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Embedding Distributions of Generalized Fan Graphs

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Abstract. Total embedding distributions have been known for a few classes of graphs. Chen, Gross, and Rieper computed it for necklaces, close-end ladders and cobblestone paths. Kwak and Shim computed it for bouquets of circles and dipoles. In this paper, a splitting theorem is generalized and the embedding distributions of generalized fan graphs are obtained.

1 Background

One enumerative aspect of topological graph theory is to count the distribution of genus of a graph. The history of genus distribution began with J. Gross in 1980s. Since then it has attracted considerable attention. See [1, 8–12, 14, 15, 18, 20, 23, 27, 28, 31–37, 39] for a sample of results in the direction. However, for the total embedding distributions, only a few classes are known. For example, Chen, Gross, and Rieper [2] computed the total embedding distribution for necklaces, close-end ladders, and cobblestone paths, Kwak and Shim [22] computed for bouquets of circles and dipoles. Chen, Liu ,and Wang [5] calculated the total embedding distributions of all graphs with maximum genus 1.

It is assumed that the reader is familiar with the basics of topological graph theory as found in Gross and Tucker [13]. A graph $G = (V(G), E(G))$ is permitted to have both loops and multiple edges. A *surface* is a compact closed 2-dimensional manifold without boundary. In topology, surfaces are classified into O_m, the *orientable surface* with $m(m \geq 0)$ handles, and N_n, the *nonorientable surface* with $n(n > 0)$ crosscaps. A graph embedding into a surface means a *cellular embedding*.

A *spanning tree* of a graph G is a tree on its edges having the same order as G. The number of co-tree edges of a spanning tree of G is called the *Betti number*, $\beta(G)$, of G. A *rotation* at a vertex v of a graph G is a cyclic order of all edges incident with v. A *pure rotation system* P of a graph G is the collection of rotations at all vertices of G. A *general rotation system* is a pair (P, λ), where P is a pure rotation system and λ is a mapping $E(G) \to \{0, 1\}$. The edge e is said to be *twisted* (respectively, *untwisted*) if $\lambda(e) = 1$ (respectively, $\lambda(e) = 0$). It is well known that every orientable embedding of a graph G can be described by a general rotation system (P, λ) with $\lambda(e) = 0$ for all $e \in E(G)$. By allowing λ to take the nonzero value, we can describe nonorientable embeddings of G; see [2, 30] for more details. A *T-rotation system* (P, λ) of G is a general rotation system (P, λ) such that $\lambda(e) = 0$, for all $e \in E(T)$.

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Theorem 1.1 (see [2, 30]) *Let T be a spanning tree of G and (P, λ) a general rotation system. Then there exists a general rotation system $(P^{'}, \lambda^{'})$ such that*

(i) $(P^{'}, \lambda^{'})$ *yields the same embedding of G as (P, λ),*
(ii) $\lambda^{'}(e) = 0$ *for all $e \in E(T)$.*

Two embeddings are considered to be the *same* if their T-rotation systems are combinatorially equivalent. Fix a spanning tree T of a graph G. Let Φ_G^T be the set of all T-rotation systems of G. It is known that $|\Phi_G^T| = 2^{\beta(G)} \prod_{v \in V(G)} (d_v - 1)!$. Suppose that in these $|\Phi_G^T|$ embeddings of G there are a_i, $i = 0, 1, \ldots$, embeddings into orientable surface O_i and b_j, $j = 1, 2, \ldots$, embeddings into nonorientable surface N_j. We call the polynomial

$$I_G^T(x) = \sum_{i=0}^{\infty} a_i x^i + \sum_{j=1}^{\infty} b_j x^{-j}$$

the *T-distribution polynomial* of G. By the *total embedding-distribution polynomial* of G, we shall mean the polynomial $I_G(x) = I_G^T(x)$. We call the first (respectively, second) part of $I_G(x)$ the *genus polynomial* (respectively, *crosscap number polynomial*) of G and denoted by $g_G(x) = \sum_{i=0}^{\infty} a_i x^i$ (respectively, $f_G(y) = \sum_{i=1}^{\infty} b_i x^{-i}$). Clearly, $I_G(x) = g_G(x) + f_G(x)$. This means the number of orientable embeddings of G is $\prod_{v \in G} (d_v - 1)!$, while the number of non-orientable embeddings of G is

