4
Maxwell theory

In this chapter we will study the quantization of the free Maxwell theory. Admittedly, this is a simple problem that certainly could be tackled with more economical techniques, and this was historically the case. However, it will prove to be a very convenient testing ground to gain intuitive feelings for results in the language of loops. It will also highlight the fact that the loop techniques actually produce the usual results of more familiar quantization techniques and guide us in the interpretation of the loop results.

We will perform the loop quantization in terms of real and Bargmann [70] coordinates. The reason for considering the complex Bargmann coordinatization is that it shares many features with the Ashtekar one for general relativity. It also provides a concrete realization of the introduction of an inner product purely as a consequence of reality conditions, a feature that is expected to be useful in the gravitational case.

The Maxwell field was first formulated in the language of loops by Gambini and Trias [62]. The vacuum and other properties are discussed in reference [63] and multiphoton states are discussed in referece [64]. The loop representation in terms of Bargmann coordinates was first discussed by Ashtekar and Rovelli [65].

The organization of this chapter is as follows: in section 4.1 we will first detail some convenient results of Abelian loop theory, which will simplify the discussion of Maxwell theory and will highlight the role that Abelian theories play in the language of loops. In section 4.2 we will discuss the classical theory. We will discuss the Fock representation in section 4.3. We will then discuss in section 4.4 the quantization of the Maxwell theory in terms of real loop variables. We will recover the usual Fock space and the photon states in terms of loops, and study the interpretation of loop observables in terms of familiar notions of field theory. We will introduce an inner product and an interpretation of the wavefunctions in

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terms of loops. In section 4.5 we will summarize the loop quantization of Maxwell theory in terms of the Bargmann representation and see how this quantization leads, perhaps more naturally, to the same results as the previous section. This will also serve as the motivation and background for the discussion of the gravitational case. Finally, we will discuss in section 4.6 the quantization in the extended loop representation in terms of loop coordinates. We will show how one can reconstruct a classical canonical theory in terms of loops, the quantization of which leads to the loop representation. We will see that the loop representation is directly related to the canonical quantization in the electric field representation.

4.1 The Abelian group of loops

Although one could formulate Maxwell theory in terms of the full group of loops, it turns out that a subgroup of it is all that is needed due to the Abelian nature of the theory. We find it convenient to discuss in some detail the properties of this subgroup since they will help us to simplify the treatment of Maxwell theory.

Let us start by considering the elements of the group of loops of the following nature:

$$\kappa = \gamma \circ \eta \circ \gamma^{-1} \circ \eta^{-1}. \tag{4.1}$$

Generically, γ and η could be composed of an arbitrary number of loops $\gamma = \gamma_1 \circ \ldots \circ \gamma_n$, $\eta = \eta_1 \circ \ldots \circ \eta_n$. These kinds of loops are usually called commutators. It is easy to check that the set of all such loops and their products form a subgroup of the group of loops. We will denote it by $\mathcal{L}_{comm}$. One can immediately see that it forms a normal subgroup, i.e., given any element κ of $\mathcal{L}_{comm}$,

$$\gamma \circ \kappa \circ \gamma^{-1} \in \mathcal{L}_{comm} \qquad \forall \gamma \in \mathcal{L}. \tag{4.2}$$

Whenever one has a normal subgroup, one can define the quotient group. In order to do this we introduce an equivalence relation,

$$\gamma \sim \eta \iff \gamma \circ \eta^{-1} = \kappa \in \mathcal{L}_{comm}. \tag{4.3}$$

The reader can check that the relation is reflexive, symmetric and transitive. We denote the quotient group $\mathcal{L}_{Abel} = \mathcal{L}/\mathcal{L}_{comm}$. Its elements are the equivalence classes determined by the relation (4.3). Again it can be readily checked that the product of equivalence classes is independent of the representative element of the class chosen to perform the calculation.

The intuitive interpretation of the equivalence relation defined is that we have identified the commutators in the group of loops with the identity.

Therefore, if γ_1 and γ_2 belong to $\mathcal{L}_{Abel}$,

$$\gamma_1 \circ \gamma_2 = \gamma_2 \circ \gamma_1, \tag{4.4}$$

and $\mathcal{L}_{Abel}$ is an Abelian group.

As we saw in chapter 1, gauge theories are simply representations of the group of loops. Let us consider representations of the Abelian subgroup that we constructed. We therefore need matrices $H(\gamma)$, such that

$$H(\gamma_1)H(\gamma_2) = H(\gamma_1 \circ \gamma_2) = H(\gamma_2 \circ \gamma_1) = H(\gamma_2)H(\gamma_1) \tag{4.5}$$

for any pair of loops γ_1, γ_2. If one wishes to consider unitary representations of the group of loops, equation (4.5) can only hold if H is a unimodular complex number, i.e., an element of $U(1)$.

As we saw in section 1.4, any representation (sufficiently regular) of the group of loops can be written locally as

$$H_A(\gamma) = \exp\left(i \oint_\gamma dy^a A_a(y)\right), \tag{4.6}$$

where $A_a(y)$ is just a real number for the Abelian case we are considering and therefore

$$H_A(\gamma) = W_A(\gamma). \tag{4.7}$$

$W_A(\gamma)$ depends on the loop γ only through the circulation of A_a. This can be written using only the simplest of the loop coordinates introduced in chapter 2, the coordinate of order one,

$$\oint_\gamma dy^a A_a(y) = \int d^3z A_a(z) X^{az}(\gamma). \tag{4.8}$$

An interesting point is that the representation depends only on the information of the loop contained in the first order loop coordinate. This implies some strong differences with the general case. For instance, $W(\pi_x^y \circ \gamma_y \circ \pi_y^x \circ \eta_x) = W(\gamma_y \circ \eta_x)$, where γ is any loop basepointed at y and π_x^y is an arbitrary path and η_x is a loop basepointed at x. This implies that for an infinitesimal deformation

$$W(\pi_o^x \circ \delta u \delta v \delta \bar{u} \delta \bar{v} \circ \pi_x^0 \circ \gamma) = W(\delta u \delta v \delta u \delta v \circ \gamma). \tag{4.9}$$

Therefore loop derivatives are no longer path dependent but just point dependent,

$$\Delta_{ab}(\pi_o^x) \longrightarrow \Delta_{ab}(x) \qquad \forall \pi_o^x. \tag{4.10}$$

As a consequence, loop derivatives in the Abelian case commute,

$$[\Delta_{ab}(x), \Delta_{cd}(y)] = 0, \tag{4.11}$$

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and the Bianchi identities can be expressed in terms of ordinary derivatives,

$$\partial_{[a}\Delta_{bc]}(x) = 0. \tag{4.12}$$

We will now study the classical Maxwell theory and the relation of the classical theory to quantities in terms of loops.

4.2 Classical theory

The classical canonical Maxwell theory can be expressed in terms of the canonical pair $\tilde{E}^a(x)$, $A_b(y)$,

$$\{A_b(y), \tilde{E}^a(x)\} = \delta^a_b \delta(x-y). \tag{4.13}$$

The only constraint of the theory is the Abelian Gauss law,

$$\partial_a \tilde{E}^a = 0. \tag{4.14}$$

The Hamiltonian of the theory is the sum of the squares of the electric and magnetic fields, integrated over space,

$$H = \int d^3x \tfrac{1}{2}\eta_{ab}(\tilde{E}^a\tilde{E}^b + \tilde{B}^a\tilde{B}^b), \tag{4.15}$$

where $\tilde{B}^a = \tilde{\eta}^{abc}F_{bc}$. Here η_{ab} is a flat Euclidean three-dimensional metric and from now on we will assume all indices are raised and lowered with it. The commutator of the electric field and the connection with the Hamiltonian gives the time evolution of the fields. These plus the Gauss law are equivalent to the usual four-dimensional Maxwell equations.

