COMMUTATIVE EXTENSION OF PARTIAL AUTOMORPHISMS OF GROUPS

by C. G. CHEHATA

(Received 5th June, 1953)

Introduction.

Let μ be an isomorphism which maps a subgroup A of the group G onto a second subgroup B (not necessarily distinct from A) of G; then μ is called a partial automorphism of G. If A coincides with G, that is if the isomorphism is defined on the whole of G, we speak of a *total automorphism*; this is what is usually called an automorphism of G. A partial (or total) automorphism μ^* *extends* or *continues* a partial automorphism μ if μ^* is defined for, at least, all those elements for which μ is defined, and moreover μ^* coincides with μ where μ is defined.

It is known (2) that any partial automorphism of a group can always be extended to a total automorphism of a supergroup, and even to an inner automorphism of a supergroup. Moreover, any number of partial automorphisms can be simultaneously extended to inner automorphisms of one and the same supergroup. In this paper conditions are investigated which ensure that two partial automorphisms can be extended to commutative (or permutable) automorphisms of a supergroup.

Sufficient conditions are derived in § 2, conditions which are too restrictive to be necessary as well, but which are sufficiently wide to give the following special case as a corollary:

If μ maps $A \subseteq G$ isomorphically onto $B \subseteq G$ and if ν maps $C \subseteq G$ isomorphically onto $D \subseteq G$, and if

$$A \cap C = B \cap C = A \cap D = B \cap D = \{1\},$$

then μ and ν can be extended to commutative automorphisms μ^* and ν^* of a supergroup G^* of G.

The principal tool throughout is the free product of two groups with one amalgamated subgroup.

I am indebted to Dr. B. H. Neumann for his advice and continuous help during the work.

§ 1. *Definitions and Lemmas.*

In this paragraph we explain what is meant by an incomplete group, group amalgams, homomorphism of an amalgam, canonic group, canonic homomorphism and generalized free products.† We then state Lemma 1 which is proved by Hanna Neumann (3), and prove some other lemmas which will be required later on.

Definition 1.

An *incomplete group* is a set of elements with a group operation defined for some pairs of its elements.

Then we call A an *amalgam* of groups G_i if A is an incomplete group consisting of the elements of the groups G_i with the product of two elements of A defined if, and only if, they both lie in (at least) one and the same group G_i. We call G_i the *constituent groups* of the amalgam and denote by $H_{ij} = H_{ji}$ $(i \neq j)$ the intersection of G_i and G_j; H_{ij} may contain the unit element alone.

† These definitions are adopted from a paper under publication by B. H. Neumann and H. Neumann for the purpose of making this paper self-contained.

https://doi.org/10.1017/S2040618500035693 Published online by Cambridge University Press

Definition 2.

A *homomorphism* of an amalgam A into a group P is a mapping ϕ of A into P such that if a and b are two elements of A whose product is defined in A, then

$$(ab)\phi = a\phi\, b\phi$$

in P.

Definition 3.

We associate with the amalgam A the group P^* which is generated by elements

$$a^* = a\phi^*$$

corresponding to the elements a of A; with the defining relations

$$a^*b^* = c^* \qquad \text{(1.1)}$$

whenever $a^* = a\phi^*$, $b^* = b\phi^*$, $c^* = c\phi^*$ and

$$ab = c$$

in A. We call P^* the *canonic group* of A and ϕ^* the *canonic homomorphism* of A.

If the mapping ϕ^* is one-to-one, then we say that P^* is the *free product* of the G_i with *amalgamated subgroups* H_{ij}.

Hanna Neumann proved the following lemma (3, Corollary 8.11):

Lemma 1.

Let P be the free product of two groups G_1 and G_2 with an amalgamated subgroup H; we use the notation

$$P = \{G_1 * G_2;\ H\}.$$

Then if A_1 and A_2 are subgroups of G_1 and G_2 respectively which have the same intersection B with H, then they generate in P the free product of A_1 and A_2 with the amalgamated subgroup B.

Lemma 2.

