Acta Numerica (2024), pp. 163–485 Printed in the United Kingdom
doi:10.1017/S0962492924000011

Adaptive finite element methods

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This is a survey of the theory of adaptive finite element methods (AFEMs), which are fundamental to modern computational science and engineering but whose mathematical assessment is a formidable challenge. We present a self-contained and up-to-date discussion of AFEMs for linear second-order elliptic PDEs and dimension $d > 1$, with emphasis on foundational issues. After a brief review of functional analysis and basic finite element theory, including piecewise polynomial approximation in graded meshes, we present the core material for coercive problems. We start with a novel *a posteriori* error analysis applicable to rough data, which delivers estimators fully equivalent to the solution error. They are used in the design and study of three AFEMs depending on the structure of data. We prove linear convergence of these algorithms and rate-optimality provided the solution and data belong to suitable approximation classes. We also address the relation between approximation and regularity classes. We finally extend this theory to discontinuous Galerkin methods as prototypes of non-conforming AFEMs, and beyond coercive problems to inf-sup stable AFEMs.

2020 Mathematics Subject Classification: Primary 65N30, 65N50
Secondary 65N35, 65N41

CONTENTS

1. Introduction: overview of AFEMs

This is a survey of the theory of adaptive finite element methods (AFEMs), which are fundamental to modern computational science and engineering. We present a self-contained and up-to-date discussion of AFEMs for linear second-order elliptic PDEs in dimension $d > 1$, with emphasis on foundational issues rather than applications of AFEMs. This paper builds on and expands the older surveys by Nochetto, Siebert and Veeser (2009) and Nochetto and Veeser (2012). In fact, we decided to incorporate several new aspects into the theory described below.

The paper develops the theory of AFEMs gradually, and is meant to be accessible to advanced students and researchers interested in learning the fundamental aspects of adaptivity and why AFEMs outperform classical FEMs. We quantify the superior performance of AFEMs with precise mathematical statements rather than simulations. We present very few numerical experiments to illustrate some key (and new) algorithmic ideas and methods, but the paper is otherwise a tour through the numerical analysis of adaptive approximation of linear elliptic PDEs.

By design, this paper goes deep into some foundational aspects of AFEMs theory, provides full discussions and proofs, as well as pointers to the main literature. We consider the following model problem on a polyhedral domain $\Omega \subset \mathbb{R}^d$ with $d \geq 2$:

$$L[u] := -\operatorname{div}(\boldsymbol{A}\nabla u) + cu = f, \tag{1.1}$$

with general variable coefficients $(\boldsymbol{A}, c)$, forcing $f \in H^{-1}(\Omega)$ and homogeneous Dirichlet boundary conditions $u = 0$ on $\partial\Omega$ mostly, but not exclusively. If $\mathbb{V} := H_0^1(\Omega)$ and $\mathcal{B}\colon \mathbb{V}\times\mathbb{V} \to \mathbb{R}$ is the bilinear form associated with (1.1), the weak form reads

$$u \in \mathbb{V}: \quad \mathcal{B}[u, v] = \langle f, v\rangle \quad \text{for all } v \in \mathbb{V}. \tag{1.2}$$

Given a conforming and shape-regular partition $\mathcal{T}$ of Ω, created by successive refinement of a coarse mesh $\mathcal{T}_0$, let $\mathbb{V}_{\mathcal{T}}$ denote the space of continuous piecewise polynomial functions of degree $n \geq 1$ over $\mathcal{T}$ vanishing on $\partial\Omega$. The Galerkin approximation $u_{\mathcal{T}}$ of u solves

$$u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}: \quad \mathcal{B}[u_{\mathcal{T}}, v] = \langle f, v \rangle \quad \text{for all } v \in \mathbb{V}_{\mathcal{T}}. \tag{1.3}$$

This is a conforming approximation because $\mathbb{V}_{\mathcal{T}} \subset \mathbb{V}$. The aim of this paper is as follows.

- To design and analyse practical ways to estimate the error $|u - u_{\mathcal{T}}|_{H_0^1(\Omega)} := \|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)}$ in terms of so-called *a posteriori* error estimators, which are computable quantities depending on the discrete solution $u_{\mathcal{T}}$ and data $\mathcal{D} = (\boldsymbol{A}, c, f)$.
- To design adaptive algorithms that equidistribute the local errors $\|\nabla(u - u_{\mathcal{T}})\|_{L^2(T)}$ for all elements $T \in \mathcal{T}$, thereby optimizing the computational effort; this is a key step that makes complex three-dimensional situations accessible computationally.
- To show that this strategy delivers a performance comparable with the best possible in terms of degrees of freedom, which is a measure of computational complexity. This is a delicate matter because it entails dealing with approximation classes and their relation to regularity classes in terms of Besov and Lipschitz spaces.
- To present and analyse the bisection method for mesh refinement, one of the most versatile techniques for local mesh refinement that guarantees shape regularity and optimal complexity; the latter is instrumental to the previous point. Our study includes conforming meshes as well as certain non-conforming meshes.
- To extend the theory to a range of important problems that fail to be conforming or coercive. The first class is that of discontinuous Galerkin methods and the second class is inf-sup stable FEMs. The former is a notorious example of non-conforming approximation, whereas the latter is non-coercive.

In achieving these goals we provide several new ideas and methods. We also refer to the pertinent literature but we do not give a full list of references or get into comparisons of various approaches. It is not our intention to be comprehensive but rather to cover basic aspects of adaptivity in depth at the expense of important topics we do not touch upon. Some of them are:

- adaptive eigenvalue approximation,
- goal-oriented error analysis,
- non-conforming discretizations (except for discontinuous Galerkin),
- coarsening or aggregation,

- anisotropic refinements,
- hp-adaptivity,
- tree approximation,
- other PDEs, e.g. convection–diffusion equations, nonlinear and evolution equations.

We devote the rest of this introduction to providing a roadmap for the rest of the paper. In doing so, we introduce notation that will be used later, and present some topics in their most primitive form to provide an early idea about how they fit and interrelate.

A posteriori error analysis. We refer to the books by Ainsworth and Oden (2000) and Verfürth (2013) for the classical theory. However, in contrast to most of the existing literature, the current theory deals with forcing $f \in H^{-1}(\Omega)$. This allows for rough data useful in applications, such as line Dirac masses, but also encompasses a new approach to error estimation that leads to an error-dominated estimator and oscillation, and prevents error overestimation; this extends Kreuzer and Veeser (2021) to (1.1) and polynomial degree $n \geq 1$. The new twist is the construction of a projection operator $P_{\mathcal{T}} \colon H^{-1}(\Omega) \to \mathbb{F}_{\mathcal{T}}$ into a space of piecewise polynomials in $\mathcal{T}$ and on its skeleton $\mathcal{F}$, namely the set of all internal faces. Such an operator happens to be locally stable on stars (or patches) ω_z of $\mathcal{T}$ for all vertices $z \in \mathcal{V}$ of $\mathcal{T}$:

$$\|P_{\mathcal{T}}\ell\|_{H^{-1}(\omega_z)} \leq C_{\mathrm{lStb}}\|\ell\|_{H^{-1}(\omega_z)} \quad \text{for all } \ell \in H^{-1}(\omega_z). \tag{1.4}$$

An important property of $P_{\mathcal{T}}$ and its range $\mathbb{F}_{\mathcal{T}}$ is that for piecewise polynomial coefficients $(\boldsymbol{A}, c)$, or in short discrete coefficients, $P_{\mathcal{T}}$ is invariant in the subspace $L[\mathbb{V}_{\mathcal{T}}]$ of $H^{-1}(\Omega)$ or equivalently

$$P_{\mathcal{T}}(L[v]) = L[v] \in \mathbb{F}_{\mathcal{T}} \quad \text{for all } v \in \mathbb{V}_{\mathcal{T}}. \tag{1.5}$$

It is worth realizing that $L[v]$ is made of two distinct parts. The first one is absolutely continuous relative to the Lebesgue measure, namely $-\operatorname{div}(\boldsymbol{A}\nabla v) + cv$ in every element $T \in \mathcal{T}$. The second part is singular and supported in the skeleton $\mathcal{F}$, namely $[\![\boldsymbol{A}\nabla v]\!] \cdot \boldsymbol{n}|_F \delta_F$ for every face $F \in \mathcal{F}$, where $[\![\cdot]\!]$ is the jump across F, $\boldsymbol{n}$ is a unit normal to F, and δ_F is the Dirac mass on F.

These two properties of $P_{\mathcal{T}}$ have the following crucial consequences. Let

$$R_{\mathcal{T}} := L[u - u_{\mathcal{T}}] = f - L[u_{\mathcal{T}}] \in H^{-1}(\Omega) \tag{1.6}$$

be the residual of the Galerkin approximation of (1.2). Using (1.5) yields

$$R_{\mathcal{T}} - P_{\mathcal{T}}R_{\mathcal{T}} = f - P_{\mathcal{T}}f.$$

This shows that $R_{\mathcal{T}}$ decomposes into a discrete, thus finite-dimensional and computable, PDE part $P_{\mathcal{T}}R_{\mathcal{T}} = P_{\mathcal{T}}f - L[u_{\mathcal{T}}]$ and an infinite-dimensional component $f - P_{\mathcal{T}}f$, the so-called data oscillation that depends on f and can only be evaluated with additional knowledge of f.

The non-local H^{-1}-norm of $R_{\mathcal{T}}$ splits into local contributions on stars, whence[1]

$$|u - u_{\mathcal{T}}|^2_{H^1_0(\Omega)} \approx \|R_{\mathcal{T}}\|^2_{H^{-1}(\Omega)} \approx \sum_{z\in\mathcal{V}} \|R_{\mathcal{T}}\|^2_{H^{-1}(\omega_z)}.$$

The discrete nature of $P_{\mathcal{T}}R_{\mathcal{T}}$ allows us to derive a computable L^2-weighted PDE estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}}, z)$ equivalent to $\|R_{\mathcal{T}}\|_{H^{-1}(\omega_z)}$, which together with the remaining data oscillation $\operatorname{osc}_{\mathcal{T}}(f, z)_{-1} := \|f - P_{\mathcal{T}}f\|_{H^{-1}(\omega_z)}$ gives the upper bound

$$|u - u_{\mathcal{T}}|^2_{H^1_0(\Omega)} \lesssim \sum_{z\in\mathcal{V}} \big(\eta_{\mathcal{T}}(u_{\mathcal{T}}, z)^2 + \operatorname{osc}_{\mathcal{T}}(f, z)^2_{-1}\big).$$

It turns out that this estimate is sharp or, in other words, that there is no overestimation of the error. To see this important and unique property of these new estimators, we invoke (1.4) to write the local lower bounds

$$\begin{aligned}\eta_{\mathcal{T}}(u_{\mathcal{T}}, z) &\approx \|P_{\mathcal{T}}R_{\mathcal{T}}\|_{H^{-1}(\omega_z)} \le C_{\mathrm{lStb}}\|R_{\mathcal{T}}\|_{H^{-1}(\omega_z)},\\ \operatorname{osc}_{\mathcal{T}}(f, z)_{-1} &= \|R_{\mathcal{T}} - P_{\mathcal{T}}R_{\mathcal{T}}\|_{H^{-1}(\omega_z)} \le (1 + C_{\mathrm{lStb}})\|R_{\mathcal{T}}\|_{H^{-1}(\omega_z)}.\end{aligned}$$

Section 4 constructs the operator $P_{\mathcal{T}}$, and derives several important properties such as its local quasi-best approximation and the above error-dominated *a posteriori* bounds. The former guarantees the inequality for the local L^2-projection $\Pi_{\mathcal{T}}$, that is,

$$\|f - P_{\mathcal{T}}f\|_{H^{-1}(\omega_z)} \lesssim \|f - \Pi_{\mathcal{T}}f\|_{H^{-1}(\omega_z)} \lesssim \|h(f - \Pi_{\mathcal{T}}f)\|_{L^2(\omega_z)},$$

which is the typical form of data oscillation provided $f \in L^2(\Omega)$. However, this L^2-weighted oscillation is not bounded above by the error and is thus responsible for potential overestimation. Section 4 proves further properties of $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ such as its reduction upon refinement and its localized discrete upper bound, as well as quasi-monotonicity of $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ upon refinement. These properties, known for the standard L^2-weighted estimator and oscillation, are thus retained by the new construction.

Section 4 also deals with the alternative error estimators that result from solving local problems, using hierarchy, or imposing flux equilibration. We show that all of them lead, essentially, to estimators equivalent to $\|R_{\mathcal{T}}\|_{H^{-1}(\omega_z)}$. Moreover, we present an optimal framework to deal with non-homogeneous Dirichlet boundary conditions as well as with Robin and Neumann boundary conditions.

Linear convergence of AFEMs. Local *a posteriori* error indicators are usually employed to mark elements (or sets of elements) with largest indicators for refinement. We are concerned with the most popular *Dörfler marking* (or bulk chasing): given a parameter $\theta \in (0, 1]$, select a set $\mathcal{M} \subset \mathcal{T}$ such that

$$\eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{M}) \ge \theta\, \eta_{\mathcal{T}}(u_{\mathcal{T}}); \tag{1.7}$$

[1] Throughout this work $A \lesssim B$ signifies $A \le CB$ with a constant C independent of the discretization parameters, and $A \approx B$ stands for $A \lesssim B$ and $B \lesssim A$.

hereafter we define

$$\eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{M})^2 := \sum_{T \in \mathcal{M}} \eta_{\mathcal{T}}(u_{\mathcal{T}}, T)^2,$$

where $\eta_{\mathcal{T}}(u_{\mathcal{T}}, T)$ is the PDE indicator associated with a generic element $T \in \mathcal{T}$. Note that $\theta = 1$ corresponds to uniform refinement. In Section 5 we present three AFEMs in increasing order of complexity regarding data $\mathcal{D} = (\boldsymbol{A}, c, f)$ and prove their linear convergence.

The simplest algorithm, so-called GALERKIN, works for discrete $\mathcal{D}$ and is the usual adaptive loop

$$\text{SOLVE} \to \text{ESTIMATE} \to \text{MARK} \to \text{REFINE}\,.$$

We assume SOLVE computes the exact Galerkin solution $u_{\mathcal{T}}$, so we refrain from addressing linear algebra issues. The module ESTIMATE computes the *a posteriori* error indicator and the module MARK implements (1.7); in most of the paper we deal with weighted L^2-error indicators $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ but we also address linear convergence for alternative estimators. The module REFINE bisects marked elements and perhaps a few more to keep meshes conforming (or Λ-admissible if they are non-conforming). We let $|||u - u_{\mathcal{T}}|||_\Omega$ denote the energy error associated with the bilinear form $\mathcal{B}$. This error is monotone with refinement but may stagnate. We thus exploit the estimator reduction property with refinement, typical of $\eta_{\mathcal{T}}(u_{\mathcal{T}})$, to show that the combined quantity

$$\zeta_{\mathcal{T}}(u_{\mathcal{T}})^2 := |||u - u_{\mathcal{T}}|||_\Omega^2 + \gamma\, \eta_{\mathcal{T}}(u_{\mathcal{T}})^2 \tag{1.8}$$

contracts in every iteration of GALERKIN for a suitable scaling parameter $\gamma > 0$. This readily leads to linear convergence of both $|u - u_{\mathcal{T}}|_{H_0^1(\Omega)}$ and $\eta_{\mathcal{T}}(u_{\mathcal{T}})$.

We next keep the coefficients $(\boldsymbol{A}, c)$ discrete but allow for a general $f \in H^{-1}(\Omega)$. This is to prevent the *multiplicative interaction* between $(\boldsymbol{A}, c)$ and u that occurs in (1.1) if we were to approximate $(\boldsymbol{A}, c)$. In contrast, the effect of f is *linear* in (1.1). We show examples where $|||u - u_{\mathcal{T}}|||_\Omega$ may stagnate because the adaptive process is dominated by oscillation $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ (pre-asymptotic regime). To compensate for this fact, we design a one-step AFEM with switch as in Kreuzer, Veeser and Zanotti (2024), the so-called AFEM-SW, that proceeds like GALERKIN provided $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ dominates and otherwise reduces $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ separately. We show that for a suitable parameter $\gamma > 0$, the combined quantity

$$\zeta_{\mathcal{T}}(u_{\mathcal{T}})^2 := |||u - u_{\mathcal{T}}|||_\Omega^2 + \gamma \eta_{\mathcal{T}}(u_{\mathcal{T}})^2 + \operatorname{osc}_{\mathcal{T}}(f)_{-1}^2 \tag{1.9}$$

contracts in every loop of AFEM-SW. This yields linear convergence of the error $|u - u_{\mathcal{T}}|_{H1_0(\Omega)}$ and the estimator $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f) = \eta_{\mathcal{T}}(u_{\mathcal{T}}) + \operatorname{osc}_{\mathcal{T}}(f)_{-1}$.

The third algorithm is the two-step AFEM, the so-called AFEM-TS, which allows for general data $\mathcal{D} = (\boldsymbol{A}, c, f)$. To handle the aforementioned nonlinear effect of $(\boldsymbol{A}, c)$ and also deal with general f, all data $\mathcal{D}$ are first approximated by a routine DATA to a desired level of accuracy, which is adjusted at every step of AFEM-TS,

and then fed to GALERKIN which handles discrete data. Suitably combining the accuracies of each intermediate module leads to linear convergence and optimal complexity of GALERKIN within each loop. The structure of AFEM-TS is flexible enough to easily handle discontinuous coefficients $(\boldsymbol{A}, c)$ with discontinuities that may not be aligned with the mesh. This is because the approximation of $(\boldsymbol{A}, c)$ by discontinuous polynomials takes place in $L^p(\Omega)$ for $p < \infty$.

It is worth stressing two important points. First, the approximation of $\mathcal{D}$ is carried out by a GREEDY algorithm, which is shown to perform optimally starting from any refinement of $\mathcal{T}_0$. Second, the discontinuous piecewise polynomial approximations $(\widehat{\boldsymbol{A}}, \widehat{c})$ of $(\boldsymbol{A}, c)$ may not respect, for polynomial degree ≥ 1, the positivity bounds associated with the coefficients. This requires a nonlinear correction of the output of GREEDY that restores positivity and does not reduce accuracy beyond a modest multiplicative constant. We postpone the discussion of these two delicate and technical processes to Section 7, which can be omitted in a first reading.

Rate-optimality of AFEMs. Showing that AFEMs outperform classical FEMs is a difficult but important matter. This reduces to proving a superior relation between the required degrees of freedom (or number of elements) for a desired accuracy; the former is in fact an acceptable measure of complexity. Showing that AFEMs deliver performance comparable with the best entails the following basic ingredients.

- *Nonlinear approximation classes.* These classify functions in terms of the best possible algebraic decay rate of approximation $e_N(v)_X$ of a given function v in a given norm X with N number of elements; roughly speaking, we say $v \in \mathbb{A}_s$ if $e_N(v)_X \lesssim N^{-s}$. These classes are related to regularity classes (Sobolev, Besov and Lipschitz) along Sobolev embedding lines.
- *Dörfler marking.* If the oscillation $\mathrm{osc}_{\mathcal{T}}(f)_{-1}$ is dominated by the PDE estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ for a given mesh $\mathcal{T}$, then any conforming refinement $\mathcal{T}_* \geq \mathcal{T}$ of $\mathcal{T}$ that reduces $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ by a substantial amount induces a refined set $\mathcal{R} := \mathcal{T} \setminus \mathcal{T}_*$ to modify $\mathcal{T}$ into $\mathcal{T}_*$ satisfying (1.7), namely $\eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{R}) \geq \theta\, \eta_{\mathcal{T}}(u_{\mathcal{T}})$.
- *Minimality of $\mathcal{M}$.* If the subset $\mathcal{M} \subset \mathcal{T}$ in (1.7) is minimal, then the cardinality of $\mathcal{M}$ compares favourably to the cardinality of the best mesh with a comparable error accuracy, thereby leading to rate-optimality of AFEM.

Together, these comprise the topic of Section 6. It is important to notice that membership of $\mathbb{A}_s$ is never used explicitly by AFEM to learn about problem (1.1) and improve its resolution. The fundamental reason for the superior performance of AFEM relative to FEM lies in nonlinear approximation theory. We now illustrate this point with the following insightful approximation example for $d = 1$ and $X = L^\infty(0, 1)$ (DeVore 1998, Kahane 1961): let $\Omega = (0, 1)$, $\mathcal{T}_N = \{[x_{j-1}, x_j]\}_{j=1}^N$ be a partition of Ω, with

$$0 = x_0 < x_1 < \cdots < x_j < \cdots < x_N = 1,$$

and let $v\colon \Omega \to \mathbb{R}$ be an absolutely continuous function to be approximated by a piecewise constant function v_N over $\mathcal{T}_N$. To quantify the difference between v and v_N we resort to the *maximum norm*, and study two cases depending on the regularity of v. We define $v_N(x) := v(x_{j-1})$ for all $x_{j-1} \le x < x_j$ and note that

$$|v(x) - v_N(x)| = |v(x) - v(x_{j-1})| \le \int_{x_{j-1}}^{x} |v'(t)|\,\mathrm{d}t.$$

- *Case 1: W^1_∞-regularity.* If $u \in W^1_\infty(0,1)$ and $x_{j-1} \le x < x_j$, then

$$|v(x) - v_N(x)| \le h_j \|v'\|_{L^\infty(x_{j-1},x_j)} \quad \Rightarrow \quad \|v - v_N\|_{L^\infty(\Omega)} \le \frac{1}{N}\|v'\|_{L^\infty(\Omega)}$$

  for a *uniform* mesh. We thus deduce a rate N^{-1} using the same integrability L^∞ on both sides of the error estimate.

- *Case 2: W^1_1-regularity.* Let $\|v'\|_{L^1(\Omega)} = 1$ and $\mathcal{T}_N$ be a *graded* partition so that

$$\int_{x_{j-1}}^{x_j} |v'(t)|\,\mathrm{d}t = \frac{1}{N}.$$

  Then, for $x \in [x_{j-1}, x_j]$,

$$|v(x) - v(x_{j-1})| \le \int_{x_{j-1}}^{x_j} |v'(t)|\,\mathrm{d}t = \frac{1}{N} \quad \Rightarrow \quad \|v - v_N\|_{L^\infty(\Omega)} \le \frac{1}{N}\|v'\|_{L^1(\Omega)}.$$

We thus conclude that we could achieve the same rate of convergence N^{-1} for rougher functions with just $\|u'\|_{L^1(\Omega)} < \infty$. Three comments are now in order. First, the contrast between Cases 1 and 2 is more dramatic for $v(x) = x^\alpha$ with $\alpha \in (0,1)$ because Case 1 only yields the suboptimal rate $\|v - v_N\|_{L^\infty(\Omega)} \le N^{-\alpha}$. Second, $\mathcal{T}_N$ in Case 2 *equidistributes* the max-error, a concept that will permeate our discussions later. Third, the optimal rate of Case 2 is due to the exchange of differentiability with integrability along the critical Sobolev embedding line between the left- and right-hand sides of the error estimate (nonlinear Sobolev scale), while Case 1 relies on the linear Sobolev scale with constant integrability.