$$(2^{\beta(G)} - 1) \prod_{v \in G} (d_v - 1)!.$$

2 The Total Embedding Distributions of Generalized Fan Graphs

A *fan graph* $F_{(1,n)}$ is defined as the graph $K_1 + P_n$, where K_1 is the empty graph on one vertex and P_n is the path graph on n vertices. A fan-type graph $F_{t_1,t_2,\ldots,t_n}$ is defined as the graph K_1 connecting t_j edges to the vertex v_j of P_n, $t_j \geq 1$, $j = 1, 2, \ldots, n$. A dipole graph D_n is a multigraph consisting of two vertices connected with n edges. Figure 1 presents the graphs $F_{(1,n)}$, $F_{2,2,\ldots,2}$, and D_n.

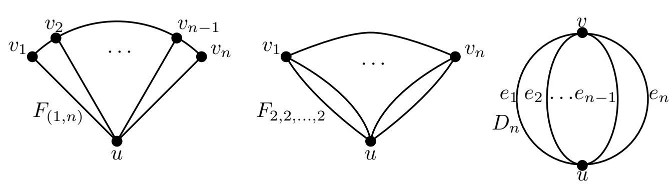

Figure 1: The graphs $F_{(1,n)}$, $F_{2,2,\ldots,2}$, and D_n

In this section, a special form of vertex-splitting of [4, 9] is generalized to the ordinary case.

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Definition 2.1 Suppose the graph $G = (V, E)$ is simple. Let u be a vertex of G of valence $\deg(u) = d + 1 > 3$ and $v, v_1, v_2, \dots, v_d$ be its neighbors. We denote the edge uv_i by e_i, for $i = 1, 2, \dots, d$, and the edge uv by f. The graph $G_{i_1,\dots,i_k}$ is called a *k-degree proper splitting* of G at u if it can be obtained from $G - u$ by adjoining $v, v_{i_1}, \dots, v_{i_k}$ to a new vertex x, adjoining all the other ex-neighbors of u to a new vertex y ($i_l \in \{1, 2, \dots, k\}$, for $l = 1, 2, \dots, k$ and $d > k \geq 1$), and finally adjoining x and y.

The new vertex x is $(k+2)$-valent for each $G_{i_1,\dots,i_k}$ and the new vertex y is $(d-k+1)$-valent. Let Λ be the set of all graphs $G_{i_1,\dots,i_k}$. Then the number of elements in Λ is $\binom{d}{k}$. It is obvious that each graph $G_{i_1,\dots,i_k}$ has the same Betti number as that of G, and they can contract the new edge xy to get the graph G. Figure 2 gives an example of a 2-degree proper splitting of G at u.

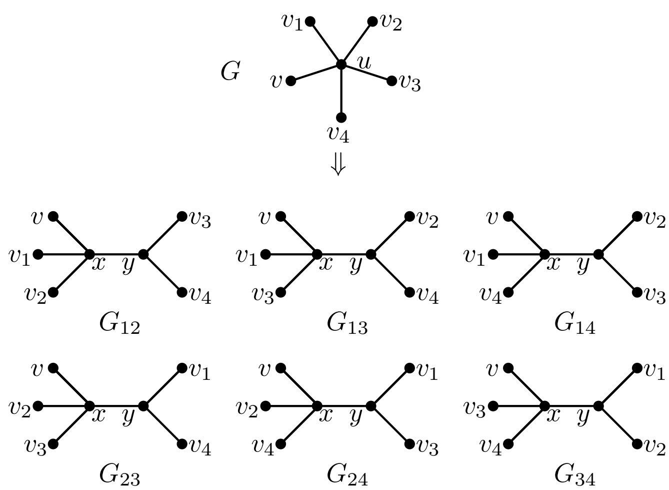

Figure 2: The 2-degree proper splitting of G at u with a designate neighbor v

Theorem 2.2 (see [9]) *Let G be a connected graph with a vertex of valence 4, and let G_j, $j \in \{1, 2, 3\}$, be graphs obtained by 1-degree properly splitting at vertex u. Then we have*