Let

$$P = \{G_1 * G_2;\ H\},$$

and let A_1 and A_2 be subgroups of G_1 and G_2 respectively with $A_1 \cap H = A_2 \cap H = B$ as in Lemma 1. Let

$$\{A_1, A_2\} = Q.$$

Then

$$Q \cap G_i = A_i;\quad i = 1, 2.$$

Proof:

By Lemma 1 we have

$$Q = \{A_1 * A_2;\ B\}.$$

Choose T_i as a set of left-hand coset representatives of A_i mod B $(i = 1, 2)$. Then $a \in A_i$ is uniquely expressed in the form

$$a = tb;\quad t \in T_i,\quad b \in B.$$

Assume that $1 \in T_i$, that is to say, B is to be represented by the unit element. Then for any $q \in Q$ there exists a unique normal form

$$q = t_1 t_2 \dots t_n b,$$

where each $t_i \in T_1 - \{1\}$ or $T_2 - \{1\}$, $b \in B$ and if $t_i \in T_\alpha$ then $T_{i+1} \bar{\in} T_\alpha$. We call n the length of q; thus $n = 0$ for elements of B and $n = 1$ for elements of $A_i - B$.

https://doi.org/10.1017/S2040618500035693 Published online by Cambridge University Press

Choose S_i as a set of left-hand coset representatives of G_i mod H such that $S_i \supseteq T_i$. This is possible because if $t_m H = t_n H$ $(m \neq n\,;\ t_m, t_n \in T_1$ say), then

$$t_m^{-1} t_n \in H,$$

but

$$t_m^{-1} t_n \in A_1,$$

thus

$$t_m^{-1} t_n \in A_1 \cap H = B,$$

contrary to the assumption that T_i is a set of left-hand coset representatives of A_i mod B.

A normal form thus exists in P with respect to S_i.

Let

$$g \in Q \cap G_i.$$

g as an element in Q has a normal form

$$g = t_1 t_2 \ldots t_n b,$$

but as an element in G_i, it can be written uniquely as

$$g = s_i h, \quad s_i \in S_i, \quad h \in H.$$

From the uniqueness of the representation we get

$$n = 1, \quad t_1 = s_i, \quad b = h.$$

Thus $g = t_1 b \in A_i$, that is to say,

$$Q \cap G_i \subseteq A_i\,;$$

but since we obviously have

$$Q \cap G_i \supseteq A_i,$$

then

$$Q \cap G_i = A_i.$$

This completes the proof.

We mention without proof the following known result (**1**) :

Lemma 3.

Let μ be a homomorphism of the amalgam A into a group Q. Then there is a homomorphism ν of the canonic group P^* of A into Q such that

$$\mu = \phi^* \nu$$

where ϕ^* is the canonic mapping of A.

As an immediate consequence of Lemma 3 we have :

Corollary :

Let G_1 and G_2 be two groups with $U = G_1 \cap G_2$, and let μ_i map G_i homomorphically onto H_i $(i = 1, 2)$. Suppose that $U\mu_1 \simeq U\mu_2 = V$ and that, more precisely, $u\mu_1 = u\mu_2$ for all $u \in U$. Then there exists a homomorphic mapping ν of

$$P_1 = \{G_1 * G_2\,;\ U\}$$

onto any group P_2 generated by H_1 and H_2 such that $H_1 \cap H_2 \supseteq V$ where ν extends μ_1 and μ_2 simultaneously.

Proof.

For we can take A as the amalgam whose constituent groups are G_1 and G_2. P^* and Q become P_1 and P_2 respectively, and the result follows immediately from Lemma 3.

Lemma 4.

Let G_1 and G_2 be two groups with $U = G_1 \cap G_2$ and let μ_i map G_i isomorphically onto H_i $(i=1, 2)$. Suppose that $U\mu_1 \cong U\mu_2 = V$ and that, more precisely, $u\mu_1 = u\mu_2$ for all $u \in U$. Then there exists an isomorphic mapping of

$$P_1 = \{G_1 * G_2 ;\ U\}$$

onto

$$P_2 = \{H_1 * H_2 ;\ V\},$$

which extends μ_1 and μ_2 simultaneously.

Proof.

Since $U\mu_1 \cong U\mu_2 = V$, then by the previous corollary there exists a homomorphic mapping ν of P_1 onto P_2 which extends both μ_1 and μ_2, that is to say

$$\left.\begin{array}{l} P_1\nu = P_2, \\ G_i\nu = G_i\mu_i = H_i \ ; \quad i=1, 2. \end{array}\right\} \qquad (1.2)$$

Since μ_i is an isomorphism, then its inverse μ_i^{-1} is an isomorphic mapping of H_i onto G_i.