We exploit and further elaborate these concepts in Section 6 to show rate-optimality of the three algorithms GALERKIN, AFEM-SW and AFEM-TS, discussed in Section 5, provided u and $\mathcal{D}$ belong to suitable approximation classes. We also investigate the relation between these approximation classes with regularity classes, allowing for discontinuous coefficients, and present a fairly complete discussion.

Mesh refinement. A key component of any adaptive algorithm, such as the three AFEMs already described, is the routine REFINE which refines a current mesh $\mathcal{T}$ into $\mathcal{T}_*$ to improve resolution. In Section 8 we study the *bisection method*, which is the most popular method to refine simplicial meshes in $\mathbb{R}^d$ for $d \ge 1$. For simplicity we focus our attention on this method, but most results apply to other refinement strategies such as quadtrees (for quadrilaterals), octrees (for hexagons)

and red–green (for simplicial meshes). We do not insist on these extensions but refer to Bonito and Nochetto (2010) for details.

Given an initial grid $\mathcal{T}_0$ with a suitable labelling, the bisection method splits a given simplex into two children. The rules for successive cutting of simplices, for instance newest vertex bisection for $d = 2$, are such that the ensuing meshes are shape-regular (with a uniform constant depending only on $\mathcal{T}_0$ and d). However, bisection may not be completely local to keep conformity. The analysis of propagation of refinement is a delicate combinatorial problem. It is easy to see by example that bisecting one element of large generation (i.e. the number of bisections needed to produce it) may require a chain of elements with length similar to the generation. Therefore the number of refined elements in one step cannot be bounded by the number of marked elements. The following amazing estimate by Binev, Dahmen and DeVore (2004) for $d = 2$ and Stevenson (2008) for $d > 2$ shows that the cumulative effect of bisection, counting all the marked elements $\mathcal{M}_j$ from $\mathcal{T}_0$, is quasi-optimal: there exists a constant $D > 0$, depending on $\mathcal{T}_0$ and d, such that

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \leq D \sum_{j=0}^{k-1} \#\mathcal{M}_j. \tag{1.10}$$

This estimate is crucial for the study of rate-optimality of AFEM and is proved in Section 8 for $d = 2$ and for both conforming refinement and Λ-admissible refinement. The latter is a systematic way to handle non-conforming meshes that goes back to Beirão da Veiga *et al.* (2023). It associates a computable global index to hanging nodes and imposes a restriction on them not to exceed a preassigned value $\Lambda \geq 0$; if $\Lambda = 0$ then the mesh is conforming. Section 8 also discusses several interesting geometric properties of Λ-admissible meshes which turn out to be crucial for discontinuous Galerkin methods. Since Section 8 is quite technical, it can be skipped in a first reading.

Discontinuous Galerkin methods. These methods, so-called dG, are the natural first step in investigating the role of non-conformity in adaptivity, namely that the discrete space of discontinuous piecewise polynomials $\mathbb{V}_{\mathcal{T}}$ is no longer a subspace of $H^1_0(\Omega)$. To this end, we study the symmetric interior penalty dG method in Section 9 on Λ-admissible partitions $\mathcal{T}$ of $\mathcal{T}_0$. Such dG methods exhibit some characteristic and novel features with respect to conforming FEMs: the most notable one is the presence of weighted jumps that stabilize the method and compensate for the lack of H^1-conformity. We consider the formulation with lifting, which allows for minimal regularity $u \in H^1_0(\Omega)$, and forcing $f \in H^{-1}(\Omega)$ despite the fact that $\mathbb{V}_{\mathcal{T}}$ is not a subspace of $H^1_0(\Omega)$. The latter is possible because, within the framework of AFEM-TS, f is approximated by a piecewise polynomial $P_{\mathcal{T}} f$ for which the pairing with functions in $\mathbb{V}_{\mathcal{T}}$ is meaningful.

The fact that jumps are not monotone upon refinement constitutes one of the main obstructions to studying adaptivity for dG. To circumvent this issue we

follow Bonito and Nochetto (2010), who in turn modified the original approach of Karakashian and Pascal (2007), and introduce the largest conforming subspace $\mathbb{V}_{\mathcal{T}}^0$ of $\mathbb{V}_{\mathcal{T}}$. It turns out that, despite being coarser, $\mathbb{V}_{\mathcal{T}}^0$ exhibits a local resolution comparable with $\mathbb{V}_{\mathcal{T}}$ because of key geometric properties of Λ-admissible meshes that control the degree of non-conformity of $\mathcal{T}$. In addition, $\mathbb{V}_{\mathcal{T}}^0$ is responsible for the scaled jumps being bounded by the PDE estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}})$. Exploiting properties of $\eta_{\mathcal{T}}(u_{\mathcal{T}})$, similar to the conforming case, leads to a quasi-orthogonality estimate for the dG norm, a dG variant of the Pythagoras equality. This is instrumental in proving a contraction property for the error plus scaled estimator and deducing convergence for both GALERKIN and AFEM-TS. Moreover, we derive rate-optimality for both algorithms provided u and $\mathcal{D}$ belong to suitable approximation classes. Such classes are the same as for conforming AFEMs: in fact we prove that the approximation classes for u using continuous and discontinuous piecewise polynomials on Λ-admissible meshes coincide.

Inf-sup stable AFEMs. The convergence and optimality theories developed in Sections 5 and 6 rely on the bilinear form in (1.2) being *coercive*. We remove this strong restriction in Section 10 and consider uniformly *inf-sup stable* FEMs on conforming refinements $\mathcal{T}_j$ of $\mathcal{T}_0$. The lack of an energy norm and its monotone behaviour upon refinement has been an obstacle to the study of this class of problems. We follow the recent work by Feischl (2022), who introduced the following form of *quasi-orthogonality* between consecutive Galerkin solutions $u_j \in \mathbb{V}_j$, originally proposed in Carstensen, Feischl, Page and Praetorius (2014) as part of an abstract set of axioms of adaptivity:

$$\sum_{k=j}^{j+N} \|u_{k+1} - u_k\|_{\mathbb{V}}^2 \le C(N)\|u - u_j\|_{\mathbb{V}}^2, \quad j \ge 0, \tag{1.11}$$

where $C(N)/N \to 0$ as $N \to \infty$. This is our departure point for developing a *variational approach* to prove linear convergence of u_j provided data $\mathcal{D}$ is discrete; the latter is reflected in an equivalence property between error and estimator (without oscillation). This is the context of a GALERKIN routine, which is next used as a building block together with DATA for an AFEM-TS that handles general data $\mathcal{D}$. Moreover, we prove rate-optimality for both algorithms, thereby extending Sections 5 and 6.

This discussion is fairly abstract. We specialize it to the Stokes equations for viscous incompressible fluids and mixed formulations of (1.1) using Raviart–Thomas–Nédélec and Brezzi–Douglas–Marini elements. We thereby obtain convergence and rate-optimality for AFEM-SW for the Stokes equations and AFEM-TS for mixed methods with variable and possibly discontinuous coefficients $(\boldsymbol{A}, c)$.

We conclude with a complete proof of (1.11) following Feischl (2022). This is a *tour de force* in applied linear algebra and is rather technical. It can be omitted in a first reading.

2. Linear elliptic boundary value problems: basic theory

In this section we examine the variational formulation of elliptic partial differential equations (PDEs). We start with a brief review of Sobolev spaces and their properties, and continue with two model boundary value problems that are instrumental to our subsequent analysis. We next present the so-called inf-sup theory, which characterizes the existence, uniqueness and stability of variational problems, and apply it to coercive and saddle point problems. These two classes will play essential roles later.

2.1. *Sobolev spaces: scaling and embedding*

Let $\Omega \subset \mathbb{R}^d$ with $d > 1$ be a Lipschitz and bounded domain, and let $k \in \mathbb{N}$, $1 \le p \le \infty$. The Sobolev space $W_p^k(\Omega)$ is defined by

$$W_p^k(\Omega) := \{v\colon \Omega \to \mathbb{R} |\ D^\alpha v \in L^p(\Omega) \text{ for all } |\alpha| \le k\},$$

and is a Banach space with the norm

$$\|v\|_{W_p^k(\Omega)} = \left(\sum_{j=1}^{k} |v|^p_{W_p^j(\Omega)}\right)^{1/p}, \quad |v|_{W_p^j(\Omega)} = \left(\int_\Omega |D^j v|^p\right)^{1/p}.$$

The space $W_p^k(\Omega;\mathbb{R}^m)$ is the space $W_p^k(\Omega)$ of vector- or matrix-valued functions. If $p = 2$ we write $H^k(\Omega) = W_2^k(\Omega)$ and note that this is a Hilbert space. We let $H_0^1(\Omega) \subset H^1(\Omega)$ denote the completion of $C_0^\infty(\Omega)$ within $H^1(\Omega)$.

Sobolev spaces $W_p^k(\Omega)$ of fractional order $k > 0$ can be defined as well, by applying the real interpolation method between $W_p^{[k]}(\Omega)$ and $W_p^{[k]+1}(\Omega)$; see Bergh and Löfström (1976) and Adams and Fournier (2003) for the details. The subsequent definitions and properties hold for Sobolev spaces of integer or fractional order.

The *Sobolev number* of $W_p^k(\Omega)$ is given by

$$\mathrm{sob}(W_p^k) := k - \frac{d}{p}. \tag{2.1}$$

This number governs the scaling properties of the seminorm $|v|_{W_p^k(\Omega)}$, because rescaling variables $\widehat{x} = x/h$ for all $x \in \Omega$, transforms Ω into $\widehat{\Omega}$ and v into $\widehat{v}$, while the corresponding norms scale as

$$|\widehat{v}|_{W_p^k(\widehat{\Omega})} = h^{\mathrm{sob}(W_p^k)} |v|_{W_p^k(\Omega)}.$$

2.2. *Properties of Sobolev spaces*

We now summarize, but do not prove, several important properties of Sobolev spaces which play a key role later. We refer to Evans (2010), Gilbarg and Trudinger (2001) and Grisvard (2011) for details. We use the notation $\hookrightarrow$ to denote a continuous embedding.

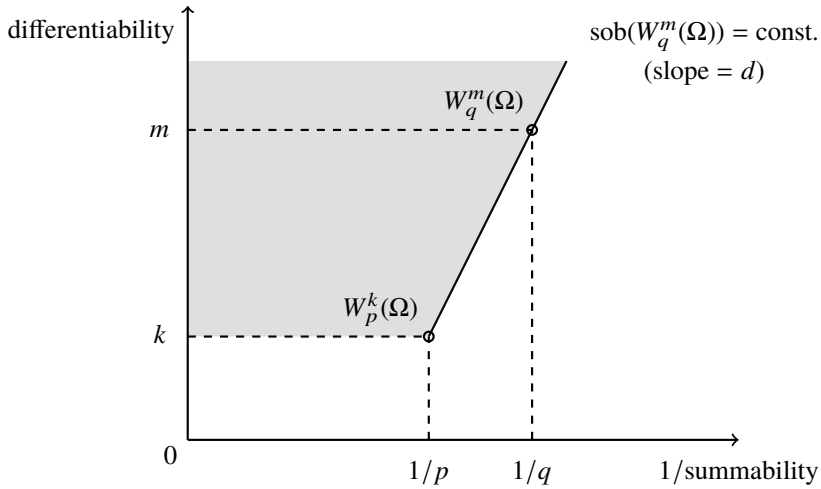


Figure 2.1. DeVore diagram (DeVore 1998). The space $W_p^k(\Omega)$ is represented by the point $(1/p, k)$ in the first quadrant. The line $\mathrm{sob}(W_q^m) = \text{const.} = \mathrm{sob}(W_p^k)$, with slope d, may be called the (critical) Sobolev embedding line for $W_p^k(\Omega)$. It represents all Sobolev spaces having the same Sobolev number as $W_p^k(\Omega)$. Sobolev spaces corresponding to points inside the grey region and on its boundary on the vertical axis are compactly embedded in $W_p^k(\Omega)$. Spaces on the oblique and horizontal lines emanating from $W_p^k(\Omega)$ are generally continuously but not compactly embedded in $W_p^k(\Omega)$ with exceptions such as $p = \infty$. Note that indices k and m may take non-integer values, corresponding to fractional Sobolev spaces.

Lemma 2.1 (Sobolev embedding). *Let $m > k \geq 0$ and assume* $\mathrm{sob}(W_q^m) > \mathrm{sob}(W_p^k)$. *Then the embedding $W_q^m(\Omega) \hookrightarrow W_p^k(\Omega)$ is compact and*

$$\|v\|_{W_p^k(\Omega)} \leq C\|v\|_{W_q^m(\Omega)} \quad \textit{for all } v \in W_q^m(\Omega),$$

where $C = C(m, k, q, p, \Omega, d)$.

We say that two Sobolev spaces are in the same nonlinear Sobolev scale if they have the same Sobolev number; see Figure 2.1. We thus note that for compactness the space $W_q^m(\Omega)$ must be above the Sobolev scale of $W_p^k(\Omega)$, i.e. $\mathrm{sob}(W_q^m) > \mathrm{sob}(W_p^k)$.

The assumption on the Sobolev number cannot be relaxed. To see this, consider Ω to be the unit ball of $\mathbb{R}^d$ for $d \geq 2$ and set $v(x) = \log\log(|x|/2)$ for $x \in \Omega \setminus \{0\}$. Then we have $v \in W_d^1(\Omega) \setminus L^\infty(\Omega)$, but

$$\mathrm{sob}(W_d^1) = 1 - d/d = 0 = 0 - d/\infty = \mathrm{sob}(L^\infty).$$

Therefore equality cannot be expected in the embedding theorem. On the other hand, consider $d = 1$ and the spaces $W_1^1(\Omega)$ and $L^\infty(\Omega)$. We see that $\mathrm{sob}(W_1^1) =$

$\mathrm{sob}(L^\infty) = 0$ but $W^1_1(\Omega)$ is compactly embedded in $L^\infty(\Omega)$ in this case. This shows that these two spaces are in the same nonlinear Sobolev scale and that the above inequality between Sobolev numbers for a compact embedding is only sufficient.

Moreover, if $0 < \alpha = \mathrm{sob}(W^k_p) \le 1$, then functions of $W^k_p(\Omega)$ are Hölder-α and

$$|v|_{C^{0,\alpha}(\overline{\Omega})} \le C|v|_{W^k_p(\Omega)} \quad \text{for all } v \in W^k_p(\Omega).$$

This allows for the use of the standard Lagrange interpolation operator. We will exploit this fact later in Section 3.

Lemma 2.2 (first Poincaré inequality). *Let $\Omega \subset \mathbb{R}^d$ be a bounded and Lipschitz domain. Then there exists a constant $C_P = C_d|\Omega|^{1/d}$ such that*

$$\|v\|_{L^2(\Omega)} \le C_P\|\nabla v\|_{L^2(\Omega)} \quad \textit{for all } v \in H^1_0(\Omega). \tag{2.2}$$

The same inequality is valid in $W^1_p(\Omega)$ for any $1 \le p \le \infty$ provided v has vanishing trace (Gilbarg and Trudinger 2001, page 158).

Lemma 2.3 (second Poincaré inequality). *There exists C_P depending on Ω such that*

$$\|v - \overline{v}\|_{L^2(\Omega)} \le C_P\|\nabla v\|_{L^2(\Omega)} \quad \textit{for all } v \in H^1(\Omega), \tag{2.3}$$

where $\overline{v} := |\Omega|^{-1}\int_\Omega v$. The best constant within the class of convex domains is $C_P = \pi^{-1}\,\mathrm{diam}(\Omega)$ (Payne and Weinberger 1960). The same inequality is valid in $W^1_p(\Omega)$ for $1 \le p \le \infty$ (Gilbarg and Trudinger 2001).

Lemma 2.4 (traces). *Let Ω be Lipschitz. There exists a unique linear operator $T\colon H^1(\Omega) \to L^2(\partial\Omega)$ such that*

$$\begin{aligned} \|Tv\|_{L^2(\partial\Omega)} &\le c(\Omega)\|v\|_{H^1(\Omega)} && \textit{for all } v \in H^1(\Omega),\\ Tv &= v_{|\partial\Omega} && \textit{for all } v \in C^0(\overline{\Omega}) \cap H^1(\Omega). \end{aligned}$$

The operator T is also well-defined on $W^1_p(\Omega)$ for $1 \le p \le \infty$ (Evans 2010, Grisvard 1985).

Since $Tv = v_{|\partial\Omega}$ for continuous functions, we write v for Tv. The image of T is a strict subspace of $L^2(\partial\Omega)$, the so-called $H^{1/2}(\partial\Omega)$. This is a Hilbert space if equipped with the norm $\|g\|_{H^{1/2}(\partial\Omega)} := \inf\{\|v\|_{H^1(\Omega)} \mid Tv = g\}$, and T is continuous with respect to this norm, since

$$\|Tv\|_{H^{1/2}(\partial\Omega)} \le \|v\|_{H^1(\Omega)} \quad \text{for all } v \in H^1(\Omega). \tag{2.4}$$

The definition of $H^1_0(\Omega)$ can be reconciled with that of traces because

$$H^1_0(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \partial\Omega\}.$$

We point out that, in view of Lemma 2.2, the seminorm $|v|_{H_0^1(\Omega)} := \|\nabla v\|_{L^2(\Omega)}$ is a norm in $H_0^1(\Omega)$. We let $H^{-1}(\Omega)$ be the dual of $H_0^1(\Omega)$, with corresponding norm

$$\|f\|_{H^{-1}(\Omega)} := \sup_{v \in H_0^1(\Omega)} \frac{\langle f, v\rangle}{|v|_{H_0^1(\Omega)}}.$$

These definitions extend to any $p \in (1, \infty)$.

Lemma 2.5 (Gauss divergence theorem). *If $\boldsymbol{n}$ is the outer unit normal to Ω, then*

$$\int_\Omega \operatorname{div} \boldsymbol{w} = \int_{\partial\Omega} \boldsymbol{w} \cdot \boldsymbol{n} \quad \textit{for all } \boldsymbol{w} \in W_1^1(\Omega; \mathbb{R}^d).$$

Lemma 2.6 (Green's formula). *We have the integration by parts formula*

$$\int_\Omega \operatorname{div} \boldsymbol{w}\, v = -\int_\Omega \boldsymbol{w} \cdot \nabla v + \int_{\partial\Omega} v\, \boldsymbol{w} \cdot \boldsymbol{n} \quad \textit{for all } v \in H^1(\Omega),\ \boldsymbol{w} \in H^1(\Omega; \mathbb{R}^d).$$

2.3. Examples of boundary value problems

We consider two model elliptic problems in this paper. We start with the *scalar diffusion–reaction equation* with variable coefficients

$$\begin{aligned} L[u] := -\operatorname{div}(\boldsymbol{A}\nabla u) + cu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.5}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary, $\boldsymbol{A} \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ is a diffusion tensor, uniformly symmetric positive definite (SPD) over Ω, that is, there exist constants $0 < \alpha_1 \le \alpha_2$ such that

$$\alpha_1 |\boldsymbol{\xi}|^2 \le \boldsymbol{\xi}^\top \boldsymbol{A}(x)\, \boldsymbol{\xi} \le \alpha_2 |\boldsymbol{\xi}|^2 \quad \text{for all } x \in \Omega,\ \boldsymbol{\xi} \in \mathbb{R}^d, \tag{2.6}$$

$c \in L^\infty(\Omega), c \ge 0$ is a reaction term, and $f \in L^2(\Omega)$ is a scalar load term.

To derive the variational formulation of (2.5) we let $\mathbb{V} = H_0^1(\Omega)$ and $\mathbb{V}^* = H^{-1}(\Omega)$. Since $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ of functions with vanishing trace, asking for $u \in \mathbb{V}$ accounts for the homogeneous Dirichlet boundary values in (2.5). We next multiply (2.5) by a test function $v \in H_0^1(\Omega)$, integrate over Ω and use Lemma 2.6 (Green's formula), provided $\boldsymbol{w} = -\boldsymbol{A}\nabla u \in H^1(\Omega; \mathbb{R}^d)$, to obtain

$$u \in \mathbb{V}: \quad \mathcal{B}[u, v] = \langle f, v\rangle \quad \text{for all } v \in \mathbb{V}. \tag{2.7}$$

Here, $\mathcal{B}\colon \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ stands for the bilinear form

$$\mathcal{B}[w, v] := \int_\Omega \nabla v \cdot \boldsymbol{A}\nabla w + cvw \quad \text{for all } v, w \in \mathbb{V} \tag{2.8}$$

and $\langle \cdot, \cdot \rangle$ stands for the $L^2(\Omega)$-scalar product and also for a duality pairing $H^{-1}(\Omega)$–$H_0^1(\Omega)$. Since $\boldsymbol{A}$ is assumed to be symmetric, the bilinear form $\mathcal{B}$ is also symmetric; however, $\boldsymbol{A}$ does not have to be symmetric in general. Note that the weak form

(2.7) allows for fluxes $\boldsymbol{w} \in L^2(\Omega;\mathbb{R}^d)$ and forcing $f \in H^{-1}(\Omega)$. We examine the existence, uniqueness and stability of (2.7) in Section 2.4 below.

We assume *homogeneous Dirichlet* boundary conditions in (2.5) for simplicity and because this will be our basic setting later. However, we could allow a *non-homogeneous Dirichlet* condition in the sense of traces, that is,

$$Tu = g \quad \text{on } \partial\Omega, \tag{2.9}$$

for any given function $g \in H^{1/2}(\partial\Omega)$. To write the companion variational formulation to (2.7), we first introduce the subspace $\mathbb{V}(g) \subset H^1(\Omega)$ of functions v with trace $Tv = g$ on $\partial\Omega$, and then rewrite (2.7) as follows:

$$u \in \mathbb{V}(g): \quad \mathcal{B}[u,v] = \langle f, v\rangle \quad \text{for all } v \in \mathbb{V}(0). \tag{2.10}$$

Moreover, we could consider a *Robin* boundary condition for given functions g and p on $\partial\Omega$:

$$\boldsymbol{A}\nabla u \cdot \boldsymbol{n} + pu = g \quad \text{on } \partial\Omega, \tag{2.11}$$

where $\boldsymbol{n}$ is the outer unit normal to Ω. To figure out the variational formulation, we now multiply the PDE in (2.5) by a test function $v \in H^1(\Omega)$ and integrate by parts to find the following variant of (2.7):

$$u \in H^1(\Omega): \quad \mathcal{B}[u,v] = \ell(v) \quad \text{for all } v \in H^1(\Omega), \tag{2.12}$$

where for all $v, w \in H^1(\Omega)$,

$$\mathcal{B}[w,v] := \int_\Omega \nabla v \cdot \boldsymbol{A}\nabla w + cvw + \int_{\partial\Omega} pvw, \quad \ell(v) := \langle f, v\rangle + \int_{\partial\Omega} gv. \tag{2.13}$$

We realize that (2.13) makes sense for $p \in L^\infty(\partial\Omega)$, $p \geq 0$ and $g \in H^{-1/2}(\partial\Omega)$, the dual space of $H^{1/2}(\partial\Omega)$, whence the last integral in (2.13) means a duality pairing. If $p = 0$, then (2.13) reduces to the weak form of the *Neumann* boundary condition.

