = 1 - i_{\Gamma^+}(t)^* i_{\Gamma^+}(t) = 0$ for all $t \in \Gamma^+$. For the reverse inclusion, suppose that ρ is a nondegenerate representation of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ on a Hilbert space H with $\ker\rho = \mathcal{I}$. Then the pair $(\pi := \rho \circ j_A, V := \overline{\rho} \circ j_{\Gamma^+})$ is a covariant partial-isometric representation of (A, Γ^+, α) on H. We claim that every V_t is an isometry. To see this, let (a_λ) be an approximate identity for A. Then

$$0 = \rho(j_A(a_\lambda)(1 - j_{\Gamma^+}(t)^* j_{\Gamma^+}(t))) = \pi(a_\lambda)(1 - V_t^* V_t) \quad \text{for all } \lambda,$$

and $\pi(a_\lambda)(1 - V_t^* V_t)$ converges strongly to $1 - V_t^* V_t$ in $B(H)$. Therefore $1 - V_t^* V_t = 0$. Consequently, the pair (π, V) is a covariant isometric representation of (A, Γ^+, α) on H, and hence there exists a nondegenerate representation ψ of $(A \times_\alpha^{\mathrm{iso}} \Gamma^+, i_A, i_{\Gamma^+})$ on H which satisfies $\psi(i_A(a)) = \rho(j_A(a))$ and $\overline{\psi}(i_{\Gamma^+}(x)) = \overline{\rho}(j_{\Gamma^+}(x))$ for all $a \in A$ and $x \in \Gamma^+$. So $\psi \circ \phi = \rho$ on the spanning elements of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$, thus $\ker\phi \subset \ker\rho$. □

Proposition 2.5. *If Γ is a subgroup of $\mathbb{R}$, then* $\ker\phi$ *is an essential ideal of the crossed product $A \times_\alpha^{\mathrm{piso}} \Gamma^+$.*

Proof. Let J be a nonzero ideal of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$. We want to show that $J \cap \ker\phi \neq \{0\}$. Assume that $\ker\phi \neq \{0\}$. Take a nondegenerate representation $\pi \times V$ of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ on H such that $\ker \pi \times V = J$. Since $J \neq \{0\}$, $\pi \times V$ is not a faithful representation. Consequently, by [10, Theorem 4.8], π does not act faithfully on $(V_s^* H)^\perp$ for some $s \in \Gamma^+ \backslash \{0\}$. So there is $a \neq 0$ in A such that $\pi(a)(1 - V_s^* V_s) = 0$. It follows from

$$0 = \pi(a)(1 - V_s^* V_s) = \pi \times V(j_A(a)(1 - j_{\Gamma^+}(s)^* j_{\Gamma^+}(s)))$$

that $j_A(a)(1 - j_{\Gamma^+}(s)^* j_{\Gamma^+}(s))$ belongs to $\ker \pi \times V = J$. Moreover, $j_A(a)(1 - j_{\Gamma^+}(s)^* j_{\Gamma^+}(s))$ is also contained in $\ker\phi$ because $\overline{\phi}(P_s) = 0$, hence it is contained in $\ker\phi \cap J$.

Next we have to clarify that $j_A(a)(1 - j_{\Gamma^+}(s)^* j_{\Gamma^+}(s))$ is nonzero. If it is zero, then $1 - j_{\Gamma^+}(s)^* j_{\Gamma^+}(s) = 0$ because $j_A(a) \neq 0$ by injectivity of j_A. Thus $j_{\Gamma^+}(s)$ is an isometry, and so is $j_{\Gamma^+}(ns)$ for every $n \in \mathbb{N}$. We claim that every $j_{\Gamma^+}(x)$ is an isometry, and consequently $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ is isomorphic to $A \times_\alpha^{\mathrm{iso}} \Gamma^+$. Therefore $\ker\phi = 0$, and $j_A(a)(1 - j_{\Gamma^+}(s)^* j_{\Gamma^+}(s))$ cannot be zero.

To justify the claim, note that if $x < s$ then $s - x < s$, and

$$\begin{aligned}
j_{\Gamma^+}(s-x)^* j_{\Gamma^+}(s) &= j_{\Gamma^+}(s-x)^* j_{\Gamma^+}(s-x) j_{\Gamma^+}(s-(s-x)) \\
&= [j_{\Gamma^+}(s-x)^* j_{\Gamma^+}(s-x)][j_{\Gamma^+}(x) j_{\Gamma^+}(x)^*] j_{\Gamma^+}(x) \\
&= [j_{\Gamma^+}(x) j_{\Gamma^+}(x)^*][j_{\Gamma^+}(s-x)^* j_{\Gamma^+}(s-x)] j_{\Gamma^+}(x) \\
&= j_{\Gamma^+}(x) j_{\Gamma^+}(s)^* j_{\Gamma^+}(s) = j_{\Gamma^+}(x).
\end{aligned}$$

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So the equation $j_{\Gamma^+}(s)^* = j_{\Gamma^+}(x)^* j_{\Gamma^+}(s-x)^*$ implies

$$1 = j_{\Gamma^+}(s)^* j_{\Gamma^+}(s) = j_{\Gamma^+}(x)^* j_{\Gamma^+}(s-x)^* j_{\Gamma^+}(s) = j_{\Gamma^+}(x)^* j_{\Gamma^+}(x).$$

Thus $j_{\Gamma^+}(x)$ is an isometry for every $x < s$. For $x > s$, by the Archimedean property of Γ, there exists $n_x \in \mathbb{N}$ such that $x < n_x s$, and since $j_{\Gamma^+}(n_x s)$ is an isometry, applying the previous arguments, we see that $j_{\Gamma^+}(x)$ is an isometry. □

3. The partial-isometric crossed product as a full corner

Suppose that (A, Γ^+, α) is a dynamical system, and consider the Hilbert A-module

$$\ell^2(\Gamma^+, A) = \Big\{ f : \Gamma^+ \to A : \sum_{x\in\Gamma^+} f(x)^* f(x) \text{ converges in the norm of } A \Big\}$$

with the module structure $(f \cdot a)(x) = f(x)a$ and $\langle f, g\rangle = \sum_{x\in\Gamma^+} f(x)^* g(x)$ for $f, g \in \ell^2(\Gamma^+, A)$ and $a \in A$. One may also wish to consider the Hilbert A-module $\ell^2(\Gamma^+) \otimes A$, the completion of the vector space tensor product $\ell^2(\Gamma^+) \odot A$, which has a right (incomplete) inner product A-module structure $(x \otimes a) \cdot b = x \otimes ab$ and $\langle x \otimes a, y \otimes b\rangle = (y \mid x) a^* b$ for $x, y \in \ell^2(\Gamma^+)$ and $a, b \in A$. The two modules are naturally isomorphic via the map defined by $\phi : x \otimes a \mapsto \phi(x \otimes a)(t) = x(t)a$ for $x \in \ell^2(\Gamma^+)$, $t \in \Gamma^+$, $a \in A$.

Let $\pi_\alpha : A \to \mathcal{L}(\ell^2(\Gamma^+, A))$ be a map of A into the C^*-algebra $\mathcal{L}(\ell^2(\Gamma^+, A))$ of adjointable operators on $\ell^2(\Gamma^+, A)$, defined by

$$(\pi_\alpha(a) f)(t) = \alpha_t(a) f(t) \quad \text{for } a \in A,\ f \in \ell^2(\Gamma^+, A).$$

It is a well-defined map as we can see that $\pi_\alpha(a) f \in \ell^2(\Gamma^+, A)$:

$$\begin{aligned}\sum_{t\in\Gamma^+} (\alpha_t(a) f(t))^* (\alpha_t(a) f(t)) &= \sum_{t\in\Gamma^+} f(t)^* \alpha_t(a^* a) f(t) \\ &\le \|\alpha_t(a^* a)\| \sum_{t\in\Gamma^+} f(t)^* f(t).\end{aligned}$$

Moreover, π_α is an injective *-homomorphism, which could be degenerate (for example, when each of endomorphism α_t acts on a unital algebra A and $\alpha_t(1) \neq 1$).

Let $S \in \mathcal{L}(\ell^2(\Gamma^+, A))$ be defined by

$$S_t(f)(i) = \begin{cases} f(i-t) & \text{if } i \ge t, \\ 0 & \text{if } i < t. \end{cases}$$

Then $S_t^* S_t = 1$, $S_t S_t^* \neq 1$, and the pair (π_α, S) satisfies the following equations for all $a \in A$, $t \in \Gamma^+$:

$$\pi_\alpha(a) S_t = S_t \pi_\alpha(\alpha_t(a)) \quad \text{and} \quad (1 - S_t S_t^*)\pi_\alpha(a) = \pi_\alpha(a)(1 - S_t S_t^*). \tag{3.1}$$

Next we consider the vector subspace of $\mathcal{L}(\ell^2(\Gamma^+, A))$ spanned by

$$\{S_x \pi_\alpha(a) S_y^* : a \in A,\ x, y \in \Gamma^+\}.$$

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Using the equations in (3.1), it is evident that this space is closed under the multiplication and adjoint, and we therefore have a C^*-subalgebra of $\mathcal{L}(\ell^2(\Gamma^+, A))$, namely

$$\mathcal{T}_\alpha := \overline{\operatorname{span}}\{S_x\pi_\alpha(a)S_y^* : a \in A,\ x, y \in \Gamma^+\}. \tag{3.2}$$

One can see that $x \in \Gamma^+ \mapsto S_x \in M(\mathcal{T}_\alpha)$ is a semigroup of nonunitary isometries, and $\pi_\alpha(A) \subseteq \mathcal{T}_\alpha$. We show in Lemma 3.1 that π_α extends to the strictly continuous homomorphism $\overline{\pi}_\alpha$ on the multiplier algebra $M(A)$, and the equations in (3.1) remain valid.