The Gauss law can be solved by considering only transverse electric fields, $\tilde{E}^a_T(x)$. The canonical theory can be reformulated entirely in terms of transverse fields (the transverse connection $A^T_a(x)$ is defined in terms of the fixed flat background metric), the canonical pair is then given in terms of Dirac brackets by

$$\{A^T_a(y), \tilde{E}^a_T(x)\} = \delta_T{}^a_b(x-y), \tag{4.16}$$

where the "transverse Dirac delta" is defined by

$$\delta_T{}^a_b(x-y) = \delta^a_b\delta(x-y) - \Delta^{-1}\partial^a\partial_b\delta^3(x-y), \tag{4.17}$$

where Δ^{-1} is the inverse of the Laplacian of the background metric on the three-manifold.

A usual simplification is to consider momentum space variables,

$$A^T_a(x) = \frac{1}{(2\pi)^{3/2}}\int d^3k \exp(i\vec{k}\cdot\vec{x})[q_1(\vec{k})e^1_a(\vec{k}) + q_2(\vec{k})e^2_a(\vec{k})], \tag{4.18}$$

$$E^a_T(x) = \frac{1}{(2\pi)^{3/2}}\int d^3k \exp(-i\vec{k}\cdot\vec{x})[p^1(\vec{k})e^a_1(\vec{k}) + p^2(\vec{k})e^a_2(\vec{k})], \tag{4.19}$$

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where $e^a_A(\vec{k}), e^A_a(\vec{k})$ are transverse vectors and their dual one-forms in momentum space are normalized such that $k^a e^A_a = 0$, $e^a_A(\vec{k})e^B_a(\vec{k}) = \delta^B_A$, $e^A_a(\vec{k}) = (e^A_a(\vec{k}))^* = e^A_a(-\vec{k})$; also $q(-\vec{k}) = q^*(\vec{k})$ and $p(-\vec{k}) = p^*(\vec{k})$. These relations can be inverted to yield

$$q_A(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3x \exp(-i\vec{k}\cdot\vec{x}) e^a_A(\vec{k}) A^T_a(x,) \tag{4.20}$$

$$p^A(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3x \exp(i\vec{k}\cdot\vec{x}) e^A_a(\vec{k}) E^a_T(x), \tag{4.21}$$

with $A = 1, 2$.

The $q_A(\vec{k}), p^A(\vec{k})$ capture the two degrees of freedom of the electromagnetic field and describe the radiative modes corresponding to the two possible helicities of the photon. One can reformulate the theory in terms of these variables. The Poisson brackets are

$$\{q_A(\vec{k}), p^B(\vec{k'})\} = \delta^B_A \delta^3(\vec{k} + \vec{k'}). \tag{4.22}$$

The Hamiltonian, written in terms of these basic variables, adopts the form of an infinite collection of harmonic oscillators, one for each $\vec{k}$,

$$H = \tfrac{1}{2} \int d^3k \left(p_A(\vec{k}) p^A(-\vec{k}) + |k|^2 q_A(\vec{k}) q^A(-\vec{k}) \right). \tag{4.23}$$

Let us now introduce the two quantities,

$$a_A(\vec{k}) = \tfrac{1}{\sqrt{2}} \left(\sqrt{|k|} q_A(\vec{k}) + i \frac{1}{\sqrt{|k|}} p_A(\vec{k}) \right), \tag{4.24}$$

$$a^*_A(\vec{k}) = \tfrac{1}{\sqrt{2}} \left(\sqrt{|k|} q_A(-\vec{k}) - i \frac{1}{\sqrt{|k|}} p_A(-\vec{k}) \right), \tag{4.25}$$

with Poisson brackets

$$\{a_A(\vec{k}), a^*_B(\vec{k'})\} = -i\delta_{AB}\delta(k - k'), \tag{4.26}$$

in terms of which the classical Hamiltonian reads

$$H = \int d^3k |k| a^*_C(\vec{k}) a^C(\vec{k}). \tag{4.27}$$

4.3 Fock quantization

The Fock quantization arises by considering the number representation for each harmonic oscillator of the Hamiltonian (4.23). Since there is a continuous infinite number of oscillators, one for each $\vec{k}$, it is convenient to consider quantization in a finite region of space ("a box") in order to have a countable infinity of modes $\vec{k}_i$. Then, the canonical commutation

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relations become

$$\{a_A(\vec{k}_i), a^*_B(\vec{k}_j)\} = -i\delta_{ij}.\delta_{AB} \tag{4.28}$$

The Hamiltonian becomes

$$H = \sum_{i=1}^{\infty} |\vec{k}_i| \; a^*_C(\vec{k}_i) a^C(\vec{k}_i). \tag{4.29}$$

One can introduce the Fock representation directly by considering the quantum representation of the algebra (4.28) in a space of functions of infinite pairs of integer variables $\Phi(\ldots, n_{i,c}, \ldots)$. Each variable represents the state of each harmonic oscillator for a given $\vec{k}_i$ and a given polarization. The representation of the algebra is as follows:

$$\hat{a^*}_C(\vec{k}_j)\Phi(\ldots, n_{i,D}, \ldots) = \sqrt{n_{j,C}}\;\Phi(\ldots, n_{j,D} - \delta_{CD}, \ldots), \tag{4.30}$$

$$\hat{a}_C(\vec{k}_j)\Phi(\ldots, n_{i,C}, \ldots) = \sqrt{n_{j,C}+1}\;\Phi(\ldots, n_{j,D} + \delta_{CD}, \ldots), \tag{4.31}$$

where the wavefunctions vanish if any of their arguments are negative numbers.

The commutation relations can be immediately derived:

$$[\hat{a}_B(\vec{k}_i), \hat{a^*}_B(\vec{k}_j)] = \delta_{ij}\delta_{AB}. \tag{4.32}$$

The next step in the quantization program is to introduce an inner product. This can be readily done:

$$< \Phi|\Psi >=$$
$$\sum_{n_{1,1}=1}^{\infty} \sum_{n_{1,2}=1}^{\infty} \cdots \sum_{n_{j,1}=1}^{\infty} \sum_{n_{j,2}=1}^{\infty} \cdots \Phi(n_{1,1}, n_{1,2}, \ldots, n_{j,1}, n_{j,2}, \ldots)^* \times$$
$$\times \Psi(n_{1,1}, n_{1,2}, \ldots, n_{j,1}, n_{j,2}, \ldots). \tag{4.33}$$

In terms of this inner product the operators $\hat{a}_C(\vec{k}_j)$ and $\hat{a^*}_C(\vec{k}_j)$ satisfy the relations

$$\hat{a}^\dagger_C(\vec{k}_j) = \hat{a^*}_C(\vec{k}_j), \tag{4.34}$$

where $\dagger$ means adjoint in the operatorial sense. One can now define the Hermitian operator $N(\vec{k}_j, C)$ by

$$\hat{N}(\vec{k}_j, C) = \hat{a}^\dagger_C(\vec{k})\hat{a}_C(\vec{k}) \tag{4.35}$$

with no summation over C.

The explicit action of the operator $\hat{N}(\vec{k}_j, A)$ is given by

$$\hat{N}(\vec{k}_j, A)\Psi(n_{1,1}, n_{1,2}, \ldots, n_{l,1}, n_{l,2}, \ldots) =$$
$$n_{j,A}\Psi(n_{1,1}, n_{1,2}, \ldots, n_{l,1}, n_{l,2}, \ldots). \tag{4.36}$$

The reader can immediately notice the resemblance with the usual harmonic oscillator: $\hat{a}$ and $\hat{a}^\dagger$ are annihilation and creation operators and $\hat{N}$ is the number operator, and we have one of each per momentum $\vec{k}_j$ and polarization C. The usual commutation relations follow:

$$[\hat{N}(\vec{k}_i, C), \hat{a}^\dagger_D(\vec{k}_j)] = \delta_{ij}\delta_{CD}\hat{a}^\dagger_D(\vec{k}_j), \tag{4.37}$$

$$[\hat{N}(\vec{k}_i, C), \hat{a}_D(\vec{k}_j)] = -\delta_{ij}\delta_{CD}\hat{a}_D(\vec{k}_j). \tag{4.38}$$