$$g_G(x) = \frac{1}{2}[g_{G_1}(x) + g_{G_2}(x) + g_{G_3}(x)].$$

Let G be a simple graph and T be a spanning tree of G such that all the edges incident with u are edges of T, *i.e.*, $E(u) = \{uv \mid v \in V(G)\} \subseteq E(T)$ and (P, λ) is a T-rotation system of G. Let $T_{i_1,i_2,\dots,i_k}$ be the spanning tree of $G_{i_1,i_2,\dots,i_k}$ such that all the edges incident with x and y are edges of $T_{i_1,i_2,\dots,i_k}$ and the other edges of $T_{i_1,i_2,\dots,i_k}$ are the same as the edges of $T - E(u)$.

hbadness=1112 In order to extend the above operation on embedding graphs, we denote the T-rotation system (P, λ) with the pure rotation system P at vertex u

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by $u\,.\,e_{i_1}e_{i_2}\cdots e_{i_d}f$, where $i_\ell \in \{1,2,\ldots,d\}$, for $\ell = 1,2,\ldots,d$ and f is the edge uv. Let $(P_{i_1,i_2,\ldots,i_k}, \lambda_{i_1,i_2,\ldots,i_k})$ be the $T_{i_1,i_2,\ldots,i_k}$-rotation system of the graph $G_{i_1,i_2,\ldots,i_k}$ with rotations $x\,.\,fe_{i_1}\cdots e_{i_k}e$ and $y\,.\,ee_{i_{k+1}}\cdots e_{i_d}$, and all other vertex rotations as in P, where e is the new edge in $G_{i_1,i_2,\ldots,i_k}$ that connects the vertices x and y. Let $(P_{i_{d-k+1},\ldots,i_d}, \lambda_{i_{d-k+1},\ldots,i_d})$ be the $T_{i_{d-k+1},\ldots,i_d}$-rotation system of the graph $G_{i_{d-k+1},\ldots,i_d}$ with rotations $x\,.\,fee_{i_{d-k+1}}e_{i_{d-k+2}}\cdots e_{i_d}$, $y\,.\,ee_{i_1}\cdots e_{i_{d-k}}$ and all other vertex rotations as in P.

Similarly, let $(P_{i_j,\ldots,i_d,i_1,\ldots,i_{j+k-d-1}}, \lambda_{i_j,\ldots,i_d,i_1,\ldots,i_{j+k-d-1}})$ be the $T_{i_j,\ldots,i_d,i_1,\ldots,i_{j+k-d-1}}$-rotation system of $G_{i_j,\ldots,i_d,i_1,\ldots,i_{j+k-d-1}}$ for $j = d-k+2,\ldots,d$, with rotations $x\,.\,e_{i_j}\cdots e_{i_d}fe_{i_1}\cdots e_{i_{k+j-d-1}}e$ and $y\,.\,ee_{i_{k+j-d}}\cdots e_{i_{j-1}}$ and all other vertex rotations as in P.

