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THE POSET OF CONJUGACY CLASSES AND DECOMPOSITION OF PRODUCTS IN THE SYMMETRIC GROUP

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ABSTRACT. The action by multiplication of the class of transpositions of the symmetric group $\mathfrak{S}_n$ on the other conjugacy classes defines a graded poset $\mathcal{T}_n$ as described by Birkhoff ([2]). In this paper, the edges of this poset are given a weight and the structure obtained is called the *poset of conjugacy classes of the symmetric group*. We use weights of chains in the posets $\mathcal{T}_n$ to obtain new formulas for the decomposition of products of conjugacy classes of the symmetric group in its group algebra as linear combinations of conjugacy classes and we derive a new identity involving partitions of n.

Introduction. Consider the conjugacy classes C_μ (μ in subscript) of the symmetric group $\mathfrak{S}_n$ indexed with partitions $\mu \vdash n$, as elements C^μ (μ in superscript) of $\mathbb{Q}[\mathfrak{S}_n]$, the group algebra of $\mathfrak{S}_n$ over the field rationals $\mathbb{Q}$; more precisely set $C^\mu = \sum_{\sigma \in C_\mu} \sigma$. The set $\{C^\mu\}_{\mu \vdash n}$ form a basis of the center $\mathcal{C}[\mathfrak{S}_n]$ of the group algebra and the product $C^\lambda * C^\mu$ of two conjugacy classes in $\mathcal{C}[\mathfrak{S}_n]$ can be decomposed as a linear combination of conjugacy classes with integer coefficients:

$$C^\lambda * C^\mu = \sum_{\gamma \vdash n} c^\gamma_{\lambda\mu} C^\gamma \tag{1}$$

The coefficients $c^\gamma_{\lambda\mu}$ in (1), also written $c^\gamma_{\lambda\mu} = C^\lambda * C^\mu|_{C^\gamma}$, count the number of ways a permutation of type γ can be decomposed as a product of a permutation of type λ with a permutation of type μ. More generally, for partitions $\lambda^1, \lambda^2, \ldots, \lambda^m$ of n, the coefficient $C^{\lambda^1} * C^{\lambda^2} * \cdots * C^{\lambda^m}|_{C^\gamma}$ is the number of ways a permutation of type γ can be written as a product of permutations of types $\lambda^1, \lambda^2, \ldots, \lambda^m$. In this paper, we use the action by multiplication of the class of transpositions on the other classes of the symmetric group obtained through the regular representation of the center of the group algebra to give a partial order on partitions. This order gives rise to a graded poset $\mathcal{T}_n$ called *the poset of conjugacy classes of the symmetric group*. The vertices of the poset $\mathcal{T}_n$ are the conjugacy classes C_μ, $\mu \vdash n$ and a directed edge (λ, μ) exists from C_λ to C_μ in the Hasse diagram ([11], p. 98) of $\mathcal{T}_n$ if a permutation $\sigma \in C_\mu$ can be written as the product of a permutation $\beta \in C_\lambda$ with a transposition τ. The posets $\mathcal{T}_n$ are graded modular posets of rank $n-1$ with maximal element $\mathbf{1} = C_{(n)}$ and minimal element $\mathbf{0} = C_{(1^n)}$; they were introduced by Birkhoff ([2]). The direct limit of the sequence $(\mathcal{T}_n)_{n\geq 1}$ is an infinite poset denoted $\mathcal{T}_\infty$

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and it has recently been studied by Ziegler ([12)]. Farahat and Higman ([5]) have also implicitly studied the posets $\mathcal{T}_n$ to prove the Nakayama conjecture, their combinatorial result ([5], lemmas 3.10 and 3.11) are special cases of our main result: theorem 3.1.

For general poset notation and for representation theory concepts, the reader is referred respectively to [11] and [9]. We are grateful to the referees and to Professor R. P. Stanley for their suggestions and corrections.