The second model problem is the *Stokes system*, which is the simplest model of a stationary viscous incompressible fluid. Given an external force $\boldsymbol{f} \in L^2(\Omega;\mathbb{R}^d)$, let the velocity–pressure pair $(\boldsymbol{u}, p)$ satisfy the momentum and incompressibility equations with no-slip boundary condition:

$$\begin{aligned} -\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.14}$$

For the variational formulation we consider two Hilbert spaces $\mathbb{V} = H_0^1(\Omega;\mathbb{R}^d)$ and $\mathbb{Q} = L_0^2(\Omega)$, where $L_0^2(\Omega)$ is the space of L^2 functions with zero mean value. The space $H_0^1(\Omega;\mathbb{R}^d)$ takes care of the no-slip boundary values of the velocity. We first multiply the momentum equation in (2.14) by $\boldsymbol{v} \in \mathbb{V}$, assume $\boldsymbol{u} \in H^2(\Omega;\mathbb{R}^d)$ and use Lemma 2.6 (Green's formula) component-wise. We next multiply the incompressibility equation in (2.14) by $q \in \mathbb{Q}$ and integrate over Ω. We end up

with the following variational formulation: find $(\boldsymbol{u}, p) \in \mathbb{V} \times \mathbb{Q}$ such that

$$\begin{aligned} a[\boldsymbol{u}, \boldsymbol{v}] + b[p, \boldsymbol{v}] &= \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \text{for all } \boldsymbol{v} \in \mathbb{V}, \\ b[q, \boldsymbol{u}] &= 0 \qquad\quad\ \ \text{for all } q \in \mathbb{Q}. \end{aligned} \tag{2.15}$$

Here the bilinear forms $a\colon \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ and $b\colon \mathbb{Q} \times \mathbb{V} \to \mathbb{R}$ read

$$a[\boldsymbol{w}, \boldsymbol{v}] := \int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{w} = \sum_{i=1}^d \int_\Omega \nabla v_i \cdot \nabla w_i \quad \text{for all } \boldsymbol{v}, \boldsymbol{w} \in \mathbb{V}$$

and

$$b[q, \boldsymbol{v}] := -\int_\Omega q \operatorname{div} \boldsymbol{v} \quad \text{for all } q \in \mathbb{Q},\ \boldsymbol{v} \in \mathbb{V}.$$

We observe that $a[\boldsymbol{w}, \boldsymbol{v}]$ does not require $\boldsymbol{w} \in H^2(\Omega; \mathbb{R}^d)$ and that (2.15) makes sense for $\boldsymbol{f} \in H^{-1}(\Omega; \mathbb{R}^d)$; note that the second equation in (2.15) is always satisfied for constant q due to Gauss's theorem, which explains the choice $q \in \mathbb{Q}$. Furthermore, (2.15) can be reformulated as (2.7), namely

$$(\boldsymbol{u}, p) \in \mathbb{V} \times \mathbb{Q}: \quad \mathcal{B}[(\boldsymbol{u}, p), (\boldsymbol{v}, q)] = \langle \boldsymbol{f}, \boldsymbol{v} \rangle \quad \text{for all } (\boldsymbol{v}, q) \in \mathbb{V} \times \mathbb{Q}, \tag{2.16}$$

with

$$\mathcal{B}[(\boldsymbol{u}, p), (\boldsymbol{v}, q)] := a[\boldsymbol{u}, \boldsymbol{v}] + b[p, \boldsymbol{v}] + b[q, \boldsymbol{u}].$$

We discuss the existence, uniqueness and stability of (2.16) in Section 2.4.

We could formulate the Stokes system with other boundary conditions. First, we may allow a non-homogeneous Dirichlet condition $\boldsymbol{u} = \boldsymbol{g}$ for a given function $\boldsymbol{g} \in H^{1/2}(\partial\Omega; \mathbb{R}^d)$ satisfying the compatibility condition $\int_\Omega \boldsymbol{g} \cdot \boldsymbol{n} = 0$ imposed by Gauss's theorem, and proceed as in the scalar case (2.10). Second, to deal with a Neumann boundary condition, we introduce the stress tensor $\boldsymbol{\sigma}(\boldsymbol{u}, p) := \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top) - p\boldsymbol{I}$ and the symmetric part of the velocity gradient $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top)$. Then instead of (2.14) we could write the strong form of the Neumann problem as

$$\begin{aligned} -\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}, p) = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} &= \boldsymbol{g} \quad \text{on } \partial\Omega, \end{aligned} \tag{2.17}$$

and its weak form as (2.15) but with

$$\mathbb{V} := \left\{ \boldsymbol{v} \in H^1(\Omega; \mathbb{R}^d) \mid \int_\Omega \boldsymbol{v} = \boldsymbol{0} \right\}, \quad \mathbb{Q} := L^2(\Omega), \tag{2.18}$$

as well as bilinear form a and right-hand side

$$a[\boldsymbol{u}, \boldsymbol{v}] := \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}), \quad \ell(\boldsymbol{v}) := \langle \boldsymbol{f}, \boldsymbol{v} \rangle + \int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{v} \tag{2.19}$$

for all $\boldsymbol{v} \in \mathbb{V}$. Again, the last integral in (2.19) is to be interpreted as a duality pairing for $\boldsymbol{g} \in H^{-1/2}(\partial\Omega; \mathbb{R}^d)$.

2.4. *Inf-sup theory*

We present a functional framework for the existence, uniqueness and stability of variational problems of the form (2.7) or (2.16). Throughout this section we let $(\mathbb{V}, \langle\cdot,\cdot\rangle_{\mathbb{V}})$ and $(\mathbb{W}, \langle\cdot,\cdot\rangle_{\mathbb{W}})$ be a pair of Hilbert spaces with induced norms $\|\cdot\|_{\mathbb{V}}$ and $\|\cdot\|_{\mathbb{W}}$. We let $\mathbb{V}^*$ and $\mathbb{W}^*$ denote their respective dual spaces equipped with norms

$$\|f\|_{\mathbb{V}^*} = \sup_{v\in\mathbb{V}} \frac{\langle f,\, v\rangle}{\|v\|_{\mathbb{V}}} \quad \text{and} \quad \|g\|_{\mathbb{W}^*} = \sup_{v\in\mathbb{W}} \frac{\langle g,\, v\rangle}{\|v\|_{\mathbb{W}}}.$$

We write $L(\mathbb{V};\mathbb{W})$ for the space of all linear and continuous operators from $\mathbb{V}$ into $\mathbb{W}$ with operator norm

$$\|B\|_{L(\mathbb{V};\mathbb{W})} = \sup_{v\in\mathbb{V}} \frac{\|Bv\|_{\mathbb{W}}}{\|v\|_{\mathbb{V}}}.$$

The following result relates a continuous bilinear form $\mathcal{B}\colon \mathbb{V}\times\mathbb{W} \to \mathbb{R}$ with an operator $B \in L(\mathbb{V};\mathbb{W})$ (Nečas 1962, Babuška 1971).

Theorem 2.7 (Banach–Nečas). *Let $\mathcal{B}\colon \mathbb{V}\times\mathbb{W} \to \mathbb{R}$ be a continuous bilinear form with norm*

$$\|\mathcal{B}\| := \sup_{v\in\mathbb{V}} \sup_{w\in\mathbb{W}} \frac{\mathcal{B}[v,w]}{\|v\|_{\mathbb{V}}\|w\|_{\mathbb{W}}}. \tag{2.20}$$

Then there exists a unique linear operator $B \in L(\mathbb{V},\mathbb{W})$ such that

$$\langle Bv,\, w\rangle_{\mathbb{W}} = \mathcal{B}[v,w] \quad \textit{for all } v\in\mathbb{V},\ w\in\mathbb{W}$$

with operator norm

$$\|B\|_{L(\mathbb{V};\mathbb{W})} = \|\mathcal{B}\|.$$

Moreover, the bilinear form $\mathcal{B}$ satisfies the following two conditions:

$$\textit{there exists } \alpha > 0 \textit{ such that } \alpha\|v\|_{\mathbb{V}} \le \sup_{w\in\mathbb{W}} \frac{\mathcal{B}[v,w]}{\|w\|_{\mathbb{W}}} \textit{ for all } v\in\mathbb{V}; \tag{2.21a}$$

$$\textit{for every } 0 \neq w \in \mathbb{W} \textit{ there exists } v\in\mathbb{V} \textit{ such that } \mathcal{B}[v,w] \neq 0, \tag{2.21b}$$

if and only if $B\colon \mathbb{V}\to\mathbb{W}$ is an isomorphism with

$$\|B^{-1}\|_{L(\mathbb{W},\mathbb{V})} \le \alpha^{-1}. \tag{2.22}$$

We now consider the abstract variational problem

$$u\in\mathbb{V}: \quad \mathcal{B}[u,v] = \langle f,\, v\rangle \quad \text{for all } v\in\mathbb{W}. \tag{2.23}$$

The following result, due to Nečas (1962, Theorem 3.3), characterizes properties of the bilinear form $\mathcal{B}$ that imply that (2.23) is well-posed.

Theorem 2.8 (Nečas). *Let $\mathcal{B}\colon \mathbb{V}\times\mathbb{W} \to \mathbb{R}$ be a continuous bilinear form. Then the variational problem* (2.23) *admits a unique solution $u\in\mathbb{V}$ for all $f\in\mathbb{W}^*$, which depends continuously on f, if and only if the bilinear form $\mathcal{B}$ satisfies one of the following equivalent* inf-sup conditions.

(1) *There exists* $\alpha > 0$ *such that*

$$\sup_{w \in \mathbb{W}} \frac{\mathcal{B}[v, w]}{\|w\|_{\mathbb{W}}} \geq \alpha \|v\|_{\mathbb{V}} \quad \textit{for some } \alpha > 0; \tag{2.24a}$$

$$\textit{for every } 0 \neq w \in \mathbb{W} \textit{ there exists } v \in \mathbb{V} \textit{ such that } \mathcal{B}[v, w] \neq 0. \tag{2.24b}$$

(2) *We have*

$$\inf_{v \in \mathbb{V}} \sup_{w \in \mathbb{W}} \frac{\mathcal{B}[v, w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}}} > 0, \qquad \inf_{w \in \mathbb{W}} \sup_{v \in \mathbb{V}} \frac{\mathcal{B}[v, w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}}} > 0. \tag{2.25}$$

(3) *There exists* $\alpha > 0$ *such that*

$$\inf_{v \in \mathbb{V}} \sup_{w \in \mathbb{W}} \frac{\mathcal{B}[v, w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}}} = \inf_{w \in \mathbb{W}} \sup_{v \in \mathbb{V}} \frac{\mathcal{B}[v, w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}}} = \alpha. \tag{2.26}$$

In addition, the solution u *of* (2.23) *satisfies the stability estimate*

$$\|u\|_{\mathbb{V}} \leq \alpha^{-1} \|f\|_{\mathbb{W}^*}. \tag{2.27}$$

The equality in (2.26) might at first seem surprising but is just a consequence of $\|B^{-*}\|_{L(\mathbb{V};\mathbb{W})} = \|B^{-1}\|_{L(\mathbb{W};\mathbb{V})}$. In general, (2.24) is simpler to verify than (2.26) and α of (2.26) is the largest possible α in (2.24a). Since (2.22) shows that $\|B^{-1}\|_{L(\mathbb{W},\mathbb{V})}$ is the best inf-sup constant α in (2.21a), we readily obtain the following result.

Corollary 2.9 (well-posedness implies inf-sup). *If the variational problem* (2.23) *admits a unique solution* $u \in \mathbb{V}$ *for all* $f \in \mathbb{W}^*$, *so that*

$$\|u\|_{\mathbb{V}} \leq C \|f\|_{\mathbb{W}^*},$$

then $\mathcal{B}$ *satisfies the inf-sup condition* (2.26) *with* $\alpha \geq C^{-1}$.

We next apply these abstract results to two special but important cases. The first class comprises problems with coercive bilinear form and the second class comprises problems of saddle point type.

Coercive problems. An existence and uniqueness result for coercive bilinear forms was established by Lax and Milgram eight years prior to the result by Nečas (Lax and Milgram 1954). Coercivity of $\mathcal{B}$ is a sufficient condition for existence and uniqueness but it is not necessary. In this case, we assume $\mathbb{V} = \mathbb{W}$.

Corollary 2.10 (Lax–Milgram). *Let* $\mathcal{B}\colon \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ *be a continuous bilinear form that is* coercive, *namely there exists* $\alpha > 0$ *such that*

$$\mathcal{B}[v, v] \geq \alpha \|v\|_{\mathbb{V}}^2 \quad \textit{for all } v \in \mathbb{V}. \tag{2.28}$$

Then (2.23) *has a unique solution that satisfies the stability estimate* (2.27).

If the bilinear form $\mathcal{B}$ is also symmetric, that is,

$$\mathcal{B}[v, w] = \mathcal{B}[w, v] \quad \text{for all } v, w \in \mathbb{V},$$

then $\mathcal{B}$ is a scalar product on $\mathbb{V}$. The norm induced by $\mathcal{B}$ is the so-called *energy norm*

$$|||v|||_\Omega := \mathcal{B}[v, v]^{1/2}.$$

For the reaction–diffusion equation (2.5), the bilinear form given in (2.8) satisfies

$$0 < \alpha_1 \le \alpha \le \|\mathcal{B}\| \le \alpha_2 + \|c\|_{L^\infty(\Omega)} C_P^2, \tag{2.29}$$

where C_P is the constant in Lemma 2.2 (first Poincaré inequality). Coercivity and continuity of $\mathcal{B}$, with constants $c_\mathcal{B} = \alpha_1$ and $C_\mathcal{B} = \|\mathcal{B}\|$, in turn imply that the *energy norm* $|||\cdot|||_\Omega$ is equivalent to the natural norm $\|\cdot\|_\mathbb{V}$ in $\mathbb{V} = H_0^1(\Omega)$:

$$c_\mathcal{B}\|v\|_\mathbb{V}^2 \le |||v|||_\Omega^2 \le C_\mathcal{B}\|v\|_\mathbb{V}^2 \quad \text{for all } v \in \mathbb{V}. \tag{2.30}$$

Moreover, it is fairly easy to show that for symmetric and coercive $\mathcal{B}$ the solution u of (2.23) is the unique minimizer of the quadratic energy

$$J[v] := \frac{1}{2}\mathcal{B}[v, v] - \langle f, v\rangle \quad \text{for all } v \in \mathbb{V},$$

that is, $u = \arg\min_{v\in\mathbb{V}} J[v]$. In particular, the energy norm and the quadratic energy play a relevant role in Sections 5, 6 and 9.

This framework applies to the scalar diffusion–reaction equation (2.7) with *homogeneous Dirichlet* condition. Since the full H^1-norm $\|\cdot\|_{H^1(\Omega)}$ and the seminorm $|\cdot|_{H_0^1(\Omega)}$ are equivalent in the space $\mathbb{V} = H_0^1(\Omega)$, according to Lemma 2.2 (first Poincaré inequality), the bilinear form $\mathcal{B}$ in (2.8) is coercive and continuous in view of (2.6) and $c \ge 0$. This framework also applies to the *non-homogeneous Dirichlet* problem (2.10), upon extending g to a function $g \in H^1(\Omega)$, rewriting the problem in terms of $w = u - g \in H_0^1(\Omega)$ and forcing $\ell = f - L[g] \in H^{-1}(\Omega)$.

On the other hand, the bilinear form $\mathcal{B}$ in (2.13) associated with a *Robin* boundary condition is coercive provided $p \ge p_0$ on $\partial\Omega$ (or at least on an open subset of $\partial\Omega$) with some constant $p_0 > 0$. This is a consequence of the norm equivalence

$$\|v\|_{H^1(\Omega)}^2 \approx |v|_{H_0^1(\Omega)}^2 + \|v\|_{L^2(\partial\Omega)}^2 \quad \text{for all } v \in H^1(\Omega). \tag{2.31}$$

For the *Neumann* problem, instead, we have $p = 0$ in $\partial\Omega$, whence $\mathcal{B}$ is coercive whenever $c > 0$ in Ω (or at least in an open subset of Ω). If $c = 0$ in Ω, then $\mathcal{B}$ is only coercive in the subspace of $H^1(\Omega)$ of functions with vanishing mean value, according to Lemma 2.3 (second Poincaré inequality). Existence, uniqueness and stability is thus guaranteed by Corollary 2.10 (Lax–Milgram) provided the forcing in (2.13) satisfies the compatibility condition $\ell(1) = 0$.

Saddle point problems. We now consider an abstract problem a bit more general than (2.15), so the following results apply to the Stokes system (2.14).

Given a pair of Hilbert spaces $(\mathbb{V}, \mathbb{Q})$, we consider two continuous bilinear forms $a\colon \mathbb{V}\times\mathbb{V} \to \mathbb{R}$ and $b\colon \mathbb{Q}\times\mathbb{V} \to \mathbb{R}$. If $f \in \mathbb{V}^*$ and $g \in \mathbb{Q}^*$, then we seek a pair

$(u, p) \in \mathbb{V} \times \mathbb{Q}$ solving the *saddle point problem*

$$a[u, v] + b[p, v] = \langle f,\ v\rangle \quad \text{for all } v \in \mathbb{V}, \tag{2.32a}$$

$$b[q, u] = \langle g,\ q\rangle \quad \text{for all } q \in \mathbb{Q}. \tag{2.32b}$$

Problem (2.32) is variational and can be rewritten in the form (2.23)

$$(u, p) \in \mathbb{V} \times \mathbb{Q}: \quad \mathcal{B}[(u, p), (v, q)] = \langle f,\ v\rangle + \langle g,\ q\rangle \quad \text{for all } (v, q) \in \mathbb{V} \times \mathbb{Q}, \tag{2.33}$$

where $\mathcal{B}$ is the bilinear form

$$\mathcal{B}[(u, p), (v, q)] := a[u, v] + b[p, v] + b[q, u]. \tag{2.34}$$

Therefore the saddle point problem (2.32) is well-posed if and only if $\mathcal{B}$ satisfies the inf-sup condition (2.26). Since $\mathcal{B}$ is defined via the bilinear forms a and b, and (2.32) has a degenerate structure, it is not that simple to show (2.26) directly. However, the result is a consequence of the inf-sup theorem for saddle point problems given by Brezzi (1974).

Theorem 2.11 (Brezzi). *The saddle point problem* (2.32) *has a unique solution* $(u, p) \in \mathbb{V} \times \mathbb{Q}$ *for all data* $(f, g) \in \mathbb{V}^* \times \mathbb{Q}^*$, *that depends continuously on data, if and only if there exist constants* $\alpha, \beta > 0$ *such that*

$$\inf_{v \in \mathbb{V}_0} \sup_{w \in \mathbb{V}_0} \frac{a[v, w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{V}}} = \inf_{w \in \mathbb{V}_0} \sup_{v \in \mathbb{V}_0} \frac{a[v, w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{V}}} = \alpha > 0, \tag{2.35a}$$

$$\inf_{q \in \mathbb{Q}} \sup_{v \in \mathbb{V}} \frac{b[q, v]}{\|q\|_{\mathbb{Q}} \|v\|_{\mathbb{V}}} = \beta > 0, \tag{2.35b}$$

where

$$\mathbb{V}_0 := \{v \in \mathbb{V} \mid b[q, v] = 0 \textit{ for all } q \in \mathbb{Q}\}.$$

In addition, there exists $\gamma = \gamma(\alpha, \beta, \|a\|)$ *such that the solution* (u, p) *is stable, that is,*

$$\left(\|u\|_{\mathbb{V}}^2 + \|p\|_{\mathbb{Q}}^2\right)^{1/2} \le \gamma \left(\|f\|_{\mathbb{V}^*}^2 + \|g\|_{\mathbb{Q}^*}^2\right)^{1/2}. \tag{2.36}$$

Combining Theorem 2.11 (Brezzi) with Corollary 2.9 (well-posedness implies inf-sup), we infer the inf-sup condition for the bilinear form $\mathcal{B}$ in (2.34).

Corollary 2.12 (inf-sup of $\mathcal{B}$). *Let the bilinear form* $\mathcal{B}\colon \mathbb{W} \to \mathbb{W}$ *be defined by* (2.34). *Then*

$$\inf_{(v,q) \in \mathbb{W}} \sup_{(w,r) \in \mathbb{W}} \frac{\mathcal{B}[(v, q), (w, r)]}{\|(v, q)\|_{\mathbb{W}} \|(w, r)\|_{\mathbb{W}}} = \inf_{(w,r) \in \mathbb{W}} \sup_{(v,q) \in \mathbb{W}} \frac{\mathcal{B}[(v, q), (w, r)]}{\|(v, q)\|_{\mathbb{W}} \|(w, r)\|_{\mathbb{W}}} \ge \gamma^{-1},$$

where γ *is the stability constant from Theorem 2.11.*

Assume that $a\colon \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ is symmetric and let (u, p) be the solution to (2.32). Then u is the unique minimizer of the energy $J[v] := \frac{1}{2} a[v, v] - \langle f,\ v\rangle$ under the

constraint $b[\cdot, u] = g$ in $\mathbb{Q}^*$. In view of this, p is the corresponding Lagrange multiplier and the pair (u, p) is the unique saddle point of the Lagrangian

$$L[v, q] := J[v] + b[q, v] - \langle g, q\rangle \quad \text{for all } v \in \mathbb{V}, q \in \mathbb{Q}.$$

Stokes system. Theorem 2.11 (Brezzi) applies to the Stokes system (2.14) and (2.15) once we verify the inf-sup property (2.35b) for the bilinear form $b[q, \boldsymbol{v}] = -\int_\Omega q \operatorname{div} \boldsymbol{v}$. This turns out to be equivalent to the following problem: for any $q \in L^2_0(\Omega)$ there exists a $\boldsymbol{w} \in H^1_0(\Omega; \mathbb{R}^d)$ such that

$$-\operatorname{div} \boldsymbol{w} = q \quad \text{in } \Omega \quad \text{and} \quad |\boldsymbol{w}|_{H^1(\Omega;\mathbb{R}^d)} \le C(\Omega)\|q\|_{L^2(\Omega)}. \tag{2.37}$$

This non-trivial result goes back to Nečas (Carroll *et al.* 1966) and a proof can be found in Galdi (1994, Theorem III.3.1), for example. This implies

$$\sup_{\boldsymbol{v} \in H^1_0(\Omega;\mathbb{R}^d)} \frac{b[q, \boldsymbol{v}]}{|\boldsymbol{v}|_{H^1(\Omega;\mathbb{R}^d)}} \ge \frac{b[q, \boldsymbol{w}]}{|\boldsymbol{w}|_{H^1(\Omega;\mathbb{R}^d)}} = \frac{\|q\|^2_{L^2(\Omega)}}{|\boldsymbol{w}|_{H^1(\Omega;\mathbb{R}^d)}} \ge C(\Omega)^{-1}\|q\|_{L^2(\Omega)}.$$

Therefore (2.35b) holds with $\beta \ge C(\Omega)^{-1}$.