Let us now introduce the quantum Hamiltonian. Rewriting (4.23) in terms of creation and annihilation operators, one gets

$$\hat{H} = \tfrac{1}{2}\sum_{j=1}^{\infty} |\vec{k}_j|(a^\dagger_C(\vec{k}_j)a^C(\vec{k}_j) + \tfrac{1}{2}) \tag{4.39}$$

and it should be realized that this corresponds to a different factor ordering than the natural one that we would have inferred from the classical expression (4.27). This expression is divergent even for the case we are considering (a finite box) since we are summing the zero point energy for each of the infinite excited modes. In order to make this expression finite, it is usual to subtract the zero modes through the procedure called "normal ordering" (denoted by enclosing expressions in colons) consisting in ordering the $a\dagger$ to the left,

$$: \hat{H} := \tfrac{1}{2}\sum_{j=1}^{\infty} |\vec{k}_j|(a^\dagger_C(\vec{k}_j)a^C(\vec{k}_j)). \tag{4.40}$$

Since $\hat{H}$ commutes with $\hat{N}(\vec{k}_i, C)$, $\forall i, C$, both operators could be diagonalized simultaneously. In the representation we are considering, this can be accomplished straightforwardly by determining the vacuum state. This is the state with minimal energy and it can be checked that such a state Φ_0 satisfies

$$\hat{a}_C(\vec{k}_j)\Phi_0 = 0 \quad \forall k, C. \tag{4.41}$$

Once this state is given, the whole space of "excited" states can be spanned by applying the "creation" operator $\hat{a}^\dagger$. One can interpret this construction in terms of particles: the application of the operator $\hat{a}^\dagger_C(\vec{k}_i)$ creates a photon with polarization C and three-momentum $\vec{k}_i$. This can be verified by computing the normal-ordered momentum operator $\hat{P}_b =: \int d^3x \hat{\tilde{E}}^a \hat{F}_{ab}$: in this state. It can be checked that : $\hat{H}^2 - \hat{P}_b\hat{P}^b := 0$ and therefore the photon is massless.

To diagonalize the Hamiltonian and number operators we introduce a

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basis of states labeled $|n_{1,1}, n_{1,2}, \ldots, n_{j,1}, n_{j,2}, \ldots >$, defined by

$$|n_{1,1}, n_{1,2}, \ldots, n_{j,1}, n_{j,2}, \ldots >= \prod_{j=1}^{\infty} \prod_{C=1}^{2} \frac{1}{\sqrt{n_{j,C}!}} (a_C^\dagger(\vec{k}_j))^{n_{j,C}} |0, \ldots, 0>, \tag{4.42}$$

where $|0, \ldots, 0>$ is the vacuum. Therefore

$$\hat{H}\, |n_{1,1}, n_{1,2}, \ldots, n_{j,1}, n_{j,2}, \ldots >= \sum_{j=1}^{\infty} \sum_{C=1}^{2} n_{j,C}\, |\vec{k}_i|\, |n_{1,1}, n_{1,2}, \ldots, n_{j,1}, n_{j,2}, \ldots >, \tag{4.43}$$

$$\hat{N}(\vec{k}_j, C)\, |n_{1,1}, n_{1,2}, \ldots, n_{j,1}, n_{j,2}, \ldots >= n_{j,C} |n_{1,1}, n_{1,2}, \ldots, n_{j,1}, n_{j,2}, \ldots > \tag{4.44}$$

and this is what is usually called the Fock basis.

It is useful to introduce a dual Fock basis through the relation,

$$< n_{1,1}, n_{1,2}, \ldots, n_{j,1}, n_{j,2}, \ldots |m_{1,1}, m_{1,2}, \ldots, m_{j,1}, m_{j,2}, \ldots >= \prod_{j=1}^{\infty} \prod_{C=1}^{2} \delta_{m_{j,C}; n_{j,C}} \tag{4.45}$$

and this relation leads naturally to the inner product (4.33).

The Fock basis describes naturally states with a definite number of incoherent photons of definite energy and momentum. These states have vanishing expectation values for the field operators $\hat{E}^c$ and $\hat{A}_c$. They therefore present a description of electromagnetism that is not naturally associated with the classical one. To be able to make contact with the classical limit more easily it is convenient to introduce a basis of states in terms of which the expectation values of both $\hat{E}^c$ and $\hat{A}_c$ are non-vanishing. The elements of this basis are called the coherent states.

The coherent states form a basis labeled by arbitrary complex numbers $\alpha_{i,c}$, associated with each mode. Their definition is

$$\hat{a}_C(\vec{k}_j)|\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{j,1}, \alpha_{j,2}, \ldots >= \alpha_{j,C}|\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{j,1}, \alpha_{j,2}, \ldots > \tag{4.46}$$

and can be written in terms of the vacuum as

$$|\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{j,1}, \alpha_{j,2}, \ldots >= \prod_{i=1}^{\infty} \prod_{C=1}^{2} \exp(-\tfrac{1}{2}|\alpha_{i,C}|^2) \exp(\tfrac{1}{2}\alpha_{i,C}\hat{a}_C^\dagger(\vec{k}_i))|0, \ldots, 0> . \tag{4.47}$$

It should be noticed that the states introduced do not strictly belong to the Fock space but to its closure, due to the infinite summation. It can be checked that these states minimize the uncertainty in both the electric field and the connection and are therefore the closest to a "classical" configuration one can get.

Since we did not impose any restrictions on the eigenvalues of annihilation and creation operators while defining the coherent basis, it follows that the basis is *overcomplete*. A given state can be expanded in terms of this basis in infinitely many different ways. We will see later a connection between this overcompleteness and that of Wilson loops.

4.4 Loop representation

In order to introduce the loop representation let us first remind the reader of some aspects of the usual connection representation of the Maxwell theory. We can particularize the steps we presented in the previous chapter for the canonical quantization of Yang–Mills theories to the Maxwell case.

The connection representation is the most natural quantization since it is based on the straightforward quantization of the canonical algebra of connections and electric fields, taking a polarization based on wavefunctionals of the configuration variables.

Let us therefore start by picking a polarization in which wavefunctions are functionals of the connection $\Psi[A]$ and promote the connection and electric field to quantum operators,

$$\hat{\tilde{E}}^{a}\Psi[A] = -i\frac{\delta}{\delta A_a}\Psi[A], \tag{4.48}$$

$$\hat{A}_a\Psi[A] = A_a\Psi[A]. \tag{4.49}$$

Notice that we are considering functionals of the full (non-transverse) connection, so we will have to enforce the Gauss law as a quantum constraint,

$$\hat{\mathcal{G}}\Psi[A] = \partial_a\frac{\delta}{\delta A_a}\Psi[A] = 0, \tag{4.50}$$

which tells us that $\Psi[A]$ has to be a gauge invariant function of A. We are imposing gauge invariance at a quantum level. This is different from what we did in the previous section where we solved the constraints at a classical level (reduced phase space quantization). Therefore there is potential for these two procedures to be inequivalent.

We can now formally write the quantum Hamiltonian,

$$\mathcal{H}\Psi[A] = \int d^3x\left(\eta^{ab}\frac{\delta}{\delta A_a}\frac{\delta}{\delta A_a} + \frac{1}{2}\eta^{ab}\eta^{cd}F_{ac}F_{bd}\right)\Psi[A], \tag{4.51}$$

though it is clear that a detailed discussion of the first term requires a regularization.

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One can solve the eigenvalue problem for this Hamiltonian (in terms of gauge invariant functions in order to satisfy the Gauss law) and determine the ground and excited states of the theory [63]. We will return to these issues in terms of other representations.

Let us now proceed to construct the loop representation. As described in the previous chapter one can introduce a loop representation either in terms of a non-canonical algebra of classical quantities or via a transform.