The algebra $\mathcal{T}_\alpha$ defined in (3.2) satisfies the following natural properties. If (A, Γ^+, α) and (B, Γ^+, β) are two dynamical systems with extendible endomorphism actions, let $S_x\pi_\alpha(a)S_y^*$ and $T_x\pi_\beta(b)T_y^*$ denote spanning elements for $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$, respectively. If $\phi : A \to B$ is a nondegenerate homomorphism such that $\phi \circ \alpha_t = \beta_t \circ \phi$ for every $t \in \Gamma^+$, then by using the identification $\ell^2(\Gamma^+, A) \otimes_A B \simeq \ell^2(\Gamma^+, B)$, we have a homomorphism $\tau_\phi : \mathcal{T}_\alpha \to \mathcal{T}_\beta$ which satisfies $\tau_\phi(S_x\pi_\alpha(a)S_y^*) = T_x\pi_\beta(\phi(a))T_y^*$ for all $a \in A$ and $x, y \in \Gamma^+$. Note that if ϕ is injective then so is τ_ϕ. This property is consistent with the extendibility of endomorphisms α_t and β_t. Since the canonical map $\iota_A : A \to M(A)$ is injective and nondegenerate, it follows that we have an injective homomorphism $\tau_{\iota_A} : \mathcal{T}_\alpha \to \mathcal{T}_{\overline{\alpha}}$ such that $\tau_{\iota_A}(\mathcal{T}_\alpha)$ is an ideal of $\mathcal{T}_{\overline{\alpha}}$. Moreover, since the nondegenerate homomorphism $\phi : A \to B$ extends to $\overline{\phi}$ on the multiplier algebras in which it satisfies $\overline{\phi} \circ \overline{\alpha}_t = \overline{\beta}_t \circ \overline{\phi}$ for all $t \in \Gamma^+$, $\overline{\phi}$ induces the homomorphism $\tau_{\overline{\phi}} : \mathcal{T}_{\overline{\alpha}} \to \mathcal{T}_{\overline{\beta}}$ and satisfies $\tau_{\overline{\phi}} \circ \tau_{\iota_A} = \tau_{\iota_B} \circ \tau_\phi$.

Lemma 3.1. *The homomorphism $\pi_\alpha : A \to M(\mathcal{T}_\alpha)$ extends to the strictly continuous homomorphism $\overline{\pi}_\alpha$ on the multiplier algebra $M(A)$, such that the pair $(\overline{\pi}_\alpha, S)$ satisfies $\overline{\pi}_\alpha(m)S_t = S_t\overline{\pi}_\alpha(\overline{\alpha}_t(m))$ and $(1 - S_tS_t^*)\overline{\pi}_\alpha(m) = \overline{\pi}_\alpha(m)(1 - S_tS_t^*)$ for all $m \in M(A)$ and $t \in \Gamma^+$.*

Proof. We want to find a projection $p \in M(\mathcal{T}_\alpha)$ such that $\pi_\alpha(a_\lambda)$ converges strictly to p in $M(\mathcal{T}_\alpha)$ for an approximate identity (a_λ) in A.

Consider the map p defined on $\ell^2(\Gamma^+, A)$ by

$$(p(f))(t) = \overline{\alpha}_t(1)f(t).$$

First we clarify that $p(f)$ belongs to $\ell^2(\Gamma^+, A)$ for all $f \in \ell^2(\Gamma^+, A)$. Let $t \in \Gamma^+$. Then

$$(p(f))(t)^*(p(f))(t) = (\overline{\alpha}_t(1)f(t))^*(\overline{\alpha}_t(1)f(t)) = f(t)^*\overline{\alpha}_t(1)f(t).$$

Since $\overline{\alpha}_t(1)$ is a positive element of $M(A)$, it follows that

$$f(t)^*\overline{\alpha}_t(1)f(t) \le \|\overline{\alpha}_t(1)\| f(t)^*f(t) \le f(t)^*f(t).$$

Consequently, $0 \le \sum_{t\in F}(p(f))(t)^*p(f)(t) \le \sum_{t\in F} f(t)^*f(t)$ for every finite set $F \subset \Gamma^+$. Moreover, the sequence of partial sums of $\sum_{t\in\Gamma^+} f(t)^*f(t)$ is Cauchy in A because $f \in \ell^2(\Gamma^+, A)$. Therefore $\sum_{t\in\Gamma^+}(p(f))(t)^*p(f)(t)$ converges in A, and hence $p(f) \in \ell^2(\Gamma^+, A)$.

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One can see from the definition of p that it is a linear map, and the computations below show it is adjointable, and such that $p^* = p$ and $p^2 = p$. So p is a projection in $\mathcal{L}(\ell^2(\Gamma^+, A))$:

$$\begin{aligned}\langle p(f), g\rangle &= \sum_{t\in\Gamma^+}(p(f)(t))^* g(t) = \sum_{t\in\Gamma^+}(\overline{\alpha}_t(1)f(t))^* g(t) = \sum_{t\in\Gamma^+} f(t)^*\overline{\alpha}_t(1)g(t)\\ &= \sum_{t\in\Gamma^+} f(t)^*(p(g)(t)) = \langle f, p(g)\rangle.\end{aligned}$$

To see that p belongs to $M(\mathcal{T}_\alpha)$, direct computations for every $f \in \ell^2(\Gamma^+, A)$ give the equations $[(p\ (S_x\pi_\alpha(a)S_y^*))\ f](t) = [S_x\pi_\alpha(\overline{\alpha}_x(1)a)S_y^*\ f](t)$ and $[((S_x\pi_\alpha(a)S_y^*)\ p)\ f](t) = [S_x\pi_\alpha(a\overline{\alpha}_y(1))S_y^*\ f](t)$. Thus p multiplies every spanning element of $\mathcal{T}_\alpha$ into itself, so $p \in M(\mathcal{T}_\alpha)$.

Now we want to prove that $(\pi_\alpha(a_\lambda))_{\lambda\in\Lambda}$ converges strictly to p in $M(\mathcal{T}_\alpha)$. For this we show that $\pi_\alpha(a_\lambda)S_x\pi_\alpha(a)S_y^*$ and $S_x\pi_\alpha(a)S_y^*\pi_\alpha(a_\lambda)$ converge in $\mathcal{T}_\alpha$ to $pS_x\pi_\alpha(a)S_y^*$ and $S_x\pi_\alpha(a)S_y^*p$, respectively. Note that $\pi_\alpha(a_\lambda)S_x\pi_\alpha(a)S_y^* = S_x\pi_\alpha(\alpha_x(a_\lambda)a)S_y^* \in \mathcal{T}_\alpha$ and $S_x\pi_\alpha(a)S_y^*\pi_\alpha(a_\lambda) = S_x\pi_\alpha(a\alpha_y(a_\lambda))S_y^* \in \mathcal{T}_\alpha$. Since $\alpha_x(a_\lambda)a \to \overline{\alpha}_x(1)a$ in A by the extendibility of α_x, it follows that $S_x\pi_\alpha(\alpha_x(a_\lambda)a)S_y^* \to S_x\pi_\alpha(\overline{\alpha}_x(1)a)S_y^* = p(S_x\pi_\alpha(a)S_y^*)$ and

$$S_x\pi_\alpha(a\alpha_y(a_\lambda))S_y^* \to S_x\pi_\alpha(a\overline{\alpha}_y(1))S_y^* = (S_x\pi_\alpha(a)S_y^*)p \quad \text{in } \mathcal{T}_\alpha.$$

Thus we have shown that π_α is extendible, and therefore $\overline{\pi}_\alpha(1_{M(A)}) = p$.

Next we want to clarify the equation $\overline{\pi}_\alpha(m)S_x = S_x\overline{\pi}_\alpha(\overline{\alpha}_x(m))$ in $M(\mathcal{T}_\alpha)$. Let (a_λ) be an approximate identity for A. The extendibility of π_α implies $\pi_\alpha(a_\lambda m) \to \overline{\pi}_\alpha(m)$ strictly in $M(\mathcal{T}_\alpha)$, and hence $\pi_\alpha(a_\lambda m)S_x \to \overline{\pi}_\alpha(m)S_x$ strictly in $M(\mathcal{T}_\alpha)$. But $\pi_\alpha(a_\lambda m)S_x = S_x\pi_\alpha(\alpha_x(a_\lambda m))$ converges strictly to $S_x\overline{\pi}_\alpha(\overline{\alpha}_x(m))$ in $M(\mathcal{T}_\alpha)$. Therefore $\overline{\pi}_\alpha(m)S_x = S_x\overline{\pi}_\alpha(\overline{\alpha}_x(m))$. Similar arguments show that $\overline{\pi}_\alpha(m)(1 - S_tS_t^*) = (1 - S_tS_t^*)\overline{\pi}_\alpha(m)$ in $M(\mathcal{T}_\alpha)$. □

We have already shown that $\pi_\alpha : A \to M(\mathcal{T}_\alpha)$ is extendible in Lemma 3.1. Therefore we have a projection $\overline{\pi}_\alpha(1_{M(A)}) = p$ in $M(\mathcal{T}_\alpha)$. Note that p is the identity of $pM(\mathcal{T}_\alpha)p$, and $\pi_\alpha(a) = \pi_\alpha(1_{M(A)}a1_{M(A)}) = p\pi_\alpha(a)p \in pM(\mathcal{T}_\alpha)p$. We claim that the homomorphism $\pi_\alpha : A \to pM(\mathcal{T}_\alpha)p$ is nondegenerate. To see this, let (a_λ) be an approximate identity for A, and $\xi := S_x\pi_\alpha(b)S_y^*$. Then $\pi_\alpha(a_\lambda)p\xi p = S_x\pi_\alpha(\alpha_x(a_\lambda)b)S_y^*p$ converges to $S_x\pi_\alpha(\overline{\alpha}_x(1)b)S_yp = p\xi p$ in $p\mathcal{T}_\alpha p$. Similar arguments show that $p\xi p\pi_\alpha(a_\lambda) \to p\xi p$ in $p\mathcal{T}_\alpha p$.

In the next proposition we show that the algebra $p\mathcal{T}_\alpha p$ is a partial-isometric crossed product of (A, Γ^+, α).

Proposition 3.2. *Suppose that (A, Γ^+, α) is a system such that every $\alpha_x \in \mathrm{End(A)}$ is extendible. Let $p = \overline{\pi}_\alpha(1_{M(A)})$, and let*

$$k_A : A \to p\mathcal{T}_\alpha p \quad \textit{and} \quad w : \Gamma^+ \to M(p\mathcal{T}_\alpha p)$$

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be the maps defined by $k_A(a) = \pi_\alpha(a)$ *and* $w_x = pS_x^* p$. *Then the triple* $(p\mathcal{T}_\alpha p, k_A, w)$ *is a partial-isometric crossed product of the system* (A, Γ^+, α), *and therefore* $\psi := k_A \times w : (A \times_\alpha^{\text{piso}} \Gamma^+, i_A, v) \to p\mathcal{T}_\alpha p$ *is an isomorphism which satisfies* $\psi(i_A(a)) = k_A(a)$ *and* $\psi(v_x) = w_x$. *Moreover,* $A \times_\alpha^{\text{piso}} \Gamma^+$ *is Morita equivalent to the algebra* $\mathcal{T}_\alpha$.

