ON CROSSED PRODUCTS OF A SFIELD

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To RICHARD BRAUER on the occasion of his 60th birthday

In the previous paper [3] the author has shown a possibility to construct a series of sfields by taking sfields of quotients of split crossed products of a sfield. In this paper the same problem is treated, and, by considering general crossed products, a generalization of the previous result is given: Let K be a sfield and G be the join of a well-ordered ascending chain of groups G_α of outer automorphisms of K such that a) G_1 is the identity automorphism group, b) G_α is a group extension of $G_{\alpha-1}$ by a torsion-free abelian group for each non-limit ordinal α, and c) $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ for each limit ordinal α. Then an arbitrary crossed product of K with G is an integral domain with a sfield of quotients Q and the commutor ring of K in Q coincides with the centre of K.

1. Let K be a sfield and G be a group of outer automorphisms of K. A crossed product P of K with the group G is defined as follows[1]:

(C_1) P is a ring extension of K and possesses a unit element which is at the same time a unit element of K.

(C_2) P is expressible as a sum $\sum_{\sigma\in G} u_\sigma K$ with the regular elements u_σ corresponding to G-elements σ.

(C_3) $a u_\sigma = u_\sigma a^\sigma$ for all K-elements a and G-elements σ.

Then it can be shown that

(P_1) $\{u_\sigma\}$ is a K-basis of P,

(P_2) if a P-element u satisfies the condition $uK = Ku$, then u is an element of a module $u_\sigma K$ for a suitable G-element σ,

(P_3) for each pair σ, τ of G-elements, $u_\sigma u_\tau = u_{\sigma\tau} c_{\sigma,\tau}$ with a non-zero element $c_{\sigma,\tau}$ of the centre of K, and the elements $c_{\sigma,\tau}$ satisfy the relation $c_{\rho,\sigma\tau} c_{\sigma,\tau} = c_{\rho\sigma,\tau} c^{\tau}_{\rho,\sigma}$, and

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[1] For detailed discussion see G. Azumaya and T. Nakayama [2].

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(P_4) P is a simple ring.

The set $C_G=\{c_{\sigma,\tau}\}$ is called a factor set of G with respect to K. Two factor sets $C_G=\{c_{\sigma,\tau}\}$ and $C'_G=\{c'_{\sigma,\tau}\}$ are said to be associated with each other, if there exists a set $\{b_\sigma\}$ of non-zero elements of the centre of K such that $c'_{\sigma,\tau}=c_{\sigma,\tau}b_\sigma^\tau b_\tau b_{\sigma\tau}^{-1}$ for every pair of elements of G. If there is an isomorphism of a crossed product of K with G onto another crossed product of K with G and every element of K is mapped on itself, then these crossed products are said to be similar to each other. It can be then shown that there is a one-to-one correspondence between the classes of associated factor sets and the classes of similar crossed products. In this sense we shall write a crossed product of K with G in the form (K, G, C_G). Of course this expression depends on the choice of the transformers $\{u_\sigma\}$. Now let H be a subgroup of G, then a crossed product (K, G, C_G) contains isomorphically the crossed product $(K, H, C_G(H))$ of K with H having the factor set $C_G(H)$ obtained by the restriction of the factor set C_G on H. In this paper the notation $C_G(H)$ will be used always to be the factor set obtained by the restriction of C_G of a group G on a subgroup H, and the crossed product $(K. H, C_G(H))$ will be considered always to be imbedded in (K, G, C_G).

LEMMA 1. *Let K be a sfield and G be a group of outer automorphisms of K which is a group extension of a subgroup H by an infinite cyclic group. Assume further that every crossed product P of K with H satisfies the following conditions:*

i) P is an integral domain and possesses a sfield of quotients Q.

ii) An automorphism of K can be extended to an inner automorphism of Q which maps P onto itself, if and only if it belongs to an automorphism class represented by an element of H.

iii) The commutor ring $V_Q(K)$ of K in Q coincides with the centre of K.

Then an arbitrary crossed product of K with G satisfies the same conditions i), ii) and iii).