In the Abelian case one can immediately find a non-canonical algebra of gauge invariant operators in terms of which one can write all physical quantities by simply considering the Wilson loop and the electric field. In order to keep the construction as close as possible to that which we will later perform for the non-Abelian cases, let us introduce the operators

$$T(\eta) = W(\eta), \tag{4.52}$$
$$T^a(\eta_x^x) = \tilde{E}^a(x) W(\eta), \tag{4.53}$$

which satisfy the non-canonical algebra

$$\{T(\eta), T(\gamma)\} = 0, \tag{4.54}$$
$$\{T^a(\gamma_x^x), T(\eta)\} = -iX^{ax}(\eta) W(\eta \circ \gamma), \tag{4.55}$$
$$\{T^a(\gamma_x^x), T^b(\eta_y^y)\} = -iX^{ax}(\eta) T^b(\gamma \circ \eta) + iX^{by}(\gamma) T^a(\eta \circ \gamma). \tag{4.56}$$

A quantum realization of this algebra in a space of loop-dependent functions is

$$\hat{T}(\eta)\Psi(\gamma) = \Psi(\eta^{-1} \circ \gamma), \tag{4.57}$$
$$\hat{T}^a(\eta_x^x)\Psi(\gamma) = X^{ax}(\gamma)\Psi(\eta^{-1} \circ \gamma), \tag{4.58}$$

and the reader can check that this realizes correctly the Poisson algebra in terms of quantum commutators. A choice of factor ordering with the functional derivatives to the right has been made.

The loop transform is given by

$$\Psi(\gamma) = \int DA \exp\left(-i \int d^3z X^{az}(\gamma) A_a(z)\right) \Psi[A] \tag{4.59}$$

and due to the Abelian nature of the connection the integral can be rigorously defined [65].

If one considers operators $\hat{T}(\gamma), \hat{T}^a(\gamma_x^x)$ in the connection representation defined by

$$\hat{T}(\gamma)\Psi[A] \equiv W_A(\gamma)\Psi[A], \tag{4.60}$$
$$\hat{T}^a(\gamma_x^x)\Psi[A] \equiv \hat{\tilde{E}}^a(x) W_A(\gamma)\Psi[A], \tag{4.61}$$

one can check that applying the transform (4.59) one obtains the operators introduced in (4.57),(4.58).

In terms of the non-canonical algebra one can express the electric field and the field tensor in the following way:

$$\hat{F}_{ab}(x) = -i\Delta_{ab}(x)T(\gamma)|_{\gamma=\iota}, \tag{4.62}$$

$$\hat{\hat{E}}^{a}(x) = T^a(\gamma_x^x)|_{\gamma=\iota}, \tag{4.63}$$

where ι is the identity loop. This allows a loop representation to be found naturally through equations (4.57),(4.58),

$$\hat{F}_{ab}(x)\Psi(\gamma) = -i\Delta_{ab}(x)\Psi(\gamma), \tag{4.64}$$

$$\hat{\hat{E}}^a(x)\Psi(\gamma) = X^{ax}(\gamma)\Psi(\gamma). \tag{4.65}$$

Therefore there is a natural interpretation of loops as lines of electric flux in this representation.

One could arrive at these expressions by using the loop transform (4.59), integrating by parts and considering the action of the fields on Wilson loops in the connection representation,

$$\hat{F}_{ab}(x)W(\gamma) = F_{ab}(x)W(\gamma) = -i\Delta_{ab}(x)W(\gamma), \tag{4.66}$$

$$\hat{\hat{E}}^a(x)W(\gamma) = X^{ax}(\gamma)W(\gamma) = \oint_\gamma dy^a\delta(x-y)W(\gamma). \tag{4.67}$$

The last expression ensures that the Gauss law is automatically satisfied in the loop representation (due to the transverse nature of the first order multitangent $X^{ax}(\gamma)$). This is a natural consequence of the fact that the loop representation is based on the quantization of an algebra of gauge invariant objects. Only gauge invariant quantities can be realized naturally in the loop representation. Gauge dependent objects could be introduced by means of the connection derivative defined in chapter 1. The gauge dependence is introduced through the path prescription used in the definition of the connection derivative.

The commutation relation of E and F,

$$[\hat{F}_{cd}(y), \hat{\hat{E}}^{a}(x)] = -i\delta^a_{[c}\partial_{d]}\delta(x-y), \tag{4.68}$$

finds its natural counterpart in the expression of the action of the loop derivative on the loop coordinate that we introduced in chapter 2,

$$\Delta_{cd}(y)X^{ax}(\gamma) = \delta^a_{[c}\partial_{d]}\delta(x-y). \tag{4.69}$$

One can now realize the Hamiltonian in terms of loops. The magnetic field portion of it is given simply in terms of loop derivatives,

$$\eta_{ab}\hat{\hat{B}}^{a}(x)\hat{\hat{B}}^{b}(x)\Psi(\gamma) = -\tfrac{1}{2}\eta^{ac}\eta^{bd}\Delta_{ab}(x)\Delta_{cd}(x)\Psi(\gamma). \tag{4.70}$$

The electric field portion is given in terms of two loop integrals, which can be reexpressed as

$$\eta_{ab}\hat{\tilde{E}}^{a}(x)\hat{\tilde{E}}^{b}(x)\Psi(\gamma) = \eta_{ab}X^{ax}(\gamma)X^{bx}(\gamma)\Psi(\gamma). \tag{4.71}$$

The Hamiltonian eigenvalue equation then reads

$$\begin{aligned}\hat{H}\Psi(\gamma) \equiv & \int d^3x \left(-\tfrac{1}{4}\eta^{ac}\eta^{bd}\Delta_{ab}(x)\Delta_{cd}(x) + \tfrac{1}{2}\eta_{ab}X^{ax}(\gamma)X^{bx}(\gamma)\right)\Psi(\gamma) \\ &= E\Psi(\gamma).\end{aligned} \tag{4.72}$$

The second term can be suggestively rewritten as

$$\int d^3x X^{ax}(\gamma)X^{bx}(\gamma)\Psi(\gamma) = \oint_\gamma dy^a \oint_\gamma dy'^b \delta(y-y')\eta_{ab}\Psi(\gamma), \tag{4.73}$$

which is proportional (through a divergent factor that needs to be regularized) to the length of the loop. Therefore the eigenvalue equation can be qualitatively interpreted as a "Laplacian" in terms of the double loop derivative and a "quadratic potential" given by the length of the loop. Notice that the other term, involving the loop derivatives, is also potentially ill defined. If one considers wavefunctions such that their loop derivative is distributional a regularization may be needed. We will not discuss the details here since for the particular case of Maxwell theory the extended representation discussed in section 4.6 furnishes a natural setting to regularize the theory.

Let us now study the vacuum and excited states of this system. One possible avenue is to take this analogy with the Hamiltonian of a harmonic oscillator seriously and propose a "Gaussian" state of the form

$$\Psi_0(\gamma) = \exp\left(-\tfrac{1}{2}\oint_\gamma dy^a \oint_\gamma dy'^b K_{ab}(y-y')\right) \tag{4.74}$$

$$\equiv \exp\left(-\tfrac{1}{2}X^{ax}(\gamma)X^{by}(\gamma)K_{ax\,by}\right) \tag{4.75}$$

and insert this expression in the eigenvalue equation for the Hamiltonian to determine K_{ab}. This course has actually been pursued in reference [63]. Here, however, we will find the vacuum by introducing the creation and annihilation operators in the loop representation and finding the state annihilated by the annihilation operator. It will turn out that this construction yields the same vacuum as that of reference [63].