Again since $V\mu_1^{-1} \cong V\mu_2^{-1} = U$, then by the previous corollary there exists a homomorphic mapping ν' of P_2 onto P_1 which extends μ_1^{-1} and μ_2^{-1} respectively, that is to say

$$\left.\begin{array}{l} P_2\nu' = P_1, \\ H_i\nu' = H_i\mu_i^{-1} = G_i \ ; \quad i=1, 2. \end{array}\right\} \qquad (1.3)$$

Applying ν' to (1.2) we get

$$P_1\nu\nu' = P_2\nu' = P_1,$$
$$G_i\nu\nu' = H_i\nu' = G_i\epsilon_i,$$

where ϵ_i is the identity mapping.

Applying ν to (1.3) we get

$$P_2\nu'\nu = P_1\nu = P_2,$$
$$H_i\nu'\nu = G_i\nu = H_i\epsilon_i.$$

Thus $\nu\nu'$ maps P_1 onto itself and on the constituent groups of P_1 it is equal to the identity mapping. Also $\nu'\nu$ maps P_2 onto itself and on the constituent groups of P_2 it is equal to the identity mapping.

Hence ν and ν' are reciprocal isomorphisms and the lemma follows.

§ 2. *Sufficient Conditions.*

Denote by A, B, C, D subgroups of a given group G and assume that μ maps A isomorphically onto B and ν maps C isomorphically onto D. Throughout this paragraph: A, B, C, D, G, μ and ν will always retain the same meaning.

For the existence of a group $G^* \supseteq G$ and two total automorphisms μ^* and ν^* of G^* such that $\mu^*\nu^* = \nu^*\mu^*$ and μ^*, ν^* extend μ, ν respectively one condition is obviously necessary, namely the following:

$$g\mu\nu = g\nu\mu,$$

whenever $g\mu$, $g\nu$, $(g\mu)\nu$, $(g\nu)\mu$ are defined, in other words whenever $g \in A \cap C$, $g\mu \in C$ and $g\nu \in A$. Because if G^*, μ^* and ν^* exist, then for such an element g we get:

$$g\mu\nu = g\mu^*\nu^*,$$
$$g\nu\mu = g\nu^*\mu^*,$$

and since μ^* commutes with ν^*, then

$$g\mu\nu = g\nu\mu.$$

https://doi.org/10.1017/S2040618500035693 Published online by Cambridge University Press

We now show the sufficiency of certain conditions by proving the following theorem :

Theorem 1.

For the existence of a supergroup $G^* \supseteq G$ with two commutative inner automorphisms s and t extending μ and ν respectively, it is sufficient that the following hold :

$$(A \cap C)\mu = B \cap C, \qquad (2.1)$$
$$(A \cap D)\mu = B \cap D, \qquad (2.2)$$
$$(A \cap C)\nu = A \cap D, \qquad (2.3)$$
$$(B \cap C)\nu = B \cap D, \qquad (2.4)$$
$$g\mu\nu = g\nu\mu, \qquad (2.5)$$

whenever $g\mu$, $g\nu$, $(g\mu)\nu$ and $(g\nu)\mu$ are defined.

Proof.

The proof of this theorem is somewhat long and it will be effected by a number of lemmas. The main steps in the procedure are these : we take a sequence ..., G_{-1}, G_0, G_1, ... of copies of the group G, that is to say a sequence of groups isomorphic to G, and construct certain free products P_{ij} defined inductively for any $i<j$ and for G_i, G_{i+1}, ..., G_j with certain amalgamated subgroups. We then prove that in P_{ij} there exist two subgroups Q_{ij} and R_{ij} which are isomorphic under a mapping ν_{ij} that extends ν. Lastly we form the union $\bigcup_{n=1}^{\infty} P_{-n,+n}$ and prove that it contains two subgroups which are isomorphic under a mapping $\bar{\nu}$ that extends ν, and that it possesses an automorphism μ^* which extends μ and commutes with $\bar{\nu}$. The proof will then be completed by applying a theorem due to G. Higman, B. H. Neumann and H. Neumann (2).