In § 1, we establish the basic properties of the posets $\mathcal{T}_n$ and exhibit the connection between weight of chains in $\mathcal{T}_n$ and the coefficients $c^{\gamma}_{\lambda\mu}$ in the decomposition given by (1). In § 2, we collect facts leading to the evaluation of weight of chains in $\mathcal{T}_n$. In § 3, we prove our main result which is a formula for the number of decompositions of an r-cycle as a product of a permutation of type μ with a permutation of type γ, the product having minimum number of transpositions in its decomposition. Finally in § 4, we derive a new formula from the previous results. Sections 1 to 3 are contained in the Ph. D. thesis of one of the authors ([6], chapt. III).

1. **Definitions and basic properties.** Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ denote a partition of n whose parts λ_i are written in increasing order and consider the set of conjugacy classes $\{C_\lambda\}_{\lambda\vdash n}$ of the symmetric group $\mathfrak{S}_n$ on n letters determined by such partitions. Let T denote the class of transpositions.

DEFINITION 1.1. The *rank* $r(\lambda)$ of the conjugacy class C_λ in the poset $\mathcal{T}_n$ is the smallest integer q for which a decomposition $\sigma = \tau_1\tau_2\cdots\tau_q$, $\tau_i \in T$, $\sigma \in C_\lambda$ as product of transpositions is possible. Observe that the number q is independent of the choice of a representative $\sigma \in C_\lambda$.

DEFINITION 1.2. *Partial order* $\leq$ (see also [11], p. 166 #54). The conjugacy class C_λ precedes the class C_μ written $\lambda \leq \mu$, if for any permutation $\sigma \in C_\mu$, there exists a permutation $\xi \in C_\lambda$ and m transpositions $\tau_1, \dots, \tau_m$ such that $\sigma = \xi\tau_1\tau_2\cdots\tau_m$ and $r(\lambda) + m = r(\mu)$.

We denote by $\mathcal{T}_n$ the graded poset defined by this partial order. The rank $r(\lambda)$ of definition 1.1 coincides with the rank function ([11], p. 99) in the poset $\mathcal{T}_n$.

The *length* $\ell(\mu) = \sum_{i=1}^n \alpha_1$ *of a partition* $\mu = 1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n} \vdash n$, written in the multiplicative notation, is the number of parts contained in μ and (because each i-cycle is a product of $i-1$ transpositions) the rank function satisfies $r(\mu) = \sum_{i=1}^n (i-1)\alpha_i$. Thus we have the relation $r(\mu) + \ell(\mu) = n$.

The *sum of two partitions* $\mu_1 = 1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}$ and $\mu_2 = 1^{\beta_1}2^{\beta_2}\cdots n^{\beta_n}$ is the partition $\mu_1 + \mu_2 = 1^{\alpha_1+\beta_1}2^{\alpha_2+\beta_2}\cdots n^{\alpha_n+\beta_n}$. The *partial order* $\lhd$ on the set of partitions is defined by the condition $\mu_1 \lhd \mu_2 \iff \alpha_i \leq \beta_i$, for all $i = 1, \dots, n$ where μ_1 and μ_2 are partitions of possibly different integers.

Each permutation $\sigma \in \mathfrak{S}_n$ gives rise in a natural way to a partition $[\sigma]$ of the set $[n] = \{1, 2, \dots, n\}$ whose parts are given by the underlying set of each cycle of σ.

A *decomposition of a permutation* $\sigma \in \mathfrak{S}_n$ *in* k *parts* is a partition of the set $[n] = \{1, 2, \dots, n\}$ in k parts refined by the partition $[\sigma]$. The decomposition is said to be *of type*

λ if it is a partition of $[n]$ of type λ. When a decomposition of $\sigma \in C_\mu$ of type λ occurs, then $\mu \leq \lambda$ in $\mathcal{T}_n$. For example the partition $\{\{1,2,4,6,7,8,11\},\{3,5,9\},\{10,13\},\{12\}\}$ is a decomposition of the permutation $\sigma = (1,4,7,8)(2,11,6)(3)(5,9)(10,13)(12)$.

