ON SEQUENCES OF INTEGRABLE FUNCTIONS

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1. Introduction

Let $f_1(x), f_2(x), \cdots$ be a sequence of functions belonging to the real or complex Banach space L, (see S. Banach: [1] (The results can be generalised to functions on any space that is the union of countably many sets of finite measure). We are concerned with various properties that such a sequence may have, and in particular with the more important kinds of convergence (strong, weak and pointwise). This article shows what relations connect the various properties considered; for instance that for strong convergence (i.e. $||f_n - f|| \to 0$) *it is necessary and sufficient firstly that the sequence* should converge weakly (i.e. if g is bounded and measurable then $\int (f_n(x) - f(x))g(x)dx \to 0$) and secondly that any sub-sequence should contain a sub-sub-sequence converging p.p. to $f(x)$.

2. Notation

Each of the four following properties is represented by the capital letter shown.

S Strong convergence to zero, i.e. $||f_n|| \to 0$

P Pointwise convergence to zero, i.e. $f_n(x) \to 0$ p.p.

W Weak convergence to zero, i.e. if $g(x)$ is any bounded measurable function then:

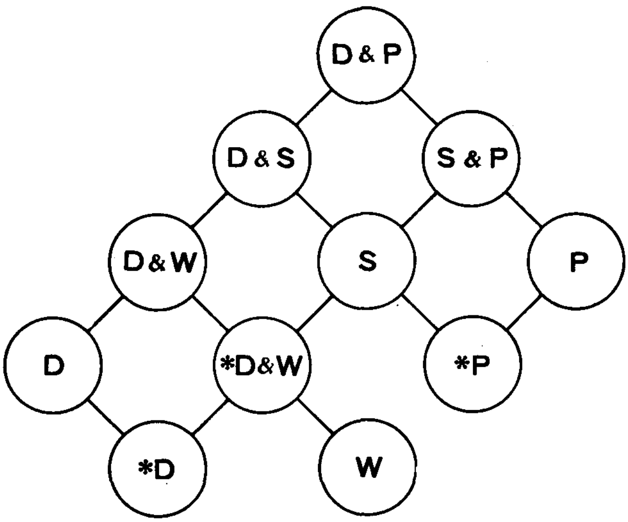
$$\int f_n(x)g(x)dx \to 0$$

D The property of being dominated, i.e. the existence of $g(x)$ such that $|f_n(x)| \leqq g(x)$ p.p. and that $\int g(x)dx$ is finite.

Now let A be one of the four properties above. We define a sequence to have the property $*A$ or "star A" if every infinite sub-sequence contains an infinite sub-sub-sequence with the property A. The idea of star-convergence goes back to P. Urysohn [2].

3. The problem

From the eight properties defined above we can form 247 others such as "W and $*P$" by conjunction. We want to find when any one of these compound properties implies any other. This task is not as formidable as it sounds, for there are several trivial relations connecting the eight primitive properties. It turns out that only 11 of the compound properties are distinct and that we can find all the relations connecting them. The properties with the relation of implication form a semi-lattice, and may be represented by the diagram below, where each property implies those below it.



4. The relations

Firstly some general remarks can be made. The first four properties mentioned above are all of the kind that is shared by all sub-sequences. Therefore if A is one of them then $A \Rightarrow *A$. Also it follows that $*(A \text{ and } B) = (*A \text{ and } *B)$ so that if A and B together imply C then $*A$ and $*B$ together will imply $*C$.

There are three trivial relations:

(1) $$*S = S \Rightarrow W = *W$$

Lebesgue's theorem on dominated convergence shows

that P and $D \Rightarrow W$.

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Consequently it follows that:

$$\text{(2)} \qquad {}^*P \text{ and } {}^*D \Rightarrow W$$

Also it has been shown (reference [3]) that:

$$\text{(3)} \qquad P \text{ and } W \Rightarrow S \Rightarrow {}^*D$$

and from this it follows similarly that:

$$\text{(4)} \qquad {}^*P \text{ and } W \Rightarrow S$$

Also it is a by-product of the Riesz-Fischer theorem that:

$$\text{(5)} \qquad S \Rightarrow {}^*P$$

From (1) (4) and (5) there follows:

$${}^*P \text{ and } W = S$$

Also from (2) and (4):

$${}^*P \text{ and } {}^*D \Rightarrow S$$

and the converse implication follows from (3) and (5), so that.

$${}^*P \text{ and } {}^*D = S$$

These considerations show that there are no more than eleven elements in the semi-lattice and that they are related as shown in the diagram. The task remains of showing that these results are the best possible, and the only difficult part is the proof that W and $^*D$ together are not the same as W. This is done in the following section.

5. A counter-example

We construct a sequence weakly convergent to zero but not star-dominated, in the space $L(-\pi, \pi)$, but with all functions defined also on the rest of the real axis by having period 2π.

Let
$$f_n(x) = |\sin nx|^{\frac{1}{2}}/\sin nx$$

To prove weak convergence let $g(x)$ be any bounded measurable function. Suppose that $|g(x)| < K$ for all x. Take any $\varepsilon > 0$. There is a bounded odd function $h(x)$ such that $||h - f_1|| < \varepsilon/K$. Now let $h_n(x) = h(nx)$ so that $||h_n - f_n|| < \varepsilon/K$. Let M be an upper bound of $|h(x)|$. There is a step-function $G(x)$ such that $||G - g|| < \varepsilon/M$.