Now take a sequence of groups

$$\ldots,\ G_{-1},\ G_0,\ G_1,\ \ldots,$$

each of which is isomorphic to G. Let under a fixed isomorphic mapping γ_i,

$$G\gamma_i = G_i\,; \quad i = 0, \pm 1, \ldots.$$

Each group G_i contains subgroups A_i, B_i, C_i, D_i, which are the images of A, B, C, D under the mapping γ_i and possesses two isomorphisms μ_i and ν_i mapping A_i onto B_i and C_i onto D_i respectively, such that if

$$a_i = a\gamma_i, \quad a \in A,$$
$$b_i = b\gamma_i, \quad b \in B,$$

then $a_i\mu_i = b_i$ if $a\mu = b$. In other words,

$$\mu_i = \gamma_i^{-1}\mu\gamma_i.$$

Similarly one defines ν_i to be

$$\nu_i = \gamma_i^{-1}\nu\gamma_i.$$

If we replace A, B, C, D, μ and ν by A_i, B_i, C_i, D_i, μ_i and ν_i respectively, then they will satisfy the relations that correspond to (2.1) – (2.5).

Form the free product of G_i and G_{i+1} amalgamating $B_i \subseteq G_i$ with $A_{i+1} \subseteq G_{i+1}$ as follows : if

$$b_i = b\gamma_i, \qquad b \in B,$$
$$a_{i+1} = a\gamma_{i+1}, \quad a \in A,$$

and if $a\mu = b$ then we put $b_i = a_{i+1}$. That is to say, the isomorphism underlying the amalgamation is $\gamma_i^{-1}\mu^{-1}\gamma_{i+1}$. $\gamma_i^{-1}\mu^{-1}\gamma_{i+1}$ is defined on B_i, moreover it is the identical mapping on B_i. Call this free product

$$P_{ii+1} = \{G_i * G_{i+1}\,; \quad B_i = A_{i+1}\}.$$

We then form the free product of P_{ii+1} and G_{i+2} amalgamating $B_{i+1} \subseteq P_{ii+1}$ with $A_{i+2} \subseteq G_{i+2}$ according to the isomorphism $\gamma_{i+1}^{-1}\mu^{-1}\gamma_{i+2}$. Call the product

$$P_{ii+2} = \{P_{ii+1} * G_{i+2}\,;\quad B_{i+1} = A_{i+2}\}.$$

More generally we define P_{ij} inductively for any $i<j$ in the following way:

$$P_{ii+1} = \{G_i \;*\; G_{i+1}\,;\quad B_i \;= A_{i+1}\},$$
$$P_{ij} \;= \{P_{ij-1} * G_j \quad;\quad B_{j-1} = A_j\},$$

where B_{j-1} is amalgamated with A_j according to the isomorphism $\gamma_{j-1}^{-1}\mu^{-1}\gamma_j$.

We note that we have:

$$P_{ij} \subseteq P_{hk}, \qquad (2.6)$$

for any $h \leqslant i < j \leqslant k$.
Also by this method of construction P_{ii+r} is isomorphic to P_{hh+r}, in an obvious way, for any i, h and positive r. This fact will be used later in Lemma 8.

Now we prove that in P_{ij} the following lemmas are true:

Lemma 5 :

In P_{ii+1} the following relations hold:

$$C_i \cap B_i = C_{i+1} \cap A_{i+1}, \qquad (2.7)$$
$$D_i \cap B_i = D_{i+1} \cap A_{i+1}. \qquad (2.8)$$

Proof.

Let
$$a_i \in C_i \cap B_i,$$
then
$$a_i\gamma_i^{-1} \in C \cap B.$$
Since $a_i \in B_i = A_{i+1}$, that is to say a_i is an amalgamated element,
then
$$a_i\gamma_i^{-1}\mu^{-1}\gamma_{i+1} = a_i,$$
or
$$a_i\gamma_i^{-1} = a_i\gamma_{i+1}^{-1}\mu\,;$$
thus
$$a_i\gamma_{i+1}^{-1}\mu \in C \cap B = (C \cap A)\mu \quad \text{by (2.1)}.$$
Since μ is an isomorphism, then
$$a_i\gamma_{i+1}^{-1} \in C \cap A,$$
$$a_i \in (C \cap A)\gamma_{i+1} = C_{i+1} \cap A_{i+1},$$
$$C_i \cap B_i \subseteq C_{i+1} \cap A_{i+1}. \qquad (2.9)$$