In $\mathcal{T}_\infty$, the number of conjugacy classes of a given rank k (Whitney numbers of the second kind; see [12]) is equal to the number of partitions of k. This fact follows from the correspondence between partitions μ of rank k and partitions $\tilde{\mu}$ obtained by subtracting one from each part of μ. Thus $|\tilde{\mu}| = n - \ell(\mu) = r(\mu)$ and the correspondence $\mu \longmapsto \tilde{\mu}$ is a bijection between the set $\{C_\mu\}_{r(\mu)=k}$ and the set $\{C_\lambda\}_{\lambda \vdash r(\mu)}$ in $\mathcal{T}_\infty$. For fixed n, the cardinality of this set is the number of partitions of n of length $n-k$.

The bijection described above gives a correspondence between the vertices of $\mathcal{T}_\infty$ and the vertices of Young's lattice (for a description of Young's lattice, see [7]).

We shall denote by $\mu^- = \{\lambda \vdash n : \mu \text{ covers } \lambda\}$, the set of conjugacy classes (partitions) that immediately precede C_μ in the poset $\mathcal{T}_n$, by $\mu^+ = \{\lambda \vdash n : \lambda \text{ covers } \mu\}$, the set of classes that are immediate successor of μ and by $\mu^\pm$, the union of μ^- and μ^+.

DEFINITION 1.3. *Weight of a directed edge.* If $\mu \in \mathcal{T}_n$ and $\lambda \in \mu^\pm$, the *weight* $\omega(\lambda,\mu)$ *of the directed edge* (λ,μ) in $\mathcal{T}_n$ is given by the number of ways a permutation of type μ can be written as a product of a permutation of type λ and a transposition: $\omega(\lambda,\mu) = \text{card}\{\xi \in C_\lambda : \exists \tau \in T, \sigma = \xi\tau\}, \sigma \in C_\mu$. This weight is independent of the choice of $\sigma \in C_\mu$.

When the class T of transpositions is considered as an element of $\mathcal{C}[\mathfrak{S}_n]$, the weight (λ,μ) is the coefficient $\omega(\lambda,\mu) = C^\lambda * T|_{C_\mu}$ of the class C^μ in the product $C^\lambda * T$.

Let $\mu = 1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}$ and $\gamma \in \mu^+$ be partitions of n. Suppose that $\xi \in C_\gamma$ is obtained as a product of $\sigma \in C_\mu$ by a transposition. This must be the result of merging two cycles of σ into one cycle of ξ, all other cycles remaining unchanged. Thus γ has one of the forms:

$$\gamma = \begin{cases} 1^{\alpha_1}\cdots i^{\alpha_i-1}\cdots j^{\alpha_j-1}\cdots(i+j)^{\alpha_{i+j}+1}\cdots n^{\alpha_n}(i\neq j) & \text{(form (a))} \\ 1^{\alpha_1}\cdots i^{\alpha_i-2}\cdots(2i)^{\alpha_{2i}+1}\cdots n^{\alpha_n} & \text{(form (b))} \end{cases} \tag{2}$$

so that we have:

PROPOSITION 1.1. *Let* $\mu = 1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}$ *and* $\gamma \in \mu^+$ *be as in (2). Then*

$$C^\mu * T|_{C^\gamma} = \begin{cases} (\alpha_{i+j}+1)(i+j) & \textit{if } \gamma \textit{ has form (a)} \\ (\alpha_{2i}+1)i & \textit{if } \gamma \textit{ has form (b)} \end{cases} \tag{3a}$$

$$C^\gamma * T|_{C^\mu} = \begin{cases} \alpha_i\alpha_j ij & \textit{if } \gamma \textit{ has form (a)} \\ \binom{\alpha_i}{2}i^2 & \textit{if } \gamma \textit{ has form (b)} \end{cases} \tag{3b}$$

PROOF. (see also [8]) We start with (3a). There are $i+j$ ways of obtaining a product of two disjoint cycles of length i and j $(i \neq j)$ from the product of a fixed $(i+j)$-cycle with any transposition; this number times the number $(\alpha_{i+j}+1)$ of cycles of length $i+j$ gives form (a) of (3a). Similarly, form (b) of (3a) is obtained by observing that there are i ways of splitting a $2i$-cycle into a product of two disjoint i-cycles by the action of a transposition (3b) is a direct consequence of the following equalities (see [3] 43, p. 59):

$$C^\mu * C^\lambda|_{C^\gamma} = C^\lambda * C^\mu|_{C^\gamma} = \frac{|C_\mu|}{|C_\gamma|} C^\lambda * C^\gamma|_{C^\mu} \tag{4}$$

We deduce from equations (3a) and (3b) the weights of the directed edges in $\mathcal{T}_\infty$. Their values are given in figure 1. We show in figure 2, the first six levels of $\mathcal{T}_\infty$ together with the weights of the directed edges $\omega(\lambda,\mu)$, $\lambda \in \mu^\pm$.