Both the creation and annihilation operators can be readily realized in loop space. To introduce them we need to realize the q and p operators, and therefore the $\hat{A}^T_a$ operator. To do so we use the relation in the classical theory

$$\partial^a F_{ab} = \Delta A^T_b, \tag{4.76}$$

where Δ is the three-dimensional Laplacian, and realize this expression in terms of the loop derivative. Then,

$$\hat{A}^T_a(x)\Psi(\gamma) = -i\frac{1}{\Delta}\partial^b\Delta_{ab}(x)\Psi(\gamma). \tag{4.77}$$

In terms of this expression, the operator $\hat{q}^A(\vec{k})$ is,

$$\hat{q}^A(\vec{k})\Psi(\gamma) = \frac{1}{(2\pi)^{3/2}}\int d^3x \exp(-i\vec{k}\cdot\vec{x})e^{aA}(\vec{k})\frac{k^b}{|\vec{k}|^2}\Delta_{ba}(x)\Psi(\gamma). \tag{4.78}$$

The operator $\hat{p}^1(\vec{k})$ can be realized immediately,

$$\hat{p}^A(\vec{k})\Psi(\gamma) = \tfrac{1}{(2\pi)^{3/2}}\int d^3x e^{-i\vec{k}\cdot\vec{x}}e^A_a(\vec{k})X^{ax}(\gamma)\Psi(\gamma). \tag{4.79}$$

Therefore the creation and annihilation operators in the loop representation have the forms

$$\hat{a}^\dagger_A(\vec{k}) = \frac{1}{(2\pi)^{3/2}}\int d^3x \left(\exp(-i\vec{k}\cdot\vec{x})e^a_A(\vec{k})\frac{k^b}{|\vec{k}|^{3/2}}\Delta_{ba}(x)\right.$$
$$\left. -i\frac{1}{|\vec{k}|^{1/2}}\exp(-i\vec{k}\cdot\vec{x})e_{aA}(\vec{k})X^{ax}(\gamma)\right), \tag{4.80}$$
$$\hat{a}_A(\vec{k}) = \frac{1}{(2\pi)^{3/2}}\int d^3x \left(\exp(-i\vec{k}\cdot\vec{x})e^a_A(\vec{k})\frac{k^b}{|\vec{k}|^{3/2}}\Delta_{ba}(x)\right.$$
$$\left. +i\frac{1}{|\vec{k}|^{1/2}}\exp(-i\vec{k}\cdot\vec{x})e_{aA}(\vec{k})X^{ax}(\gamma)\right). \tag{4.81}$$

We now apply (4.81) to (4.74). The application of the first term in $\hat{a}$ yields,

$$-\frac{i}{(2\pi)^{3/2}}\int d^3x \exp(-i\vec{k}\cdot\vec{x})e^a_A(\vec{k})\sqrt{|k|}X^{by}(\gamma)K_{ax\,by}\Psi_0(\gamma). \tag{4.82}$$

We must now determine $K_{ax\,by}$ so that this terms cancels the second one. It can be straightforwardly checked that if one takes,

$$K_{ax\,by} = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3q}{|q|}\exp\left(-i\vec{q}\cdot(\vec{x}-\vec{y})\right) \tag{4.83}$$

the two terms actually cancel. The expression for K is that of the homogeneous symmetric propagator of Maxwell theory.

It is now immediate to find the excited states, simply by operating with $\hat{a}^\dagger$ on the vacuum. The first excited state is given by

$$\Psi_1^{(A,\vec{k})}(\gamma) = \frac{1}{(2\pi)^{3/2}}\int d^3x\frac{1}{|\vec{k}|^{1/2}}\exp(-i\vec{k}\cdot\vec{x})e_{aA}(\vec{k})X^{ax}(\gamma)\Psi_0(\gamma). \tag{4.84}$$

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This expression can be more compactly written in $\vec{k}$ space. Introducing the Fourier transform of the multitangent,

$$X^{ak}(\gamma) = \frac{1}{(2\pi)^{3/2}} \int d^3x \exp(-i\vec{k}\cdot\vec{x}) X^{ax}(\gamma), \tag{4.85}$$

the first excited state is

$$\Psi_1^{(A,\vec{k})}(\gamma) = \frac{1}{|\vec{k}|^{1/2}} e_{aA}(\vec{k}) X^{ak} \Psi_0(\gamma). \tag{4.86}$$

This state corresponds to a photon of momentum $\vec{k}$ and polarization A. The objects $X^{ak}(\gamma)$ are usually called "form factors" of the loop. The form factors are transverse,

$$k_a X^{ak}(\gamma) = 0. \tag{4.87}$$

and therefore their only relevant components are the projections on the polarization vectors.

The n-photon state is given by,

$$\Psi_n^{(A_1,\vec{k}_1,\ldots,A_n,\vec{k}_n)}(\gamma) = \left(\frac{1}{|\vec{k_1}|^{1/2}} e_{aA_1}(\vec{k}_1) X^{ak_1} \cdots \frac{1}{|\vec{k_n}|^{1/2}} e_{aA_n}(\vec{k}_n) X^{ak_n} \right) \Psi_0(\gamma). \tag{4.88}$$

An appealing fact is the form of the coherent states in this representation. They are given by

$$\Psi^{(\alpha)}(\gamma) = W(\gamma, A)\Psi_0(\gamma), \tag{4.89}$$

where $W(\gamma, A)$ is the Wilson loop along the loop γ of a given connection A. It can be readily checked that these states are eigenvectors of the annihilation operator. When one operates with (4.81) on the state the first term (involving the loop derivative) acts both on the Wilson loop and on $\Psi_0(\gamma)$. The action on $\Psi_0(\gamma)$ cancels the contribution from the second term of (4.81) as we observed when deriving the vacuum. The action of the loop derivative on the Wilson loop gives the field tensor F_{ab} of the given connection, as we showed in chapter 1. The eigenvalue α is therefore given in terms of the connection as

$$\alpha = \frac{i}{|\vec{k}|^{3/2}} e_A^a(\vec{k}) k^b F_{ab}(k). \tag{4.90}$$

The field tensor so introduced actually has a physical meaning. It corresponds to the expectation value of the spatial part of the Maxwell field tensor in the coherent state in question.

Up to now we have operated with the Hamiltonian in a formal fashion, ignoring the issues of regularization. As a result, the eigenfunctions we find are really ill defined. This can be readily seen from the expression of the vacuum (4.74) since the propagator diverges quadratically when $x \longrightarrow y$.

A suitable regularization for the second term of the Hamiltonian is to replace the delta function by a function $f_\epsilon(y-y')$ such that

$$\lim_{\epsilon \longrightarrow 0} f_\epsilon(y-y') = \delta(y-y'). \tag{4.91}$$

Explicitly,

$$f_\epsilon(y-y') = \frac{1}{(2\pi)^{3/2}} \int d^3q\, r(|q|\epsilon) \exp(i\vec{q}\cdot(\vec{y}-\vec{y'})), \tag{4.92}$$

where the function r is defined such that

$$\int_0^\infty r(x)dx = 0 \tag{4.93}$$

and explicit examples of such a function are

$$r(x) = (1-x)\exp(-x), \tag{4.94}$$
$$r(x) = (1-\tfrac{1}{2}x)\Theta(1-x). \tag{4.95}$$

If one now repeats the procedure that led to the vacuum taking into account the regularization, one finds that the vacuum of the regularized Hamiltonian is also given by a Gaussian,

$$\Psi_0^\epsilon(\gamma) = \exp\left(-\tfrac{1}{2}\oint_\gamma dy^a \oint_\gamma dy'^b K^\epsilon_{ab}(x-y)\right), \tag{4.96}$$

where the regularized propagator is given by

$$K^\epsilon_{ab}(x-y) = \delta_{ab}\frac{1}{(2\pi)^{3/2}}\int d^3q\, \frac{r(\epsilon|q|)}{|q|}\exp(-i\vec{q}\cdot(\vec{y}-\vec{y'})), \tag{4.97}$$

where $r(x)$ is the function that we introduced while regularizing the Hamiltonian. Other regularizations for this same problem have been considered in reference [64].

Finally, we can introduce an inner product. We define a normalized form factor as

$$C^a(\vec{k}) = \frac{1}{|\vec{k}|^{1/2}} X^{ak}(\gamma), \tag{4.98}$$

in terms of which we introduce an inner product,

$$< \Phi_1(\gamma)|\Phi_2(\gamma) >= \int DCDC^*\, \Phi_1^*(C,C^*)\Phi_2(C,C^*). \tag{4.99}$$

The integrals on C and its complex conjugate are functional integrals. Note that the functional integrals defined above can only be computed in practice if one assumes that the normalized multitangents are arbitrary

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transverse fields, not necessarily associated with a loop. Therefore one is really going to an extension of the representation in order to perform it. We will return to these issues when we discuss the extended loop representation in section 4.6. The vacuum (4.74) is normalized with this inner product,

$$\int dC dC^* \Psi_0(C)^* \Psi_0(C) = \int DCDC^* \exp\left(-\int d^3k C^{*a}(\vec{k}) C^b(\vec{k}) \delta_{ab}\right) = 1. \tag{4.100}$$

Because in this representation excited states are proportional to the vacuum, the factor $\exp\left(-\int d^3k C^{*a}(\vec{k}) C^b(\vec{k}) \delta_{ab}\right)$ acts as a Gaussian measure in the inner product and the vacuum is simply represented as a constant and the excited states by the projections of form factors on the polarization vectors. We will see that a similar feature arises naturally in the Bargmann representation.

