ON THE INSTABILITY OF STATIONARY SPHERICAL MODELS WITH PURELY RADIAL MOTION

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Models of spherical, self-gravitating stellar systems in which all the trajectories pass initially near the center are interesting to examine from different perspectives. Such models are compatible not only with the idea of explosive cosmogony, but also with the concept of formation of stars from a gaseous, spherical cloud, if they are not affected during formation by significant peculiar motions and thus "fall" into the center. Another question is whether such a system is capable of remaining stable for a long period of time. We will show that such a system is unstable with respect to regular forces and must be reconstructed during the course of one period of revolution of an individual star.

The system may be represented as being composed of separate radial streams, each with its own energy integral E. For the sake of convenience, let us assume that there exists a finite value N of such streams, although this is not important. It is easy to see that the absolute value of the velocity in each stream is given by the formula

$$|V_\nu| = \sqrt{2\,(E_\nu - \Phi(r))},$$

whereas the partial density of the matter is

$$\rho_\nu = \frac{h_\nu}{r^2 |V_\nu(r)|},$$

where ν is the number of the stream, E_ν and h_ν are certain constants characteristic of it, and $\Phi(r)$ is the gravitational potential of the system as a whole. $\rho_\nu = 0$ should be considered to be the turning point. Taking into account Poisson's equation, we derive the basic equation for determining the model structure:

$$\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right) = 4\pi G \sum_{\nu=1}^{N} \frac{h_\nu}{\sqrt{2\,(E_\nu - \Phi(r))}}, \tag{1}$$

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which already appeared in the work of Agekyan [1] for the case of a single stream. There, the asymptotic behavior in the center was observed to be

$$\Phi \sim |\ln r|^{2/3},$$

which is also correct in the general case. This singularity is rather weak and presents no obstacles to the following reasoning. We cannot solve equation (1) in its general form, but we do not require a specific kind of solution. In order to examine the instability, we will make use of the quasi-classical approximation. Only the different orientation of the constant-phase surfaces distinguishes the present problem from the problem of the development of annular perturbations in a cylinder [2]. Namely, let us take the variation of the potential proportional to the sectorial harmonic $Y_n(\theta, \phi)$, i.e.

$$\delta\Phi = \sin^n\theta \cos n\phi \cdot \chi(r), \tag{2}$$

where $\chi(r)$ is a certain slowly-varying function. Then, Poisson's equation may be written in the following form:

$$n(n+1)\delta\Phi = \sin^n\theta \cos n\phi \frac{d}{dr}\left(r^2\frac{d\chi}{dr}\right) - 4\pi G r^2 \delta\rho. \tag{3}$$

In the approximation under consideration, $n >> 1$, and on the right hand side of (3), we must leave the terms which are proportional to n^2. The additional radial displacement of particles, like the force $\partial\delta\Phi/\partial r$ which acts in this direction, does not increase at all with n. On the basis of general properties of symmetry [3], it may be stated immediately that the transverse displacement must depend on the angular coordinate like grad $Y_n(\theta, \phi)$, and in the zone $|\theta - \frac{\pi}{2}| \sim n^{-1/2}$ where expression (2) is significantly different from zero, $\partial Y_n/\partial\theta$ is of a smaller order than $\partial Y_n/\partial\phi$. In calculating the divergence, which is necessary for deriving the continuity equation, this difference will increase even further. Thus, it is sufficient to limit oneself to displacements along θ near the equator. In relating the linear displacement x_ν to the particles of each stream, we derive the simplified form of the continuity equation:

$$\delta\rho_\nu = -\frac{\rho_\nu}{2r}\left(\frac{\partial x_\nu^+}{\partial\phi} + \frac{\partial x_\nu^-}{\partial\phi}\right). \tag{4}$$

The superscripts + and − in (4), just as below, are brought in to distinguish between the two halves of the stream, as their velocities V_ν are of opposite sign. Meanwhile, Poisson's equation, after the truncation of low-order terms, has the following form:

$$n^2\delta\Phi = -\frac{\partial^2\delta\Phi}{\partial\phi^2} = -4\pi G r^2 \sum_{\nu=1}^{N} \delta\rho_\nu. \tag{5}$$

Using (4) and (5), it is now an easy task to derive an equation of motion along θ in terms of the transverse deflection $x_\nu(r)$ and the angular momentum per unit mass $J_\nu(r)$:

$$\frac{dx_\nu^+}{dt} = \frac{J_\nu^+ + V_\nu x_\nu^+}{r}, \quad \frac{dx_\nu^-}{dt} = \frac{J_\nu^- - V_\nu x_\nu^-}{r},$$

$$\frac{dJ_\nu^+}{dt} = \frac{dJ_\nu^-}{dt} = 2\pi G r \sum_{\mu=1}^{N} \rho_\mu (x_\mu^+ + x_\mu^-). \tag{6}$$

Instability is proved by constructing the Lyapunov functional

$$H(t) = \sum_{\nu=1}^{N} \int \frac{x_\nu^+ J_\nu^+ + x_\nu^- J_\nu^-}{r} dm_\nu, \tag{7}$$

where $dm_\nu = r^2 \rho_\nu dr$ stands for the mass element along a certain diameter (more precisely, along the elementary cone), whereas the integrals are taken between the turning point of the corresponding stream. Differentiating (7) and keeping in mind the invariance of dm_ν, we get after some calculations

$$\frac{dH}{dt} = \sum_{\nu=1}^{N} \int \frac{(J_\nu^+)^2 + (J_\nu^-)^2}{r^2} dm_\nu + 2\pi G r^2 \int \left[\sum_{\nu=1}^{N} \rho_\nu (x_\nu^+ + x_\nu^-) \right]^2 dr > 0.$$

This means that even in our scheme there exist no stable modes at all.

Earlier [4] we had proven the stability of a wide range of models with a spherically-symmetric velocity distribution. It would be interesting to trace the loss of stability in response to a gradual increase in the radial prolateness of the velocity ellipsoid. At present, the difficulty lies in the relative complexity of the known models.

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DISCUSSION

A. M. Fridman: These results appear obvious to me. If during the course of motion along circular orbits, there exists an analog to elastic forces, then during the course of radial motion, there is nothing to prevent an adhesion to neighboring orbits.

V. A. Antonov: From time to time, one should not forego the opportunity to support intuition with solid evidence. In matters of this kind, certainty is quite relative, and only becomes established after careful consideration.

A. M. Fridman: The only thing that bothers me is that you are allowing for perturbed motion in a single direction only.

V. A. Antonov: This is not an arbitrary assumption. It follows with certainty from the comparison of components of grad $\delta\Phi$ with varying directions when $n >> 1$.

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