Before we prove the proposition, we show the following lemma.

Lemma 3.3. *The pair* (k_A, w) *forms a covariant partial-isometric representation of* (A, Γ^+, α) *in* $p\mathcal{T}_\alpha p$, *and the homomorphism* $\varphi := k_A \times w : A \times_\alpha^{\text{piso}} \Gamma^+ \to p\mathcal{T}_\alpha p$ *is injective.*

Proof. Each of w_x is a partial isometry: $w_x = pS_x^* p = \overline{\pi}_\alpha(\overline{\alpha}_x(1))S_x^* \Rightarrow w_x w_x^* w_x = \overline{\pi}_\alpha(\overline{\alpha}_x(1))S_x^* = w_x$, and for $x, y \in \Gamma^+$ we have

$$w_x w_y = \overline{\pi}_\alpha(\overline{\alpha}_x(1))S_x^* \overline{\pi}_\alpha(\overline{\alpha}_y(1))S_y^* = \overline{\pi}_\alpha(\overline{\alpha}_x(1))\overline{\pi}_\alpha(\overline{\alpha}_{x+y}(1))S_{x+y}^* = w_{x+y}.$$

The computations below show that (k_A, w) satisfies the partial-isometric covariance relations:

$$\begin{aligned} w_x k_A(a) w_x^* &= \overline{\pi}_\alpha(\overline{\alpha}_x(1))S_x^*[\pi_\alpha(a)S_x]\overline{\pi}_\alpha(\overline{\alpha}_x(1)) \\ &= \overline{\pi}_\alpha(\overline{\alpha}_x(1))\pi_\alpha(\alpha_x(a))\overline{\pi}_\alpha(\overline{\alpha}_x(1)) = \pi_\alpha(\alpha_x(a)) = k_A(\alpha_x(a)) \end{aligned}$$

and

$$\begin{aligned} w_x^* w_x k_A(a) &= S_x \overline{\pi}_\alpha(\overline{\alpha}_x(1))S_x^* \pi_\alpha(a) = S_x \pi_\alpha(\overline{\alpha}_x(1)\alpha_x(a))S_x^* \\ &= S_x \pi_\alpha(\alpha_x(a)\overline{\alpha}_x(1))S_x^* = S_x \pi_\alpha(\alpha_x(a))\overline{\pi}_\alpha(\overline{\alpha}_x(1))S_x^* \\ &= \pi_\alpha(a)S_x \overline{\pi}_\alpha(\overline{\alpha}_x(1))S_x^* = \pi_\alpha(a)w_x^* w_x = k_A(a)w_x^* w_x. \end{aligned}$$

So there exists a nondegenerate homomorphism $\varphi := k_A \times w : A \times_\alpha^{\text{piso}} \Gamma^+ \to p\mathcal{T}_\alpha p$. We want to see if it is injective. Put $p\mathcal{T}_\alpha p$ by a faithful and nondegenerate representation γ into a Hilbert space H. Then we want to prove that the representation $\gamma \circ \varphi$ of $(A \times_\alpha^{\text{piso}} \Gamma^+, i_A, v)$ on H is faithful. Let $\sigma = \gamma \circ \varphi \circ i_A$ and $t = \overline{\gamma \circ \varphi} \circ v$. By [10, Theorem 4.8], we have to show that σ acts faithfully on the range of $(1 - t_x^* t_x)$ for every $x > 0$ in Γ^+. If $x > 0$ in Γ^+, $a \in A$, and $\sigma(a)|_{\text{range}(1-t_x^* t_x)} = 0$, then we want to see that $a = 0$. First note that $\sigma(a)(1 - t_x^* t_x) = \gamma \circ \varphi(i_A(a)(1 - v_x^* v_x))$, and

$$\begin{aligned} \varphi(i_A(a)(1 - v_x^* v_x)) &= \varphi(i_A(a))(\overline{\varphi}(1) - \varphi(v_x^* v_x)) = \varphi(i_A(a))(p - \overline{\varphi}(v_x^*)\overline{\varphi}(v_x)) \\ &= k_A(a)(p - w_x^* w_x) \\ &= \pi_\alpha(a)(\overline{\pi}_\alpha(1) - S_x \overline{\pi}_\alpha(\overline{\alpha}_x(1))\overline{\pi}_\alpha(\overline{\alpha}_x(1))S_x^*) \\ &= \pi_\alpha(a)(\overline{\pi}_\alpha(1) - \overline{\pi}_\alpha(1)S_x S_x^* \overline{\pi}_\alpha(1)) \\ &= \pi_\alpha(a)(1 - S_x S_x^*)\overline{\pi}_\alpha(1) = \pi_\alpha(a)\overline{\pi}_\alpha(1)(1 - S_x S_x^*) \\ &= \pi_\alpha(a)(1 - S_x S_x^*). \end{aligned}$$

So $\sigma(a)(1 - t_x^* t_x) = 0$ implies $\pi_\alpha(a)(1 - S_x S_x^*) = 0$ in $\mathcal{L}(\ell^2(\Gamma^+, A))$. But for $f \in \ell^2(\Gamma^+, A)$,

$$((1 - S_x S_x^*)f)(y) = \begin{cases} 0 & \text{for } y \geq x > 0, \\ f(y) & \text{for } y < x. \end{cases}$$

Thus evaluating the operator $\pi_\alpha(a)(1 - S_x S_x^*)$ on a chosen element $f \in \ell^2(\Gamma^+, A)$ where $f(y) = a^*$ for $y = 0$ and $f(y) = 0$ for $y \neq 0$,

$$(\pi_\alpha(a)(1 - S_x S_x^*)(f))(y) = \begin{cases} \alpha_y(a)f(y) & \text{for } y = 0 \\ 0 & \text{for } y \neq 0 \end{cases} = \begin{cases} aa^* & \text{for } y = 0 \\ 0 & \text{for } y \neq 0. \end{cases}$$

Therefore $aa^* = 0 \in A$, and hence $a = 0$. □

Proof of Proposition 3.2. Let (ρ, W) be a covariant partial-isometric representation of (A, Γ^+, α) on a Hilbert space H. We want to construct a nondegenerate representation Φ of $p\mathcal{T}_\alpha p$ on H such that $\Phi(pS_i\pi_\alpha(a)S_j^*p) = W_i^*\rho(a)W_j$ for all $a \in A$, $i, j \in \Gamma^+$. It follows from this equation that $\Phi(k_A(a)) = \rho(a)$ for all $a \in A$, and $\overline{\Phi}(w_i) = W_i$ for $i \in \Gamma^+$ because $\Phi(p\pi_\alpha(a_\lambda)S_i^*p) = \rho(a_\lambda)W_i$ for all $i \in \Gamma^+$, $\rho(a_\lambda)W_i$ converges strongly to W_i in $B(H)$, and $\Phi(p\pi_\alpha(a_\lambda)S_i^*p) = \Phi(\pi_\alpha(a_\lambda))\overline{\Phi}(pS_i^*p) = \rho(a_\lambda)\overline{\Phi}(pS_i^*p)$ converges strongly to $\overline{\Phi}(pS_i^*p)$ in $B(H)$.

So we want the representation Φ to satisfy

$$\Phi\Big(\sum \lambda_{i,j} pS_i\pi_\alpha(a_{i,j})S_j^*p\Big) = \sum \lambda_{i,j}\Phi(pS_i\pi_\alpha(a_{i,j})S_j^*p) = \sum \lambda_{i,j}W_i^*\rho(a_{i,j})W_j.$$

We prove that this formula gives a well-defined linear map Φ on $\operatorname{span}\{pS_i\pi_\alpha(a)S_j^*p : a \in A, i, j \in \Gamma^+\}$, and simultaneously Φ extends to $p\mathcal{T}_\alpha p$ by showing that

$$\Big\|\sum \lambda_{i,j}W_i^*\rho(a_{i,j})W_j\Big\| \leq \Big\|\sum \lambda_{i,j}pS_i\pi_\alpha(a_{i,j})S_j^*p\Big\|.$$

Note that the nondegenerate representation $\rho \times W$ of $(A \times_\alpha^{\text{piso}} \Gamma^+, i_A, v)$ on H satisfies $\rho \times W(v_i^* i_A(a)v_j) = W_i^*\rho(a)W_j$, and the injective homomorphism $\varphi : (A \times_\alpha^{\text{piso}} \Gamma^+, i_A, v) \to p\mathcal{T}_\alpha p$ in Lemma 3.3 satisfies $\varphi(v_i^* i_A(a)v_j) = w_i^* k_A(a)w_j = pS_i\pi_\alpha(a)S_j^*p$. Now we compute

$$\begin{aligned}\Big\|\sum_{i,j\in\Gamma^+} \lambda_{i,j}W_i^*\rho(a_{i,j})W_j\Big\| &= \Big\|\rho \times W\Big(\sum \lambda_{i,j}v_i^* i_A(a_{i,j})v_j\Big)\Big\| \\ &\leq \Big\|\sum \lambda_{i,j}v_i^* i_A(a_{i,j})v_j\Big\| \\ &= \Big\|\varphi\Big(\sum \lambda_{i,j}v_i^* i_A(a_{i,j})v_j\Big)\Big\| \quad \text{by injectivity of } \varphi \\ &= \Big\|\sum \lambda_{i,j}pS_i\pi_\alpha(a_{i,j})S_j^*p\Big\|.\end{aligned}$$

Next we verify that Φ is a *-homomorphism. It certainly preserves the adjoint, and we claim by our arguments below that it also preserves the multiplication. Note that

$$\begin{aligned}\xi :&= (pS_i\pi_\alpha(a)S_j^*p)\,(pS_n\pi_\alpha(b)S_m^*p) \\ &= \begin{cases} pS_i\pi_\alpha(a\overline{\alpha}_j(1)b)S_m^*p & \text{for } j = n, \\ pS_i\pi_\alpha(a\alpha_{j-n}(\overline{\alpha}_n(1)b))S_{j-n+m}^*p & \text{for } j > n, \\ pS_{i+n-j}\pi_\alpha(\alpha_{n-j}(a)\overline{\alpha}_n(1)b)S_m^*p & \text{for } j < n. \end{cases}\end{aligned}$$

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Then the covariance of (ρ, W) gives $\Phi(\xi) = (W_i^*\rho(a)W_j)(W_n^*\rho(a)W_m)$ for all cases of j and n. So Φ preserves the multiplication. Thus Φ is a representation of $p\mathcal{T}_\alpha p$ on H.