The inf-sup condition is also satisfied for the spaces $\mathbb{V}$ and $\mathbb{Q}$ defined in (2.18), which are appropriate for the weak formulation (2.17) of the Neumann boundary value problem. Indeed, given any $q \in L^2(\Omega)$, we can split it as $q = (q - \hat{q}) + \hat{q}$ with $\hat{q} := |\Omega|^{-1} \int_\Omega q$. Let $\boldsymbol{w}_0 \in H^1_0(\Omega; \mathbb{R}^d)$ be the function defined as in (2.37) with q replaced by $q - \hat{q}$, and let $\hat{\boldsymbol{w}} = \frac{1}{2}(\hat{q}x, \hat{q}y)$. Then it is easily checked that the function $\boldsymbol{w} = \bar{\boldsymbol{w}} - |\Omega|^{-1} \int_\Omega \bar{\boldsymbol{w}}$ with $\bar{\boldsymbol{w}} = \boldsymbol{w}_0 + \hat{\boldsymbol{w}}$ belongs to $\mathbb{V}$ and satisfies

$$-\operatorname{div} \boldsymbol{w} = q \quad \text{in } \Omega \quad \text{and} \quad \|\boldsymbol{w}\|_{H^1(\Omega;\mathbb{R}^d)} \le C\|q\|_{L^2(\Omega)}.$$

2.5. W^1_p-regularity of reaction–diffusion equation

It will be useful later in Lemma 5.20 to know whether L^∞-coefficients $(\boldsymbol{A}, c)$ allow for enhanced regularity beyond $H^1_0(\Omega)$ for solutions u of (2.7). We can reformulate this question as an extension of the Lax–Milgram theory, which states that the solution operator L^{-1} of (2.5) is an isomorphism between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$; see Corollary 2.10 (Lax–Milgram).

This issue is well understood for the Laplace operator, i.e. $\boldsymbol{A} = \boldsymbol{I}$ and $c = 0$. It is known that for Lipschitz domains $\Omega \subset \mathbb{R}^d$, there exists $p_0 = p_0(\Omega) > 2$ such that

$$\|\nabla u\|_{L^p(\Omega)} \le K\|f\|_{W^{-1}_p(\Omega)} \quad \text{for all } p \in [2, p_0], \tag{2.38}$$

where K depends on p (see e.g. Jerison and Kenig 1995); in particular, $p_0 > 4$ for $d = 2$ and $p_0 > 3$ for $d = 3$. Hereafter, $W^{-1}_p(\Omega)$ denotes the dual space of $\mathring{W}^1_q(\Omega)$, i.e. functions in $W^1_q(\Omega)$ with zero trace and $q = p/(p-1)$. For $\boldsymbol{A} \in L^\infty(\Omega, \mathbb{R}^{d\times d})$ and $c = 0$, estimate (2.38) was first derived by Meyers (1963) as a perturbation result for the Laplacian; see also Brenner and Scott (2008). We now present a

simple proof for $p > 2$ following Bonito, DeVore and Nochetto (2013*b*). Let

$$\theta(p) := \frac{1/2 - 1/p}{1/2 - 1/p_0} \quad \text{for all } p \in [2, p_0], \tag{2.39}$$

and note that $\theta(p)$ increases strictly from 0 at $p = 2$ to 1 at $p = p_0$. Let K_0 be the constant K in (2.38) for $p = p_0$ and, for any $t \in (0, 1)$, define

$$p_*(t) := \max\{p \in [2, p_0] \mid K_0^{\theta(p)}(1 - t) \le 1\}. \tag{2.40}$$

Lemma 2.13 (W_p^1**-regularity).** *Let $\boldsymbol{A} \in L^\infty(\Omega, \mathbb{R}^{d\times d})$ satisfy (2.6) with $\alpha_1 \le \alpha_2$, let $c \in L^\infty(\Omega)$ be non-negative, and let Ω be Lipschitz. If $f \in W_p^{-1}(\Omega)$ for some*

$$p \in \left[2, \min\left\{p_*\left(\frac{\alpha_1}{\alpha_2}\right), \frac{2d}{d-2}\right\}\right),$$

then the solution $u \in H_0^1(\Omega)$ of (2.7) satisfies

$$\|\nabla u\|_{L^p(\Omega)} \le C(p)\big(1 + c_{\mathcal{B}}^{-1} C(\Omega)\|c\|_{L^\infty(\Omega)}\big)\|f\|_{W_p^{-1}(\Omega)} \tag{2.41}$$

with constants

$$C(p) = \frac{1}{\alpha_2}\frac{K_0^{\theta(p)}}{1 - K_0^{\theta(p)}\big(1 - \frac{\alpha_1}{\alpha_2}\big)}$$

and $C(\Omega) = C_s C_P$, where C_s is the constant in Lemma 2.1 (Sobolev embedding) and C_P is the constant in Lemma 2.2 (first Poincaré inequality).

Proof. We first consider the principal part of the operator L in (2.5), namely we take $c = 0$. In fact, let the operator $S\colon \mathring{W}_p^1(\Omega) \to W_p^{-1}(\Omega)$ be defined by $Sv := -\operatorname{div}(\alpha_2{}^{-1}\boldsymbol{A}\nabla v)$. In order to prove (2.41) for S, we resort to a perturbation argument for the Laplace operator $Tv := -\Delta v$.

The first task is to bound K in (2.38) in terms of K_0 and p. To this end, we recall that the operator $T\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism with norm $\|T^{-1}\|_2 = 1$. Moreover, $T\colon \mathring{W}_p^1(\Omega) \to W_p^{-1}(\Omega)$ is also an isomorphism with norm $\|T^{-1}\|_{p_0} = K_0$ according to (2.38) for $p = p_0$, provided we adopt the norm $\|\nabla \cdot\|_{L^p(\Omega)}$ in $\mathring{W}_p^1(\Omega)$. By the real method of interpolation, we know that $\mathring{W}_p^1(\Omega) = \big[H_0^1(\Omega), \mathring{W}_{p_0}^1(\Omega)\big]_{\theta(p),p}$ is the interpolation space between $H_0^1(\Omega)$ and $\mathring{W}_{p_0}^1(\Omega)$ with parameter $\theta(p)$ given by (2.39). Hence operator interpolation theory implies that $T\colon \mathring{W}_p^1(\Omega) \to W_p^{-1}(\Omega)$ is an isomorphism with

$$K = \|T^{-1}\|_p = K_0^{\theta(p)}.$$

We regard S as a perturbation of T, define the operator $Q := T - S\colon \mathring{W}_p^1(\Omega) \to W_p^{-1}(\Omega)$, and observe that $\|Q\|_p \le 1 - \alpha_1/\alpha_2$ because

$$\langle Qv, w\rangle = \int_\Omega \left(I - \frac{1}{\alpha_2}\boldsymbol{A}\right)\nabla v \nabla w \le \left(1 - \frac{\alpha_1}{\alpha_2}\right)\|\nabla v\|_{L^p(\Omega)}\|\nabla w\|_{L^q(\Omega)}, \quad w \in \mathring{W}_q^1(\Omega).$$

Therefore the operator $T^{-1}Q\colon \mathring{W}^1_p(\Omega) \to \mathring{W}^1_p(\Omega)$ satisfies

$$\|T^{-1}Q\|_p \le \|T^{-1}\|_p \|Q\|_p \le K_0^{\theta(p)}\left(1 - \frac{\alpha_1}{\alpha_2}\right)$$

as well as $\|T^{-1}Q\|_p < 1$ for any $p \in \big[2, p_*(\alpha_1/\alpha_2)\big)$ in view of definition (2.40) of $p_*(t)$. We conclude by the Neumann theorem that the operator $S = T(I - T^{-1}Q)\colon \mathring{W}^1_p(\Omega) \to W^{-1}_p(\Omega)$ is invertible and its norm is bounded by

$$\|S^{-1}\|_p \le \|T^{-1}\|_p \|I - T^{-1}Q\|_p \le \frac{\|T^{-1}\|_p}{1 - \|T^{-1}Q\|_p} \le \frac{K_0^{\theta(p)}}{1 - K_0^{\theta(p)}(1 - \alpha_1/\alpha_2)}.$$

This yields the asserted estimate for $S = -\operatorname{div}({\alpha_2}^{-1}\boldsymbol{A}\nabla v)$.

Finally, we consider the operator L in (2.5) with $c \neq 0$. If $u \in H^1_0(\Omega)$ is the solution of (2.7) given by Corollary 2.10 (Lax–Milgram), rewrite (2.7) as

$$Su = -\operatorname{div}\left(\frac{1}{\alpha_2}\boldsymbol{A}\nabla u\right) = \frac{1}{\alpha_2}(f - cu) = \frac{1}{\alpha_2}g,$$

and apply the preceding estimate for S to infer that

$$\|S^{-1}g\|_{\mathring{W}^1_p(\Omega)} = \|\nabla u\|_{L^p(\Omega)} \le C(p)\|g\|_{W^{-1}_p(\Omega)} \le C(p)\big(\|f\|_{W^{-1}_p(\Omega)} + \|cu\|_{W^{-1}_p(\Omega)}\big).$$

It remains to estimate the last term on the right-hand side. Using the Cauchy–Schwarz inequality in conjunction with Lemma 2.2 (first Poincaré inequality), i.e. $\|u\|_{L^2(\Omega)} \le C_P\|\nabla u\|_{L^2(\Omega)}$, and Lemma 2.1 (Sobolev embedding), i.e. $\|w\|_{L^2(\Omega)} \le C\|w\|_{\mathring{W}^1_q(\Omega)} \le C_s\|\nabla w\|_{L^q(\Omega)}$ for $q = p/(p-1) \ge 2d/(d+2)$, we see that

$$\begin{aligned}\langle cu, w\rangle &\le C_P C_s \|c\|_{L^\infty(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|\nabla w\|_{L^q(\Omega)}\\ &\le \frac{C_P C_s}{C_B}\|c\|_{L^\infty(\Omega)}\|f\|_{H^{-1}(\Omega)}\|\nabla w\|_{L^q(\Omega)}\end{aligned}$$

because of the stability estimate (2.27) with constant $\alpha = c_{\mathcal{B}}$. Since $\|f\|_{H^{-1}(\Omega)} \le |\Omega|^{(p-2)/(2p)}\|f\|_{W^{-1}_p(\Omega)}$, the asserted estimate (2.41) for $c \neq 0$ follows immediately. □

3. *A priori* approximation theory

We devote this section to discussing basic concepts about piecewise polynomial approximation in Sobolev spaces over graded meshes in any dimension d. We start by introducing the Petrov–Galerkin method in an abstract setting (Section 3.1); this motivates our interest in approximation results in Sobolev spaces. Hence we briefly discuss the construction of finite element spaces in Section 3.2, along with polynomial interpolation of functions in Sobolev spaces in Section 3.3. This provides local estimates suitable for the comparison of quasi-uniform and graded meshes for $d > 1$. We exploit them in developing the so-called error equidistribution

principle in Section 3.4 and the construction of suitably graded meshes via a greedy algorithm in Section 3.6. We conclude that graded meshes can deliver optimal interpolation rates for certain classes of singular functions, and thus supersede quasi-uniform refinement.

In the second part of the section, we explore the geometric aspects of mesh refinement for conforming meshes in Section 3.5 and non-conforming meshes in Section 3.7, but postpone a full and rather technical discussion to Section 8. We include a statement about complexity of the refinement procedure, which turns out to be instrumental later and will be proved in Section 8.

3.1. The Galerkin method: best approximation

In order to make the variational problem (2.23) computable, we let $\mathbb{V}_N \subset \mathbb{V}$ and $\mathbb{W}_N \subset \mathbb{W}$ be subspaces with the same dimension $N < \infty$ and consider the Petrov–Galerkin approximation

$$u_N \in \mathbb{V}_N : \quad \mathcal{B}[u_N, w] = \langle f, w \rangle \quad \text{for all } w \in \mathbb{W}_N . \tag{3.1}$$

If $\mathbb{V}_N = \mathbb{W}_N$, this is called Galerkin approximation. Since (3.1) is a square algebraic system, the existence and uniqueness of $u_N \in \mathbb{V}_N$ are equivalent to the kernel of the corresponding linear discrete operator being trivial. This leads to the following equivalent conditions for unique solvability (Nečas 1962, Babuška 1971); see also Nochetto *et al.* (2009, Proposition 1).

Lemma 3.1 (discrete inf-sup condition). *The following statements are equivalent.*

(1) *For every* $0 \neq v \in \mathbb{V}_N$ *there exists* $w \in \mathbb{W}_N$ *such that* $\mathcal{B}[v, w] \neq 0$.

(2) *For every* $0 \neq w \in \mathbb{W}_N$ *there exists* $v \in \mathbb{V}_N$ *such that* $\mathcal{B}[v, w] \neq 0$.

(3) *The following discrete inf-sup condition holds with a constant* $\beta_N > 0$*:*

$$\inf_{v \in \mathbb{V}_N} \sup_{w \in \mathbb{W}_N} \frac{\mathcal{B}[v, w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}}} = \inf_{w \in \mathbb{W}_N} \sup_{v \in \mathbb{V}_N} \frac{\mathcal{B}[v, w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}}} = \beta_N . \tag{3.2}$$

(4) $\displaystyle \inf_{v \in \mathbb{V}_N} \sup_{w \in \mathbb{W}_N} \frac{\mathcal{B}[v, w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}}} > 0.$

(5) $\displaystyle \inf_{w \in \mathbb{W}_N} \sup_{v \in \mathbb{V}_N} \frac{\mathcal{B}[v, w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}}} > 0.$

This is a discrete version of Theorem 2.8 (Nečas) and leads to the stability bound

$$\|u_N\|_{\mathbb{V}} \leq \frac{1}{\beta_N} \|f\|_{\mathbb{W}^*} . \tag{3.3}$$

Therefore β_N^{-1} acts as a stability constant for (3.1), and it is thus desirable for it to be uniformly bounded below away from zero. This is always the case for *coercive*

problems because (2.28) is inherited within the subspaces $\mathbb{V}_N = \mathbb{W}_N \subset \mathbb{V}$, whence $\beta_N \geq \alpha > 0$. In contrast, a uniform lower bound for *saddle point* problems,

$$\beta_N \geq \beta > 0, \tag{3.4}$$

requires compatibility between the subspaces $\mathbb{V}_N$ and $\mathbb{W}_N$ (Boffi, Brezzi and Fortin 2013).

If we now subtract (3.1) from (2.23), we obtain *Galerkin orthogonality*:

$$\mathcal{B}[u - u_N, w] = 0 \quad \text{for all } w \in \mathbb{W}_N. \tag{3.5}$$

This relation is instrumental in deriving the following best approximation property as well as developing *a posteriori* error estimates in Section 4.

Proposition 3.2 (quasi-best-approximation property). *Let $\mathcal{B}\colon \mathbb{V} \times \mathbb{W} \to \mathbb{R}$ be continuous and satisfy (3.2). Then the error $u - u_N$ satisfies the bound*

$$\|u - u_N\|_{\mathbb{V}} \leq \frac{\|\mathcal{B}\|}{\beta_N} \min_{v \in \mathbb{V}_N} \|u - v\|_{\mathbb{V}}. \tag{3.6}$$

Proof. We give a simplified proof, which follows Babuška (1971) and Babuška and Aziz (1972), and yields the constant $1 + \|\mathcal{B}\|/\beta_N$. The asserted constant is due to Xu and Zikatanov (2003).

Combining (3.2), (3.5) and the continuity of $\mathcal{B}$, we derive for all $v \in \mathbb{V}_N$

$$\beta_N \|u_N - v\|_{\mathbb{V}} \leq \sup_{w \in \mathbb{W}_N} \frac{\mathcal{B}[u_N - v, w]}{\|w\|_{\mathbb{W}}} = \sup_{W \in \mathbb{W}_N} \frac{\mathcal{B}[u - v, w]}{\|w\|_{\mathbb{W}}} \leq \|\mathcal{B}\| \|u - v\|_{\mathbb{V}},$$

whence

$$\|u_N - v\|_{\mathbb{V}} \leq \frac{\|\mathcal{B}\|}{\beta_N} \|u - v\|_{\mathbb{V}}.$$

Using the triangle inequality yields

$$\|u - u_N\|_{\mathbb{V}} \leq \|u - v\|_{\mathbb{V}} + \|v - u_N\|_{\mathbb{V}} \leq \left(1 + \frac{\|\mathcal{B}\|}{\beta_N}\right) \|u - v\|_{\mathbb{V}}$$

for all $v \in \mathbb{V}_N$. It just remains to minimize in $\mathbb{V}_N$. □

Corollary 3.3 (quasi-monotonicity). *Let $\mathcal{B}\colon \mathbb{V} \times \mathbb{W} \to \mathbb{R}$ be continuous and satisfy (3.2). If $\mathbb{V}_M$ is a subspace of $\mathbb{V}_N$, then for all $v \in \mathbb{V}_M$*

$$\|u - u_N\|_{\mathbb{V}} \leq \frac{\|\mathcal{B}\|}{\beta_N} \|u - v\|_{\mathbb{V}}. \tag{3.7}$$

Moreover, if $\mathbb{V} = \mathbb{W}$ and $\mathcal{B}$ is symmetric and coercive with constants $c_{\mathcal{B}} \leq C_{\mathcal{B}}$, then for all $v \in \mathbb{V}_M$

$$|\!|\!| u - u_N |\!|\!|_{\Omega} \leq |\!|\!| u - v |\!|\!|_{\Omega}, \quad \|u - u_N\|_{\mathbb{V}} \leq C_{\text{Céa}} \|u - v\|_{\mathbb{V}}, \tag{3.8}$$

where $C_{\text{Céa}} := \sqrt{C_{\mathcal{B}}/c_{\mathcal{B}}}$.

Proof. Inequality (3.7) is a consequence of the previous bound (3.6) upon taking $v \in \mathbb{V}_M$ instead of $\mathbb{V}_N$. To show the left inequality in (3.8), we combine (2.28) and (3.5):

$$|||u - u_N|||_\Omega^2 = \mathcal{B}[u - u_N, u - v] \le |||u - u_N|||_\Omega |||u - v|||_\Omega \quad \text{for all } v \in \mathbb{V}_M.$$

This together with the norm equivalence (2.30) gives the remaining inequality. □

The significance of (3.6) is that we need to construct discrete spaces $\mathbb{V}_N$ with good approximation properties. Next we introduce piecewise polynomial approximation, which gives rise to the finite element method.

3.2. Finite element spaces

In this section we focus on the construction of the discrete spaces $\mathbb{V}_N$ and $\mathbb{W}_N$. We consider the bilinear forms $\mathcal{B}$ introduced in Section 2.3 with emphasis on the diffusion–reaction case (2.8). We assume that Ω is a bounded polyhedral domain $\Omega \subset \mathbb{R}^d$ and is partitioned into a conforming or non-conforming mesh $\mathcal{T}$ made of simplices T, which are assumed to be closed with non-overlapping interiors; thus

$$\overline{\Omega} = \bigcup \{T \mid T \in \mathcal{T}\}.$$

The reference element is denoted by

$$T_d := \left\{ x = (x_1, \dots, x_d) \in \mathbb{R}^d \mid 0 \le x_i \le 1,\ i = 1, \dots, d,\ \sum_{i=1}^d x_i \le 1 \right\}.$$

We will discuss the construction of conforming meshes in Section 3.5 by the bisection method and that of non-conforming meshes (constrained to the fulfilment of an *admissibility condition*) in Section 3.7, both for $d = 2$. We will embark on a thorough discussion in Section 8. We assume for the moment that $\mathcal{T}$ is an element of a (possibly infinite) class $\mathbb{T}$ of conforming *shape-regular* meshes. To define this geometric concept, we let $\overline{h}_T$ denote the diameter of $T \in \mathcal{T}$, let $\underline{h}_T$ denote the diameter of the largest ball contained in T, and impose the restriction

$$\sigma := \sup_{\mathcal{T} \in \mathbb{T}} \sup_{T \in \mathcal{T}} \frac{\overline{h}_T}{\underline{h}_T} < \infty. \tag{3.9}$$

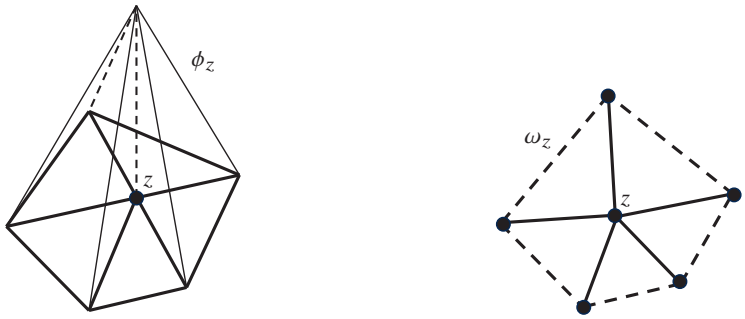
The constant σ is referred to as the shape regularity constant of $\mathbb{T}$.

Given a shape-regular mesh $\mathcal{T} \in \mathbb{T}$, we define the finite element space of *discontinuous* piecewise polynomials of total degree up to $n \ge 1$,

$$\mathbb{S}_{\mathcal{T}}^{n,-1} := \{v \in L^2(\Omega) \mid v|_T \in \mathbb{P}_n(T) \text{ for all } T \in \mathcal{T}\},$$

and its globally *continuous* counterpart

$$\mathbb{S}_{\mathcal{T}}^{n,0} := \mathbb{S}_{\mathcal{T}}^{n,-1} \cap C^0(\overline{\Omega}).$$



(a) (b)

Figure 3.1. (a) Piecewise linear basis function ϕ_z corresponding to interior node z. (b) Support ω_z of ϕ_z and skeleton γ_z (solid line)

Note that $\mathbb{S}_{\mathcal{T}}^{n,0} \subset H^1(\Omega)$, which makes it adequate for (2.7)–(2.8). We refer to Braess (2007), Brenner and Scott (2008), Ciarlet (2002) and Siebert (2012) for a discussion of the local construction of this space (i.e. Lagrange elements of degree $n \geq 1$) along with its properties. We let

$$\mathbb{V}_{\mathcal{T}} := \mathbb{S}_{\mathcal{T}}^{n,0} \cap H_0^1(\Omega) \tag{3.10}$$

denote the subspace of finite element functions which vanish on $\partial\Omega$. Note that we do not explicitly refer to the polynomial degree, which will be clear in each context.

We focus on the conforming piecewise linear case $n = 1$ (Courant elements), but most results extend to non-conforming meshes or $n > 1$. In this vein, global continuity can be simply enforced by imposing continuity at the set $\mathcal{V}$ of vertices z of $\mathcal{T}$, the so-called *nodal values*. However, the following local construction leads to global continuity. If T is a generic simplex of $\mathcal{T}$, namely the convex hull of $\{z_i\}_{i=0}^d$, then we associate to each vertex z_i a *barycentric coordinate* λ_i^T, which is the linear function in T with nodal value 1 at z_i and 0 at the other vertices of T. Upon pasting together the barycentric coordinates λ_z^T of all simplices T containing the vertex $z \in \mathcal{V}$, we obtain a continuous piecewise linear function $\phi_z \in \mathbb{S}_{\mathcal{T}}^{1,0}$, as depicted in Figure 3.1 for $d = 2$.