Definition 2.3 We say that the general rotation systems $(P_{i_1,i_2,\ldots,i_k}, \lambda_{i_1,i_2,\ldots,i_k})$, $(P_{i_{d-k+1},\ldots,i_d}, \lambda_{i_{d-k+1},\ldots,i_d})$ and $(P_{i_j,\ldots,i_d,i_1,\ldots,i_{j+k-d-1}}, \lambda_{i_j,\ldots,i_d,i_1,\ldots,i_{j+k-d-1}})$, for $j = d-k+2,\ldots,d$, are obtained by *k-degree proper splitting* the vertex u in the rotation system (P,λ) with the designated neighbor v. We also say that the rotation system (P,λ) is obtained by contracting the rotation system $(P_{i_1,i_2,\ldots,i_k}, \lambda_{i_1,i_2,\ldots,i_k})$, $(P_{i_{d-k+1},\ldots,i_d}, \lambda_{i_{d-k+1},\ldots,i_d})$ or $(P_{i_j,\ldots,i_d,i_1,\ldots,i_{j+k-d-1}}, \lambda_{i_j,\ldots,i_d,i_1,\ldots,i_{j+k-d-1}})$, for $j = d-k+2,\ldots,d$, on the edge e.

Let $\Phi^{T_{i_1,i_2,\ldots,i_k}}_{G_{i_1,i_2,\ldots,i_k}}$ be the set of all $T_{i_1,i_2,\ldots,i_k}$-rotation systems of $G_{i_1,i_2,\ldots,i_k}$ and Φ^T_G be the set of all T-rotation systems of G. Let

$$\Phi_u = \bigcup_{i_1,i_2,\ldots,i_k} \Phi^{T_{i_1,i_2,\ldots,i_k}}_{G_{i_1,i_2,\ldots,i_k}}.$$

Theorem 2.4 *Let G be a connected graph with a vertex of valence $d+1$, $d \geq 3$, $G_{i_1,i_2,\ldots,i_k}$, $i_j \in \{1,2,\ldots,d\}$, be a graph obtained by k-degree proper splitting at vertex u, and let Λ be the set of all such graphs $G_{i_1,i_2,\ldots,i_k}$. Then we have*

$$I_G(x) = \frac{1}{k+1} \sum_{G_{i_1,i_2,\ldots,i_k} \in \Lambda} I_{G_{i_1,i_2,\ldots,i_k}}(x).$$

Proof Suppose the designated neighbor is v and the T-rotation system (P,λ) with the pure rotation system P at vertex u is $u\,.\,e_{i_1}e_{i_2}\cdots e_{i_d}f$, where $i_j \in \{1,2,\ldots,d\}$, for $j = 1,2,\ldots,d$ and f is the edge uv. We make the following two assertions.

(i) It induces a $(k+1)$-to-1 correspondence from the set of Φ_u to onto the set Φ^T_G. Moreover, every general rotation system of Φ_u is uniquely contractible on the edge xy to a T-rotation system of $\Phi^{T_{i_1,i_2,\ldots,i_k}}_{G_{i_1,i_2,\ldots,i_k}}$.

(ii) It preserves the genus of the surface.

In regard to assertion (i), by the definition, the rotation system (P,λ) can be obtained only by contracting the edge e in the rotation system $(P_{i_1,i_2,\ldots,i_k}, \lambda_{i_1,i_2,\ldots,i_k})$, $(P_{i_{d-k+1},\ldots,i_d}, \lambda_{i_{d-k+1},\ldots,i_d})$ and $(P_{i_j,\ldots,i_d,i_1,\ldots,i_{j+k-d-1}}, \lambda_{i_j,\ldots,i_d,i_1,\ldots,i_{j+k-d-1}})$ defined above, for $j = d-k+2,\ldots,d$. Moreover, if (P,λ) can be obtained by contracting the edge e in a rotation system $(P_{i_1,i_2,\ldots,i_k}, \lambda_{i_1,i_2,\ldots,i_k})$, not only the rotations of all vertices, including the vertices x and y, but also the label of all cotree edges of $G_{i_1,i_2,\dots,i_k}$ are determined. It is obvious that each of them is uniquely contractible on the edge e to the rotation system (P,λ).

In regard to assertion (ii), we observe that a contraction operation decreases the numbers of vertices and edges, each by 1, and preserves the number of faces. This implies that the genus or crosscap number of the embedding surface is unchanged.