If, on the other hand,
$$a_{i+1} \in C_{i+1} \cap A_{i+1},$$
then
$$a_{i+1}\gamma_{i+1}^{-1} \in C \cap A,$$
$$a_{i+1}\gamma_{i+1}^{-1}\mu \in (C \cap A)\mu = C \cap B, \quad \text{by (2.1)}.$$
Since $a_{i+1} \in A_{i+1} = B_i$, that is to say a_{i+1} is an amalgamated element, then:
$$a_i\gamma_i^{-1}\mu^{-1}\gamma_{i+1} = a_i,$$
or
$$a_i\gamma_i^{-1} = a\gamma_{i+1}^{-1}\mu \in C \cap B,$$
$$a_i \in (C \cap B)\gamma_i = C_i \cap B_i,$$
thus
$$C_{i+1} \cap A_{i+1} \subseteq C_i \cap B_i. \qquad (2.10)$$
(2.9) and (2.10) together give
$$C_i \cap B_i = C_{i+1} \cap A_{i+1},$$

which proves (2.8). Similarly one proves (2.9), and the lemma follows.

https://doi.org/10.1017/S2040618500035693 Published online by Cambridge University Press

Lemma 5 and relation (2.6) give the following :

Corollary :

In P_{ij} the following relations hold :

$$C_t \cap B_t = C_{t+1} \cap A_{t+1}, \quad \text{......................................(2.11)}$$

$$D_t \cap B_t = D_{t+1} \cap A_{t+1}, \quad \text{......................................(2.12)}$$

for all $t = i, i+1, \ldots, j-1$.

Lemma 6.

If $h \leqslant i < j \leqslant k$ and if we denote by Q_{ij} the subgroup of P_{hk} generated by $C_i, C_{i+1}, \ldots, C_j - Q_{ii}$, in particular, is $C_i -$, and if we similarly denote by R_{ij} the subgroup of P_{hk} generated by $D_i, D_{i+1}, \ldots, D_j - R_{ii}$, in particular, is $D_i -$, then

$$Q_{ij} = \{Q_{ij-1} * C_j \,; \quad C_{j-1} \cap B_{j-1} = C_j \cap A_j\},$$

$$R_{ij} = \{R_{ij-1} * D_j \,; \quad D_{j-1} \cap B_{j-1} = D_j \cap A_j\}.$$

Proof.

It is sufficient to prove the lemma for Q_{ij}, the proof for R_{ij} will be on similar lines.

Now we shall prove by induction that

$$\left.\begin{array}{l} Q_{ij-1} \cap B_{j-1} = C_j \cap A_j, \\ Q_{ij} = \{Q_{ij-1} * C_j \,; \quad C_{j-1} \cap B_{j-1} = C_j \cap A_j\}, \end{array}\right\} , \quad \text{........................(2.13)}$$

for any $i < j$. The induction is on j.

For $j = i+1$ we have

$$\begin{aligned} Q_{ii} \cap B_i &= C_i \cap B_i \\ &= C_{i+1} \cap A_{i+1} \quad \text{by Lemma 5.} \end{aligned}$$

Thus by Lemma 1, C_i and C_{i+1} generate in P_{ii+1} their free product with the amalgamated subgroup $C_i \cap B_i$:

$$Q_{ii+1} = \{C_i * C_{i+1} \,; \quad C_i \cap B_i = C_{i+1} \cap A_{i+1}\}.$$

Assume that (2.13) is true for j.

Applying Lemma 2 with $P_{ij}, P_{ij-1}, G_j, B_{j-1}, Q_{ij}, Q_{ij-1}, C_j, C_{j-1} \cap B_{j-1}$ in the place of $P, G_1, G_2, H, Q, A_1\ A_2, B$ respectively we get, because of the first inductive relation of (2.13) :

$$Q_{ij} \cap G_j = C_j.$$

Intersecting both sides with B_j we get

$$\begin{aligned} Q_{ij} \cap B_j &= C_j \cap B_j \\ &= C_{j+1} \cap A_{j+1} \quad \text{by (2.11).} \end{aligned}$$

Because of this equation, we can apply Lemma 1, with

$$P_{ij+1}, \quad P_{ij}, \quad G_{j+1}, \quad B_j, \quad Q_{ij}, \quad C_{j+1}$$

in the place of

$$P, \quad G_1, \quad G_2, \quad H, \quad A_1, \quad A_2,$$

and thus Q_{ij} and C_{j+1} generate in P_{ij+1} their free product with the amalgamated subgroup $C_j \cap B_j = C_{j+1} \cap A_{j+1}$. This completes the proof of Lemma 6.

Lemma 7.