Proof. First we mention that G is a split group extension of H by an infinite cyclic group, i.e. G contains an infinite cyclic subgroup $\{\varphi\}$ such that $G=\{\varphi\}\cdot H$, $\{\varphi\}\cap H=1$ and $\varphi^{-1}H\varphi=H$. Now let P^* be a crossed product of the form (K, G, C_G) and the set of transformers be $\{u_\sigma\}$. We choose another set of transformers $\{v_\sigma\}$ connected with the set $\{u_\sigma\}$ by the relations: $v_{\varphi^i} = u_\varphi^i$, $v_{\varphi^i\rho} = u_\varphi^i u_\rho$ for $-\infty < i < +\infty$ and $v_\rho = u_\rho$ for H-elements ρ. Then we get a new factor set C_G' associated with C_G. It can be easily seen that $c'_{\varphi^i, \varphi^j} = 1$ for all elements of the cyclic group $\{\varphi\}$. Further we see readily that the crossed product P^* can be expressed as a direct sum $\sum_i v_{\varphi^i} P$, where $P = (K, H, C_G(H))$ is the crossed product of K with H obtained by the restriction of the factor set C_G on the subgroup H. The elements v_{φ^i} induce automorphisms φ^i on P, and these automorphisms can be extended on the sfield of quotients Q of P whose existence is assumed. The automorphisms of Q thus obtained are not inner automorphisms of Q. For otherwise an automorphism φ^i of K would be extended to an inner automorphism of Q, and the latter maps P onto itself. Hence by the assumption the automorphism φ^i of K would belong to an automorphism class represented by an element of H. But since G is a group of outer automorphisms, this would imply that the automorphism φ^i belong to H. This is a contradiction. Now we consider the split crossed product $\overline{P}$ of Q with the cyclic group $\{\varphi\}$ of automorphisms of Q. $\overline{P}$ is expressible in the form $\sum_i w_{\varphi^i} Q$, where $w_{\varphi^i} w_{\varphi^j} = w_{\varphi^{i+j}}$ and $A w_{\varphi^i} = w_{\varphi^i} A^{\varphi^i}$ for Q-elements A. It can be easily seen that P^* can be isomorphically imbedded into $\overline{P}$. In fact the subring $\sum_i w_{\varphi^i} P$ is isomorphic to P^* under the correspondence f: $f(v_{\varphi^i}) = w_{\varphi^i}$ and $f(A) = A$ for P-element A. As is easily seen, the split crossed product $\overline{P}$ is an integral domain and possesses a sfield of quotients $\overline{Q}$. Since P^* is isomorphic to a subring of $\overline{P}$, it is an integral domain. We shall show now further that $\overline{Q}$ is a sfield of quotients of the ring $\sum_i w_{\varphi^i} P$. Since $\overline{Q}$ is a sfield of quotients of $\overline{P}$, there are two elements A and B for each element X of $\overline{Q}$ such that AX and XB belong to $\overline{P}$. Let the elements A, B, AX and XB be of the form $\sum_i w_{\varphi^i} a_i$, $\sum_i w_{\varphi^i} b_i$, $\sum_i w_{\varphi^i} a_i'$ and $\sum_i w_{\varphi^i} b_i'$ respectively, where only a finite number of Q-elements a_i, b_j, a_k' and b_k' is different from zero. Then, since Q is a sfield of quotients of P, we can find two P-elements c and d so that $c(\sum_i w_{\varphi^i} a_i)$, $c(\sum_i w_{\varphi^i} a_i')$, $(\sum_i w_{\varphi^i} b_i)d$ and $(\sum_i w_{\varphi^i} b_i')d$ belong to the ring $\sum_i w_{\varphi^i} P$. This tells us that $\overline{Q}$ is a sfield of quotients of $\sum_i w_{\varphi^i} P$. Thus the crossed product P^* possesses a sfield of quotients which is isomorphic to $\overline{Q}$ and will be denoted by Q^*. Suppose now that an automorphism τ of K could be extended to an inner automorphism of Q^* which maps P^* onto itself. Let the extension of τ on Q^* be given by the transformation with a Q^*-element u. Then from the assumption $uP^* = P^*u$, and the element u induces an automorphism τ on P^*: $Cu = uC^\tau$ for P^*-elements C. Now let S be the set of all P^*-elements A such that Au belongs to P^*. Since Q^* is a sfield of quotients of P^*, the set S contains at least one non-zero P^*-element. Obviously S forms an additive module. Moreover, if A is an element of S, then, since $u(AC)^\tau$ belongs to P^* for any P^*-element C, it follows that AC belongs to S for every P^*-element C. On the other hand DA belongs to S for every P^*-element D. Thus the set S is a two-sided ideal of P^*. But P^* is a simple ring, therefore the non-zero ideal S must be equal to P^*. Consequently the element u must be a P^*-element. But, since the element u induces the automorphism τ on K, it follows, by the property (P_2) of crossed products, that the element u is an element of $u_\sigma K$ for a suitable G-element σ. Thus the automorphism τ belongs to an automorphism class represented by a G-element. Conversely, if an automorphism τ of K belongs to an automorphism class represented by a G-element, then it can be obviously extended to an inner automorphism of Q^* which maps P^* onto itself. Now let A be an element of the commutor ring $V_{Q^*}(K)$ of K in Q^*. Since Q^* is isomorphic to $\bar{Q}$, we may consider $\bar{Q}$ instead of Q^*. The sfield $\bar{Q}$ is a sfield of quotients of the split crossed product of a sfield Q with an infinite cyclic group of automorphisms of Q. Hence, by Hilfssatz 1 in [3], the commutor ring $V_{\bar{Q}}(K)$ is equal to the commutor ring $V_Q(K)$ of K in Q. But Q was a sfield of quotients of a crossed product P of K with H, therefore, by the assumption, $V_{\bar{Q}}(K)$ coincides with the centre of K. Thus the proof is completed.