The set $\{\phi_z\}_{z\in\mathcal{V}}$ of all such functions is the nodal basis of $\mathbb{S}_{\mathcal{T}}^{1,0}$, or Courant basis. We let $\omega_z := \operatorname{supp}(\phi_z)$ denote the support of ϕ_z, from now on called the *star* associated to z, and let γ_z be the interior skeleton of ω_z, namely the union of all the faces containing z.

In view of the definition of ϕ_z, we have the following unique representation of any function $v \in \mathbb{S}^{1,0}(\mathcal{T})$:

$$v(x) = \sum_{z\in\mathcal{V}} v(z)\phi_z(x).$$

The functions ϕ_z are non-negative and satisfy the *partition of unity* property

$$\sum_{z\in\mathcal{V}} \phi_z(x) = 1 \quad \text{for all } x \in \Omega. \tag{3.11}$$

If we further impose $v(z) = 0$ for all $z \in \partial\Omega \cap \mathcal{V}$, then $v \in H^1_0(\Omega)$.

For each simplex $T \in \mathcal{T}$, generated by vertices $\{z_i\}_{i=0}^d$, the *dual functions* $\{\lambda_i^*\}_{i=0}^d \subset \mathbb{P}_1(T)$ to the barycentric coordinates $\{\lambda_i\}_{i=0}^d$ satisfy the bi-orthogonality relation $\int_T \lambda_i^*\lambda_j = \delta_{ij}$, and are given by

$$\lambda_i^* = \frac{(1+d)^2}{|T|}\lambda_i - \frac{1+d}{|T|}\sum_{j\neq i}\lambda_j \quad \text{for all } \ 0 \le i \le d.$$

The *Courant dual basis* $\phi_z^* \in \mathbb{S}^{1,-1}(\mathcal{T})$ is formed by the functions over $\mathcal{T}$ given by

$$\phi_z^* = \frac{1}{V_z}\sum_{T\ni z}(\lambda_z^T)^*\chi_T \quad \text{for all } \ z \in \mathcal{V},$$

where $V_z \in \mathbb{N}$ is the valence of z (number of elements of $\mathcal{T}$ containing z) and χ_T is the characteristic function of T. These functions have the same support ω_z as the nodal basis ϕ_z and satisfy the global bi-orthogonality relation

$$\int_\Omega \phi_z^*\phi_y = \delta_{zy} \quad \text{for all } \ z, y \in \mathcal{V}.$$

Finally, we let $\mathcal{N}$ denote the Lagrange nodes of order n of a mesh $\mathcal{T}$, and let $\psi_z \in \mathbb{S}^{n,0}_{\mathcal{T}}$ be the corresponding Lagrange basis of $\mathbb{S}^{n,0}_{\mathcal{T}}$; hence $\mathbb{S}^{n,0}_{\mathcal{T}} = \operatorname{span}\{\psi_z\}_{z\in\mathcal{N}}$.

3.3. *Polynomial interpolation in Sobolev spaces*

We wish to use the space $\mathbb{V}_{\mathcal{T}}$ defined in (3.10) as the discrete space $\mathbb{V}_N$ in the Galerkin method (3.1). If the bilinear form $\mathcal{B}$ satisfies an inf-sup condition with constant $\beta_{\mathcal{T}} > 0$, we find a discrete solution $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$ which satisfies the error bound (3.6), that is,

$$\|u - u_{\mathcal{T}}\|_{\mathbb{V}} \le \frac{\|\mathcal{B}\|}{\beta_{\mathcal{T}}} \min_{v\in\mathbb{V}_{\mathcal{T}}} \|u - v\|_{\mathbb{V}}.$$

In turn, the minimum on the right-hand side can be bounded from above by the quantity $\|u - v\|_{\mathbb{V}}$ for any chosen $v \in \mathbb{V}_{\mathcal{T}}$. This motivates the search for quasi-best approximations of u in the norm of $\mathbb{V}$. One classical tool to generate approximations to any given function is *interpolation*. Interpolation in $\mathbb{V}_{\mathcal{T}}$ is discussed next.

If $v \in C^0(\overline{\Omega})$, we define the *Lagrange interpolant* $I_{\mathcal{T}}v$ of v as follows,

$$I_{\mathcal{T}}v(x) = \sum_{z\in\mathcal{N}} v(z)\psi_z(x),$$

and note that $I_{\mathcal{T}} v = v$ for all $v \in \mathbb{S}^{n,0}_{\mathcal{T}}$ (i.e. $I_{\mathcal{T}}$ is invariant in $\mathbb{S}^{n,0}_{\mathcal{T}}$). For functions without point values, such as functions in $H^1(\Omega)$ for $d > 1$, we need to determine nodal values by averaging. For any conforming mesh $\mathcal{T} \in \mathbb{T}$, the averaging process extends beyond nodes and so gives rise to the discrete neighbourhood

$$\omega_{\mathcal{T}}(T) := \bigcup_{\substack{T' \in \mathcal{T} \\ T' \cap T \neq \emptyset}} T'$$

for each element $T \in \mathcal{T}$, along with the *local quasi-uniformity* properties

$$\max_{T \in \mathcal{T}} \#\omega_{\mathcal{T}}(T) \leq C(\sigma), \qquad \max_{T' \subset \omega_{\mathcal{T}}(T)} \frac{|T|}{|T'|} \leq C(\sigma),$$

where σ is the shape regularity coefficient defined in (3.9). We will often write ω_T if there is no confusion about the underlying mesh $\mathcal{T}$. We shall also need a smaller subset, namely the set of elements sharing a face with a given element T:

$$\widetilde{\omega}_T := \widetilde{\omega}_{\mathcal{T}}(T) := \bigcup_{\substack{T' \in \mathcal{T} \\ T' \cap T \in \mathcal{F}}} T', \tag{3.12}$$

where $\mathcal{F}$ is the set of all $(d-1)$-dimensional faces of the mesh $\mathcal{T}$.

We now introduce one such operator $I_{\mathcal{T}}$ due to Scott and Zhang (Brenner and Scott 2008, Scott and Zhang 1990), from now on called a *quasi-interpolation operator*. We focus on polynomial degree $n = 1$, but the construction is valid for any n; see Brenner and Scott (2008) and Scott and Zhang (1990) for details. We recall that $\{\phi_z\}_{z \in \mathcal{V}}$ is the global Lagrange basis of $\mathbb{S}^{1,0}_{\mathcal{T}}$, $\{\phi^*_z\}_{z \in \mathcal{V}}$ is the global dual basis, and $\operatorname{supp} \phi^*_z = \operatorname{supp} \phi_z$ for all $z \in \mathcal{V}$. We thus define $I_{\mathcal{T}} \colon L^1(\Omega) \to \mathbb{S}^{1,0}_{\mathcal{T}}$ to be

$$I_{\mathcal{T}} v := \sum_{z \in \mathcal{V}} \langle v, \phi^*_z \rangle \phi_z, \tag{3.13}$$

If $0 \leq s \leq 2$ (integer) is a regularity index and $1 \leq p \leq \infty$ is an integrability index, then we would like to prove the *quasi-local error estimate*

$$|v - I_{\mathcal{T}} v|_{W^t_q(T)} \lesssim h_T^{\operatorname{sob}(W^s_p) - \operatorname{sob}(W^t_q)} |v|_{W^s_p(\omega_T)} \tag{3.14}$$

for all $T \in \mathcal{T}$, provided $0 \leq t \leq s$, $1 \leq q \leq \infty$ are such that $\operatorname{sob}(W^s_p) > \operatorname{sob}(W^t_q)$.

We first recall that $I_{\mathcal{T}}$ is invariant in $\mathbb{S}^{1,0}_{\mathcal{T}}$, namely,

$$I_{\mathcal{T}} w = w \quad \text{for all } w \in \mathbb{S}^{1,0}_{\mathcal{T}}.$$

Since the averaging process giving rise to the values of $I_{\mathcal{T}} v$ for each element $T \in \mathcal{T}$ takes place in the neighbourhood ω_T, we also deduce the local invariance

$$I_{\mathcal{T}} w|_T = w \quad \text{for all } w \in \mathbb{P}_1(\omega_T),$$

as well as the local stability estimate for any $1 \leq q \leq \infty$,

$$\|I_{\mathcal{T}} v\|_{L^q(T)} \lesssim \|v\|_{L^q(\omega_T)}.$$

We may thus write

$$v - I_{\mathcal{T}} v|_T = (v - w) - I_{\mathcal{T}}(v - w)|_T \quad \text{for all } T \in \mathcal{T},$$

where $w \in \mathbb{P}_{s-1}$ is arbitrary ($w = 0$ if $s = 0$). It now suffices to prove (3.14) in the reference element $\widehat{T}$ and scale back and forth to T; the definition (2.1) of Sobolev number accounts precisely for this scaling. We keep the notation T for $\widehat{T}$, apply the inverse estimate for linear polynomials $|I_{\mathcal{T}} v|_{W^t_q(T)} \lesssim \|I_{\mathcal{T}} v\|_{L^q(T)}$ to $v - w$ instead of v, and use the above local stability estimate, to infer that

$$|v - I_{\mathcal{T}} v|_{W^t_q(T)} \lesssim \|v - w\|_{W^t_q(\omega_T)} \lesssim \|v - w\|_{W^s_p(\omega_T)}.$$

The last inequality is a consequence of the inclusion $W^s_p(\omega_T) \subset W^t_q(\omega_T)$ because $\mathrm{sob}(W^s_p) > \mathrm{sob}(W^t_q)$ and $t \le s$. Estimate (3.14) now follows from the Bramble–Hilbert lemma (see Brenner and Scott (2008, Lemma 4.3.8), Ciarlet (2002, Theorem 3.1.1), Dupont and Scott (1980)) or Proposition 6.34 below:

$$\inf_{w \in \mathbb{P}_{s-1}(\omega_T)} \|v - w\|_{W^s_p(\omega_T)} \lesssim |v|_{W^s_p(\omega_T)}. \tag{3.15}$$

This proves (3.14) for $n = 1$. The construction of $I_{\mathcal{T}}$ and ensuing estimate (3.14) is still valid for any $n > 1$ (Brenner and Scott 2008, Scott and Zhang 1990).

Proposition 3.4 (quasi-interpolant without boundary values). *Let s, t be regularity indices with $0 \le t \le s \le n+1$, and let $1 \le p, q \le \infty$ be integrability indices so that $\mathrm{sob}(W^s_p) > \mathrm{sob}(W^t_q)$. Then there exists a quasi-interpolation operator $I_{\mathcal{T}} \colon L^1(\Omega) \to \mathbb{S}^{n,0}_{\mathcal{T}}$, which is invariant in $\mathbb{S}^{n,0}_{\mathcal{T}}$ and satisfies*

$$|v - I_{\mathcal{T}} v|_{W^t_q(T)} \lesssim h_T^{\mathrm{sob}(W^s_p) - \mathrm{sob}(W^t_q)} |v|_{W^s_p(\omega_T)} \quad \textit{for all } T \in \mathcal{T}. \tag{3.16}$$

The hidden constant in (3.14) depends on the shape coefficient of $\mathcal{T}_0$ and d.

To impose a vanishing trace on $I_{\mathcal{T}} v$ we may suitably modify the averaging process for boundary nodes. We thus define a set of dual functions with respect to an L^2-scalar product over $(d-1)$-subsimplices contained on $\partial\Omega$; see again Brenner and Scott (2008) and Scott and Zhang (1990) for details. This retains the invariance property of $I_{\mathcal{T}}$ on $\mathbb{S}^{n,0}(\mathcal{T})$ and guarantees that $I_{\mathcal{T}} v$ has a zero trace if $v \in W^1_1(\Omega)$ does. Hence the above argument applies, and (3.16) follows provided $s \ge 1$.

Proposition 3.5 (quasi-interpolant with boundary values). *Let s, t, p, q be as in Proposition 3.4. There exists a quasi-interpolation operator $I_{\mathcal{T}} \colon W^1_1(\Omega) \to \mathbb{S}^{n,0}_{\mathcal{T}}$ invariant in $\mathbb{S}^{n,0}_{\mathcal{T}}$ which satisfies (3.16) for $s \ge 1$ and preserves the boundary values of v provided they are piecewise polynomial of degree $\le n$. In particular, if $v \in W^1_1(\Omega)$ has a vanishing trace on $\partial\Omega$, then so does $I_{\mathcal{T}} v$.*

Remark 3.6 (fractional regularity). We observe that (3.16) does not require the regularity indices t and s to be integer. The proof follows along the same lines but replaces the polynomial degree n with the greatest integer smaller than s; the generalization of (3.15) can be taken from Dupont and Scott (1980).

Remark 3.7 (local error estimate for Lagrange interpolant). Let the regularity index s and integrability index $1 \le p \le \infty$ satisfy $s - d/p > 0$. This implies that $\mathrm{sob}(W^s_p) > \mathrm{sob}(L^\infty)$, whence $W^s_p(\Omega) \subset C(\overline{\Omega})$ and the Lagrange interpolation operator $I_{\mathcal{T}} \colon W^s_p(\Omega) \to \mathbb{S}^{n,0}_{\mathcal{T}}$ is well-defined and satisfies the *local error estimate*

$$|v - I_{\mathcal{T}} v|_{W^t_q(T)} \lesssim h_T^{\mathrm{sob}(W^s_p) - \mathrm{sob}(W^t_q)} |v|_{W^s_p(T)}, \tag{3.17}$$

provided $0 \le t \le s$, $1 \le q \le \infty$ are such that $\mathrm{sob}(W^s_p) > \mathrm{sob}(W^t_q)$. We point out that ω_T in (3.14) is now replaced by T in (3.17). We also remark that if v vanishes on $\partial\Omega$ then so does $I_{\mathcal{T}} v$. The proof of (3.17) proceeds along the same lines as that of Proposition 3.4, except that the nodal evaluation does not extend beyond the element $T \in \mathcal{T}$, and the inverse and stability estimates over the reference element are replaced by

$$|I_{\mathcal{T}} v|_{W^t_q(\widehat{T})} \lesssim \|I_{\mathcal{T}} v\|_{L^q(\widehat{T})} \lesssim \|v\|_{L^\infty(\widehat{T})} \lesssim \|v\|_{W^s_p(\widehat{T})}.$$

The following global interpolation error estimate builds on Proposition 3.4 and relates to Figure 2.1 (DeVore diagram).

Theorem 3.8 (global interpolation error estimate). *Let $1 \le s \le n+1$, $t = 0, 1$, $t < s$ and $1 \le p \le q$ satisfy $r := \mathrm{sob}(W^s_p) - \mathrm{sob}(W^t_q) > 0$. If $v \in W^s_p(\Omega)$, then*

$$|v - I_{\mathcal{T}} v|_{W^t_q(\Omega)} \lesssim \left(\sum_{T \in \mathcal{T}} h_T^{rp} |v|^p_{W^s_p(\omega_T)} \right)^{1/p}. \tag{3.18}$$

Proof. Use Proposition 3.4 along with the elementary property of series $\sum_n a_n \le (\sum_n a_n^{p/q})^{q/p}$ for $0 < p/q \le 1$. □

Continuous vs. discontinuous approximation of gradients. The preceding discussion might induce us to believe that when dealing with Sobolev functions $v \in W^1_p(\Omega)$ without point values, namely $1 \le p \le d$, global continuity of the quasi-interpolant $I_{\mathcal{T}} v$ might degrade the approximation quality relative to discontinuous approximations. The following instrumental result shows that this is not the case (Veeser 2016, Theorem 2). It hinges on a new geometric concept: we say that a star ω_z is $(d-1)$*-face-connected* if, for any element $T \subset \omega_z$ and $(d-1)$-face $F \subset \omega_z$ containing z, there exists a sequence $(T_i)_{i=0}^m$ such that

- any T_i is an element of ω_z for $0 \le i \le m$,
- any intersection $T_i \cap T_{i+1}$ is a $(d-1)$-face of ω_z for $0 \le i \le m-1$,
- T_0 contains F and $T_m = T$.

Note that a star ω_z is $(d-1)$-face-connected if the set $\omega_z \cap \Omega$ is connected.

Proposition 3.9 (approximation of gradients). *Let $v \in W^1_p(\Omega)$ for $1 \le p \le d$. Let $\mathcal{T}$ be a conforming mesh such that its stars are $(d-1)$-face-connected. Then there exists a constant $C(\sigma)$ depending on the shape regularity coefficient σ of*

(3.9), *the dimension d and the polynomial degree $n \geq 1$, such that*

$$1 \leq \frac{\min_{w \in \mathbb{S}_{\mathcal{T}}^{n,0}} \|\nabla(v-w)\|_{L^p(\Omega)}}{\min_{w \in \mathbb{S}_{\mathcal{T}}^{n,-1}} \|\nabla(v-w)\|_{L^p(\Omega;\mathcal{T})}} \leq C(\sigma), \tag{3.19}$$

where $\|\nabla w\|_{L^p(\Omega;\mathcal{T})}$ stands for the broken norm over $\mathcal{T}$.

The left inequality in (3.19) is obvious because $\mathbb{S}_{\mathcal{T}}^{n,0} \subset \mathbb{S}_{\mathcal{T}}^{n,-1}$. In contrast, the right inequality is delicate and relies on examining the quasi-interpolant (3.13) (Veeser 2016, Theorems 1 and 22). An important consequence of (3.19) is the following localized version of (3.18).

Proposition 3.10 (localized quasi-interpolation estimate). *Let $1 < s \leq n+1$, $1 \leq p \leq d$ and $r = \mathrm{sob}(W_p^s) - \mathrm{sob}(W_q^1) > 0$. If $v \in W_q^1(\Omega)$, then*

$$\|\nabla(v - I_{\mathcal{T}} v)\|_{L^q(\Omega)} \lesssim \left(\sum_{T \in \mathcal{T}} h_T^{pr} |v|_{W_p^s(T)}^p \right)^{1/p}, \tag{3.20}$$

Proof. Since $v - I_{\mathcal{T}} v = (v-w) - I_{\mathcal{T}}(v-w)$ for any $w \in \mathbb{S}_{\mathcal{T}}^{n,0}$, combine Proposition 3.9 with Proposition 6.34 (Bramble–Hilbert for Sobolev spaces) to write

$$\|\nabla(v - I_{\mathcal{T}} v)\|_{L^q(\Omega)} \lesssim \min_{w \in \mathbb{S}_{\mathcal{T}}^{n,-1}} \|\nabla(v-w)\|_{L^q(\Omega)} \lesssim \left(\sum_{T \in \mathcal{T}} h_T^{pr} |v|_{W_p^s(T)}^p \right)^{1/p}.$$

This concludes the proof. □

The crucial difference between (3.20) and (3.18) is that the function $v \in W_q^1(\Omega)$ does not have to belong to $W_p^s(\Omega)$ globally but rather locally, namely $v \in W_p^s(T)$ for every $T \in \mathcal{T}$, to get optimal *a priori* error estimates. This property will find several applications later. A special case of (3.20) for $p = q = 2$ and $s = 2$ reads

$$|v - I_{\mathcal{T}} v|_{H^1(\Omega)}^2 \lesssim \sum_{T \in \mathcal{T}} h_T^2 |v|_{H^2(T)}^2,$$

for $v \in H^2(\Omega;\mathcal{T}) := \{w \in H^1(\Omega) \mid w_{|T} \in H^2(T) \text{ for all } T \in \mathcal{T}\}$.

Quasi-uniform meshes. We now apply Theorem 3.8 to *quasi-uniform* meshes, namely meshes $\mathcal{T} \in \mathbb{T}$ for which all its elements are of comparable size h, whence

$$h \approx (\#\mathcal{T})^{-1/d} |\Omega|^{1/d} \approx (\#\mathcal{T})^{-1/d}.$$

Corollary 3.11 (quasi-uniform meshes). *Let $1 < s \leq n+1$ and $v \in H^s(\Omega)$. If $\mathcal{T} \in \mathbb{T}$ is quasi-uniform, then*

$$\|\nabla(v - I_{\mathcal{T}} v)\|_{L^2(\Omega)} \lesssim |v|_{H^s(\Omega)} (\#\mathcal{T})^{-(s-1)/d}. \tag{3.21}$$

Remark 3.12 (optimal rate). If $s = n+1$, and so v has the maximal regularity $v \in H^{n+1}(\Omega)$, then we obtain the optimal convergence rate in a linear Sobolev scale

$$\|\nabla(v - I_{\mathcal{T}} v)\|_{L^2(\Omega)} \lesssim |v|_{H^{n+1}(\Omega)} (\#\mathcal{T})^{-n/d}. \tag{3.22}$$

Table 3.1. Rate of convergence s in term of uniform mesh size h. We observe an asymptotic error decay of about $h^{2/3}$ (i.e. $s = 2/3$), or equivalently $(\#\mathcal{T})^{-1/3}$, irrespective of the polynomial degree n. This provides a lower bound for $\|v - I_{\mathcal{T}} v\|_{L^2(\Omega)}$ and thus shows that (3.21) is sharp.

h	linear ($n = 1$)	quadratic ($n = 2$)	cubic ($n = 3$)
1/4	1.14	9.64	9.89
1/8	0.74	0.67	0.67
1/16	0.68	0.67	0.67
1/32	0.66	0.67	0.67
1/64	0.66	0.67	0.67
1/128	0.66	0.67	0.67

The order $-n/d$ is just dictated by the polynomial degree n and cannot be improved upon assuming either higher regularity than $H^{n+1}(\Omega)$ or a graded mesh $\mathcal{T}$. The presence of d in the exponent is referred to as the *curse of dimensionality*.

Example (corner singularity in two dimensions). To explore the effect of a geometric singularity on (3.21), we let $\Omega = (-1, 1)^2 \setminus [0, 1]^2$ be an L-shaped domain and let $v \in H^1(\Omega)$ be

$$v(r, \theta) = r^{2/3} \sin(2\theta/3) - r^2/4.$$

This function $v \in H^1(\Omega)$ exhibits the typical corner singularity of the solution of $-\Delta v = f$ with suitable Dirichlet boundary condition: $v \in H^s(\Omega)$ for $s < 5/3$. Table 3.1 displays the best approximation error for polynomial degree $n = 1, 2, 3$ and a sequence of *uniform* refinements in the seminorm $|\cdot|_{H^1(\Omega)} = \|\nabla \cdot\|_{L^2(\Omega)}$. This gives a *lower* bound for the interpolation error in (3.21).

Even though s is fractional, the error estimate (3.21) is still valid, as stated in Remark 3.6. In fact, for uniform refinement, (3.21) can be derived by space interpolation between $H^1(\Omega)$ and $H^{n+1}(\Omega)$. The asymptotic rate $(\#\mathcal{T})^{-1/3}$ reported in Table 3.1 is consistent with (3.21) and independent of the polynomial degree n; this shows that (3.21) is sharp. It is also suboptimal as compared with the optimal rate $(\#\mathcal{T})^{-n/2}$ of Remark 3.12.

The question arises whether the rate $(\#\mathcal{T})^{-1/3} \approx h^{2/3}$ in Table 3.1 is just a consequence of uniform refinement or unavoidable. It is important to realize that $v \notin H^s(\Omega)$ for $s \geq 5/3$ and thus (3.21) is not applicable. However, the problem is not that second-order derivatives of v do not exist but rather that they are not square-integrable. In particular, it is true that $v \in W^2_p(\Omega)$ if $1 \leq p < 3/2$. We may therefore apply Theorem 3.8 with, for example, $n = 1$, $s = 2$ and $p \in [1, 3/2)$, and then ask whether the structure of (3.18) can be exploited, for example by

compensating the local singular behaviour of v with the local mesh size h. This enterprise naturally leads to *graded* meshes adapted to v.