4.5 Bargmann representation

In 1962 Bargmann introduced a complex coordinatization for the harmonic oscillator. It is based on using as canonical coordinates $z \equiv q + ip$ and z^*, its complex conjugate. The resulting formulation is very elegant, wavefunctions are holomorphic, and the inner product is determined, fixing the reality of the relevant operators. This formulation has several analogous elements to Ashtekar's formulation of general relativity in which one of the canonical coordinates is complex and the other real. The hope is that similar analytic properties will help determine the inner product of quantum gravity. In this section we will present a Bargmann-like formulation of Maxwell theory in terms of both traditional variables and loops. This formulation naturally fixes the inner product to be the complex measure introduced a bit arbitrarily in the previous section. This treatment follows closely that of Ashtekar and Rovelli [65].

4.5.1 The harmonic oscillator

The canonical formulation of the harmonic oscillator is given in terms of coordinates q, p and the Hamiltonian is $H = p^2 + \omega^2 q^2$. Quantization is achieved through wavefunctions $\Psi(q)$ and the eigenvalue equation for the Hamiltonian is $(-\partial^2/\partial q^2 + \omega^2 q^2)\Psi(q) = E\Psi(q)$. The eigenstates of the system are given by a Gaussian in q times the Hermite polynomials.

Normally, as mentioned above, the Bargmann representation involves both real and complex coordinates. Discussion of the harmonic oscillator in those coordinates can be seen in reference [2] and in Bargmann's

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original paper [70]. Here, however, we will explore a fully complex representation which is better geared for comparison with what was done in reference [65] for the Maxwell case. One could also treat the Maxwell case in a mixed polarization and then it would more resemble Bargmann's original treatment.

Assume now that a complex coordinatization given by the variables and $z = \frac{1}{\sqrt{2}}(\omega q - ip)$ and $z^* = \frac{1}{\sqrt{2}}(\omega q + ip)$ is introduced. The Poisson bracket is $\{z, z^*\} = i\omega$. The variables satisfy reality conditions that say that they are complex conjugates of each other. One can then construct a representation of the canonical algebra on holomorphic functions $\Psi(z)$,

$$\hat{z}\Psi(z) = z\Psi(z), \tag{4.101}$$

$$\hat{z^*}\Psi(z) = \omega\frac{d\Psi(z)}{dz}. \tag{4.102}$$

An inner product is introduced that translates the reality conditions into operatorial relations:

$$\hat{z}^\dagger = \hat{z^*}, \tag{4.103}$$

$$\hat{z^*}^\dagger = \hat{z}. \tag{4.104}$$

We will now use these relations to determine the inner product. Let us start with a generic inner product,

$$< \Phi|\Psi >= \int dz \int d\bar{z}\mu(z,\bar{z})\bar{\Phi}(z)\Psi(z), \tag{4.105}$$

and if one now requires that the operatorial relations be satisfied this fixes the measure uniquely to be

$$\mu(z,\bar{z}) = \exp(-z\bar{z}). \tag{4.106}$$

In terms of these variables the quantum Hamiltonian of the harmonic oscillator is

$$\hat{\mathcal{H}}\Psi(z) = \tfrac{1}{2}\omega\left(z\frac{\partial}{\partial z} + 1\right)\Psi(z) \tag{4.107}$$

where we have chosen a symmetric factor ordering in z and z^*. This ordering corresponds in the traditional variables to $\hat{\mathcal{H}} = \hat{p}^2 + \omega^2\hat{q}^2$. The vacuum is simply $\Psi_0(z) = 1$ and the excited states are polynomials in z. With the given measure, polynomial states are normalizable.

This is attractive because just by requiring the reality of the classical operators the inner product is uniquely fixed. Since Maxwell theory is just a collection of harmonic oscillators, it is immediate to construct a Bargmann representation. Since the reality conditions are a structure that is present in other theories (e.g. gravity) where other structures that one could use to build an inner product (e.g. Lorentz invariance) are absent, this gives some hope that a similar construction could yield the inner product for those theories. It is certainly reassuring that this construction at least yields the correct result for Maxwell theory as we will discuss in the next section.

4.5.2 Maxwell–Bargmann quantization in terms of loops

For the kind of calculation that we will perform in this section, it is convenient to introduce circular polarization. We express the fields as

$$A_a^T(x)=\frac{1}{(2\pi)^{3/2}}\int d^3k \exp(i\vec{k}\cdot\vec{x}) \left(q_1^{(c)}(\vec{k})m_a(\vec{k})+q_2^{(c)}(\vec{k})m_a^*(\vec{k})\right), \quad (4.108)$$

$$E_T^a(x)=\frac{-1}{(2\pi)^{3/2}}\int d^3k \exp(i\vec{k}\cdot\vec{x}) \left(p_1^{(c)}(\vec{k})m^a(\vec{k})+p_2^{(c)}(\vec{k})m^{*a}(\vec{k})\right), \quad (4.109)$$

where the complex polarization vectors satisfy*

$$k^a m_a(\vec{k})=0, \quad m^a(\vec{k})m_a(\vec{k})=0, \quad (4.110)$$

$$m_a(-\vec{k})=-m^{*a}(\vec{k}), m^a(\vec{k})m_a^*(\vec{k})=1. \quad (4.111)$$

Given a conjugate pair A_a^T and E_T^a of Maxwell theory, one could decompose it into positive and negative frequency (for instance, by evolving it and decomposing the resulting spacetime solution). Examining the canonical commutation relations one finds that the positive frequency connection and the negative frequency electric field form a conjugate pair, given by,

$$\begin{aligned} A_a^+(x) &= \frac{1}{\sqrt{2}}(A_a^T(x)+i\Delta^{-1/2}(E_T)^b\eta_{ab}) \\ &= \int\frac{d^3k}{(2\pi)^{3/2}|k|}\exp(i\vec{k}\cdot\vec{x})(\zeta_1(\vec{k})m_a(\vec{k})+\zeta_2(\vec{k})m_a^*(\vec{k})), \quad (4.112) \\ \tilde{E}^{b-}(x) &= \frac{1}{\sqrt{2}}(\tilde{E}_T^b(x)+i\Delta^{1/2}A_a^T\eta^{ab}) \\ &= \int\frac{d^3k}{(2\pi)^{3/2}}\exp(i\vec{k}\cdot\vec{x})(\zeta_1^*(-\vec{k})m_a(\vec{k})+\zeta_2^*(-\vec{k})m_a^*(\vec{k})), \quad (4.113) \end{aligned}$$

where $\zeta(\vec{k})_i=\frac{1}{\sqrt{2}}(|k|q_i^{(c)}(\vec{k})-ip_i^{(c)}(\vec{k}))$. The definition of the ζs embodies exactly the same construction that we performed for the harmonic oscillator. The true degrees of freedom of the Maxwell field are now embodied in the two complex ζ fields. They provide a complex coordinatization on

* If one translates back to the language we used in section 4.2 by considering that the vector $m_a(\vec{k})=\frac{1}{\sqrt{2}}(e_a^1(\vec{k})+ie_a^2(\vec{k}))$ one finds that $e_a^1(\vec{k})=-e_a^1(-\vec{k})$ as before but $e_a^2(\vec{k})=e_a^2(-\vec{k})$. These conventions are also used by Bjorken and Drell [71].

the phase space of Maxwell theory. The canonical commutation relations for the ζs are

$$\{\zeta_B(\vec{k}), \zeta_C^*(\vec{k'})\} = i|k|\delta_{BC}\delta(\vec{k} - \vec{k'}) \tag{4.114}$$

and we see the close relation between the ζ variables and the a, a^* variables that were introduced for the Fock representation.