Form (a)

$$\begin{array}{c} 1^{\alpha_1}\cdots i^{\alpha_i}\cdots j^{\alpha_j}\cdots n^{\alpha_n} \\ {\scriptstyle (\alpha_{i+j}+1)(i+j)}\downarrow\uparrow{\scriptstyle \alpha_i\alpha_j ij} \\ 1^{\alpha_1}\cdots i^{\alpha_i-1}\cdots j^{\alpha_j-1}\cdots(i+j)^{\alpha_{i+j}+1}\cdots n^{\alpha_n} \end{array}$$

Form (b)

$$\begin{array}{c} 1^{\alpha_1}\cdots i^{\alpha_i}\cdots n^{\alpha_n}\colon \mu \\ {\scriptstyle (\alpha_{2i}+1)i}\downarrow\uparrow{\scriptstyle \binom{\alpha_i}{2}i^2} \\ 1^{\alpha_1}\cdots i^{\alpha_i-2}\cdots(2i)^{\alpha_{2i}+1}\cdots n^{\alpha_n}\colon \gamma \in \mu^+ \end{array}$$

FIGURE 1. Weight of edges in $\mathcal{T}_n$

REMARK. When the transpose of the adjacency matrix of the poset $\mathcal{T}_n$ is seen as the matrix representation of the action of the class of transpositions on the other classes of $\mathfrak{S}_n$, we deduce from character theory, that the eigenvalues of this matrix are the central characters $\binom{n}{2}\chi^\lambda_{1^{n-2}2}/f^\lambda$, $\lambda \vdash n$ and all the eigenvectors are rows of the character table $\{\chi^\lambda_\mu\}$. Evaluating these eigenvalues provides interesting identities. For instance, from $\binom{n}{2}\chi^{1,n-1}_{1^{n-2}2}/f^{1,n-1} = \binom{n}{2} - n$ (use [9] p. 64, examples 5 and 7), we get the relations:

$$\sum_{\mu\in\lambda^\pm}\omega(\mu,\lambda)\chi^{1,n-1}_\mu = \chi^{1,n-1}_\lambda[\binom{n}{2}-n] \iff \sum_{\mu\in\lambda^\pm}\omega(\mu,\lambda)(\alpha_1-1) = (\beta_1-1)[\binom{n}{2}-n]$$

where $\mu = 1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}$, $\lambda = 1^{\beta_1}2^{\beta_2}\cdots n^{\beta_n}$.

DEFINITION 1.4. Let $(\lambda = \lambda^0, \lambda^1, \lambda^2, \ldots, \lambda^m = \mu)$ be an $(m+1)$-tuple of partitions of n with each partition $\lambda^i \in \lambda^{(i+1)^-}$. This $(m+1)$-tuple is a saturated chain of length m in $\mathcal{T}_n$ from C_λ to C_μ. The weight of $(\lambda = \lambda^0, \lambda^1, \lambda^2, \ldots, \lambda^m = \mu)$ is the product $\prod_{i=0}^{m-1}\omega(\lambda^i, \lambda^{i+1})$ of the weights of each directed edge $(\lambda^1, \lambda^{i+1})$ and the weight $\omega(\lambda,\mu)$ of the ordered couple (λ,μ) is the sum of weights of all saturated chains of length m from λ to μ.