We want to see that Φ is nondegenerate. The representation ρ of A is nondegenerate and $\rho(a) = \Phi(\pi_\alpha(a))$, therefore

$$\begin{aligned} H &= \overline{\operatorname{span}}\{\rho(a)h : a \in A, h \in H\} \\ &\subset \overline{\operatorname{span}}\{\Phi(pS_i\pi_\alpha(a)S_j^*p)h : a \in A, i, j \in \Gamma^+, h \in H\}, \end{aligned}$$

so Φ is nondegenerate. The C^*-algebra $p\mathcal{T}_\alpha p$ is spanned by $\{w_i^* i_A(a)w_j : a \in A, i, j \in \Gamma^+\}$ because $w_i^* i_A(a)w_j = pS_i p\pi_\alpha(a)pS_j^*p = pS_i\pi_\alpha(a)S_j^*p$. Thus $p\mathcal{T}_\alpha p$ and $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ are isomorphic.

Finally, we prove the fullness of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ in $\mathcal{T}_\alpha$. It is enough by [15, Example 3.6] to show that $\mathcal{T}_\alpha p\mathcal{T}_\alpha$ is dense in $\mathcal{T}_\alpha = \overline{\operatorname{span}}\{S_i\pi_\alpha(a)S_j^* : i, j \in \Gamma^+, a \in A\}$. Take a spanning element $S_i\pi_\alpha(a)S_j^* \in \mathcal{T}_\alpha$ and an approximate identity (a_λ) for A. Then $S_i\pi_\alpha(a)S_j^* = \lim_\lambda S_i\pi_\alpha(aa_\lambda)S_j^*$, and since $S_i\pi_\alpha(aa_\lambda)S_j^* = S_i\pi_\alpha(a)S_0^*pS_0\pi_\alpha(a_\lambda)S_j^* \in \mathcal{T}_\alpha p\mathcal{T}_\alpha$, a linear combination of spanning elements in $\mathcal{T}_\alpha$ can be approximated by elements of $\mathcal{T}_\alpha p\mathcal{T}_\alpha$. Thus $\overline{\mathcal{T}_\alpha p\mathcal{T}_\alpha} = \mathcal{T}_\alpha$. □

Remark 3.4. When dealing with systems (A, Γ^+, α) in which $\overline{\alpha}_t(1) = 1$, $p = \overline{\pi}_\alpha(1)$ is the identity of $\mathcal{L}(\ell^2(\Gamma^+, A))$, and the assertion of Proposition 3.2 says that $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ is isomorphic to $\mathcal{T}_\alpha$.

4. The partial-isometric crossed product of a system by a single endomorphism

In this section we consider a system $(A, \mathbb{N}, \alpha)$ of a (nonunital) C^*-algebra A and an action α of $\mathbb{N}$ by extendible endomorphisms of A. The module $\ell^2(\mathbb{N}, A)$ is the vector space of sequences (x_n) such that the series $\sum_{n\in\mathbb{N}} x_n^* x_n$ converges in the norm of A, with the module structure $(x_n) \cdot a = (x_n a)$ and the inner product $\langle (x_n), (y_n)\rangle = \sum_{n\in\mathbb{N}} x_n^* y_n$.

The homomorphism $\pi_\alpha : A \to \mathcal{L}(\ell^2(\mathbb{N}, A))$ defined by $\pi_\alpha(a)(x_n) = (\alpha_n(a)x_n)$ is injective, and together with the nonunitary isometry $S \in \mathcal{L}(\ell^2(\mathbb{N}, A))$,

$$S(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots),$$

satisfies the equation

$$\pi_\alpha(a)S_i = S_i\pi_\alpha(\alpha_i(a)) \quad \text{for all } a \in A, i \in \mathbb{N}.$$

Note that $S_n\pi_\alpha(ab^*)(1 - SS^*)S_m^* = \theta_{f,g}$ where $f(n) = a$ and $f(i) = 0$ for $i \neq n$, $g(m) = b$ and $g(i) = 0$ for $i \neq m$. So the C^*-algebra $\mathcal{K}(\ell^2(\mathbb{N}, A))$ is

$$\overline{\operatorname{span}}\{S_n\pi_\alpha(ab^*)(1 - SS^*)S_m^* : n, m \in \mathbb{N}, a, b \in A\}.$$

Let $(A \times_\alpha^{\mathrm{iso}} \mathbb{N}, j_A, T)$ be the isometric crossed product of $(A, \mathbb{N}, \alpha)$, and consider the natural homomorphism $\phi = (i_A \times T) : A \times_\alpha^{\mathrm{piso}} \mathbb{N} \to A \times_\alpha^{\mathrm{iso}} \mathbb{N}$. From Proposition 2.3, we know that

$$\ker \phi = \overline{\operatorname{span}}\{v_m^* i_A(a)(1 - v^*v)v_n : a \in A, m, n \in \mathbb{N}\}. \tag{4.1}$$

We show in the next theorem that the ideal $\ker \phi$ is a corner in $A \otimes K(\ell^2(\mathbb{N}))$.

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Theorem 4.1. *Suppose that* $(A, \mathbb{N}, \alpha)$ *is a dynamical system in which every* $\alpha_n := \alpha^n$ *extends to a strictly continuous endomorphism on the multiplier algebra* $M(A)$ *of* A*. Let* $p = \overline{\pi}_\alpha(1_{M(A)}) \in \mathcal{L}(\ell^2(\mathbb{N}, A))$*. Then the isomorphism* $\psi : A \times_\alpha^{\text{piso}} \mathbb{N} \to p\mathcal{T}_\alpha p$ *in Proposition 3.2 takes the ideal* $\ker\phi$ *of* $A \times_\alpha^{\text{piso}} \mathbb{N}$ *given by (4.1) isomorphically to the full corner* $p[K(\ell^2(\mathbb{N}, A))]p$*. So there is a short exact sequence of* C^**-algebras,*

$$0 \longrightarrow p[K(\ell^2(\mathbb{N}, A))]p \xrightarrow{\ \Psi\ } A \times_\alpha^{\text{piso}} \mathbb{N} \xrightarrow{\ \phi\ } A \times_\alpha^{\text{iso}} \mathbb{N} \longrightarrow 0 \tag{4.2}$$

where $\Psi(pS_m\pi_\alpha(a)(1 - SS^*)S_n^*p) = v_m^* i_A(a)(1 - v^*v)v_n$*.*

Proof. We compute the image $\psi(\mu)$ of a spanning element $\mu := v_m^* i_A(a)(1 - v^*v)v_n$ of $\ker\phi$:

$$\psi(\mu) = pS_m p\pi_\alpha(a)\psi(1 - v^*v)pS_n^*p = pS_m\pi_\alpha(a)(p - pSpS^*p)pS_n^*p,$$

$$\begin{aligned} pSpS^* &= (\overline{\pi}_\alpha(1)S)\overline{\pi}_\alpha(1)S^* = S\overline{\pi}_\alpha(\overline{\alpha}(1))S^* \\ &= S(S\overline{\pi}_\alpha(\overline{\alpha}(1))^* = S(\overline{\pi}_\alpha(1)S)^* = SS^*p \end{aligned}$$

and

$$\begin{aligned} pS_n^*p &= \overline{\pi}_\alpha(1)(\overline{\pi}_\alpha(1)S_n)^* = \overline{\pi}_\alpha(1)(S_n\overline{\pi}_\alpha(\overline{\alpha_n}(1)))^* \\ &= \overline{\pi}_\alpha(\overline{\alpha_n}(1))S_n^* = (\overline{\pi}_\alpha(1)S_n)^* = S_n^*p. \end{aligned}$$

Therefore

$$\psi(v_m^* i_A(a)(1 - v^*v)v_n) = p\,(S_m\pi_\alpha(a)(1 - SS^*)S_n^*)\,p. \tag{4.3}$$

Since $S_m\pi_\alpha(a)(1 - SS^*)S_n^* = \lim_\lambda S_m\pi_\alpha(aa_\lambda^*)(1 - SS^*)S_n^*$ where (a_λ) is an approximate identity in A, and $S_m\pi_\alpha(aa_\lambda^*)(1 - SS^*)S_n^* = \theta_{\xi,\eta_\lambda}$ for which $\xi, \eta_\lambda \in \ell^2(\mathbb{N}, A)$ are given by $\xi(m) = a$ and $\xi(i) = 0$ for $i \neq m$, $\eta_\lambda(n) = a_\lambda$ and $\eta_\lambda(i) = 0$ for $i \neq n$, it follows that $\psi(\mu) \in p[\mathcal{K}(\ell^2(\mathbb{N}, A))]\,p$. Thus $\psi(\ker\phi) \subset p[K(\ell^2(\mathbb{N}, A))]p$.

Conversely, by computations similar to those that lead to (4.3), $pS_m\pi_\alpha(ab^*)(1 - SS^*)S_n^*p = \psi(v_m^* i_A(ab^*)(1 - v^*v)v_n)$. Hence $p[K(\ell^2(\mathbb{N}, A))]p \subset \psi(\ker\phi)$. This is full because $K(\ell^2(\mathbb{N}, A))pK(\ell^2(\mathbb{N}, A))$ is dense in $K(\ell^2(\mathbb{N}, A))$: for an approximate identity (a_λ) in A,

$$S_m\pi_\alpha(a)(1 - SS^*)S_n^* = \lim_\lambda S_m\pi_\alpha(aa_\lambda)(1 - SS^*)S_n^*$$

and $S_m\pi_\alpha(aa_\lambda)(1 - SS^*)S_n^* = (S_m\pi_\alpha(a)(1 - SS^*)S_0^*)p(S_0\pi_\alpha(a_\lambda)(1 - SS^*)S_n^*$ is contained in $K(\ell^2(\mathbb{N}, A))pK(\ell^2(\mathbb{N}, A))$. □

Remark 4.2. The external tensor product $\ell^2(\mathbb{N}) \otimes A$ and $\ell^2(\mathbb{N}, A)$ are isomorphic as Hilbert A-modules [15, Lemma 3.43], and the isomorphism is given by

$$\varphi(f \otimes a)(n) = (f(0)a, f(1)a, f(2)a, \ldots) \quad \text{for } f \in \ell^2(\mathbb{N}) \text{ and } a \in A.$$

The isomorphism $\psi : T \in \mathcal{L}(\ell^2(\mathbb{N}, A)) \mapsto \varphi^{-1}T\varphi \in \mathcal{L}(\ell^2(\mathbb{N}) \otimes A)$ satisfies $\psi(\theta_{\xi,\eta}) = \varphi^{-1}\theta_{\xi,\eta}\varphi = \theta_{\varphi^{-1}(\xi),\varphi^{-1}(\eta)}$ for all $\xi, \eta \in \ell^2(\mathbb{N}, A)$. Therefore $\psi(\mathcal{K}(\ell^2(\mathbb{N}, A))) = \mathcal{K}(\ell^2(\mathbb{N}) \otimes A)$.