There exists an isomorphic mapping ν_{ij}, say, of Q_{ij} onto R_{ij} such that ν_{ij} extends $\nu_i, \nu_{i+1}, \ldots, \nu_j$.

Proof.

The proof is by induction on j. We first show that in Q_{ii+1} the map by ν_i of an element in $C_i \cap C_{i+1}$ is equal to its map by ν_{i+1}. By the construction of P_{ii+1} we have

$$G_i \cap G_{i+1} = B_i,$$

intersecting both sides with $C_i \cap C_{i+1}$ we get

$$\begin{aligned} C_i \cap C_{i+1} &= B_i \cap C_i \cap C_{i+1} \\ &= A_{i+1} \cap C_{i+1} \\ &= B_i \cap C_i \quad \text{by (2.7).} \end{aligned}$$

Let
$$c_i \in C_i \cap C_{i+1} = A_{i+1} \cap C_{i+1} = B_i \cap C_i,$$

then
$$c_i\nu_i = c_i\gamma_i^{-1}\nu\gamma_i \in B_i \cap D_i, \quad \text{.................................(2.14)}$$

$$c_i\nu_{i+1} = c_i\gamma_{i+1}^{-1}\nu\gamma_{i+1} \in A_{i+1} \cap D_{i+1}. \quad \text{..........................(2.15)}$$

Since c_i is an amalgamated element, then

$$c_i\gamma_i^{-1}\mu^{-1}\gamma_{i+1} = c_i$$

or
$$c_i\gamma_i^{-1} = c_i\gamma_{i+1}^{-1}\mu.$$

$c_i\gamma_i^{-1} \in B \cap C$, thus applying ν to the last equation we get

$$\begin{aligned} c_i\gamma_i^{-1}\nu &= c_i\gamma_{i+1}^{-1}\mu\nu \\ &= c_i\gamma_{i+1}^{-1}\nu\mu \quad \text{by (2.5).} \quad \text{..........................(2.16)} \end{aligned}$$

But (2.14) and (2.15) give

$$c_i\gamma_i^{-1}\nu = c_i\nu_i\gamma_i^{-1},$$

$$c_i\gamma_{i+1}^{-1}\nu = c_i\nu_{i+1}\gamma_{i+1}^{-1}.$$

Substituting in (2.16) we get

$$c_i\nu_i\gamma_i^{-1} = c_i\nu_{i+1}\gamma_{i+1}^{-1}\mu,$$

or
$$(c_i\nu_i)\gamma_i^{-1}\mu^{-1}\gamma_{i+1} = (c_i\nu_{i+1}).$$

Thus, according to the rule of amalgamation, we get

$$c_i\nu_i = c_i\nu_{i+1}.$$

This together with the fact that

$$(C_i \cap B_i)\nu_i = D_i \cap B_i,$$

enables us to apply Lemma 4, thus proving the existence of an isomorphic mapping ν_{ii+1}, say, of Q_{ii+1} onto R_{ii+1} such that ν_{ii+1} extends ν_i and ν_{i+1}.

Suppose that ν_{ij} maps Q_{ij} isomorphically onto R_{ij} such that ν_{ij} extends $\nu_i, \nu_{i+1}, \ldots, \nu_j$. For any element

$$c \in Q_{ij} \cap C_{j+1},$$

we get $c \in C_j \cap B_j = C_{j+1} \cap A_{j+1}$, since by Lemma 6 we have

$$Q_{ij} \cap C_{j+1} = C_j \cap B_j = C_{j+1} \cap A_{j+1}.$$

For such an element c, we proved above that

$$c\nu_j = c\nu_{j+1}.$$

https://doi.org/10.1017/S2040618500035693 Published online by Cambridge University Press

Since ν_{ij} extends ν_j, that is to say, since

$$c\nu_j = c\nu_{ij},$$

then

$$c\nu_{ij} = c\nu_{j+1}.$$

Thus it is legitimate to apply Lemma 4 which proves the existence of an isomorphic mapping ν_{ij+1}, say, of

$$Q_{ij+1} = \{Q_{ij} * C_{j+1}\,;\quad C_j \cap B_j = C_{j+1} \cap A_{j+1}\}$$

onto

$$R_{ij+1} = \{R_{ij} * D_{j+1}\,;\quad D_j \cap B_j = D_{j+1} \cap A_{j+1}\},$$

which extends ν_{ij} and ν_{j+1} and thus extends $\nu_i, \nu_{i+1}, \ldots, \nu_{j+1}$. This completes the proof of the lemma.