3.4. *Principle of error equidistribution.*

We investigate the relation between local interpolation error and regularity for the design of optimal graded meshes adapted to a given function $v \in H^1(\Omega)$ for $d = 2$. We recall that $W_1^2(\Omega)$ is in the same nonlinear Sobolev scale of $H^1(\Omega)$, namely $\mathrm{sob}(W_1^2) = \mathrm{sob}(H^1)$, but $W_1^2(\Omega) \subset C^0(\overline{\Omega})$ (Brenner and Scott 2008, Lemma 4.3.4), and the *Lagrange interpolant* $I_{\mathcal{T}} v$ is well-defined and satisfies

$$\|\nabla(v - I_{\mathcal{T}} v)\|_{L^2(T)} \le C|v|_{W_1^2(T)} =: e_{\mathcal{T}}(v, T) \quad \text{for all } T \in \mathcal{T}. \tag{3.23}$$

We formulate a discrete minimization problem on the surrogate quantity $\boldsymbol{e} := (e_{\mathcal{T}}(v, T))_{T \in \mathcal{T}} \in \mathbb{R}^N$ with $N = \#\mathcal{T}$: minimize the square of the total H^1-error $E_{\mathcal{T}}(v)$,

$$E_{\mathcal{T}}(v)^2 := \sum_{T \in \mathcal{T}} e_{\mathcal{T}}(v, T)^2,$$

subject to the constraint

$$\sum_{T \in \mathcal{T}} e_{\mathcal{T}}(v, T) = C|v|_{W_1^2(\Omega)}.$$

We idealize the problem upon allowing $e_{\mathcal{T}}(v, T)$ to attain any non-negative real value, despite the fact that shape regularity of $\mathcal{T}$ entails geometric restrictions between adjacent elements. We next form the Lagrangian

$$\mathcal{L}[\boldsymbol{e}, \lambda] := \sum_{T \in \mathcal{T}} e_{\mathcal{T}}(v, T)^2 - \lambda \left(\sum_{T \in \mathcal{T}} e_{\mathcal{T}}(v, T) - C|v|_{W_1^2(\Omega)} \right),$$

with Lagrange multiplier $\lambda \in \mathbb{R}$. We thus realize that the optimality condition reads

$$e_{\mathcal{T}}(v, T) = \frac{\lambda}{2} \quad \text{for all } T \in \mathcal{T}$$

or that $e_{\mathcal{T}}(v, T)$ is constant over $\mathcal{T}$. We rewrite this insightful conclusion as follows:

A graded mesh is quasi-optimal if the local error is equidistributed. (3.24)

This calculation yields

$$E_{\mathcal{T}}(v)^2 = \frac{\lambda^2}{4} N, \quad C|v|_{W_1^2(\Omega)} = \frac{\lambda}{2} N,$$

whence

$$E_{\mathcal{T}}(v) = C|v|_{W_1^2(\Omega)} N^{-1/2} \tag{3.25}$$

is the optimal decay rate but with regularity $v \in W_1^2(\Omega)$ rather than $H^2(\Omega)$. This is the second instance of nonlinear approximation, namely mesh design tailored to

the specific function v at hand; the first one was in Section 1. The principle of error equidistribution (3.24) was originally derived by Babuška and Rheinboldt (1978) for $d = 1$, and extended to $d = 2$ by Nochetto and Veeser (2012, Section 1.6), using an idealized continuous minimization problem involving a mesh size function. The current formulation is closer to applications and does not require a positive power of h_T in (3.23).

Remark 3.13 (point singularities). Corner singularities (Grisvard 1985) as well as singularities due to intersecting interfaces (Kellogg 1974/75) are of the form

$$v(x) \approx r(x)^\gamma, \quad 0 < \gamma < 1, \tag{3.26}$$

for $d = 2$. This implies $v \in W_1^2(\Omega)$ for all γ and the decay rate (3.25) provided $\mathcal{T}$ equidistributes the H^1-error. Babuška, Kellogg and Pitkäranta (1979) and Grisvard (1985) designed such meshes for corner singularities using weighted H^2-regularity. The preceding approach is more powerful in that it does not require any characterization of the singularities, rather than $v \in W_1^2(\Omega)$, and also applies to line discontinuities for $d = 2$. We will come back to this point in Section 6.

We now consider an important abstract variant of the discrete minimization process leading to (3.24), which will be instrumental in understanding the success of greedy algorithms later. Suppose that $0 < q, p \le \infty$, $v \in L^q(\Omega)$, $X_p^t(\Omega)$ is an abstract regularity space with $t = 1/p - 1/q > 0$, and $E_{\mathcal{T}}(v)_q$ and $e_{\mathcal{T}}(v, T)_q$ are global and local L^q-interpolation error indicators of v that satisfy the following two abstract properties.

- *Summability in ℓ^q.* There exists a constant $C_1 > 0$ such that

$$E_{\mathcal{T}}(v)_q^q \le C_1^q \sum_{T \in \mathcal{T}} e_{\mathcal{T}}(v, T)_q^q. \tag{3.27}$$

- *Summability in ℓ^p.* There exists a constant $C_2 > 0$ such that

$$\sum_{T \in \mathcal{T}} e_{\mathcal{T}}(v, T)_q^p = C_2^p |v|_{X_p^t(\Omega)}^p. \tag{3.28}$$

We intend to find conditions on a mesh $\mathcal{T}$ that minimize the global L^q-error $E_{\mathcal{T}}(v)_q$ of v subject to the constraint (3.28). We again propose a Lagrangian

$$\mathcal{L}[\boldsymbol{e}, \lambda] := \sum_{T \in \mathcal{T}} e_{\mathcal{T}}(v, T)_q^q - \lambda \left(\sum_{T \in \mathcal{T}} e_{\mathcal{T}}(v, T)_q^p - C_2^p |v|_{X_p^t(\Omega)}^p \right).$$

The optimality condition for $\boldsymbol{e}$ reads

$$e_{\mathcal{T}}(v, T)_q = \left(\lambda \frac{p}{q} \right)^{1/(q-p)} \quad \text{for all } T \in \mathcal{T},$$

which is a third instance of error equidistribution and thus consistent with (3.24).

We now resort to (3.27) and (3.28), to arrive at

$$\sum_{T\in\mathcal{T}} e_{\mathcal{T}}(v,T)_q^q = N\left(\lambda\frac{p}{q}\right)^{q/(q-p)}, \quad C_2^p|v|^p_{X^t_p(\Omega)} = N\left(\lambda\frac{p}{q}\right)^{p/(q-p)},$$

whence

$$E_{\mathcal{T}}(v)_q \le C_1C_2|v|_{X^t_p(\Omega)}N^{1/q-1/p}. \tag{3.29}$$

We see that the decay rate in (3.29) is $-t = 1/q - 1/p < 0$ and is just dictated by the different summabilities of (3.27) and (3.28). In the applications of (3.29) below, $t = s/d$ will be proportional to a differentiability index s, and the condition

$$0 = t - \frac{1}{p} + \frac{1}{q} = \frac{s}{d} - \frac{1}{p} + \frac{1}{q}$$

will correspond to the spaces $L^q(\Omega)$ and $X^s_p(\Omega)$ being on the same nonlinear Sobolev scale. This minimization process is an idealization that does not account for mesh regularity of $\mathcal{T}$, which in turn entails some geometric constraints in the construction of $\mathcal{T}$. A key question is whether estimates of the form (3.29) can be achieved under conditions that are practical but weaker than (3.24). In Section 3.5 we will study the bisection method, a flexible technique for conforming mesh refinement with optimal complexity. In Section 3.6 we will present and analyse GREEDY, a practical algorithm that implements these ideas and constructs quasi-optimal conforming bisection meshes under the slightly stronger assumption

$$s - \frac{d}{p} + \frac{d}{q} > 0. \tag{3.30}$$

Moreover, in Section 3.7 we will extend this analysis to non-conforming meshes.

We realize from (3.29) that in order to maximize the error decay rate we would like to have p as small as possible, even $0 < p < 1$. The range of q, p does not matter in the argument above and, despite the fact that $q \ge 1$ in all applications below, the range of p is only limited by that of s, which in turn depends on the polynomial degree $n \ge 1$ in that $0 < s \le n+1$.

We now return to the special case (3.23), namely $q = 2$, $p = 1$ and $\nabla v \in L^2(\Omega)$. As already shown in (3.23), in the nonlinear Sobolev scale

$$\mathrm{sob}(W^2_1) - \mathrm{sob}(H^1) = \left(2 - \frac{2}{1}\right) - \left(1 - \frac{2}{2}\right) = 0,$$

we expect the best error decay

$$\|\nabla(v - I_{\mathcal{T}}v)\|_{L^2(\Omega)} \lesssim |v|_{W^2_1(\Omega)}(\#\mathcal{T})^{-1/2},$$

whereas the linear Sobolev scale yields the reduced order

$$\|\nabla(v - I_{\mathcal{T}}v)\|_{L^2(\Omega)} \lesssim |v|_{H^s(\Omega)}(\#\mathcal{T})^{-(s-1)/2}$$

for $s < 1 + \gamma < 2$ and v satisfying (3.26), where $I_{\mathcal{T}}$ is the Lagrange interpolation

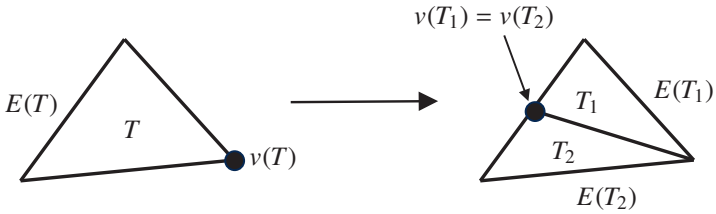


Figure 3.2. Triangle $T \in \mathcal{T}$ with vertex $v(T)$ and opposite refinement edge $E(T)$. The bisection rule for $d = 2$ consists of connecting $v(T)$ with the midpoint of $E(T)$, thereby giving rise to children T_1, T_2 with common vertex $v(T_1) = v(T_2)$, the newly created vertex, and opposite refinement edges $E(T_1)$, $E(T_2)$.

operator. The nonlinear Sobolev scale entails a trade of differentiability with integrability: we gain up to differentiability $s = 2$ at the expense of lower integrability $p = 1$ for polynomial degree $n = 1$. This trade-off is at the heart of the optimal estimate (3.25) and is represented in the so-called *DeVore diagram* in Figure 2.1.

If the polynomial degree is $n \geq 2$, then the largest differentiability index is $s = n + 1$, which for $d = 2$ leads to integrability index $p < 1$:

$$\left(s - \frac{2}{p}\right) - \left(1 - \frac{2}{2}\right) = 0 \quad \Rightarrow \quad p = \frac{2}{n+1} < 1. \tag{3.31}$$

To measure regularity of v, the corresponding Sobolev space must be replaced by the Besov space $B^{n+1}_{p,p}(\Omega)$ or the Lipschitz space $\mathrm{Lip}^{n+1}_p(\Omega)$. We will introduce and study these spaces in Section 6.8.

3.5. *Conforming meshes: the bisection method*

In order to approximate functions in $W^k_p(\Omega)$ by piecewise polynomials, we decompose Ω into simplices. We briefly discuss the *bisection* method, an elegant and versatile technique for subdividing Ω in any dimension into a conforming mesh. We also briefly discuss non-conforming meshes in Section 3.7. We present complete proofs, especially of the complexity of bisection, later in Section 8.

We focus on $d = 2$ and follow Binev *et al.* (2004), but the results carry over to any dimension $d > 2$ (Stevenson 2008). We refer to Nochetto *et al.* (2009) for a fairly complete discussion for $d \geq 2$.

Let $\mathcal{T}$ denote a *mesh* (triangulation or grid) made of simplices T, and let $\mathcal{T}$ be *conforming* (edge-to-edge). Each element is labelled, namely it has an edge $E(T)$ assigned for refinement (and an opposite vertex $v(T)$ for $d = 2$); see Figure 3.2.

The bisection method consists of a suitable *labelling* of the initial mesh $\mathcal{T}_0$ and a rule to assign the refinement edge to the two children. For $d = 2$ we consider the *newest vertex bisection* as depicted in Figure 3.2. For $d > 2$ the situation is more complicated and we need the concepts of type and vertex order (Nochetto *et al.* 2009, Stevenson 2008). More precisely, we identify a simplex T with the set of its

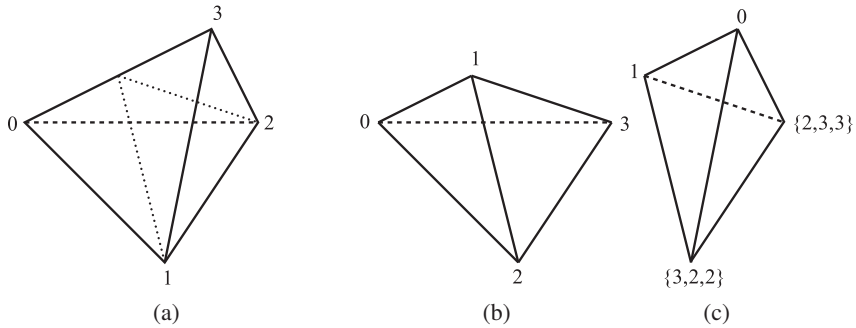


Figure 3.3. Refinement of a single tetrahedron T of type t. The child T_1 in (b) has the same node ordering regardless of type. In contrast, for the child T_2 in (c) a triple is appended to two nodes. The local vertex index is given for these nodes by the tth component of the triple.

ordered vertices and its *type* t by

$$T = \{z_0, z_1, \dots, z_d\}_t,$$

with $t \in \{0, \dots, d-1\}$. Given such a d-simplex T, we use the following bisection rule to split it in a unique fashion and to impose both vertex order and type on its children. The edge $\overline{z_0 z_d}$ connecting the first and last vertex of T is the *refinement edge* of T, and its midpoint $\bar{z} = (z_0 + z_d)/2$ becomes the new vertex. Connecting the new vertex $\bar{z}$ to the vertices of T other than z_0, z_d determines the common face $S = \{\bar{z}, z_1, \dots, z_{d-1}\}$ shared by the two children T_1, T_2 of T. The *bisection rule* dictates the following vertex order and type for T_1, T_2:

$$\begin{aligned} T_1 &:= \{z_0, \bar{z}, \underbrace{z_1, \dots, z_t}_{\rightarrow}, \underbrace{z_{t+1}, \dots, z_{d-1}}_{\rightarrow}\}_{(t+1) \bmod d}, \\ T_2 &:= \{z_d, \bar{z}, \underbrace{z_1, \dots, z_t}_{\rightarrow}, \underbrace{z_{d-1}, \dots, z_{t+1}}_{\leftarrow}\}_{(t+1) \bmod d}, \end{aligned} \tag{3.32}$$

with the convention that arrows point in the direction of increasing indices and $\{z_1, \dots, z_0\} = \emptyset$, $\{z_d, \dots, z_{d-1}\} = \emptyset$. For instance, in three dimensions the children of $T = \{z_0, z_1, z_2, z_3\}_t$ are (see Figure 3.3)

$$\begin{aligned} t = 0: &\quad T_1 = \{z_0, \bar{z}, z_1, z_2\}_1 \quad \text{and} \quad T_2 = \{z_3, \bar{z}, z_2, z_1\}_1, \\ t = 1: &\quad T_1 = \{z_0, \bar{z}, z_1, z_2\}_2 \quad \text{and} \quad T_2 = \{z_3, \bar{z}, z_1, z_2\}_2, \\ t = 2: &\quad T_1 = \{z_0, \bar{z}, z_1, z_2\}_0 \quad \text{and} \quad T_2 = \{z_3, \bar{z}, z_1, z_2\}_0. \end{aligned}$$

Note that the vertex labelling of T_1 is type-independent, whereas that of T_2 is the same for type $t = 1$ and $t = 2$. To account for this fact, the vertices z_1 and z_2 of T are tagged $\{3, 2, 2\}$ and $\{2, 3, 3\}$ in Figure 3.3. The type of T then dictates which component of the triple is used to label the vertex.

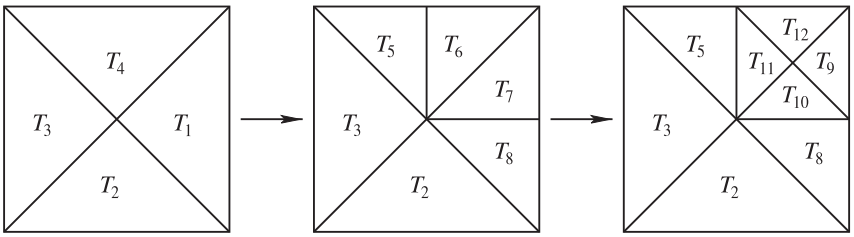


Figure 3.4. Sequence of bisection meshes $\{\mathcal{T}_k\}_{k=0}^2$ starting from the initial mesh $\mathcal{T}_0 = \{T_i\}_{i=1}^4$ with longest edges labelled for bisection. Mesh $\mathcal{T}_1$ is created from $\mathcal{T}_0$ upon bisecting T_1 and T_4, whereas mesh $\mathcal{T}_2$ arises from $\mathcal{T}_1$ upon refining T_6 and T_7. The bisection rule is described in Figure 3.2.

Bisection creates a *unique* master forest $\mathbb{F}$ of binary trees with infinite depth, where each node is a simplex (a triangle in two dimensions), its two successors are the two children created by bisection, and the roots of the binary trees are the elements of the initial conforming partition $\mathcal{T}_0$. It is important to realize that, no matter how an element arises in the subdivision process, its associated newest vertex is unique and depends only on the labelling of $\mathcal{T}_0$, so the edge $E(T)$ assigned for refinement (and the opposite vertex $v(T)$ for $d = 2$) are independent of the order of the subdivision process for all $T \in \mathbb{F}$; see Lemma 8.1. Therefore $\mathbb{F}$ is unique.

A finite subset $\mathcal{F} \subset \mathbb{F}$ is called a *forest* if $\mathcal{T}_0 \subset \mathcal{F}$ and the nodes of $\mathcal{F}$ satisfy

- all nodes of $\mathcal{F} \setminus \mathcal{T}_0$ have a predecessor;
- all nodes in $\mathcal{F}$ have either two successors or none.

Any node $T \in \mathcal{F}$ is thus uniquely connected with a node T_0 of the initial triangulation $\mathcal{T}_0$, that is, T belongs to the infinite tree $\mathbb{F}(T_0)$ emanating from T_0. Furthermore, any forest may have *interior nodes*, i.e. nodes with successors, as well as *leaf nodes*, i.e. nodes without successors. The set of leaves corresponds to a mesh (or triangulation, grid, partition) $\mathcal{T} = \mathcal{T}(\mathcal{F})$ of $\mathcal{T}_0$, which may not be conforming or edge-to-edge.

We thus introduce the set $\mathbb{T}$ of all *conforming* refinements of $\mathcal{T}_0$:

$$\mathbb{T} := \{\mathcal{T} = \mathcal{T}(\mathcal{F}) \mid \mathcal{F} \subset \mathbb{F} \text{ is finite and } \mathcal{T}(\mathcal{F}) \text{ is conforming}\}.$$

If $\mathcal{T}_* = \mathcal{T}(\mathcal{F}_*) \in \mathbb{T}$ is a conforming refinement of $\mathcal{T} = \mathcal{T}(\mathcal{F}) \in \mathbb{T}$, we write $\mathcal{T}_* \geq \mathcal{T}$ and understand this inequality in the sense of trees, namely $\mathcal{F} \subset \mathcal{F}_*$.

Example. Consider $\mathcal{T}_0 = \{T_i\}_{i=1}^4$ and the longest edge to be the refinement edge. Figure 3.4 displays a sequence of conforming meshes $\mathcal{T}_k \in \mathbb{T}$ created by bisection. Each element T_i of $\mathcal{T}_0$ is a root of a finite tree emanating from T_i, which together form the forest $\mathcal{F}_2$ corresponding to mesh $\mathcal{T}_2 = \mathcal{T}(\mathcal{F}_2)$. Figure 3.5 displays $\mathcal{F}_2$, whose leaf nodes are the elements of $\mathcal{T}_2$.

Figure 3.5. Forest $\mathcal{F}_2$ corresponding to the grid sequence $\{\mathcal{T}_k\}_{k=0}^2$ of Figure 3.4. The roots of $\mathcal{F}_2$ form the initial mesh $\mathcal{T}_0$ and the leaves of $\mathcal{F}_2$ constitute the conforming bisection mesh $\mathcal{T}_2$. Moreover, each level of $\mathcal{F}_2$ corresponds to all elements with generation equal to the level.

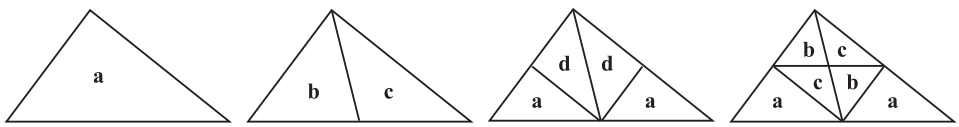
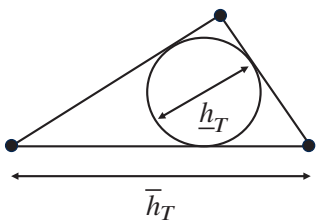


Figure 3.6. Bisection produces at most four similarity classes for any triangle.

Properties of bisection. We now discuss several crucial geometric properties of bisection. We start by recalling the concept of shape regularity. For any $T \in \mathcal{T}$, we define



$$\overline{h}_T := \operatorname{diam}(T)$$
$$h_T := |T|^{1/d}$$
$$\underline{h}_T := 2\sup\{r > 0 \mid B(x,r) \subset T \text{ for } x \in T\}.$$

Then

$$\underline{h}_T \le h_T \le \overline{h}_T \le \sigma \underline{h}_T \quad \text{for all } T \in \mathcal{T},$$

where $\sigma > 1$ is the shape regularity constant of (3.9). The next lemma guarantees that bisection keeps σ bounded.

Lemma 3.14 (shape regularity). *The partitions $\mathcal{T}$ generated by newest vertex bisection satisfy a uniform minimal angle condition, or equivalently σ is uniformly bounded, depending only on the initial partition $\mathcal{T}_0$.*

Proof. Each $T \in \mathcal{T}_0$ gives rise to a fixed number of similarity classes, namely four for $d = 2$ according to Figure 3.6. This, combined with the fact that $\#\mathcal{T}_0$ is finite, yields the assertion. □

We define the *generation (or level)* $g(T)$ of an element $T \in \mathcal{T}$ as the number of bisections needed to create T from its ancestor $T_0 \in \mathcal{T}_0$. Since bisection splits an element into two children with equal measure, we realize that

$$h_T = 2^{-g(T)/2} h_{T_0} \quad \text{for all } T \in \mathcal{T}. \tag{3.33}$$

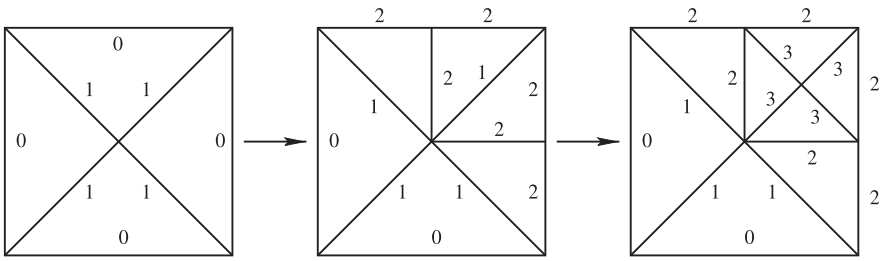


Figure 3.7. Initial labelling and its evolution for the sequence of conforming refinements $\mathcal{T}_0 \leq \mathcal{T}_1 \leq \mathcal{T}_2$ of Figure 3.4.