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So $\psi(p) = \varphi^{-1} p \varphi =: \tilde{p}$ is a projection in $\mathcal{L}(\ell^2(\mathbb{N}) \otimes A)$. To see how $\tilde{p}$ acts on $\ell^2(\mathbb{N}) \otimes A$, let $f \in \ell^2(\mathbb{N})$, $a \in A$ and $\{e_n\}$ be the usual orthonormal basis in $\ell^2(\mathbb{N})$. Then $\tilde{p}(f \otimes a) = \varphi^{-1}(p\varphi(f \otimes a))$, and

$$p\varphi(f \otimes a) = (f(i)\overline{\alpha}_i(1)a)_{i\in\mathbb{N}} = \lim_{k\to\infty} \varphi\Big(\sum_{i=0}^{k} f(i)e_i \otimes \overline{\alpha}_i(1)a\Big).$$

Therefore

$$\tilde{p}(f \otimes a) = \varphi^{-1}(p\varphi(f \otimes a)) = \lim_{k\to\infty} \sum_{i=0}^{k} f(i)e_i \otimes \overline{\alpha}_i(1)a,$$

and hence $p[\mathcal{K}(\ell^2(\mathbb{N}, A))]p \simeq \tilde{p}[\mathcal{K}(\ell^2(\mathbb{N}) \otimes A)]\tilde{p}$.

Example 4.3. We now want to compare our results with [10, Section 6]. Consider a system consisting of the C^*-algebra $\mathbf{c} := \overline{\text{span}}\{1_n : n \in \mathbb{N}\}$ of convergent sequences, and the action τ of $\mathbb{N}$ generated by the usual forward shift (nonunital endomorphism) on $\mathbf{c}$. The ideal $\mathbf{c_0} := \overline{\text{span}}\{1_x - 1_y : x < y \in \mathbb{N}\}$, of sequences in $\mathbf{c}$ convergent to 0, is an extendible τ-invariant in the sense of [1, 5]. So we can also consider the systems $(\mathbf{c_0}, \mathbb{N}, \tau)$ and $(\mathbf{c}/\mathbf{c_0}, \mathbb{N}, \tilde{\tau})$, where the action $\tilde{\tau}_n$ of the quotient $\mathbf{c}/\mathbf{c_0}$ is given by $\tilde{\tau}_n(1_x + \mathbf{c_0}) = \tau_n(1_x) + \mathbf{c_0}$. We show that the three rows of exact sequences in [10, Theorem 6.1], are given by applying our results to $(\mathbf{c}, \mathbb{N}, \tau)$, $(\mathbf{c_0}, \mathbb{N}, \tau)$ and $(\mathbf{c}/\mathbf{c_0}, \mathbb{N}, \tilde{\tau})$.

The crossed product $\mathbf{c} \times_{\tau}^{\text{piso}} \mathbb{N}$ of $(\mathbf{c}, \mathbb{N}, \tau)$ is, by [10, Proposition 5.1], the universal algebra generated by a power partial isometry v: a covariant partial-isometric representation (i_c, v) of $(\mathbf{c}, \mathbb{N}, \tau)$ is defined by $i_c(1_n) = v_n v_n^*$. Let $p = \pi_\tau(1)$ be the projection in $\mathcal{T}_{\mathbf{c},\tau}$, and the partial-isometric representation $w : n \mapsto w_n = pS_n^* p$ of $\mathbb{N}$ in $p\mathcal{T}_{\mathbf{c},\tau}p$ gives a representation π_w of $\mathbf{c}$ where $\pi_w(1_x) = w_x w_x^*$, such that (π_w, w) is a covariant partial-isometric representation of $(\mathbf{c}, \mathbb{N}, \tau)$ in $p\mathcal{T}_{\mathbf{c},\tau}p$. This π_w is the homomorphism $k_{\mathbf{c}} : \mathbf{c} \to p\mathcal{T}_{\mathbf{c},\tau}p$ defined by Proposition 3.2, and the covariant representation (π_w, w) is $(k_{\mathbf{c}}, w)$. So $\pi_w \times w = k_{\mathbf{c}} \times w$ is an isomorphism of $\mathbf{c} \times_{\tau}^{\text{piso}} \mathbb{N}$ onto the C^*-algebra $p\mathcal{T}_{\mathbf{c},\tau}p$.

Moreover, the injective homomorphism $\Psi : p\,[K(\ell^2(\mathbb{N}, \mathbf{c}))]\,p \to (\mathbf{c} \times_{\tau}^{\text{piso}} \mathbb{N}, i_{\mathbf{c}}, v)$ in Theorem 4.1 satisfies

$$\Psi(pS_i \pi_\tau(1_n)(1 - SS^*)S_j^* p) = v_i^* i_{\mathbf{c}}(1_n)(1 - v^*v)v_j = v_i^* v_n v_n^*(1 - v^*v)v_j,$$

and the latter is a spanning element $g_{i,j}^n$ of $\ker \varphi_T$ by [10, Lemma 6.2]. Consequently, the ideal $p[K(\ell^2(\mathbb{N}, \mathbf{c}))]p$, in our Theorem 4.1, is the C^* algebra $\mathcal{A} = \pi^*(\ker \varphi_T)$ of [10, Proposition 6.9], where the homomorphism $\varphi_T : \mathbf{c} \times_{\tau}^{\text{piso}} \mathbb{N} \to \mathcal{T}(\mathbb{Z})$ is induced by the Toeplitz representation $n \mapsto T_n$. Now the Toeplitz (isometric) representation $T : n \mapsto T_n$ on $\ell^2(\mathbb{N})$ gives the isomorphism of $\mathbf{c} \times_{\tau}^{\text{iso}} \mathbb{N}$ onto the Toeplitz algebra $\mathcal{T}(\mathbb{Z})$, and $\mathbf{c_0} \times_{\tau}^{\text{iso}} \mathbb{N}$ onto the algebra $K(\ell^2(\mathbb{N}))$ of compact operators on $\ell^2(\mathbb{N})$. Then the second row exact sequence in [10, Theorem 6.1] follows from the commutative

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diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p\,[K(\ell^2(\mathbb{N},\mathbf{c}))]p & \xrightarrow{\Psi} & \mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N} & \xrightarrow{\phi} & \mathbf{c}\times_\tau^{\mathrm{iso}}\mathbb{N} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle T} & & \\
0 & \longrightarrow & \ker(\varphi_T)\overset{\pi^*}{\simeq}\mathcal{A} & \xrightarrow{(\pi^*)^{-1}} & \mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N} & \xrightarrow{\varphi_T} & \mathcal{T}(\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

Next we proceed similarly for $(\mathbf{c_0},\mathbb{N},\tau)$ and $(\mathbf{c}/\mathbf{c_0},\mathbb{N},\tilde{\tau})$ to get the first and third row exact sequences of diagram (6.1) in [10, Theorem 6.1]. We know from [5, Theorem 2.2] that $\mathbf{c_0}\times_\tau^{\mathrm{piso}}\mathbb{N}$ embeds in $(\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N}, i_c, v)$ as the ideal $D=\overline{\operatorname{span}}\{v_i^* i_c(1_s-1_t)v_j : s<t,\, i,j\in\mathbb{N}\}$, such that the quotient $(\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N})/(\mathbf{c_0}\times_\tau^{\mathrm{piso}}\mathbb{N})\simeq \mathbf{c}/\mathbf{c_0}\times_{\tilde{\tau}}^{\mathrm{piso}}\mathbb{N}$. Then the isomorphism Φ in [5, Corollary 3.1] together with the isomorphism π in [10, Proposition 6.9] give the relations $\mathbf{c_0}\times_\tau^{\mathrm{piso}}\mathbb{N}\overset{\Phi}{\simeq}\ker(\varphi_{T^*})\overset{\pi}{\simeq}\mathcal{A}$, where the homomorphism $\varphi_{T^*}:\mathbf{c}\times_\tau^{\mathrm{piso}}\mathbb{N}\to\mathcal{T}(\mathbb{Z})$ is associated to the partial-isometric representation $n\mapsto T_n^*$.

Let $q=\overline{\pi}_\tau(1_{M(\mathbf{c_0})})$ be the projection in $M(\mathcal{T}_{\mathbf{c_0},\tau})$. Then

$$q[K(\ell^2(\mathbb{N},\mathbf{c_0}))]q=\overline{\operatorname{span}}\{qS_i\pi_\tau(1_m-1_{m+1})(1-SS^*)S_j^*q : i,j\le m\}$$

and

$$\xi_{ijm}:=\Psi(qS_i\pi_\tau(1_m-1_{m+1})(1-SS^*)S_j^*q)=g_{i,j}^m-g_{i,j}^{m+1}=f_{m-i,m-j}^m-f_{m-i,m-j}^{m+1}$$

where $g_{i,j}^m$ and $f_{i,j}^m$ are defined in [10, Lemma 6.2]. So ξ_{ijm} is, by [10, Lemma 6.4], the spanning element of the ideal $\mathcal{I}:=\ker(\varphi_{T^*})\cap\ker(\varphi_T)$. We use the isomorphism π given by [10, Proposition 6.5] to identify $\mathcal{I}$ with $\mathcal{A}_0$, leading to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & q[K(\ell^2(\mathbb{N},\mathbf{c_0}))]q & \xrightarrow{\Psi} & \mathbf{c_0}\times_\tau^{\mathrm{piso}}\mathbb{N} & \xrightarrow{\phi} & \mathbf{c_0}\times_\tau^{\mathrm{iso}}\mathbb{N} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle T} & & \\
0 & \longrightarrow & \mathcal{I}\overset{\pi}{\simeq}\mathcal{A}_0 & \xrightarrow{\mathrm{id}} & \ker(\varphi_{T^*})\overset{\pi}{\simeq}\mathcal{A} & \xrightarrow{\epsilon_\infty} & \mathcal{K}(\ell^2(\mathbb{N})) & \longrightarrow & 0
\end{array}$$