Now we form

$$P^* = \bigcup_{n=1}^{\infty} P_{-n,+n}.$$

Define the mapping μ^* as follows : For any $g_i \in G_i$, if $g_i\gamma_i^{-1} = g$ and $g\gamma_{i+1} = g_{i+1}$, then we put

$$g_i\mu^* = g_{i+1},$$

in other words, on G_i,

$$g_i\mu^* = g_i\gamma_i^{-1}\gamma_{i+1}.$$

We prove then the following

Lemma 8.

The mapping μ^* generates an automorphism of P^* which extends every μ_i.

Proof.

That μ^* generates a mapping of P^* onto itself is obvious. It is also consistent for if

$$g_i \in G_i \cap G_{i+1} = B_i = A_{i+1},$$

and if

$$g_i\gamma_i^{-1} = g,$$

$$g\gamma_{i+1} = g_{i+1},$$

then

$$g_i\mu^* = g_{i+1} \in B_{i+1}.$$

If, on the other hand,

$$g_i\gamma_{i+1}^{-1} = g',$$

$$g'\gamma_{i+2} = g_{i+2},$$

then

$$g_i\mu^* = g_{i+2} \in A_{i+2}.$$

Since g_i is an amalgamated element, then

$$g_i\gamma_i^{-1}\mu^{-1}\gamma_{i+1} = g_i.$$

But

$$g_i\gamma_i^{-1} = g = g_{i+1}\gamma_{i+1}^{-1},$$

and

$$g_i\gamma_{i+1}^{-1} = g' = g_{i+2}\gamma_{i+2}^{-1}.$$

These last three equations together give

$$g_{i+1}\gamma_{i+1}^{-1}\mu^{-1}\gamma_{i+2} = g_{i+2}.$$

Thus g_{i+1} is to be amalgamated with g_{i+2}. This proves the consistency of μ^*.

That μ^* generates an automorphism of P^* follows from the fact that P_{ii+1} is isomorphic to P_{i+1i+2} under the mapping generated by $\gamma_i^{-1}\gamma_{i+1}$ and $\gamma_{i+1}^{-1}\gamma_{i+2}$.

To prove that μ^* extends every μ_i, we take an arbitrary element $a_i \in A_i$, then by the definition of μ^* and μ_i we get

$$a_i\mu^* = a_i\gamma_i^{-1}\gamma_{i+1} \in A_{i+1},$$

$$a_i\mu_i = a_i\gamma_i^{-1}\mu\gamma_i \in B_i.$$

Since

$$(a_i\gamma_i^{-1}\mu\gamma_i)\,\gamma_i^{-1}\mu^{-1}\gamma_{i+1} = a_i\gamma_i^{-1}\gamma_{i+1},$$

$$(a_i\mu_i)\,\gamma_i^{-1}\mu^{-1}\gamma_{i+1} = (a_i\mu^*),$$

then by the rule of amalgamation we get

$$a_i\mu_i = a_i\mu^*.$$

This completes the proof of the lemma.

The group P^* contains the subgroups

$$C^* = \{\dots, C_{-1}, C_0, C_1, \dots\},$$

$$D^* = \{\dots, D_{-1}, D_0, D_1, \dots\}.$$

It is clear that μ^* maps C^* onto itself and maps D^* onto itself.

Define the mapping $\bar{\nu}$ of C^* onto D^* as follows: for $c \in C^*$, that is to say $c \in Q_{ij}$ for some suitable i and j, put

$$c\bar{\nu} = c\nu_{ij}.$$

$\bar{\nu}$ is thus an isomorphism of C^* onto D^*, which extends the ν_i since the ν_{ij} does.

It is clear that for any $c^* \in C^*$, $c^*\bar{\nu}\mu^*$ is defined, and since $c^*\mu^* \in C^*$ then $c^*\mu^*\bar{\nu}$ is also defined. Moreover, we have:

Lemma 9.

$$c^*\bar{\nu}\mu^* = c^*\mu^*\bar{\nu}, \qquad (2.17)$$

for any $c^* \in C^*$.

Proof.