Referring to Figure 3.5, we observe that the leaf nodes $T_9, T_{10}, T_{11}, T_{12}$ have generation 2, whereas T_5, T_8 have generation 1 and T_2, T_3 have generation 0.

The following geometric property is a simple consequence of (3.33).

Lemma 3.15 (element size vs. generation). *There exist constants* $0 < D_1 < D_2$*, depending only on* $\mathcal{T}_0$*, such that*

$$D_1 2^{-g(T)/2} \leq h_T < \overline{h}_T \leq D_2 2^{-g(T)/2} \quad \textit{for all } T \in \mathcal{T}. \tag{3.34}$$

Labelling and bisection rule. Whether the recursive application of bisection does not lead to inconsistencies depends on a suitable initial labelling of edges and a bisection rule. For $d = 2$ they are simple to state (Binev *et al.* 2004). Given $T \in \mathcal{T}$ with generation $g(T) = i$, we assign the label $(i+1, i+1, i)$ to T with i corresponding to the refinement edge $E(T)$. The following rule dictates how the labelling changes with refinement: the side i is bisected and both new sides as well as the bisector are labelled $i + 2$ whereas the remaining labels do not change. To guarantee that the label of an edge is independent of the elements sharing this edge, we need a special labelling for $\mathcal{T}_0$, due to Mitchell (1989, Theorem 2.9) and Binev *et al.* (2004, Lemma 2.1), for $d = 2$:

$$\textit{Edges of } \mathcal{T}_0 \textit{ have labels } 0 \textit{ or } 1 \textit{ and all elements } T \in \mathcal{T} \textit{ have exactly two edges with label } 1 \textit{ and one with label } 0. \tag{3.35}$$

There is a variant for $d > 2$ due to Stevenson (2008, Section 4). It is not obvious that labelling (3.35) exists, but if it does then all elements of $\mathcal{T}_0$ can be split into pairs of compatibly divisible elements. We refer to Figure 3.7 for an example of initial labelling of $\mathcal{T}_0$ satisfying (3.35) and the way it evolves for two successive refinements $\mathcal{T}_2 \geq \mathcal{T}_1 \geq \mathcal{T}_0$ corresponding to Figure 3.4.

To guarantee (3.35) we can proceed as follows: given a coarse mesh of elements T, we can bisect each T twice and label the four grandchildren as indicated in Figure 3.8, for the resulting mesh $\mathcal{T}_0$ to satisfy the initial labelling (Binev *et al.* 2004).

For $d \geq 3$ a general strategy of initial labelling is due to Stevenson (2008, Section 4, Condition (b)), who in turn improves upon Maubach (1995) and Traxler (1997) and shows how to impose it upon further refining each element of $\mathcal{T}_0$.

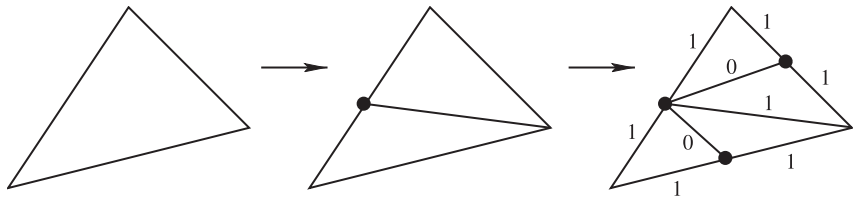


Figure 3.8. Bisecting each triangle of $\mathcal{T}_0$ twice and labelling edges in such a way that all boundary edges have label 1 yields an initial mesh satisfying (3.35).

We refer to the survey by Nochetto *et al.* (2009) for a discussion of this condition. A key consequence is:

$$\textit{Every uniform refinement of } \mathcal{T}_0 \textit{ gives a conforming bisection mesh.} \tag{3.36}$$

Condition (3.35) is still valid, and can be fulfilled by a construction by successive bisections, similar to but much trickier than the one described for $d = 2$; yet for $d = 3$ the number of elements increases by an order of magnitude, which indicates that (3.35) is a severe restriction in practice. Finding alternative, more practical conditions is an important problem.

Initialization of arbitrary triangulations. A novel initialization procedure that can be applied to *any* conforming triangulation $\mathcal{T}_0$ has recently been proposed by Diening, Gehring and Storn (2023); hereafter we present a short account of it.

The key concept is that of *colouring* the vertices of $\mathcal{T}_0$. A coloured initial triangulation in $\mathbb{R}^d$ is a pair $(\mathcal{T}_0, c)$, where $c\colon \mathcal{V}_{\mathcal{T}_0} \to \{0, \dots, d\}$ is such that the colours of all vertices of each $T \in \mathcal{T}_0$ are distinct. The colour map c allows us to sort the vertices of each initial element $T = \{z_0, \dots, z_d\}_t \in \mathcal{T}_0$ so that

$$c(z_j) = j, \quad j \in \{0, \dots, d\}.$$

To refine a marked $T \in \mathcal{T}_0$, we apply the Maubach bisection rule leading to (3.32), and possibly add a recursive closure, which is proved to terminate, to guarantee the conformity of the final triangulation. The colouring property is conserved in this process, and the conclusion of Theorem 3.16 below holds true, starting from any initially coloured triangulation $\mathcal{T}_0$.

Unfortunately, not every initial triangulation can be coloured. For instance, consider in dimension $d = 2$ a patch of triangles sharing a common vertex. If colour 0 is assigned to such an inner vertex, then the outer vertices must take colours 1 and 2 successively, but if the number of triangles in the patch is odd, there will be a vertex that is not colourable.

To overcome this obstruction, Diening *et al.* (2023) propose using more colours, and introduce the concept of *generalized colouring*: a pair $(\mathcal{T}_0, c)$ is an $(N + 1)$-coloured triangulation if there exists an integer $N \geq d$ and a mapping $c\colon \mathcal{V}_{\mathcal{T}_0} \to \{0, \dots, N\}$ such that the colours of all vertices of each $T \in \mathcal{T}_0$ are distinct. Any

initial $\mathcal{T}_0$ can be coloured in this generalized sense: indeed, after the initialization $c(z) = +\infty$ for all $z \in \mathcal{V}_{\mathcal{T}_0}$, we define

$$c(z) := \min(\mathbb{N}_0 \setminus \{c(w) \mid [z, w] \text{ is an edge of } \mathcal{T}_0\}), \quad z \in \mathcal{V}_{\mathcal{T}_0},$$

as the smallest colour not already attained by a neighbouring vertex. Then, we set $N := \max\{c(z) \mid z \in \mathcal{V}_{\mathcal{T}_0}\}$, and note that N is bounded by the maximal number of edges connected to a vertex of $\mathcal{T}_0$.

A generalized $(N+1)$-coloured triangulation $(\mathcal{T}_0, c)$ in $\mathbb{R}^d$ can be seen as a collection of d-simplices contained in a virtual, coloured triangulation $\mathcal{T}_0^+$ in $\mathbb{R}^N$. It suffices to add $N - d$ virtual nodes to each simplex in $\mathcal{T}_0$, so that it becomes a N-simplex, and attribute to these nodes the remaining $N - d$ colours. Note that these virtual simplices are only connected via their d-subsimplices belonging to $\mathcal{T}_0$. In the example mentioned above of a patch of triangles sharing a vertex, a $(3+1)$-coloured triangulation is defined as follows: a tetrahedron is built on top of each triangle; the previously uncolourable vertex takes the new colour 3, whereas colour 2 is attributed to the new vertices of the two tetrahedra sharing that vertex; the new vertex of any other tetrahedron takes the colour 3.

With the new triangulation $\mathcal{T}_0^+$ at hand, one could apply the Mauback bisection rule to it, which as a by-product would refine the initial triangulation $\mathcal{T}_0$. However, Diening, Gehring and Storn (2023) suggest a short-cut that directly refines $\mathcal{T}_0$ by invoking an algorithm that bisects a k-simplex in dimension $m > k$. A further round of recursive refinements may be needed to guarantee conformity. Diening *et al.* prove that the recursion terminates. In addition, for any $(N+1)$-coloured initial triangulation, the conclusion of Theorem 3.16 below holds true in this case too, with a constant D satisfying $D \lesssim N^d$.

The procedure REFINE. Given $\mathcal{T} \in \mathbb{T}$ and a selected subset $\mathcal{M} \subset \mathcal{T}$ (the set of *marked* elements), the procedure

$$[\mathcal{T}_*] = \mathsf{REFINE}(\mathcal{T}, \mathcal{M})$$

creates a new conforming refinement $\mathcal{T}_*$ of $\mathcal{T}$ by bisecting all elements of $\mathcal{M}$ at least once and perhaps additional elements to keep conformity.

Conformity is a constraint in the refinement procedure that prevents it from being completely local. The propagation of refinement beyond the set of marked elements $\mathcal{M}$ is a rather delicate matter, which we discuss later in Section 8. For instance, we show that a naive estimate of the form

$$\#\mathcal{T}_* - \#\mathcal{T} \le D\,\#\mathcal{M}$$

is *not* valid with an absolute constant D independent of the refinement level. This can be repaired upon considering the cumulative effect for a sequence of conforming bisection meshes $\{\mathcal{T}_k\}_{k=0}^{\infty}$. This is expressed in the following crucial complexity result due to Binev *et al.* (2004) for $d = 2$ and Stevenson (2008) for $d > 2$. We present a complete proof later in Section 8.

Theorem 3.16 (complexity of REFINE). *If $\mathcal{T}_0$ satisfies the initial labelling* (3.35) *for $d = 2$, or that in Stevenson (2008, Section 4) for $d > 2$, then there exists a constant $D > 0$ depending only on $\mathcal{T}_0$ and d such that, for all $k \geq 1$,*

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \leq D \sum_{j=0}^{k-1} \#\mathcal{M}_j.$$

If elements $T \in \mathcal{M}$ are to be bisected $b \geq 1$ times, then the procedure REFINE can be applied recursively, and Theorem 3.16 remains valid with D also depending on b.

Mesh overlay. For the subsequent discussion it will be convenient to merge (or superpose) two conforming meshes $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{T}$, thereby giving rise to the so-called *overlay* $\mathcal{T}_1 \oplus \mathcal{T}_2$. This operation corresponds to the union in the sense of trees (Cascón, Kreuzer, Nochetto and Siebert 2008, Stevenson 2007). We next bound the cardinality of $\mathcal{T}_1 \oplus \mathcal{T}_2$ in terms of that of $\mathcal{T}_1$ and $\mathcal{T}_2$.

Lemma 3.17 (mesh overlay). *Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{T}$. The overlay $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2 \in \mathbb{T}$ is conforming and*

$$\#\mathcal{T} \leq \#\mathcal{T}_1 + \#\mathcal{T}_2 - \#\mathcal{T}_0. \tag{3.37}$$

For a proof we refer to Cascón *et al.* (2008, Lemma 3.7), and to Proposition 8.15 below for a more general situation.

3.6. Constructive approximation

We now construct graded bisection meshes $\mathcal{T}$ for $n = 1, d = 2$ that achieve the optimal decay rate $(\#\mathcal{T})^{-1/2}$ of (3.25) under the global regularity assumption

$$v \in W^2_p(\Omega), \quad p > 1. \tag{3.38}$$

Therefore $W^2_p(\Omega)$ is strictly above the Sobolev line for the space $H^1(\Omega)$: $\mathrm{sob}(W^2_p) = 2 - 2/p > 0 = \mathrm{sob}(H^1)$. Note that $s = 1$, $p > 1$ and $q = 2$ obey the restriction (3.30) for $\nabla v \in L^2(\Omega)$. In particular, $W^2_p(\Omega)$ is compactly embedded into $H^1(\Omega)$ according to Lemma 2.1 (Sobolev embedding).

Following Binev, Dahmen, DeVore and Petrushev (2002) and Gaspoz and Morin (2014), we use a greedy algorithm that is based on the knowledge of the element errors and on bisection. The algorithm hinges on (3.24): if $\delta > 0$ is a given tolerance, the element error is equidistributed and within tolerance $e_{\mathcal{T}}(v, T) \approx \delta$, and the global error decays with maximum rate $(\#\mathcal{T})^{-1/2}$, then

$$\delta^2 \#\mathcal{T} \approx \sum_{T \in \mathcal{T}} e_{\mathcal{T}}(v, T)^2 = |v - I_{\mathcal{T}} v|^2_{H^1(\Omega)} \lesssim (\#\mathcal{T})^{-1}$$

whence $\#\mathcal{T} \lesssim \delta^{-1}$; here $I_{\mathcal{T}}$ stands for the Lagrange interpolation operator. With this in mind, we impose $e_T(v) \leq \delta$ as a threshold to stop refining and expect $\#\mathcal{T} \lesssim \delta^{-1}$. The following algorithm implements this idea.

Algorithm 3.18 (greedy algorithm). Given a tolerance $\delta > 0$ and a conforming mesh $\mathcal{T}_0$, GREEDY finds a conforming refinement $\mathcal{T} \geq \mathcal{T}_0$ of $\mathcal{T}_0$ by bisection such that $e_{\mathcal{T}}(v, T) \leq \delta$ for all $T \in \mathcal{T}$: let $\mathcal{T} = \mathcal{T}_0$ and

$[\mathcal{T}] = \mathsf{GREEDY}(\mathcal{T}, \delta, v)$
  while $\mathcal{M} := \{T \in \mathcal{T} \mid e_{\mathcal{T}}(v, T) > \delta\} \neq \emptyset$
    $\mathcal{T} := \mathsf{REFINE}(\mathcal{T}, \mathcal{M})$
  return $\mathcal{T}$

Since $W^2_p(\Omega) \subset C^0(\overline{\Omega})$, because $p > 1$, we can use the Lagrange interpolant and local estimate (3.17) with $r = \mathrm{sob}(W^2_p) - \mathrm{sob}(H^1) = 2 - 2/p > 0$. We deduce

$$e_{\mathcal{T}}(v, T) \lesssim h_T^r \, \|D^2 v\|_{L^p(T)}. \tag{3.39}$$

We assess the quality of the resulting mesh in a slightly more general setting, following Bonito *et al.* (2016, Proposition 1 and Corollary 1), needed later in Sections 6 and 7 for solution and data approximation for any polynomial degree.

An abstract greedy algorithm. We consider a generic (possibly vector-valued) function $v \in L^q(\Omega, \mathbb{R}^M)$, with $M \geq 1$ and $1 \leq q \leq \infty$, let $e_{\mathcal{T}}(v, T) = \|v - \Pi_{\mathcal{T}} v\|_{L^q(T)}$ denote the abstract L^q-local error for $T \in \mathcal{T}$ used in the GREEDY procedure, and let $E_{\mathcal{T}}(v) = \|v - \Pi_{\mathcal{T}} v\|_{L^q(\Omega)}$ denote the global L^q-interpolation error by either continuous or discontinuous piecewise polynomials (the definition of $\Pi_{\mathcal{T}} v$ is irrelevant now). We formulate the following assumptions.

- *Summability in ℓ^q.* The errors $\{e_{\mathcal{T}}(v, T)\}_{T \in \mathcal{T}}$ satisfy

$$\|v\|^q_{L^q(\Omega)} \lesssim \sum_{T \in \mathcal{T}} e_{\mathcal{T}}(v, T)^q. \tag{3.40}$$

Rather than (3.38), we assume that v belongs to an abstract space $X^s_p(\Omega; \mathcal{T}_0)$ of functions with differentiability index $s \in (0, n]$ and integrability index $p \in (0, \infty]$ piecewise over $\mathcal{T}_0$ with two crucial properties.

- *Local error estimate.* For $r = s - d/p + d/q > 0$ and all $T \in \mathcal{T}$,

$$e_{\mathcal{T}}(v, T) \lesssim h_T^r |v|_{X^s_p(T)}. \tag{3.41}$$

- *Norm subadditivity.* For $p < \infty$, and obvious modification for $p = \infty$,

$$\sum_{T \in \mathcal{T}} |v|^p_{X^s_p(T)} \lesssim |v|^p_{X^s_p(\Omega; \mathcal{T}_0)}. \tag{3.42}$$

The space $X^s_p(\Omega; \mathcal{T}_0)$ will later be either a Sobolev space $W^s_p(\Omega; \mathcal{T}_0)$, a Besov space $B^s_{p,p}(\Omega; \mathcal{T}_0)$ or a Lipschitz space $\mathrm{Lip}^s_p(\Omega; \mathcal{T}_0)$, with piecewise regularity over $\mathcal{T}_0$; the latter two will allow $0 < p < 1$. For the moment we do not need to be specific and just rely on the two properties above.

Proposition 3.19 (abstract greedy error). *Let $\mathcal{T}_0$ be an initial subdivision of $\Omega \subset \mathbb{R}^d$ satisfying the initial labelling property* (3.35) *for $d = 2$, or its variant for*

$d > 2$. Let $M \geq 1, 0 < q, p \leq \infty$ and $s - d/p + d/q > 0$. Let $v \in L^q(\Omega, \mathbb{R}^M)$ satisfy (3.40), (3.41) and (3.42). Then GREEDY$(\mathcal{T}_0, \delta, v)$ *terminates in a finite number of iterations with local errors verifying $e_{\mathcal{T}}(v, T) \leq \delta$ for all $T \in \mathcal{T}$, and there is a constant $C = C(p, q, s, d, \Omega, \mathcal{T}_0)$ such that the output $\mathcal{T} \in \mathbb{T}$ satisfies*

$$\|v - \Pi_{\mathcal{T}} v\|_{L^q(\Omega)} \leq C |v|_{X^s_p(\Omega;\mathcal{T}_0)} (\#\mathcal{T} - \#\mathcal{T}_0)^{-s/d}. \tag{3.43}$$

Proof. We proceed in several steps.

$\boxed{1}$ *Termination.* Since h_T decreases monotonically to 0 with bisection, so does $e_{\mathcal{T}}(v, T)$ in view of (3.41). Consequently, GREEDY terminates in a finite number $k \geq 1$ of iterations. Upon termination, the local errors satisfy $e_{\mathcal{T}}(v, T) \leq \delta$ for all $T \in \mathcal{T}$ by construction, whence (3.40) implies

$$\|v - \Pi_{\mathcal{T}} v\|_{L^q(\Omega)} \lesssim \delta (\#\mathcal{T})^{1/q}.$$

$\boxed{2}$ *Counting.* Let $\mathcal{M} = \mathcal{M}_0 \cup \cdots \cup \mathcal{M}_{k-1}$ be the set of marked elements. We organize the elements in $\mathcal{M}$ by size in a way that allows for a counting argument. Let $\mathcal{P}_j$ be the set of elements T of $\mathcal{M}$ with size

$$2^{-(j+1)} \leq |T| < 2^{-j} \quad \Rightarrow \quad 2^{-(j+1)/d} \leq h_T < 2^{-j/d},$$

because $h_T = |T|^{1/d}$ for shape-regular meshes $\mathcal{T} \in \mathbb{T}$.

We first observe that all the T in $\mathcal{P}_j$ are *disjoint*. This is because if $T_1, T_2 \in \mathcal{P}_j$ and $\mathring{T}_1 \cap \mathring{T}_2 \neq \emptyset$, then one of them is contained in the other, say $T_1 \subset T_2$, due to the bisection procedure which works in any dimension $d \geq 1$; see Section 8. Hence

$$|T_1| \leq \frac{1}{2} |T_2|,$$

contradicting the definition of $\mathcal{P}_j$. This implies the first bound

$$2^{-(j+1)} \#\mathcal{P}_j \leq |\Omega| \quad \Rightarrow \quad \#\mathcal{P}_j \leq |\Omega| \, 2^{j+1}. \tag{3.44}$$

In light of (3.41), we have for $T \in \mathcal{P}_j$

$$\delta \leq e_{\mathcal{T}}(v, T) \lesssim 2^{-jr/d} |v|_{X^s_p(T)}.$$

Therefore, accumulating these quantities in ℓ^p and invoking (3.42) yields

$$\delta^p \, \#\mathcal{P}_j \lesssim 2^{-jrp/d} \sum_{T \in \mathcal{P}_j} |v|^p_{X^s_p(T)} \lesssim 2^{-jrp/d} \, |v|^p_{X^s_p(\Omega;\mathcal{T}_0)}$$

and gives rise to the second bound

$$\#\mathcal{P}_j \lesssim \delta^{-p} \, 2^{-jrp/d} \, |v|^p_{X^s_p(\Omega;\mathcal{T}_0)}. \tag{3.45}$$

$\boxed{3}$ *Cardinality.* The two bounds for $\#\mathcal{P}$ in (3.44) and (3.45) are complementary. The first one is good for j small whereas the second is suitable for j large (think of $\delta \ll 1$). The crossover takes place for j_0 such that

$$2^{j_0+1} |\Omega| \approx \delta^{-p} \, 2^{-j_0 rp/d} |v|^p_{X^s_p(\Omega;\mathcal{T}_0)} \quad \Rightarrow \quad 2^{j_0} \approx \left(|\Omega|^{-1} \delta^{-p} |v|^p_{X^s_p(\Omega;\mathcal{T}_0)} \right)^{d/(d+rp)}.$$

We now compute

$$\#\mathcal{M} = \sum_j \#\mathcal{P}_j \lesssim \sum_{j\le j_0} 2^j|\Omega| + \delta^{-p}\,|v|^p_{X^s_p(\Omega)} \sum_{j>j_0} (2^{-rp/2})^j.$$

Since

$$\sum_{j\le j_0} 2^j \approx 2^{j_0}, \qquad \sum_{j>j_0} (2^{-rp/d})^j \lesssim 2^{-rpj_0/d},$$

we can write

$$\#\mathcal{M} \lesssim |\Omega|^{1-d/(d+rp)} \big(\delta^{-1}|v|_{X^s_p(\Omega;\mathcal{T}_0)}\big)^{dp/(d+rp)}.$$

We finally apply Theorem 3.16 (complexity of REFINE), to arrive at

$$\#\mathcal{T} - \#\mathcal{T}_0 \lesssim \#\mathcal{M} \lesssim |\Omega|^{rp/(d+rp)} \big(\delta^{-1}|v|_{X^s_p(\Omega;\mathcal{T}_0)}\big)^{dp/(d+rp)},$$

or equivalently

$$\delta \lesssim |\Omega|^{r/d}|v|_{X^s_p(\Omega;\mathcal{T}_0)}(\#\mathcal{T} - \#\mathcal{T}_0)^{-(d+rp)/(dp)}.$$

$\boxed{4}$ *Total error.* Since

$$\frac{d+rp}{dp} = \frac{s}{d} + \frac{1}{q},$$

we deduce from step $\boxed{1}$ that

$$\|v - \Pi_{\mathcal{T}} v\|_{L^q(\Omega)} \lesssim \delta(\#\mathcal{T})^{1/q} \lesssim |\Omega|^{r/d}|v|_{X^s_p(\Omega;\mathcal{T}_0)}(\#\mathcal{T} - \#\mathcal{T}_0)^{-s/d},$$

which is the desired estimate. □

The output mesh $\mathcal{T}$ of GREEDY$(\mathcal{T}_0, \delta, v)$ starting from $\mathcal{T}_0$ satisfies $\#\mathcal{T} \ge c_0\#\mathcal{T}_0$ for some $c_0 > 1$, whence $\#\mathcal{T} - \#\mathcal{T}_0 \ge (1 - 1/c_0)\#\mathcal{T}$ and (3.43) yields

$$\|v - \Pi_{\mathcal{T}} v\|_{L^q(\Omega)} \lesssim C|v|_{X^s_p(\Omega;\mathcal{T}_0)}(\#\mathcal{T})^{-s/d}, \tag{3.46}$$

where C depends on c_0. In many applications of GREEDY, to be discussed later in Sections 6 and 7, it will be convenient for the starting mesh to be a conforming refinement of $\mathcal{T}_0$ to enhance its efficiency. We will prove in Section 7.1 that (3.46) remains valid.