Finally, for the system $(\mathbf{c}/\mathbf{c_0},\mathbb{N},\tilde{\tau})$, we first note that it is equivariant to $(\mathbb{C},\mathbb{N},\mathrm{id})$. So in this case, we have $rK(\ell^2(N,\mathbb{C}))r=K(\ell^2(\mathbb{N}))$, and $\mathbb{C}\times_{\mathrm{id}}^{\mathrm{piso}}\mathbb{N}\overset{\rho}{\simeq}\mathcal{T}(\mathbb{Z})$ where the isomorphism ρ is given by the partial-isometric representation $n\mapsto T_n^*$, and identify $(\mathbb{C}\times_{\mathrm{id}}^{\mathrm{iso}}\mathbb{N}, j_\mathbb{N})\simeq\mathbb{C}\times_{\mathrm{id}}\mathbb{Z}\simeq(C^*(\mathbb{Z}),u)$ with the algebra $C(\mathbb{T})$ of continuous functions on $\mathbb{T}$ using $\delta: j_\mathbb{N}(n)\mapsto u_{-n}\in C^*(\mathbb{Z})\mapsto(z\mapsto\overline{z}^n)\in C(\mathbb{T})$. Then we get the third row exact sequence of diagram (6.1) of [10, Theorem 6.1]:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K(\ell^2(\mathbb{N})) & \xrightarrow{\Psi} & \mathbb{C}\times_{\mathrm{id}}^{\mathrm{piso}}\mathbb{N} & \xrightarrow{\phi} & \mathbb{C}\times_{\mathrm{id}}^{\mathrm{iso}}\mathbb{N} & \longrightarrow & 0\\
 & & & \searrow & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\delta} & \nearrow & \\
 & & & & \mathcal{T}(\mathbb{Z}) & \xrightarrow{\psi_T} & C(\mathbb{T}) & &
\end{array}$$

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Remark 4.4. We have seen in Example 4.3 the three row exact sequences of [10, Diagram 6.1] computed from our results. The three column exact sequences can actually be obtained by [5, Theorem 2.2, Corollary 3.1]. Although these do not imply the commutativity of all rows and columns (because we have not obtained the analogous theorem of [5, Theorem 2.2] for the algebra $\mathcal{T}_{(A,\mathbb{N},\alpha)}$), nevertheless it follows from our results that the algebras $\mathcal{A}$ and $\mathcal{A}_0$ appearing in [10, Diagram 6.1] are Morita equivalent to $\mathbf{c} \otimes K(\ell^2(\mathbb{N}))$ and $\mathbf{c_0} \otimes K(\ell^2(\mathbb{N}))$, respectively. This is helpful in particular for describing the primitive ideal space of $\mathbf{c} \times_{\tau}^{\text{piso}} \mathbb{N}$.

Example 4.5. If $(A, \mathbb{N}, \alpha)$ is a system of a C^*-algebra for which $\overline{\alpha}(1) = 1$, then (4.2) is the exact sequence of [7, Theorem 1.5]. This is because $p = \overline{\pi}_\alpha(1)$ is the identity of $\mathcal{T}_{(A,\mathbb{N},\alpha)}$, so $A \times_\alpha^{\text{piso}} \mathbb{N}$ is isomorphic to $\mathcal{T}_{(A,\mathbb{N},\alpha)}$ and $p[\mathcal{K}(\ell^2(\mathbb{N}, A))]p = \mathcal{K}(\ell^2(\mathbb{N}, A))$. Let $(A_\infty, \beta^n)_n$ be the limit of the direct sequence (A_n) where $A_n = A$ for every n and $\alpha_{m-n} : A_n \to A_m$ for $n \leq m$. All the bonding maps $\beta^i : A_i \to A_\infty$ extend trivially to the multiplier algebras and preserve the identity. Therefore $(A \times_\alpha^{\text{iso}} \mathbb{N}, j_A, j_\mathbb{N}) \simeq (A_\infty \times_{\alpha_\infty} \mathbb{Z}, i_\infty, u)$ in which the isomorphism is given by $\iota(j_\mathbb{N}(n)^* j_A(a) j_\mathbb{N}(m) = u_n^* i_\infty(\beta^0(a)) u_m$, and then the commutative diagram follows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p[K(\ell^2(\mathbb{N}, A))]p & \xrightarrow{\Psi} & A \times_\alpha^{\text{piso}} \mathbb{N} & \xrightarrow{\phi} & A \times_\alpha^{\text{iso}} \mathbb{N} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & K(\ell^2(\mathbb{N}, A)) & \xrightarrow{\text{id}} & \mathcal{T}_{(A,\mathbb{N},\alpha)} & \xrightarrow{q} & A_{\alpha_\infty} \times_{\alpha_\infty} \mathbb{Z} & \longrightarrow & 0
\end{array}$$

5. The partial-isometric crossed product of a system by a semigroup of automorphisms

Suppose that (A, Γ^+, α) is a system of an action $\alpha : \Gamma^+ \to \text{AutA}$ by automorphisms on A, and consider the distinguished system $(B_{\Gamma^+}, \Gamma^+, \tau)$ of the commutative C^*-algebra B_{Γ^+} by a semigroup of endomorphisms $\tau_x \in \text{End}(B_{\Gamma^+})$. Then $x \mapsto \tau_x \otimes \alpha_x^{-1}$ defines an action γ of Γ^+ by endomorphisms of $B_{\Gamma^+} \otimes A$. So we have a system $(B_{\Gamma^+} \otimes A, \Gamma^+, \gamma)$ by a semigroup of endomorphisms. We prove in the proposition below that the isometric crossed product $(B_{\Gamma^+} \otimes A) \times_\gamma^{\text{iso}} \Gamma^+$ is $A \times_\alpha^{\text{piso}} \Gamma^+$.

Proposition 5.1. *Suppose that* $\alpha : \Gamma^+ \to \text{AutA}$ *is an action by automorphisms on a* C^*-*algebra* A *of the positive cone* Γ^+ *of a totally ordered abelian group* Γ. *Then the partial-isometric crossed product* $A \times_\alpha^{\text{piso}} \Gamma^+$ *is isomorphic to the isometric crossed product* $((B_{\Gamma^+} \otimes A) \times_\gamma^{\text{iso}} \Gamma^+, j)$. *More precisely, the* C^*-*algebra* $(B_{\Gamma^+} \otimes A) \times_\gamma^{\text{iso}} \Gamma^+$ *together with a pair of homomorphisms* $(k_A, k_{\Gamma^+}) : (A, \Gamma^+, \alpha) \to M((B_{\Gamma^+} \otimes A) \times_\gamma^{\text{iso}} \Gamma^+)$ *defined by* $k_A(a) = j_{B_{\Gamma^+} \otimes A}(1 \otimes a)$ *and* $k_{\Gamma^+}(x) = j_{\Gamma^+}(x)^*$ *is a partial-isometric crossed product for* (A, Γ^+, α).

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PROOF. Every $k_{\Gamma^+}(x)$ satisfies $k_{\Gamma^+}(x)k_{\Gamma^+}(x)^* = j_{\Gamma^+}(x)^* j_{\Gamma^+}(x) = 1$, and (k_A, k_{Γ^+}) is a partial-isometric covariant representation for (A, Γ^+, α):

$$\begin{aligned} j_{B_{\Gamma^+}\otimes A}(1 \otimes \alpha_x(a)) &= j_{\Gamma^+}(x)^* j_{\Gamma^+}(x) j_{B_{\Gamma^+}\otimes A}(1 \otimes \alpha_x(a)) j_{\Gamma^+}(x)^* j_{\Gamma^+}(x) \\ &= j_{\Gamma^+}(x)^* j_{B_{\Gamma^+}\otimes A}(\tau_x \otimes \alpha_x^{-1}(1 \otimes \alpha_x(a))) j_{\Gamma^+}(x) \\ &= j_{\Gamma^+}(x)^* j_{B_{\Gamma^+}\otimes A}(1_x \otimes a) j_{\Gamma^+}(x) \\ &= j_{\Gamma^+}(x)^* j_{B_{\Gamma^+}}(1_x) j_A(a) j_{\Gamma^+}(x) \\ &= j_{\Gamma^+}(x)^* j_{\Gamma^+}(x) j_{\Gamma^+}(x)^* j_A(a) j_{\Gamma^+}(x) \\ &= j_{\Gamma^+}(x)^* j_{B_{\Gamma^+}\otimes A}(1 \otimes a) j_{\Gamma^+}(x), \end{aligned}$$

and $j_{\Gamma^+}(x) j_{\Gamma^+}(x)^* j_{B_{\Gamma^+}\otimes A}(1 \otimes a) = j_{B_{\Gamma^+}\otimes A}(1 \otimes a) j_{\Gamma^+}(x) j_{\Gamma^+}(x)^*$ because $j_{B_{\Gamma^+}\otimes A}(1_x \otimes a) = j_A(a) j_{B_{\Gamma^+}}(1_x)$.

Suppose that (π, V) is a partial-isometric covariant representation of (A, Γ^+, α) on H. We want a nondegenerate representation $\pi \times V$ of the isometric crossed product $(B_{\Gamma^+} \otimes A) \times_\gamma^{\mathrm{iso}} \Gamma^+$ which satisfies $(\pi \times V) \circ k_A(a) = \pi(a)$ and $\overline{(\pi \times V)} \circ k_{\Gamma^+}(x) = V_x$ for all $a \in A$ and $x \in \Gamma^+$.