Let

$$c^* = \prod_i c_i, \quad c_i \in C_i.$$

Since $\bar{\nu}$ extends all the ν_i then

$$\begin{aligned} c^*\bar{\nu} &= \prod_i c_i\nu_i \\ &= \prod_i c_i\gamma_i^{-1}\nu\gamma_i, \\ c^*\bar{\nu}\mu^* &= \prod_i c_i\gamma_i^{-1}\nu\gamma_i\gamma_i^{-1}\gamma_{i+1} \\ &= \prod_i c_i\gamma_i^{-1}\nu\gamma_{i+1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} c^*\mu^*\bar{\nu} &= \prod_i c_i\gamma_i^{-1}\gamma_{i+1}\nu_{i+1} \\ &= \prod_i c_i\gamma_i^{-1}\gamma_{i+1}\gamma_{i+1}^{-1}\nu\gamma_{i+1} \\ &= \prod_i c_i\gamma_i^{-1}\nu\gamma_{i+1}. \end{aligned}$$

Thus $c^*\bar{\nu}\mu^* = c^*\mu^*\bar{\nu}$, and the lemma follows.

https://doi.org/10.1017/S2040618500035693 Published online by Cambridge University Press

Now form the group generated by P^* and an element s, call this group

$$\widehat{P}=\{P^*, s\},$$

and define

$$s^{-1}p^*s=p^*\mu^* \quad \text{for all } p^* \in P^*;$$

s thus induces an inner automorphism of $\widehat{P}$.

Equation (2.17) then translates into

$$s^{-1}(c^*\bar{\nu})s=(s^{-1}c^*s)\bar{\nu},$$

that is, the inner automorphism s commutes with the partial automorphism $\bar{\nu}$.

If we, moreover, define

$$s\bar{\nu}=s,$$

then $\bar{\nu}$ becomes an isomorphism of

$$\widehat{C}=\{C^*, s\}$$

onto

$$\widehat{D}=\{D^*, s\},$$

and it also commutes with s.

At this stage we mention a theorem proved by G. Higman, B. H. Neumann and H. Neumann (2, Theorem 1), namely, the following

Theorem 2.

Let μ be an isomorphism of a subgroup A of a group G onto a second subgroup B of G. Then there exists a group H containing G, and an element t of H, such that the transform by t of any element of A is its image under μ:

$$t^{-1}at=a\mu \quad \text{for all } a \in A.$$

Applying this theorem with $\bar{\nu}, \widehat{C}, \widehat{D}, \widehat{P}$ taking the place of μ, A, B, G respectively, we can embed P in a group

$$G^*=\{\widehat{P}, t\}$$

such that

$$t^{-1}\hat{c}t = \hat{c}\bar{\nu} \text{ for any } \hat{c} \in \widehat{C}, \qquad \text{(2.18)}$$

that is to say t induces an inner automorphism of G which extends $\bar{\nu}$ and thus extends ν.

Putting $c=s$ in (2.18) we get

$$\begin{aligned} t^{-1}st &= s\bar{\nu} \\ &= s \quad \text{by definition.} \end{aligned}$$

Thus s and t commute and G^* has the required properties. This completes the proof of Theorem 1.

Remark. We note that conditions (2.1)–(2.4) of Theorem 1 are not necessary; for if we take the subgroups A, B, C, D of G such that

$$A=C\neq\{1\}, \quad B=D, \quad A \cap B=\{1\},$$

and take ν to be the same mapping as μ, then the conclusion of the theorem is known to be valid (by Theorem 2), although $(A \cap D)\mu\neq B \cap D$.

A special case of Theorem 1 is the following:

Corollary:

Sufficient for μ and ν to be extendable to two commutative inner automorphisms of one and the same group is that

$$A \cap C=B \cap C=A \cap D=B \cap D=\{1\}.$$

For then equations (2.1)–(2.5) will be trivially satisfied.

https://doi.org/10.1017/S2040618500035693 Published online by Cambridge University Press

REFERENCES

(1) Baer, Reinhold, " Free sums of groups and their generalizations. An analysis of the associative law," *Amer. J. Math.*, **71**, (1949), 706–742.

(2) Higman, G., Neumann, B. H., and Neumann, H., " Embedding theorems for groups," *J. London Math. Soc.*, **24** (1949), 247–254.

(3) Neumann, Hanna, " Generalized free products with amalgamated subgroups," *Amer. J. Math.*, **70**, (1948), 590–625.

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY
MANCHESTER
and
FACULTY OF SCIENCE
THE UNIVERSITY
ALEXANDRIA