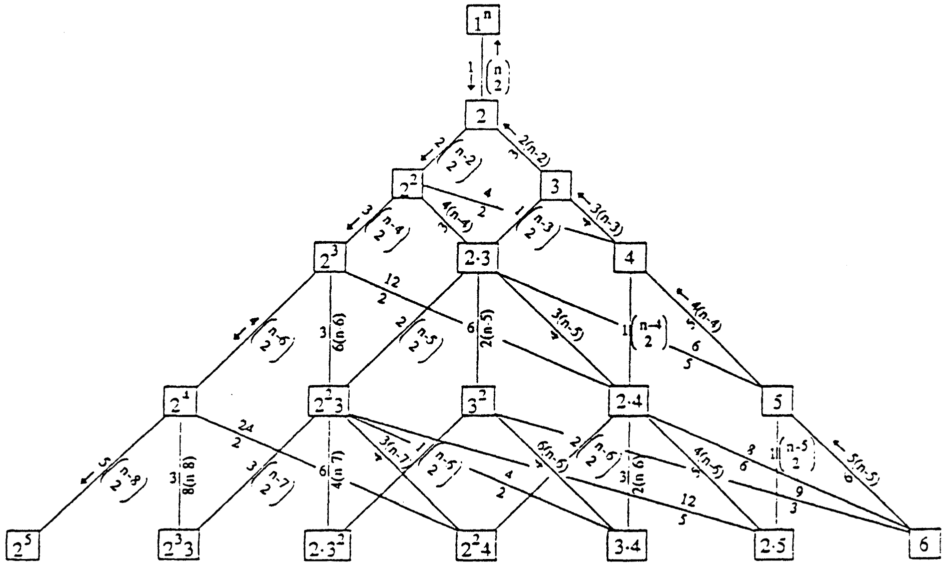

FIGURE 2. The Poset $\mathcal{T}_\infty$ of Conjugacy Classes of $\mathfrak{S}_n$

Thus we have $\omega(\lambda,\mu) = C^\lambda * T^m|_{C^\mu}$ where m is the smallest number of transpositions carrying a permutation of type λ to one of type μ. The weight of (μ,λ) is similarly defined so that we have $\omega(\mu,\lambda) = C^\mu * T^m|_{C^\lambda}$

In figure 2, the 1^{α_1} part in the vertices has been omitted. The weights on the right side are for the upward edges and the ones on the left side for the downward edges.

2. **Weight of chains in $\mathcal{T}_\infty$.** The graphical model presented in [1] (p. 119) for the representation of a product of transpositions associates with a given product of transpositions $\tau_1\tau_2\cdots\tau_k$, a graph $G = ([n], R)$ where $R = \{\tau_1,\tau_2,\ldots,\tau_k\}$ is the set of edges. For any decomposition $\sigma = \tau_1\tau_2\cdots\tau_{n-1}$, $\tau_i \in T$, we know that σ is an n-cycle if and only if the graph corresponding to $\{\tau_1,\tau_2,\ldots,\tau_{n-1}\}$ is a tree. Hence the number of sequences $(\tau_1,\tau_2,\ldots,\tau_{n-1})$ of transpositions such that $\sigma = \tau_1\tau_2\cdots\tau_{n-2}$ is n^{n-1}([4]).

Since a permutation is the product of disjoint cycles we easily get the number of decompositions of any permutation σ as a product of a minimal number of transpositions from the following observation. A given decomposition of a permutation $\sigma = \sigma_1\sigma_2$ which is the product of two disjoint cycles σ_1 and σ_2 is obtained from the shuffle of a decomposition of σ_1 and a decomposition of σ_2, i.e. we mix the two sequences of transpositions as long as the relative order in each sequence is respected. Generalizing to permutation σ of type μ, Dénes ([4]) obtained an expression giving the number of decompositions in terms of the components of μ:

$$\omega(1^n,\mu) = \frac{r(\mu)!}{1!^{\alpha_2}2!^{\alpha_3}\cdots(n-1)!^{\alpha_n}}\prod_{i=2}^{n}[\omega(1^n,1^{n-i}i)]^{\alpha_i}$$