Since $V_x V_x^* = 1$ for all $x \in \Gamma^+$, $x \mapsto V_x^*$ is an isometric representation of Γ^+, and therefore $\pi_{V^*}(1_x) = V_x^* V_x$ defines a representation π_{V^*} of B_{Γ^+} such that (π_{V^*}, V^*) is an isometric covariant representation of $(B_{\Gamma^+}, \Gamma^+, \tau)$. Moreover, π_{V^*} commutes with π because

$$\pi_{V^*}(1_x)\pi(a) = V_x^* V_x \pi(a) = \pi(a) V_x^* V_x = \pi(a)\pi_{V^*}(1_x).$$

Thus $\pi_{V^*} \otimes \pi$ is a nondegenerate representation of $B_{\Gamma^+} \otimes A$ on H, and $\pi_{V^*} \otimes \pi(1_y \otimes a) = \pi_{V^*}(1_y)\pi(a) = \pi(a)\pi_{V^*}(1_y)$. We clarify that $(\pi_{V^*} \otimes \pi, V^*)$ is in fact an isometric covariant representation of the system $(B_{\Gamma^+} \otimes A, \Gamma^+, \gamma)$:

$$\begin{aligned} \pi_{V^*} \otimes \pi(\tau_x \otimes \alpha_x^{-1}(1_y \otimes a)) &= \pi_{V^*}(\tau_x(1_y))\pi(\alpha_x^{-1}(a)) = V_x^* \pi_{V^*}(1_y) V_x \pi(\alpha_x^{-1}(a)) \\ &= V_x^* \pi_{V^*}(1_y) \pi(\alpha_x(\alpha_x^{-1}(a))) V_x \quad \text{by piso covariance of } (\pi, V) \\ &= V_x^* \pi_{V^*}(1_y) \pi(a) V_x = V_x^* (\pi_{V^*} \otimes \pi)(1_y \otimes a) V_x. \end{aligned}$$

Then $\rho := (\pi_{V^*} \otimes \pi) \times V^*$ is a nondegenerate representation of $(B_{\Gamma^+} \otimes A) \times_\gamma^{\mathrm{iso}} \Gamma^+$ which satisfies the requirements

$$\rho(k_A(a)) = \rho(j_{B_{\Gamma^+}\otimes A}(1 \otimes a)) = \pi_{V^*} \otimes \pi(1 \otimes a) = \pi(a)$$

and $\overline{\rho}(k_{\Gamma^+}(x)) = \overline{\rho}(j_{\Gamma^+}(x)^*) = V_x$. Finally, the span of $\{k_{\Gamma^+}(x)^* k_A(a) k_{\Gamma^+}(y)\}$ is dense in $(B_{\Gamma^+} \otimes A) \times_\gamma^{\mathrm{iso}} \Gamma^+$ because

$$k_{\Gamma^+}(x)^* k_A(a) k_{\Gamma^+}(y) = j_{\Gamma^+}(y)^* j_{B_{\Gamma^+}\otimes A}(1_{x+y} \otimes \alpha_{x+y}^{-1}(a)) j_{\Gamma^+}(x).$$

This concludes the proof. □

Proposition 5.1 gives an isomorphism $k : (A \times_\alpha^{\mathrm{piso}} \Gamma^+, i) \to ((B_{\Gamma^+} \otimes A) \times_\gamma^{\mathrm{iso}} \Gamma^+, j)$ which satisfies $k(i_{\Gamma^+}(x)) = j_{\Gamma^+}(x)^*$ and $k(i_A(a)) = j_{B_{\Gamma^+}\otimes A}(1 \otimes a)$. This isomorphism maps the ideal $\ker \phi$ of $A \times_\alpha^{\mathrm{piso}} \Gamma^+$ in Proposition 2.3 isomorphically onto the ideal

$$\mathcal{I} := \overline{\mathrm{span}}\{j_{B_{\Gamma^+\otimes A}}(1 \otimes a) j_{\Gamma^+}(x)[1 - j_{\Gamma^+}(t) j_{\Gamma^+}(t)^*] j_{\Gamma^+}(y)^* : a \in A,\, x, y, t \in \Gamma^+\}$$

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of $(B_{\Gamma^+} \otimes A) \times_{\gamma}^{\text{iso}} \Gamma^+$. We identify this ideal in Lemma 5.2. First we need to recall from [1] the notion of extendible ideals. It was shown there that

$$B_{\Gamma^+,\infty} := \overline{\text{span}}\{1_x - 1_y : x < y \in \Gamma^+\}$$

is an extendible τ-invariant ideal of B_{Γ^+}. Thus $B_{\Gamma^+,\infty} \otimes A$ is an extendible γ-invariant ideal of $B_{\Gamma^+} \otimes A$. We can therefore consider the system $(B_{\Gamma^+,\infty} \otimes A), \Gamma^+, \gamma)$. Extendibility of ideal is required to ensure that the crossed product $(B_{\Gamma^+,\infty} \otimes A) \times_{\gamma}^{\text{iso}} \Gamma^+$ embeds naturally as an ideal of $(B_{\Gamma^+} \otimes A) \times_{\gamma}^{\text{iso}} \Gamma^+$ such that the quotient is the crossed product of the quotient algebra $B_{\Gamma^+} \otimes A/B_{\Gamma^+,\infty} \otimes A$ [1, Theorem 3.1].

Lemma 5.2. *The ideal $\mathcal{I}$ is $(B_{\Gamma^+,\infty} \otimes A) \times_{\gamma}^{\text{iso}} \Gamma^+$.*

Proof. We know from [1, Theorem 3.1] that the ideal $(B_{\Gamma^+,\infty} \otimes A) \times_{\gamma}^{\text{iso}} \Gamma^+$ is spanned by

$$\{j_{\Gamma^+}(v)^* j_{B_{\Gamma^+ \otimes A}}((1_s - 1_t) \otimes a) j_{\Gamma^+}(w) : s < t, v, w \text{ in } \Gamma^+, a \in A\}.$$

So to prove the lemma, it is enough to show that $\mathcal{I}$ and $(B_{\Gamma^+,\infty} \otimes A) \times_{\gamma}^{\text{iso}} \Gamma^+$ contain each other.

We compute on their generator elements in next paragraph using the fact that the covariant representation $(j_{B_{\Gamma^+} \otimes A}, j_{\Gamma^+})$ gives a unital homomorphism $j_{B_{\Gamma^+}}$ which commutes with the nondegenerate homomorphism j_A, and that the pair $(j_{B_{\Gamma^+}}, j_{\Gamma^+})$ is a covariant representation of $(B_{\Gamma^+}, \Gamma^+, \tau)$. Each isometry $j_{\Gamma^+}(x)$ is not a unitary, so the pair (j_A, j_{Γ^+}) fails to be a covariant representation of $(A, \Gamma^+, \alpha^{-1})$. However, it satisfies the equation $j_A(\alpha_x^{-1}(a)) j_{\Gamma^+}(x) = j_{\Gamma^+}(x) j_A(a)$ for all $a \in A$ and $x \in \Gamma^+$.

Let ξ be a spanning element of $\mathcal{I}$. If $x < y$ and t are in Γ^+, then $j_{\Gamma^+}(y)^* = j_{\Gamma^+}(x)^* j_{\Gamma^+}(y - x)^*$ and

$$\begin{aligned} &j_{\Gamma^+}(x)[1 - j_{\Gamma^+}(t) j_{\Gamma^+}(t)^*] j_{\Gamma^+}(y)^* \\ &\quad = (j_{\Gamma^+}(x) j_{\Gamma^+}(x)^* - j_{\Gamma^+}(x+t) j_{\Gamma^+}(x+t)^*) j_{\Gamma^+}(y-x)^* \\ &\quad = \overline{j}_{B_{\Gamma^+} \otimes A}((1_x - 1_{x+t}) \otimes 1_{M(A)}) j_{\Gamma^+}(y-x)^*, \end{aligned}$$

so

$$\begin{aligned} \xi &= j_{B_{\Gamma^+} \otimes A}((1_x - 1_{x+t}) \otimes a) j_{\Gamma^+}(y-x)^* \\ &= j_{\Gamma^+}(y-x)^* j_{B_{\Gamma^+} \otimes A}(\gamma_{y-x}((1_x - 1_{x+t}) \otimes a)) \\ &= j_{\Gamma^+}(y-x)^* j_{B_{\Gamma^+} \otimes A}((1_y - 1_{y+t}) \otimes \alpha_{y-x}^{-1}(a)). \end{aligned}$$

If $x \geq y$, then $j_{\Gamma^+}(x) = j_{\Gamma^+}(x-y) j_{\Gamma^+}(y)$ and

$$\begin{aligned} &j_{\Gamma^+}(x)[1 - j_{\Gamma^+}(t) j_{\Gamma^+}(t)^*] j_{\Gamma^+}(y)^* \\ &\quad = j_{\Gamma^+}(x-y)[j_{\Gamma^+}(y) j_{\Gamma^+}(y)^* - j_{\Gamma^+}(y+t) j_{\Gamma^+}(y+t)^*] \\ &\quad = j_{\Gamma^+}(x-y) \overline{j}_{B_{\Gamma^+} \otimes A}((1_y - 1_{y+t}) \otimes 1_{M(A)}) j_{\Gamma^+}(x-y)^* j_{\Gamma^+}(x-y) \\ &\quad = \overline{j}_{B_{\Gamma^+} \otimes A}((1_x - 1_{x+t}) \otimes 1_{M(A)}) j_{\Gamma^+}(x-y), \end{aligned}$$

so $\xi = j_{B_{\Gamma^+}\otimes A}((1_x - 1_{x+t})\otimes a)j_{\Gamma^+}(x-y)$, and therefore $\mathcal{I}$ is contained in $(B_{\Gamma^+,\infty}\otimes A)\times_\gamma^{\mathrm{iso}}\Gamma^+$.

For the reverse inclusion, let $\eta = j_{B_{\Gamma^+}\otimes A}((1_s - 1_t)\otimes a)j_{\Gamma^+}(x)$ be a generator of $(B_{\Gamma^+,\infty}\otimes A)\times_\gamma^{\mathrm{iso}}\Gamma^+$. Then $\eta = j_A(a)[j_{\Gamma^+}(s)j_{\Gamma^+}(s)^* - j_{\Gamma^+}(t)j_{\Gamma^+}(t)^*]j_{\Gamma^+}(x)$, and a similar computation shows that

$$\begin{aligned}
&[j_{\Gamma^+}(s)j_{\Gamma^+}(s)^* - j_{\Gamma^+}(t)j_{\Gamma^+}(t)^*]j_{\Gamma^+}(x)\\
&\quad = \begin{cases} j_{\Gamma^+}(s)[1 - j_{\Gamma^+}(t-s)j_{\Gamma^+}(t-s)^*]j_{\Gamma^+}(s-x)^* & \text{for } x \le s < t,\\ j_{\Gamma^+}(x)[1 - j_{\Gamma^+}(t-x)j_{\Gamma^+}(t-x)^*] & s < x < t,\\ 0 & \text{for } t = x \text{ or } s < t < x,\end{cases}
\end{aligned}$$

which implies that $\eta \in \mathcal{I}$. □

An isometric crossed product is isomorphic to a full corner in the ordinary crossed product by a dilated action. The action $\tau : \Gamma^+ \to \mathrm{End}(\mathrm{B}_{\Gamma^+})$ is dilated to the action $\tau : \Gamma \to \mathrm{Aut}(\mathrm{B}_\Gamma)$ where $\tau_s(1_x) = 1_{x+s}$ acts on the algebra $B_\Gamma = \overline{\mathrm{span}}\{1_x : x \in \Gamma\}$. We refer to [3, Lemma 3.2] to see that a dilation of $(B_{\Gamma^+}\otimes A, \Gamma^+, \gamma)$ gives the system $(B_\Gamma\otimes A, \Gamma, \gamma_\infty)$, in which $\gamma_\infty = \tau\otimes\alpha^{-1}$ acts by automorphisms on the algebra $B_\Gamma\otimes A$. The bonding homomorphism h_s for $s \in \Gamma^+$ is given by

$$h_s : (1_x\otimes a) \in B_{\Gamma^+}\otimes A \mapsto (1_x\otimes a) \in \overline{\mathrm{span}}\{1_y : y \ge -s\}\otimes A \hookrightarrow B_\Gamma\otimes A.$$

This homomorphism extends to the multiplier algebras, which we write as $\overline{h}_0$, and it carries the identity $1_0\otimes 1_{M(A)} \in M(B_{\Gamma^+}\otimes A)$ into the projection $\overline{h}_0(1_0\otimes 1_{M(A)}) \in M(B_\Gamma\otimes A)$. Let

$$p := \overline{j}_{B_\Gamma\otimes A}(\overline{h}_0(1_0\otimes 1_{M(A)}))$$

be the projection in the crossed product $M((B_\Gamma\otimes A)\times_{\gamma_\infty}\Gamma)$. Then it follows from [1, Theorem 2.4] or [8, Theorem 2.4] that $(B_{\Gamma^+}\otimes A)\times_\gamma^{\mathrm{iso}}\Gamma^+$ is isomorphic onto the full corner $p\,[(B_\Gamma\otimes A)\times_{\gamma_\infty}\Gamma]\,p$.