By the two assertions, the theorem follows. ■

In [22], Kwak and Shim calculated the embedding distribution of dipole D_n. Let Let $I_{D_n}(x) = \sum_{i=-\infty}^{\infty} g_m(D_n)x^i$ be total embedding-distribution polynomial of D_n. They obtained the following result.

Theorem 2.5 (See [22]) *If $m \geq 0$, then $g_m(D_n)$ equals the coefficient of y^{n-2m} in $\widehat{i}[D_n](y)$; otherwise $g_m(D_n)$ equals the coefficient of $y^{-(n+m)}$ in $\widehat{i}[D_n](y)$, where $\widehat{i}[D_n](y)$ is as defined in* [22].

Theorem 2.6 *Let G be a graph obtained by replacing one vertex r of the dipole D_n (with vertices r, u) by a tree R, so that G consists of R and n additional edges $e_1, e_2, \dots, e_n$ each joining some vertex of R to a vertex u. Then the total embedding-distribution polynomial of G is a constant multiple of the total embedding-distribution polynomial of D_n.*

Proof To prove Theorem 2.6, take the spanning tree $T = R \cup e_1$. For any pure rotation scheme P of G, let P_R be the restriction of P to $V(R) = V(G) - u$. Thus, P_R describes the rotations of all edges (not just edges of R, but $e_1, e_2, \dots, e_n$ as well) at vertices of R. Let Π_R be the set of all possible P_R. Now for each fixed $Q \in \Pi_R$ the total genus distribution of T-rotation schemes (P,λ) with $P_R = Q$ is just $\frac{1}{(n-1)!}I_{D_n}(x)$. This is because the total genus distribution is the same as when we contract R down to a single vertex r, preserving the relative positions of the edges $e_1, e_2, \dots, e_n$ in the course of contraction. The contraction does not change the surface at all. So we get the distribution for e_1-rotation schemes of D_n with the rotation around vertex r fixed in a way that is prescribed by Q. However, by symmetry this is just $\frac{1}{(n-1)!}I_{D_n}(x)$. Therefore $I_G(x) = \frac{|\Pi_R|}{(n-1)!}I_{D_n}(x)$. ■

By Theorem 2.6, the following two results easily follow.

Corollary 2.7 *Let $F_{1,n}$ be the fan graph and D_n be the dipole graph. Then the total embedding-distribution polynomial of the fan graph is given by $I_{F_{1,n}}(x) = \frac{2^{n-2}}{(n-1)!}I_{D_n}(x)$.*

Corollary 2.8 *Let $F_{t_1,t_2,\dots,t_n}$ be the fan-type graph and D_n be the dipole graph. Then there exists a constant C such that the total embedding-distribution polynomial of the fan-type graph is given by $I_{F_{t_1,t_2,\dots,t_n}}(x) = C \cdot I_{D_n}(x)$.*

An even more general result of Theorem 2.6 can be obtained as follows. The proof is just a generalization of the proof of Theorem 2.6.

Theorem 2.9 *Let H be a graph with a vertex r such that $(\operatorname{Aut} H)_r$, the stabilizer of r in $\operatorname{Aut} H$, acts as the symmetric group on the edges incident with r. Let G be a graph obtained by replacing r by a tree R, so that each edge ur of H is replaced by an edge uv for* *some $v \in V(R)$. Then the total embedding-distribution polynomial of G is a constant multiple of the total embedding-distribution polynomial of H.*

3 Conclusions and further problems

J. Gross [1, 12]conjectured that the genus distribution is strongly unimodal. Stahl [34] posed a stronger conjecture which states that the zeros of genus polynomial are real. Some counter examples were later presented in [3, 6], thus disproving Stahl's conjecture. However, Gross's conjecture is still open. So far, none of the crosscap number distributions of those mentioned classes of graphs have proved to be strongly unimodal. Thus, checking the crosscap number distribution of a graph G to be strongly unimodal is a possible task. Unanswered questions include: Are the crosscap number distributions of the necklaces, the closed-end ladders L_n, cobblestone path and fan-type graphs strongly unimodal?

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