where

$$\omega(1^n,1^{n-i}i) = \omega(1^i,i) = i^{i-2}.$$

We also readily obtain the coefficient $\omega\big((n),\mu\big) = C^{(n)} * T^k|_{C^\mu}$ for $r(\mu)+k = n-1$. This coefficient is the number of ways of multiplying a fixed permutation $\sigma\in C_\mu$ with k transpositions, in any order, to obtain a full cycle. The k transpositions that successively multiply $\sigma\in C_\mu$ always join two cycles together. Thus each transposition is characterised by the choice of elements i,j in different cycles. This can be represented in a graph by an edge between nodes i and j. If we "shrink" each cycles of σ into vertices x_i, the graph obtained from the k transpositions must be a tree in order to obtain a full cycle $\xi\in C_{(n)}$. Now if we choose a representation $X_1,X_2,\ldots,X_{k+1}$ of the $k+1$ cycles of σ as trees, the coefficient $\omega\big((n),\mu\big)$ is the number of trees on $[n]$ having as subgraphs the disjoint trees $X_1,\ldots,X_{k+1}$. Moon's lemma ([10], theorem 2) states that this number is equal to $n^{k-1}1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}$. Thus, after ordering of the k edges and using (4) we obtain the dual formulas:

$$\omega\big((n),\mu\big) = C^{(n)} * T^k|_{C^\mu} = k!\,n^{k-1}1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n} \tag{5a}$$

$$\omega\big(\mu,(n)\big) = C^\mu * T^k|_{C(n)} = \frac{n^k}{\alpha_1!\cdots\alpha_n!}k!. \tag{5b}$$

The generalization of formulas (5a) and (5b) to obtain $\omega(\mu,\lambda)$ leads to recursive algorithms which are not, to the knowledge of the authors, reducible to closed formulas. Nevertheless we get the special cases (see [6] p. 58):

$$\begin{aligned}\omega(1^{\alpha_1}i^{\alpha_i},1^{\beta_1}i^{\beta_i}) &= \binom{\beta_i}{\alpha_i}\omega(1^{\alpha_1},1^{\beta_1}i^{\beta_i-\alpha_i})\\ &= \binom{\beta_i}{\alpha_i}\left[\frac{i^{i-2}}{(i-1)!}\right]^{B_i-\alpha_i}\big((i-1)(\beta_i-\alpha_i)\big)!\end{aligned} \tag{6}$$

$$\omega(1^{n-r}r,\mu) = \sum_{i=r}^{n}\alpha_i i^{i-r}\omega(1^{n-i\alpha_i},\mu-i^{\alpha_i}). \tag{7}$$

3. **A formula for $C^{(n)}*C^\mu|_{C^\gamma}$ when $r(\gamma)+r(\mu)=n-1$.** Set $\mu = 1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}\vdash n$, $\gamma = 1^{\beta_1}2^{\beta_2}\cdots n^{\beta_n}\vdash n$. We now want to travel, in the poset $\mathcal{T}_n$, from the class C_γ to the class $C_{(n)}$ via permutations of a specific type μ, the three classes satisfying $r(\gamma)+r(\mu) = n-1$ so that we are not computing weights of chains in $\mathcal{T}_n$ anymore.

THEOREM 3.1. *Let the partitions* $\gamma, \mu \vdash n$ *satisfy* $r(\gamma) + r(\mu) = n - 1$, *then*

$$C^{(n)} * C^{\mu}|_{C^{\gamma}} = \frac{1^{\beta_1} 2^{\beta_2} \cdots n^{\beta_n} r(\gamma)!\, r(\mu)!}{\prod_{i=1}^{n} \alpha_i!} \tag{8}$$

PROOF. By induction on $\ell(\tilde{\mu}) = \ell(\mu) - \alpha_1$ where $\tilde{\mu} = 1^{\alpha_2} \cdots (n-1)^{\alpha_n} \vdash r(\mu)$.

STEP 1. $\ell(\tilde{\mu}) = 1$. The identity

$$C^{(n)} * C^{1^{n-k_k}}|_{C^{\gamma}} = 1^{\beta_1} 2^{\beta_2} \cdots n^{\beta_n} (k-1)! \tag{9}$$

is true because this coefficient is obtained by choosing one number in each one of the k parts of the set partition $[\sigma]$ of type γ induced by σ. Then, with the k numbers chosen, we form a k-cycle in $(k-1)!$ ways. Moreover, formula (9) satisifes (8).