Corollary 5.3. *There is an isomorphism of $A\times_\alpha^{\mathrm{piso}}\Gamma^+$ onto the full corner $p[(B_\Gamma\otimes A)\times_{\gamma_\infty}\Gamma]p$ of the crossed product $(B_\Gamma\otimes A)\times_{\gamma_\infty}\Gamma$, such that the ideal* $\ker\phi$ *of $A\times_\alpha^{\mathrm{piso}}\Gamma^+$ in Proposition 3.2 is isomorphic onto the ideal $p[(B_{\Gamma,\infty}\otimes A)\times_{\gamma_\infty}\Gamma]\ p$, where $B_{\Gamma,\infty} = \overline{\mathrm{span}}\{1_s - 1_t : s < t \in \Gamma\}$.*

Corollary 5.4. *Suppose that $\alpha : \Gamma^+ \to \mathrm{Aut}(A)$ is the trivial action $\alpha_x = \mathrm{identity}$ for all x, and let C_Γ denote the commutator ideal of the Toeplitz algebra $\mathcal{T}(\Gamma)$. Then there is a short exact sequence*

$$0 \longrightarrow A\otimes C_\Gamma \longrightarrow A\times_\alpha^{\mathrm{piso}}\Gamma^+ \xrightarrow{\ \phi\ } A\times_\alpha\Gamma \longrightarrow 0 \tag{5.1}$$

Proof. We have already identified in Lemma 5.2 that the ideal $\mathcal{I}$ is $(B_{\Gamma^+,\infty}\otimes A)\times_{\tau\otimes\mathrm{id}}^{\mathrm{iso}}\Gamma^+$. We know that we have a version of [17, Lemma 2.75] for isometric crossed products, which says that if (C, Γ^+, γ) is a dynamical system and D is any C^*-algebra,

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then $(C \otimes_{\max} D) \times^{\mathrm{iso}}_{\gamma\otimes\mathrm{id}} \Gamma^+$ is isomorphic to $(C \times^{\mathrm{iso}}_{\gamma} \Gamma^+) \otimes_{\max} D$. Applying this to the system $(B_{\Gamma^+,\infty}, \Gamma^+, \tau)$ and the C^*-algebra A,

$$(B_{\Gamma^+,\infty} \otimes A) \times^{\mathrm{iso}}_{\tau\otimes\mathrm{id}} \Gamma^+ \simeq (B_{\Gamma^+,\infty} \times^{\mathrm{iso}}_{\tau} \Gamma^+) \otimes A \simeq \mathcal{C}_\Gamma \otimes A$$

and hence we obtain the exact sequence. □

Remark 5.5. Note that

$$A \times^{\mathrm{piso}}_{\mathrm{id}} \Gamma^+ \simeq (B_{\Gamma^+} \otimes A) \times^{\mathrm{iso}}_{\tau\otimes\mathrm{id}} \Gamma^+ \simeq (B_{\Gamma^+} \times^{\mathrm{iso}}_{\tau} \Gamma^+) \otimes A \simeq \mathcal{T}(\Gamma) \otimes A,$$

and $A \times^{\mathrm{iso}}_{\mathrm{id}} \Gamma^+ \simeq A \times_{\mathrm{id}} \Gamma \simeq A \otimes C^*(\Gamma) \simeq A \otimes C(\hat{\Gamma})$. So (5.1) is the exact sequence

$$0 \longrightarrow A \otimes \mathcal{C}_\Gamma \longrightarrow A \otimes \mathcal{T}(\Gamma) \xrightarrow{\ \phi\ } A \otimes C(\hat{\Gamma}) \longrightarrow 0$$

which is the (maximal) tensor product with the algebra A to the well-known exact sequence $0 \to \mathcal{C}_\Gamma \to \mathcal{T}(\Gamma) \to C(\hat{\Gamma}) \to 0$.

5.1. The Pimsner–Voiculescu extension. Consider a system (A, Γ^+, α) in which every α_x is an automorphism of A. Let $(A \times_\alpha \Gamma, j_A, j_\Gamma)$ be the corresponding group crossed product. The Toeplitz algebra $\mathcal{T}(\Gamma)$ is the C^*-algebra generated by the semigroup $\{T_x : x \in \Gamma^+\}$ of nonunitary isometries T_x, and the commutator ideal $\mathcal{C}_\Gamma$ of $\mathcal{T}(\Gamma)$ generated by the elements $T_sT_s^* - T_tT_t^*$ for $s < t$ is given by $\overline{\mathrm{span}}\{T_r(1 - T_uT_u^*)T_t^* : r, u, t \in \Gamma^+\}$ of $\mathcal{T}(\Gamma)$.

Consider the C^*-subalgebra $\mathcal{T}_{PV}(\Gamma)$ of $M((A \times_\alpha \Gamma) \otimes \mathcal{T}(\Gamma))$ generated by $\{j_A(a) \otimes I : a \in A\}$ and $\{j_\Gamma(x) \otimes T_x : x \in \Gamma^+\}$. Let $\mathcal{S}(\Gamma)$ be the ideal of $\mathcal{T}_{PV}(\Gamma)$ generated by $\{j_A(a) \otimes (T_sT_s^* - T_tT_t^*) : s < t \in \Gamma^+, a \in A\}$.

We claim that $(A \times^{\mathrm{piso}}_{\alpha^{-1}} \Gamma^+, i_A, i_{\Gamma^+}) \simeq \mathcal{T}_{PV}(\Gamma)$, and the isomorphism takes the ideal $\ker(\phi)$ onto $\mathcal{S}(\Gamma)$. To see this, let $\pi(a) := j_A(a) \otimes I$ and $V_x := j_\Gamma(x)^* \otimes T_x^*$. Then (π, V) is a partial-isometric covariant representation of $(A, \Gamma^+, \alpha^{-1})$ in the C^*-algebra $M((A \times_\alpha \Gamma) \otimes \mathcal{T}(\Gamma))$. So we have a homomorphism $\psi : A \times^{\mathrm{piso}}_{\alpha^{-1}} \Gamma^+ \to (A \times_\alpha \Gamma) \otimes \mathcal{T}(\Gamma)$ such that

$$\psi(i_A(a)) = j_A(a) \otimes I \text{ and } \overline{\psi}(i_{\Gamma^+}(x)) = j_\Gamma(x)^* \otimes T_x^* \quad \text{for } a \in A,\ x \in \Gamma^+.$$

Moreover, for $a \in A$ and $x > 0$,

$$\begin{aligned}
\pi(a)(1 - V_x^*V_x) &= (j_A(a) \otimes I)(1 - (j_\Gamma(x) \otimes T_x)(j_\Gamma(x)^* \otimes T_x^*)) \\
&= (j_A(a) \otimes I) - (j_A(a) \otimes I)(j_\Gamma(x) \otimes T_x)(j_\Gamma(x)^* \otimes T_x^*) \\
&= (j_A(a) \otimes I) - (j_A(a) \otimes T_xT_x^*) \\
&= j_A(a) \otimes (I - T_xT_x^*).
\end{aligned}$$

Since $T_xT_x^* \neq I$, the equation $\pi(a)(1 - V_x^*V_x) = 0$ must imply $j_A(a) = 0$ in $A \times_\alpha \Gamma$, and hence $a = 0$ in A. So by [10, Theorem 4.8] the homomorphism ψ is faithful. Thus $A \times^{\mathrm{piso}}_{\alpha^{-1}} \Gamma^+ \simeq \psi(A \times^{\mathrm{piso}}_{\alpha^{-1}} \Gamma^+) = \mathcal{T}_{PV}(\Gamma)$.

The isomorphism $\psi : A \times^{\mathrm{piso}}_{\alpha^{-1}} \Gamma^+ \to \mathcal{T}_{PV}(\Gamma)$ takes the ideal $\ker\phi$ of $A \times_{\alpha^{-1}} \Gamma^+$ to the algebra $\mathcal{S}(\Gamma)$.

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Corollary 5.6 (The Pimsner–Voiculescu extension). *Let* $(A, \mathbb{N}, \alpha)$ *be a system in which* $\alpha \in \mathrm{Aut(A)}$. *Then there is an exact sequence* $0 \to A \otimes \mathcal{K}(\ell^2(\mathbb{N})) \to \mathcal{T}_{PV} \to A \times_\alpha \mathbb{Z} \to 0$.

Proof. Apply Theorem 4.1 to the system $(A, \mathbb{N}, \alpha^{-1})$, and then use the identifications $A \times^{\mathrm{piso}}_{\alpha^{-1}} \mathbb{N} \simeq \mathcal{T}_{PV}(\mathbb{Z})$, $\ker \phi \simeq \mathcal{S}(\Gamma) \simeq \mathcal{K}(\ell^2(\mathbb{N}, A))$ and $A \times_\alpha \mathbb{Z} \simeq A \times_{\alpha^{-1}} \mathbb{Z}$. □

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SRIWULAN ADJI, Institute of Mathematical Sciences. University Malaya, Kuala Lumpur, Malaysia
e-mail: sriwulan.adji@yahoo.com,sriwulan.adji@gmail.com

SAEID ZAHMATKESH, School of Mathematical Sciences, Universiti Sains Malaysia, Penang, Malaysia
e-mail: zahmatkesh.s@gmail.com

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