By induction on the rank of finitely generated free abelian groups, we get readily the following

LEMMA 2. *Let K be sfield and G be a group of outer automorphisms of K which is a group extension of a subgrup H by a finitely generated free abelian group. If every crossed products of K with H satisfies the conditions i), ii) and iii) in Lemma 1, then every crossed product of K with G satisfies the same conditions i), ii) and iii).*

LEMMA 3. *Let K be a sfield and G be a group of outer automorphisms of K. Assume further that G is the join of subgroups G_α $(\alpha \in I)$, and that every* *crossed product of K with G_α satisfies the conditions i), ii) and iii) in Lemma 1. Then every crossed product of K with G satisfies the same conditions i), ii) and iii).*

Proof. Let a crossed product P of K with G be of the form (K, G, C_G) and the set of transformers be $\{u_\sigma\}$. Then the crossed product $P_\alpha = (K, G_\alpha, C_G(G_\alpha))$ of K with each G_α is contained in P, and, by the assumption, it satisfies the conditions i), ii) and iii). First it can be easily seen that P is the join of P_α, therefore each element of P belongs to P_α with a suitable index α. From this fact it follows immediately that P is an integral domain. To see the existence of a sfield of quotients of P, it is sufficient, by the Asano's criterium,[2] to show the existence of two pairs of non-zero P-elements A', B' and A'', B'' for each pair of non-zero P-elements A and B such that $AA' = BB'$ and $A''A = B''B$. As was mentioned above, the elements A and B belong to P_α with a suitable index α, and P_α possesses a sfield of quotients. Therefore, again by the Asano's criterium, we can find desired pairs of elements in P_α. Thus P possesses a sfield of quotients, which will be denoted by Q. Now the sfield Q contains a sfield of quotients Q_α of each crossed product P_α. In fact the sfield Q_α can be characterized as the set of all such Q-elements X that XA belongs to P_α for a suitable element A of P_α. Since P is the join of the subrings P_α, the join of the sfields Q_α is a sfield of quotients of P, consequently it coincides with Q. Thus Q is the join of the sfields Q_α. Now we determine the intersection of P and Q_α. Let $G = \sum_\tau G_\alpha \cdot \tau$ be the decomposition of G into left cosets with respect to the subgroup G_α, where $\{\tau\}$ is a set of representatives of the cosets. Then an element A of P can be written in the form $\sum_{\rho \in G_\alpha} u_\rho a_\rho + \sum_{\rho \in G_\alpha} u_{\rho\tau} a_{\rho\tau} + \sum_{\rho \in G_\alpha} u_{\rho\tau'} a_{\rho\tau'} + \dots$, where $1, \tau, \tau', \dots$ are representatives of cosets and only a finite number of the coefficients is different from zero. Suppose now that A belongs to the intersection $P \cap Q_\alpha$. Then, since Q_α is a sfield of quotients of P_α, there is a P_α-element A such that $A_\alpha A$ belongs to P_α. The expansion of the product $A_\alpha A$ will be of the form $\sum_{\rho \in G_\alpha} u_\rho a'_\rho + \sum_{\rho \in G_\alpha} u_{\rho\tau} a'_{\rho\tau} + \sum_{\rho \in G_\alpha} u_{\rho\tau'} a'_{\rho\tau'} + \cdots$. Since this is in P_α, the coefficients $a'_{\rho\tau}$ other than the coefficients of u_ρ corresponding to G_α-elements are all zero. Therefore the