STEP 2. Suppose (8) is satisfied by all the partitions $\mu \vdash n-1$ such that $\ell(\tilde{\mu}) = \ell$. Take the partition $\big(\mu + (i)\big)$ that is different from μ by a part equal to $i \neq 1$. We compute the coefficient $C^{(n)} * C^{(\mu+(i))}|_{C^{\gamma}}$ in two steps. We start from $\sigma \in C_{\gamma}$ and take any set of i parts of $[\sigma]$ and call it $[\gamma^1]_{\sigma}$ to identify it as a set partition and we build an i-cycle on that structure as in step 1. Secondly, we consider the set partition $[\gamma^2]_{\sigma}$ obtained from $[\sigma]$ by deleting $[\gamma^1]_{\sigma}$ and adding the single part made of the elements of $[\gamma^1]_{\sigma}$ except the i elements already chosen: $[\gamma^2]_{\sigma} = [\sigma] - [\gamma^1]_{\sigma} + \big[[\gamma^1]_{\sigma} - i\big]$. Note that we have $\gamma^2 = 1^{\beta_1} 2^{\beta_2} \cdots n^{\beta_n} r(\gamma^1) \vdash n - i$. Then, using $[\gamma^2]$, we build a permutation of type μ by multiplying transpositions whose elements are chosen one in each part of $[\gamma^2]$ as for the i-cycle of step 1. Thus

$$C^{(n)} * C^{(\mu+(i))}|_{C^{\gamma}} = \frac{1}{\alpha_i + 1} \sum_{\substack{[\gamma^1] \subseteq [\sigma] \\ \ell(\gamma^1) = i}} C^{(|\gamma^1|)} * C^{1^{|\gamma|-i_i}}|_{C^{\gamma^1}} C^{(n-i)} * C^{\mu}|_{C^{\gamma - \gamma^1 + (|\gamma^1| - i)}} r^{(\gamma^1)} \tag{10}$$

In the right hand side of (10), we divide by $\alpha_i + 1$ to correct the choice of a specific i-cycle among $\alpha_i + 1$ in the process. Using (9) and the induction hypothesis, (10) is then transformed into:

$$C^{(n)} * C^{(\mu+(i))}|_{C^{\gamma}} = \frac{1^{\beta_1} 2^{\beta_2} \cdots n^{\beta_n}}{\alpha_i + 1} \sum_{\substack{[\gamma^1] \subseteq [\sigma] \\ \ell(\gamma^1) = i}} (i-1)! \frac{r(\mu)!\, r(\gamma^2)!\, r(\gamma^1)}{\prod_{j=1}^{n} \alpha_j!} \tag{11}$$

Since $r(|\gamma^1| - i) = |\gamma^1| - i - 1 = r(\gamma^1) - 1$, we have:

$$r(\gamma^2) = r\big(\gamma - \gamma^1 + (|\gamma^1| - i)\big) = r(\gamma) - r(\gamma^1) + r(|\gamma^1| - i) = r(\gamma) - 1,$$

and (11) becomes

$$C^{(n)} * C^{(\mu+(i))}|_{C^{\gamma}} = \frac{1^{\beta_1} \cdots n^{\beta_n} (i-1)!\, r(\mu)!\, (r(\gamma) - 1)!}{(\alpha_i + 1) \prod_{j=1}^{n} \alpha_j!} \sum_{\substack{[\gamma^1] \subseteq [\sigma] \\ \ell(\gamma^1) = i}} r(\gamma^1) \tag{12}$$

Then from

$$\sum_{\substack{[\gamma^1]\subseteq[\sigma]\\ \ell(\gamma^1)=i}} r(\gamma^1) = n\binom{\ell(\gamma)-1}{i-1} - i\binom{\ell(\gamma)}{i} = \binom{\ell(\gamma)-1}{i-1}\big(n-\ell(\gamma)\big)$$
$$= \binom{\ell(\gamma)-1}{i-1} r(\gamma) = \binom{r\big(\mu+(i)\big)}{i-1} r(\gamma)$$

the right side of (12) becomes

$$\frac{1^{\beta_1}\cdots n^{\beta_n}(i-1)!\, r(\mu)!\, r(\gamma)!}{(\alpha_i+1)\prod_{j=1}^n \alpha_j!}\binom{r\big(\mu+(i)\big)}{i-1}$$

which is theorem 3.1. ∎

The special case

$$C^\mu * C^{1^{n-r_r}}|_{C^{(n)}} = \frac{n(r-1)!}{\prod_{i=1}^n \alpha_i!}$$

of theorem 3.1 is used in [5] to show that the set $\{C(m) = \sum_{r(C_\mu)=m} C^\mu\}_{m\geq 0}$ of sums of class sums of rank m in $\mathcal{T}_\infty$ and $\mathcal{T}_n$ is a set of algebraically independent generators for $\mathcal{C}[\mathfrak{S}_\infty]$.