Let us now quantize the theory by promoting the variables to quantum operators:

$$\hat{\zeta}_B(\vec{k})\Psi(\zeta) = \zeta_B(\vec{k})\Psi(\zeta), \tag{4.115}$$

$$\hat{\zeta^*}_B(\vec{k})\Psi(\zeta) = |k|\frac{\delta\Psi(\zeta)}{\delta\zeta_B(\vec{k})} \tag{4.116}$$

where the wavefunctions to be holomorphic functionals of the arguments.

One would like the fact that ζ and ζ^* are conjugate to each other translate itself into an operatorial relation of the kind

$$\hat{\zeta}_B^\dagger(\vec{k}) = \hat{\zeta^*}_B, \tag{4.117}$$

where by $\dagger$ we mean the operatorial adjoint under a suitable inner product. This relation implies that explicitly in terms of the inner product

$$< \Phi|\zeta_B(\vec{k})\Psi >= |k|\left\langle \frac{\delta}{\delta\zeta_B(\vec{k})}\Phi|\Psi \right\rangle. \tag{4.118}$$

To find an inner product that satisfies this condition, one can simply propose an explicit expression

$$< \Psi|\Phi >= \int d\zeta d\zeta_j^* \mu(\zeta, \zeta^*)\Psi^*(\zeta)\Phi(\zeta), \tag{4.119}$$

where $\mu(\zeta, \zeta^*)$ is the measure to be determined. It is easy to check that the condition (4.118) uniquely implies [70],

$$\mu(\zeta, \zeta^*) = \exp\left(-\int \frac{d^3k}{|k|}(|\zeta_1(\vec{k})|^2 + |\zeta_2(\vec{k})|^2)\right). \tag{4.120}$$

So we again get a Gaussian measure. Since the wavefunctions we are considering are holomorphic, we immediately conclude that this representation is essentially the same at the level of inner product and wavefunctions as the real connection representation that we introduced before. Again, we should notice that we found the Gaussian measure without any reference to Lorentz invariance. This therefore makes the method attractive for tackling cases in which such invariances are not present, such as in gravity.

In this representation, the normal ordered Hamiltonian is

$$\hat{\mathcal{H}} = \int d^3k|k| \sum_{B=1}^{2} \zeta_B(\vec{k}) \frac{\delta}{\delta\zeta_B(\vec{k})}, \tag{4.121}$$

The ground state, $\Psi(\zeta) = 1$ is equivalent to the Fock vacuum. A one-photon state with given polarization and momentum $\vec{k}_0$ is given by a linear function $\Psi = \zeta(\vec{k}_0)$. A generic one-photon state with given polarization is given by a superposition in momenta,

$$\Psi_1(\zeta) = \int \frac{d^3k}{|k|} f(\vec{k})\zeta_1(\vec{k}) \tag{4.122}$$

with obvious generalizations for the n-photon states.

We now proceed to construct the loop representation. As usual we could proceed by quantizing an algebra of non-canonical loop-based gauge invariant quantities or via a loop transform. Since we have given examples of the first kind of construction before and in this particular case it leads to the same results, we will simply proceed with the transform. This will also allow us to show how the transform is explicitly defined for an Abelian theory. As we said before, for the Maxwell case the loop transform is well defined. In terms of the Bargmann coordinates, it reads

$$\Psi(\gamma) = \int \prod_B D\zeta_B D\zeta_B^* \exp\left(-\int \frac{d^3k}{|\vec{k}|}|\zeta_B(\vec{k})|^2\right) \exp\left(\oint dy^a A_a^+\right)^* \Psi(\zeta_B). \tag{4.123}$$

Notice that in the definition of the loop transform introduced in chapter 3 the complex conjugate of the Wilson loop appears. For the real case which we considered before this amounts to a change of sign due to the i that appears in the definition of the holonomy. Here it implies the complex conjugate of the connection,

$$\begin{aligned}\oint dy^a A_a^+ &= \int \frac{d^3k}{|(2\pi)^{3/2}\vec{k}|} \left(\zeta_1(\vec{k})m_a(\vec{k})(X^{ak})^* + \zeta_2(\vec{k})m_a^*(\vec{k})(X^{ak})^*\right) \\ &= \int \frac{d^3k}{|\vec{k}|} \sum_{B=1}^{2} \zeta_B(\vec{k})X_B^{*k},\end{aligned} \tag{4.124}$$

where

$$X_1^{*k} = (2\pi)^{-3/2}m_a(\vec{k})(X^{ak})^*, \quad X_2^{*k} = (2\pi)^{-3/2}m_a^*(\vec{k})(X^{ak})^*. \tag{4.125}$$

Also, as we said in chapter 3, the introduction of a loop transform requires the introduction of an inner product in terms of connections. Since we have the Gaussian inner product given by the reality conditions, we use it in the definition of the transform.

Therefore the expression for the loop transform for this particular case is given by

$$\Psi(\gamma) = \int \prod_B D\zeta_B\, D\zeta_B^* \exp\left(-\int \frac{d^3k}{|\vec{k}|}|\zeta_B(\vec{k})|^2\right)$$
$$\times \exp\left(\int \frac{d^3k}{|\vec{k}|} \sum_{B=1}^{2} \zeta_B^*(\vec{k}) X_B^k(\gamma)\right) \Psi(\zeta). \tag{4.126}$$

Let us now evaluate this explicitly for some states. Generically the n-photon states are going to be polynomials in ζ. It is easy to transform such states. Simply expand the exponential $\exp(\zeta_B^* X_B)$ and note that the $\zeta^n/\sqrt{n!}$ are an orthonormal basis with the Gaussian measure. Then the loop transform of any state $\Psi(\zeta) = \sum_n C_n(\zeta)^n$ is simply given by $\Psi(\gamma) = \sum_n C_n(X)^n$ with immediate generalizations for states depending on several ζ_Bs. The vacuum, in particular, is $\Psi(\gamma) = 1$ and the one-photon state with helicity B and momentum $\vec{k}$ given by

$$\Psi_1(\gamma) = X_B^k(\gamma). \tag{4.127}$$

With this we end the discussion of this representation. Let us now compare the results obtained with the loop representation constructed from real variables. The first thing to notice is that the use of the reality conditions in the Bargmann case fixes a non-trivial inner product in terms of connections and therefore a non-trivial measure in the loop transform. Historically, this was not done with the real loop representation since the intention was to recover the Fock space structure (which, in turn, is determined by Poincaré invariance). However, it is very easy to check that if one constructs a connection representation for the real case in terms of q and p and requires the quantum operators $\hat{q}$ and $\hat{p}$ to be real, the inner product given by the trivial measure in q, p appears as a result. This, in turn, implies the trivial measure in the As which is the one we used in section 4.4 to compute the loop transform.

The appearance in the Bargmann case of a non-trivial measure in the inner product and the loop transform implies certain important differences in the two representations. To start with, the vacuum is just a constant. The Gaussian factor that appeared in the real representation is "absorbed in the non-trivial measure". Although one may consider this point irrelevant from a practical point of view, it has implications in the rigorous definition of the space of states. In fact, while in the real case we needed the introduction of a regularization to have a well defined vacuum and space of states, in the Bargmann case the states are well defined without the introduction of a regularization.

Have we gained something from nothing? That is, can we forget the regularization issues altogether by considering a non-trivial measure in the loop transform? The answer is negative. If one wishes to complete the quantization in the loop representation, one would like to introduce an inner product in terms of loops, as was done in section (4.4). If one does so in the Bargmann case, one notices that now a non-trivial Gaussian measure in the Fs appears in the loop representation. This measure coincides exactly with the expression of the vacuum in the real case. If one wants to define an inner product only in terms of loops, the expression of the measure is illdefined. If one wants to proceed as in section 4.4 and "extend" the inner product to all Xs then the difficulties disappear at the price of extending the notion of loops.

Let us now study the extended representation.