2) K. Asano [1].

products $A_\alpha(\sum_{\rho\in G_\alpha} u_{\rho\tau} a_{\rho\tau})$ are all zero for all representatives τ not belonging to G_α. But P is an integral domain, hence the element A must be of the form: $A = \sum_{\rho\in G_\alpha} u_\rho a_\rho$, i.e. A is an element of P_α. Thus we have shown that $P \cap Q_\alpha = P_\alpha$. Now let τ be an automorphism of K. Suppose that τ could be extended to an inner automorphism of Q which maps P onto itself. Let the extension of τ on Q be given by the transformation with a Q-element u. Then, since Q is the join of the sfield Q_α, the element u belongs to a sfield Q_α with a suitable index α. Thus the automorphism τ of K can be extended to an inner automorphism of Q_α. Since the intersection $P \cap Q_\alpha$ is P_α, the automorphism τ thus extended on Q_α maps P_α onto itself. Therefore, by the assumption, the automorphism τ of K must belong to an automorphism class represented by an element of G_α. Conversely, if an automorphism τ of K belongs to an automorphism class represented by an element σ of G, then it can be realized by the transformation with an element of the form $u_\sigma a$, where a is a suitable K-element. Obviously the transformation with the element $u_\sigma a$ induces an inner automorphism of Q which maps P onto itself. Now let A be an element of the commutor ring $V_Q(K)$ of K in Q. Since Q is the join of the sfields Q_α, the element A belongs to a sfield Q_α with a suitable index α, and, by the assumption, it belongs to the centre of K. Thus the proof is completed.

As is well known, a torsion-free abelian group is the join of a ascending chain of groups of linear forms,[3] therefore we get the following Lemma from Lemma 2 and Lemma 3.

LEMMA 4. *Let K be a sfield and G be a group of outer automorphisms of K which is a group extension of a subgroup by a torsion-free abelian group. If every crossed product of K with H satisfies the conditions i), ii) and iii) in Lemma 1, then every crossed product of K satisfies the same conditions i), ii) and iii).*

Finally, applying Lemma 3 and Lemma 4, we can prove, by transfinite induction, the following

THEOREM. *Let K be a sfield and G be the join of a well-ordered ascending chain of groups G_α of outer automorphisms of K. Assume further that the*

3) A. G. Kurosch [4].

groups satisfy the following conditions:

a) *G_1 is the identity automorphism group.*

b) *If α is non-limit ordinal, then G_α is a group extension of $G_{\alpha-1}$ by a torsion-free abelian group.*

c) *If α is a limit ordinal, then $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$.*

Then every crossed product P of K with G satisfies the following conditions:

i) *P is an integral domain and possesses a sfield of quotients Q.*

ii) *An automorphism of K can be extended to an inner automorphism of Q which maps P onto itself, if and only if it belongs to an automorphism class represented by an element of G.*

iii) *The commutor ring of K in Q coincides with the centre of K.*

COROLLARY. *Let K be a sfield and G be a soluble group of outer automorphisms of K. If every factor group of successive terms in the commutator series of G is a torsion-free abelian group, then every crossed product of K with G satisfies the conditions i), ii) and iii).*

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