4. **Derivation of a Formula.** Set $\omega(n,\gamma,\mu) = C^{(n)} * C^\gamma|_{C^\mu}$ where the condition $r(\gamma)+r(\mu) = n-1$ is satisfied. Since $r(\mu) = n-k-1$ implies $r(\gamma) = k$, we have

$$(13) \qquad \sum_{\substack{\gamma\vdash n\\ r(\gamma)=k}} \omega(1^n,\gamma)\,\omega(n,\gamma,\mu) = \omega(n,\mu)$$

Comparing identities (5a) and (8), we obtain:

$$(14) \qquad \sum_{\substack{\gamma\vdash n\\ r(\gamma)=k}} \omega(1^n,\gamma)\frac{r(\gamma)!\, r(\mu)!}{\prod_{i=1}^n \beta_i!} = k!\, n^{k-1}$$

for which deduce the following.

PROPOSITION 4.1. *Let $0 < k \leq n$, be two positive integers, then:*

$$(15) \qquad \sum_{\substack{\gamma\vdash n\\ \ell(\gamma)=n-k}} \frac{\omega(1^n,\gamma)}{n-k}\binom{n-k}{\gamma} = n^{k-1}$$

where $\binom{n-k}{\gamma} = \frac{(n-k)!}{\beta_1!\cdots\beta_n!}$ is a multinomial coefficient.

COROLLARY 4.2. *For any integer $n > 1$, we have:*

$$(16) \qquad \sum_{\gamma\vdash n} \omega(1^n,\gamma)\binom{\ell(\gamma)}{\gamma} = \sum_{\gamma\vdash n}\prod_{i=2}^n i^{(i-2)\beta_i}\binom{r(\gamma)}{\tilde\gamma}\binom{\ell(\gamma)}{\gamma} = \frac{n(n^{n-1}-1)}{(n-1)^2}$$

where $\binom{r(\gamma}{\tilde\gamma} = \frac{r(\gamma)!}{1!^{\beta_2}2!^{\beta_3}\cdots(n-1)!^{\beta_n}}$ is also a multinomial coefficient.

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PROOF. *This is a direct consequence of (15) and the identity*

$$\sum_{k=0}^{n-1}(n-k)n^{k-1} = \frac{n(n^{n-1}-1)}{(n-1)^2} \qquad \blacksquare$$

CONCLUSION. When the class of n-cycles is replaced in theorem 3.1 by other classes, no simple equivalent formula is known to the authors. For instance the formula

$$C^{(n_1,n_2)} * C^{\mu}|_{C^{\gamma}} = \sum_{\substack{\gamma^1 \lhd \gamma \\ \gamma^1 \vdash n_1}} \sum_{\substack{\mu^1 \lhd \mu \\ \mu^1 \vdash n_1}} \prod_{i=1}^{n} \binom{\beta_i}{\beta_i^1} C^{(n_1)} * C^{\mu^1}|_{C^{\gamma^1}} C^{(n_2)} * C^{\mu-\mu^1}|_{C^{\gamma-\gamma^1}} \tag{17}$$

where $r(\gamma) + r(\mu) = n-2$, $\gamma^1 = 1^{\beta_1^1} \cdots n^{\beta_n^1}$, $r(\mu^1) + r(\gamma^1) = n_1 - 1$ is essentially a recurrence formula and could not be simplified (by the authors). Other identities of this type are given in [6].

Combinatorial demonstrations of formulas (16) and (8) remain to be done and the determination of weights of zigzag paths in $\mathcal{T}_\infty$ is also an open problem.

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