4.6 Extended loop representation

We will now explore the consequences of introducing an "extended loop representation", a representation based on the loop coordinates introduced in chapter 2. We will immediately see that such a representation presents computational economy, technical cleanliness and also allows us to view in a conceptually different way the problem of loop quantization. We will see that regularization difficulties are better dealt with in terms of loop coordinates. We will also see that we are also able to determine the classical canonical theory that underlies the loop representation. In the particular case of Maxwell theory we will see that the extended loop representation coincides with the electric field representation. This, however, does not generalize to non-Abelian fields and in those cases the extended loop representation is a new representation that contains the loop representation as a limiting case. As a bonus we will find a way of writing the action for electromagnetism purely in terms of loops. This version of the action is amenable to lattice Monte Carlo techniques and has the potential to offer new insights into non-perturbative QED problems. The fact that so much is gained in the Maxwell case by going to an extended loop representation clearly suggests that a similar avenue should be pursued in the non-Abelian cases and especially gravity.

Let us start by replacing in our formalism the usual loop holonomy by its extended counterpart in terms of the loop coordinates,

$$H_A(\gamma) \rightarrow H_A(\mathbf{X}) = \exp\left(i \int d^3x A_{ax} X^{ax}\right). \tag{4.128}$$

Because of the Abelian nature of the theory we only need the first order multitensor, which can be simply viewed as a divergence-free vector density on the three-manifold,

$$\partial_{ax} X^{ax} = 0. \tag{4.129}$$

One can now introduce a loop coordinate representation by means of the transform

$$\Psi(\mathbf{X}) = \int DA\Psi[A]\exp\left(-ig\int d^3xA_a(x)X^{ax}\right). \tag{4.130}$$

In this representation, wavefunctions are functionals of the smooth vector density X. In terms of this representation we can realize the operators $\hat{F}_{ab}$, $\hat{\tilde{E}}^{ax}$ and the Wilson loop $\hat{W}_A(\mathbf{X})$ through

$$\hat{W}_A(\mathbf{X}_0)\,\Psi(\mathbf{X}) = \Psi(\mathbf{X}-\mathbf{X}_0), \tag{4.131}$$

$$\hat{E}^{ax}\,\Psi(\mathbf{X}) = X^{ax}\Psi(\mathbf{X}), \tag{4.132}$$

$$\hat{F}_{ab}(x)\,\Psi(\mathbf{X}) = i\,\partial_{[a}\frac{\delta}{\delta X^{b]x}}\,\Psi(\mathbf{X}). \tag{4.133}$$

As a consequence, the quantum Hamiltonian reads

$$\hat{H}\,\Psi(\mathbf{X}) = \int d^3x\left[\frac{1}{2}\hat{X}^{ax}\hat{X}^{ax} + \frac{1}{4}\,(\partial_{[a}\hat{P}_{b]x})^2\right]\,\Psi(\mathbf{X}), \tag{4.134}$$

where

$$\hat{P}_{bx} = i\,\frac{\delta}{\delta X^{bx}}. \tag{4.135}$$

From these equations, one realizes, making the identifications

$$\hat{P}_{bx} \longrightarrow A_{bx}, \tag{4.136}$$

$$X^{ax} \longrightarrow \tilde{E}^{ax}, \tag{4.137}$$

that the representation we have just introduced is nothing but the electric field representation of electromagnetism, and the vector density X^a is just the electric field. This is in agreement with the picture that we introduced before in which the loops played the role of lines of electric flux.

A remarkable fact is that one can go back to the loop representation through the substitution $X^{ax} \to X^{ax}(\gamma)$. For instance, if one finds a physical state in the extended representation one can find a physical state in the loop representation by evaluating it on multitangents (since it is a function of multitensors, it has a definite value for multitangents). Care should be exercised in general since multitangents are distributional and limits could be ill defined. For the particular case of Maxwell theory it can easily be checked that the converse property also holds: if one replaces multitangents by multitensors in the physical states of the loop representation, one obtains the physical states for the extended representation. This does not, in general, hold for non-Abelian fields.

Using this correspondence we can immediately write the expression for the vacuum in the extended loop representation,

$$\Psi(\mathbf{X}) = \exp\left(-\tfrac{1}{2}\int d^3x\int d^3yX^{ax}X^{ay}D_1(x-y)\right). \tag{4.138}$$

Here we observe a crucial feature of the extended representation. While the vacuum in terms of loops is, as we pointed out, a singular divergent quantity that only makes sense after a regularization procedure has been introduced, the vacuum in the extended representation is automatically well defined. It is an analogous situation to the one that appears in classical electrostatics: if one tries to formulate the theory in terms of point charges one needs to regularize it, whereas the theory is automatically well defined if one considers smooth charge distributions. It is natural to expect that a similar behavior will appear in non-Abelian theories and quantum gravity. This is one of the main features that make the extended representation attractive. The loop representation only appears as a singular limit, in the same spirit as the electrostatics of point charges appears through a limiting procedure from the electrostatics of smooth charge distributions.

The existence of the extended loop representation is an illustration of a property pointed out by Ashtekar and Isham [73]: that there exist possibly non-equivalent representations of quantum theories. One can introduce an inner product in the loop representation in terms of extended loop coordinates that allows a Fock interpretation as we did in section 4.4. This is the natural inner product in the extended representation and corresponds exactly to the inner product one introduces in the connection representation to implement the reality conditions of the theory. One can also introduce a representation in terms of usual loops and a discrete inner product which seems to describe naturally the states of a Type II superconductor [73].

Since we have a theory written in terms of usual smooth tensorial quantities with a well defined Hamiltonian it is immediate (in this simple Abelian case) to introduce a *classical action* in terms of which one can formulate the theory. This is by no means trivial. Whereas usually the loop representation has been viewed as a "mysterious" construction that either arises indirectly via a transform or through an unusual non-canonical quantization, the extended representation teaches us that one can actually find a *canonical* classical theory in terms of which a straightforward quantization leads to the loop representation. This construction can actually be generalized to the non-Abelian cases, although it presents more subtleties than the Abelian case we are examining here.

Let us therefore write the classical action which yields the quantum theory corresponding to the extended loop representation,

$$S = \int dt \left\{ P_{ax}\dot{X}^{ax} - \left[\frac{1}{2} X^{ax}X^{ax} + \frac{1}{4}\left(\partial_{[a}P_{b]x}\right)^2\right] + \lambda_x X^{ax}_{,a} \right\}. \quad (4.139)$$

We immediately recognize the action for classical electromagnetism if we identify the loop coordinate with the electric field and the momentum

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with the connection as we did before.

This action could be rewritten in terms of loops,

$$S = \int dt \left\{ \int_{\gamma_t} dy^a \dot{A}_a(y) + \tfrac{1}{2} \int d^3x F_{ab}(x) F_{ab}(x) \right.$$
$$\left. + \int_{\gamma_t} dy^a \int_{\gamma_t} dy'^a f_\epsilon(y - y') \right\}, \tag{4.140}$$

where f_ϵ is a regularization of the delta function and the loops γ_t belong to the surface $t =$ constant. This action could also be presented in second order form (modulo regularization difficulties). It could also be regularized by considering the theory on a lattice. This has been pursued in detail in reference [72] and it has been found to lead to the usual Kogut–Susskind formulation [74].

4.7 Conclusions

The example discussed in this chapter, due to its simplicity, allows us to illustrate in an explicit fashion several properties that are important for the program of quantization of the gravitational field and cannot be proved for that case.

We have shown that the language of loops is adequate to describe the free quantum Maxwell field. We have shown that the use of loops is inherently associated with regularization difficulties which can be cured by considering the extended loop representation. The loop representation is totally equivalent in this case to the traditional Fock quantization. The Wilson loop functional appears as naturally related to coherent states. The loop transform in this case is rigorously defined through the inner product in the connection representation. This inner product can be determined through the reality conditions of the theory, as we proved for the Bargmann case. We also showed that the loop representation can also be constructed for a complex coordinatization of phase space similar to the one that the Ashtekar variables introduce for gravity.

In the next chapter we will discuss the quantization in terms of loops of non-Abelian fields.