The integral of $h_n(x)$ over any interval of length $2\pi/n$ is zero, so that over any interval the integral tends to zero, and therefore:

$$\int h_n(x) G(x) \to 0$$

Therefore for all sufficiently large n:

$$|\int f_n(x)g(x)dx| \leqq$$
$$|\int (f_n(x) - h_n(x))g(x)dx| + \int |h_n||g - G|dx + |\int h_n(x)G(x)dx|$$
$$\leqq K\,\varepsilon/K + M\,\varepsilon/M + \varepsilon = 3\varepsilon$$

It remains now to show that the sequence is not star-dominated. It is sufficient to show that no sub-sequence is dominated.

Let $m(i)$ (for $i = 1, 2, \cdots$) be a strictly increasing sequence of positive integers. We shall show that:

$$\limsup |f_{m(n)}(x)| = \infty \text{ p.p.}$$

Take any $C > 1$. The set $E(n)$ of all x in $(-\pi, \pi)$ at which $|f_n(x)| > C$ consists of $2n + 1$ equally spaced open intervals; their total length $k = 4 \arcsin C^{-2}$ depends on C but not on n; they are of length $k/2n$ except for the two end ones which are half the length of the others.

Now take any interval (a, b) where $-\pi < a < b < \pi$. The mean density of $E(n)$ in this interval will tend to $k/2\pi$ as $n \to \infty$. Now if E is the union all $E(m(k))$ $(k = 1, 2, \cdots)$ the mean density of E in (a, b) will be $\geqq k/2\pi$. Therefore the lower density of E at any point of $(-\pi, \pi)$ must be $\geqq k/2\pi$. However E is measurable, so that it has density 0 or 1 p.p., therefore the density of E must be 1 p.p. so that almost all x in $(-\pi, \pi)$ are in E.

Thus l.u.b. $|f_{m(k)}(x)| \geqq C$ p.p. and since C was arbitrary it follows that:

$$\text{l.u.b. } |f_{m(k)}(x)| = \infty \text{ p.p.}$$

so that the sub-sequence cannot be dominated.

The proof that the sequence is not star-dominated may also be regarded as a proof of the following result on rational approximations to real numbers. For any infinite set N of positive integers, for almost all x, for all $\varepsilon > 0$, for arbitrarily large n in N, for some integer m:

$$|x - m/n| < \varepsilon/n$$

The fact that "almost all x" cannot be strengthened to "all x" is shown by the example of $N = \{1!, 2!, 3!, \cdots\}$ and $x = 1/(2e)$.

6. A theorem on weak convergence

THEOREM. Suppose that $g_n(x)$ $(n = 1, 2, \cdots)$ is a sequence of continuous functions and c is such that $g_n(c)$ is unbounded as $n \to \infty$. Then there is $f(x)$ absolutely integrable such that $\int f(x)g_n(x)dx$ is unbounded, the integral being taken over any closed interval including c.

The proof depends on the following:

LEMMA. Suppose that $g_1(x), g_2(x), \cdots$ is a sequence of continuous real or complex functions on the closed interval $[0, 1]$, and $g_n(0) = 3^n$; then there is $f(x)$; absolutely integrable, such that $\int_0^1 f(x)g_n(x)dx$ does not tend to zero.

PROOF Choose $\alpha_1 > \alpha_2 > \cdots > 0$ such that:

$$|g_n(x) - g_n(0)| \leq 3^n/5 \quad \text{if} \quad 0 < x < \alpha_n$$

For each n let $f_n(x)$ be the function that is $3^{-n}\beta_n/\alpha_n$ in the interval $[0, \alpha_n]$ and is 0 elsewhere, where β_n is a complex number of modulus 1 to be fixed later.

$$\int_0^1 f_n(x)g_n(x)dx = \beta_n + 3^{-n}\beta_n/\alpha_n \int_0^{\alpha_n} g_n(x) - g_n(0)dx$$

The second term on the right is of modulus $< 1/5$, so that the left hand side is of modulus $> 4/5$.

Now the coefficients β_n can be chosen by induction. Suppose that $\beta_1, \cdots, \beta_{n-1}$ have all been chosen, then β_n is taken such that $\int f_n(x)g_n(x)dx$ differs by at least $4/5$ from:

$$-\int (f_1(x) + \cdots + f_{n-1}(x))g_n(x)dx \text{ i.e. } \left|\int \sum_1^n f_i(x)g_n(x)dx\right| > 4/5$$

Also

$$\left|\int \sum_{n+1}^{\infty} f_i(x)g_n(x)dx\right| \leqq 6/5\, g_n(0) \sum ||f_i|| \leqq (6/5)3^n(3^{-n-1} + 3^{-n-2} + \cdots) = 3/5$$

Now let $f(x) = f_1(x) + f_2(x) + \cdots$

$$\left|\int f(x)g_n(x)dx\right| \geqq 4/5 - 3/5 = 1/5$$

This proves the lemma, and the theorem follows without difficulty.

7. An application

The theorem above may be used to show that a Fourier series need not converge weakly. Let $g(x)$ be a continuous function such that the partial sums $g_n(x)$ of its Fourier series diverge at the origin. The existence of such a function was shown by Du Bois Reymond (reference [4]). Let $f(x)$ be the absolutely integrable function on $(0, 2\pi)$ given by the theorem above, and let $f_n(x)$ be the partial sums of its Fourier series. Then by the symmetry of the Dirichlet kernel:

$$\int f_n(x)g(x)dx = \int f(x)g_n(x)dx$$

which diverges.

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