It is instructive to realize that GREEDY is a practical algorithm that hinges on the different summabilities of (3.40) and (3.42), and delivers a global L^q-error consistent with (3.29) of Section 3.4. Moreover, the outcome graded grid $\mathcal{T}$ is quasi-optimal but may not equidistribute the error, not even approximately.

We are now in a position to show that GREEDY constructs optimal graded meshes for the interpolation error in $H^1(\Omega)$ alluded to at the beginning of this section. To this end, we let $I_{\mathcal{T}}$ be the Lagrange interpolation operator for $d = 2$.

Corollary 3.20 (optimal H^1-convergence rate). *If $v \in H^1(\Omega) \cap W^2_p(\Omega)$ for $1 < p \le 2$ and $d = 2$, then* GREEDY *yields graded bisection meshes $\mathcal{T}$ so that*

$$|v - I_{\mathcal{T}} v|_{H^1(\Omega)} \lesssim |\Omega|^{1-1/p}\|D^2 v\|_{L^p(\Omega)}(\#\mathcal{T})^{-1/2}.$$

Proof. We invoke Proposition 3.19 (abstract greedy error) and equation (3.46) for $\nabla v \in L^2(\Omega, \mathbb{R}^2)$ with $\Pi_{\mathcal{T}} \nabla v = \nabla I_{\mathcal{T}} v$ and $s = 1$, $q = 2$, $p > 1$, whence $s - d/p + d/q > 0$. □

Remark 3.21 (piecewise W_p^2-smoothness). Since (3.39) is completely local for $d = 2$, we see from (3.42) that it suffices for $v \in H^1(\Omega)$ to be piecewise in W_p^2 over the initial partition $\mathcal{T}_0$, namely $W_p^2(\Omega; \mathcal{T}_0)$. It turns out that this statement is valid for any dimension $d \geq 2$ in view of Proposition 3.9 (approximation of gradients). We will revisit this issue in Section 6.8.

Remark 3.22 (case $p < 1$). We now consider polynomial degree $n \geq 1$. The integrability p corresponding to differentiability $n+1$ results from equating Sobolev numbers:

$$n + 1 - \frac{d}{p} = \mathrm{sob}(H^1) = 1 - \frac{d}{2} \quad \Rightarrow \quad p = \frac{2d}{2n + d}.$$

Depending on $d \geq 2$ and $n \geq 1$, this may lead to $0 < p < 1$, in which case $W_p^{n+1}(\Omega)$ is to be replaced by the Besov space $B_{p,p}^s(\Omega)$ for $s < n + 1$ or the Lipschitz space $\mathrm{Lip}_p^{n+1}(\Omega)$ (DeVore 1998). We will discuss this matter in Section 6.8 and make the abstract greedy setting precise.

Remark 3.23 (isotropic vs. anisotropic elements). Since geometric singularities are of the form (3.26) for $d = 2$, Corollary 3.20 (optimal H^1-convergence rate) shows that isotropic graded meshes are able to deliver optimal convergence rates for $d = 2$. Unfortunately this is no longer the case for $d > 2$, and anisotropic meshes are necessary for optimal meshes. This topic is delicate in several respects. Deriving reliable and efficient *a posteriori* error estimators is largely open for anisotropic meshes; this is the subject of Section 4 for isotropic meshes. Even having such estimators, building a theory of adaptivity is open; this is the subject of Sections 5, 6 and 10 for isotropic meshes. Finally, constructing anisotropic meshes based on *a posteriori* information alone and that easily allow for refinement and coarsening is problematic. For these reasons we do not dwell on anisotropic refinement in this survey.

3.7. Non-conforming meshes

More general subdivisions of Ω than those in Section 3.5 are used in practice. If the elements of $\mathcal{T}_0$ are quadrilaterals for $d = 2$, or their multidimensional variant for $d > 2$, then it is natural to allow for improper or *hanging nodes* for the resulting refinements $\mathcal{T}$ to be graded; see Figure 3.9(a). On the other hand, if $\mathcal{T}_0$ is made of triangles for $d = 2$, or simplices for $d > 2$, then red refinement without green completion also gives rise to graded meshes with hanging nodes; see Figure 3.9(b). In both cases, the presence of hanging nodes is inevitable to enforce mesh grading. Finally, bisection may produce meshes with hanging nodes, as depicted in Figure 3.9(c), if the completion process is incomplete. All three

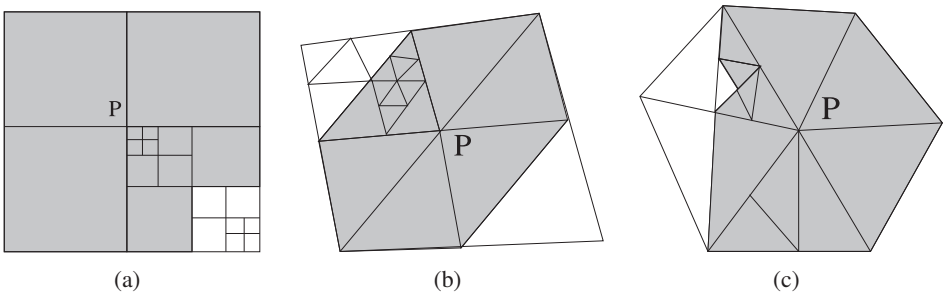


Figure 3.9. Non-conforming meshes made of quadrilaterals (a), triangles with red refinement (b) and triangles with bisection (c). The shaded regions depict the domain of influence of a proper or conforming node P.

refinements maintain shape regularity, but for both practice and theory, they cannot be arbitrary: we need to restrict the level of non-conformity. We discuss this next, starting with the case of polynomial degree $n = 1$ (Bonito and Nochetto 2010, Beirão da Veiga *et al.* 2023).

We say that a node P of $\mathcal{T}$ is a *proper* (or conforming) node if it is a vertex of all elements containing P; otherwise, we say that P is an *improper* (non-conforming or hanging) node. The set $\mathcal{N}$ of all nodes of $\mathcal{T}$ is thus partitioned into the set $\mathcal{P}$ of proper nodes, and the set $\mathcal{H} = \mathcal{N} \setminus \mathcal{P}$ of hanging nodes.

A useful notion in dealing with hanging nodes is the *global index* of a node, introduced in Beirão da Veiga *et al.* (2023): it measures the number of non-conforming refinements needed to generate a hanging node from proper nodes. To define it, for any $x \in \mathcal{H}$ which has been generated by the bisection of an edge $[x', x'']$, let us set $\mathcal{B}(x) = \{x', x''\}$.

Definition 3.24 (global index of a node). The global index $\lambda(x)$ of a node $x \in \mathcal{N}$ is defined recursively as follows:

- if $x \in \mathcal{P}$, set $\lambda(x) = 0$;
- if $x \in \mathcal{H}$ and $\mathcal{B}(x) = \{x', x''\}$, set $\lambda(x) = \max(\lambda(x'), \lambda(x'')) + 1$.

The set of all nodes of $\mathcal{T}$ is thus partitioned according to the value of the global index: for any integer $l \geq 0$, we set $\mathcal{H}_l = \{x \in \mathcal{N} \mid \lambda(x) = l\}$. Note that $\mathcal{H}_0 = \mathcal{P}$. An example of distribution of global indices for $d = 2$ is shown in Figure 3.10.

We define the global index of the triangulation $\mathcal{T}$ by $\lambda(\mathcal{T}) := \max_{x \in \mathcal{N}} \lambda(x)$. The level of non-conformity of the triangulations we are dealing with is controlled by the following condition of admissibility.

Definition 3.25 (Λ-admissibility). Let $\Lambda \geq 0$ be an integer. A refinement $\mathcal{T}$ of $\mathcal{T}_0$ is Λ-admissible if

$$\lambda(\mathcal{T}) \leq \Lambda. \tag{3.47}$$

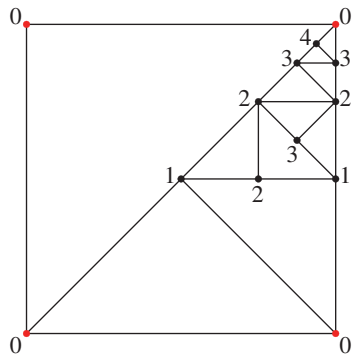


Figure 3.10. Example of distributions of proper nodes (red) and hanging nodes (black), with associated global indices λ.

If $\lambda(\mathcal{T}) \geq 1$, then $\mathcal{T}$ is non-conforming, but otherwise $\mathcal{T} \in \mathbb{T}$ is conforming if $\lambda(\mathcal{T}) = 0$. The collection of all Λ-admissible partitions is denoted by $\mathbb{T}^{\Lambda}$.

Λ-admissibility has the following basic implications.

Proposition 3.26 (properties of Λ-admissible partitions). *Let T be any element of a Λ-admissible partition $\mathcal{T}$.*

(i) *If $e \subset \partial T$ is an edge of T, then e may contain at most $2^{\Lambda} - 1$ hanging nodes.*

(ii) *If $e \subset \partial T$ is an edge of some other element T', then $h_{T'} \simeq h_T$, where the hidden constants depend only on the shape of the initial triangulation $\mathcal{T}_0$ and possibly on Λ.*

Proof. Case (i) stems from the fact that the edge may contain at most 2^{k-1} hanging nodes of level k for $1 \leq k \leq \Lambda$. To prove (ii) we observe that the length ratio $|\bar{e}|/|e|$, where $\bar{e}$ is the edge of T containing e, is at most 2^{Λ}, and we conclude by invoking the shape regularity of the partition. □

In the space $\mathbb{V}_{\mathcal{T}}$ of continuous piecewise linear maps over $\mathcal{T}$, functions are uniquely defined by their values at the proper nodes of $\mathcal{T}$. So it is natural to introduce the canonical continuous piecewise linear basis functions ϕ_P associated with any proper node P. They satisfy

$$v = \sum_{P \in \mathcal{P}} v(P)\phi_P \quad \text{for all } \ v \in \mathbb{V}_{\mathcal{T}}, \tag{3.48}$$

and are defined by the conditions $\phi_P \in \mathbb{V}_{\mathcal{T}}$ and

- $\phi_P(z) = 1$ if $z = P$, $\phi_P(z) = 0$ if $z \in \mathcal{P} \setminus \{P\}$.

The values of ϕ_P at the hanging nodes, hence everywhere in the domain, can be reconstructed by linear interpolation as follows: assuming that ϕ_P has been defined

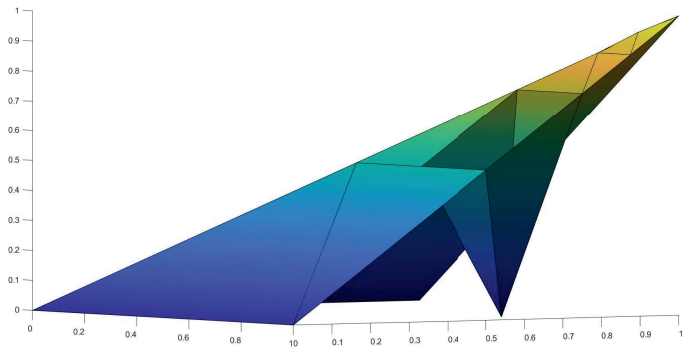


Figure 3.11. Plot of the basis function ϕ_P on the non-conforming triangulation shown in Figure 3.10, for P equal to the upper right corner of the domain, after using bisection to convert the lowest hanging node with global index 3 into a proper node.

at all nodes of global index $< l$, if $z \in \mathcal{H}_l$ and $\mathcal{B}(z) = \{z', z''\}$, then

$$\phi_P(z) = \frac{1}{2}(\phi_P(z') + \phi_P(z'')).$$

An example of basis function ϕ_P on a non-conforming triangulation is provided in Figure 3.11.

The *domain of influence* of a proper node P is the set

$$\omega_{\mathcal{T}}(P) = \operatorname{supp}(\phi_P),$$

highlighted in grey in Figure 3.9; this notion was introduced in Babuška and Miller (1987) in the context of K-meshes; see also Bonito and Nochetto (2010). To identify elements $T \in \mathcal{T}$ contained in $\omega_{\mathcal{T}}(P)$, we introduce for any node $x \in \mathcal{N}$ the set $\mathcal{P}(x)$ of the *proper nodes influencing* x, which is defined recursively as follows:

- initialize $\mathcal{P}(x) = \{x\}$;
- while $\mathcal{P}(x) \cap \mathcal{H} \neq \emptyset$, if $y \in \mathcal{P}(x) \cap \mathcal{H}$ replace $\mathcal{P}(x)$ with $(\mathcal{P}(x) \setminus \{y\}) \cup \mathcal{B}(y)$.

Then $T \subseteq \omega_{\mathcal{T}}(P)$ if and only if P influences some vertex of T, that is, there exists a vertex v of T such that $P \in \mathcal{P}(v)$.

One of the consequences of the Λ-admissibility assumption of $\mathcal{T}$ is the following result, which says that all elements T contained in $\omega_{\mathcal{T}}(P)$ have comparable size.

Proposition 3.27 (size of the domain of influence). *There exists a positive constant $C = C(\mathcal{T}_0, \Lambda)$, depending only on the shape of the initial triangulation $\mathcal{T}_0$ and possibly on Λ, such that for any $P \in \mathcal{P}$*

$$\operatorname{diam} \omega_{\mathcal{T}}(P) \leq C\, h_T \quad \textit{for all } T \in \mathcal{T},\ T \subseteq \omega_{\mathcal{T}}(P).$$

Proof. Elements in $\omega_{\mathcal{T}}(P)$ having P as a vertex share in pairs an edge or a portion of an edge, hence – as noted above – Λ-admissibility implies the existence

of a characteristic size, say h_P, which is comparable to the diameter of each of them. On the other hand, any $T \subset \omega_{\mathcal{T}}(P)$ not containing P has at least one vertex $v_T \in \mathcal{H}$ such that $P \in \mathcal{P}(v_T)$. Thus there exists a sequence $\{y_k \mid 0 \le k \le K\}$ of vertices satisfying $y_0 = v_T$, $y_K = P$ and $y_{k+1} \in \mathcal{B}(y_k)$ for $0 \le k < K$; since $\lambda(y_k) \ge \lambda(y_{k+1}) + 1$, necessarily $K \le \Lambda$. Correspondingly, we can find a chain of at most K elements, starting at T and ending at an element containing P, which share in pairs an edge or a portion of an edge. We deduce that $h_T \simeq h_P$, and $\operatorname{dist}(T, P) \simeq h_T$, where the hidden constants may depend only on the shape of the initial triangulation and on Λ. The conclusion easily follows from these results. □

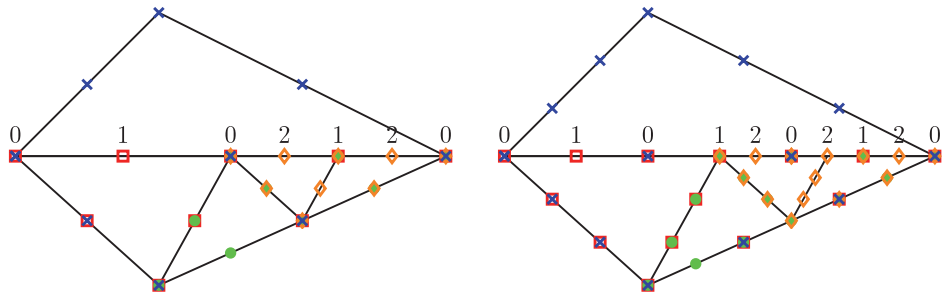
We now turn to the case of polynomial degree $n > 1$; we refer to Canuto and Fassino (2023) for more details. The concept of hanging node is no longer solely related to the geometry of the mesh, but also to the distribution of degrees of freedom along the edges of the elements. For instance, consider a full edge e shared by two triangles T and T', and bisect T' to create two new elements T_1 and T_2 having e as a vertex. If we use quadratic Lagrangian elements, the midpoint x of e carries a degree of freedom for the three elements that share it, so we do not consider it as a hanging node; on the other hand, the nodes at distance $\frac{1}{4}|e|$ and $\frac{3}{4}|e|$ from an endpoint of e are hanging nodes (despite being vertices of no triangle) since they do not carry a degree of freedom for the element T. If we move to cubic Lagrangian elements, then x becomes a hanging node, together with the nodes at distance $\frac{1}{6}|e|$ and $\frac{5}{6}|e|$ from an endpoint of e, whereas the nodes at distance $\frac{1}{3}|e|$ and $\frac{2}{3}|e|$ are not hanging nodes, since they carry a degree of freedom for each triangle they belong to.

In general, for a partition $\mathcal{T}$ made of classical affine Lagrangian or Hermitian elements, the hanging nodes are defined as follows.

- Given an element $T \in \mathcal{T}$, the set $\mathcal{P}_T$ of the *proper nodes of* T is made of all images of the reference n-lattice via the affine transformation. The set $\mathcal{H}_T$ of the *hanging nodes of* T collects the points of ∂T that are not proper nodes of T, but are proper nodes of some other contiguous element T'. The set of all nodes of T is $\mathcal{N}_T := \mathcal{P}_T \cup \mathcal{H}_T$.
- At the global level, if $\mathcal{N} = \bigcup_{T \in \mathcal{T}} \mathcal{N}_T$ is the set of all nodes of $\mathcal{T}$, the set $\mathcal{P} \subseteq \mathcal{N}$ of the *proper nodes of* $\mathcal{T}$ contains those nodes that are proper nodes for all elements they belong to. The complementary set $\mathcal{H} := \mathcal{N} \setminus \mathcal{P}$ is the set of the *hanging nodes of* $\mathcal{T}$.

In other words, a hanging node of $\mathcal{T}$ is a point that carries a degree of freedom for some but not all elements it belongs to. With this definition of proper nodes, representation (3.48) of continuous piecewise linear maps extends to $n > 1$.

The *global index* $\lambda(x)$ of a node $x \in \mathcal{N}$ is precisely defined as in Definition 3.24. The set $\mathcal{B}(x) \subset \mathcal{N}$ collects the endpoints of an interval $[x', x'']$, contained in the skeleton of $\mathcal{T}$, that has been bisected when x has been created, and contains no other node inside.



(a) (b)

Figure 3.12. Triangulation after the three refinements in the case $n = 2$ (a) and in the case $n = 3$ (b). Blue crosses represent the original degrees of freedom on the initial conforming mesh. Red squares, green circles and orange triangles are used for the degrees of freedom of the first, second and third refinement, respectively. All nodes are proper, except those on the horizontal line, whose global index is reported.

Figure 3.12 provides two examples, for $n = 2$ and $n = 3$, of distributions of hanging nodes and corresponding global indices, created by successive bisections starting from an initial conforming partition.

The concept of Λ-admissibility, given in Definition 3.25, remains unchanged for $n > 1$. The statements in Proposition 3.26, too, extend to the higher-order case; the maximum number of hanging nodes on an edge now being $O(n2^{\Lambda})$. Consequently, the conclusion of Proposition 3.27 remains valid when $n > 1$ as well: there is a constant $C = C(\mathcal{T}_0, \Lambda)$ such that

$$\operatorname{diam} \omega_{\mathcal{T}}(P) \leq C\, h_T \quad \text{for all } T \in \mathcal{T},\ T \subseteq \omega_{\mathcal{T}}(P). \tag{3.49}$$

Remark 3.28 (quadrilateral and hexahedral partitions). It is readily seen that the definitions of global index of a node and Λ-admissible partition extend seamlessly to shape-regular meshes made of quadrilaterals refined by a quadtree strategy ($d = 2$) (see Figure 3.9(a) for an example), or by hexahedra refined by an octree strategy ($d = 3$). The same holds for heterogeneous partitions made of a combination of simplices and hexahedra. All results reported above are valid for such partitions. We refer to Bonito and Nochetto (2010) for details.

Λ-admissible meshes under refinement. Given a Λ-admissible grid $\mathcal{T}$, a subset $\mathcal{M}$ of elements marked for refinement, and a desired number $b \geq 1$ of subdivisions to be performed in each marked element, the procedure

$$\mathcal{T}_* = \mathsf{REFINE}(\mathcal{T}, \mathcal{M}, \Lambda)$$

creates a minimal Λ-admissible mesh $\mathcal{T}_* \geq \mathcal{T}$ such that all the elements of $\mathcal{M}$ are subdivided at least b times. In order for $\mathcal{T}_*$ to be Λ-admissible, perhaps other elements not in $\mathcal{M}$ must be partitioned. Despite the fact that admissibility is a

constraint on the refinement procedure weaker than conformity, it cannot avoid the propagation of refinements beyond $\mathcal{M}$. The complexity of REFINE is again an issue which we discuss in Section 8.2: we show that Theorem 3.16 extends to this case.

Theorem 3.29 (complexity of REFINE for Λ-admissible meshes). *Let $\mathcal{T}_0$ be an arbitrary conforming partition of Ω, except for the bisection algorithm in which case $\mathcal{T}_0$ satisfies the labelling (3.35) for $d = 2$ or its higher-dimensional counterpart (Stevenson 2008). Then the estimate*

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \leq D \sum_{j=0}^{k-1} \#\mathcal{M}_j \quad \text{for all } k \geq 1$$

holds with a constant D depending on $\mathcal{T}_0$, d, n and Λ.

The following result about uniform refinements of Λ-admissible partitions will be used below. *The uniform refinement $\mathcal{T}_*$ of a partition $\mathcal{T} \in \mathbb{T}^\Lambda$ is the partition obtained by bisecting each element of $\mathcal{T}$ d times.* This implies, in particular, that each edge of $\mathcal{T}$ is bisected once.

Proposition 3.30 (Λ-admissibility of uniform refinements). *If $\mathcal{T} \in \mathbb{T}^\Lambda$ is a Λ-admissible partition and $\mathcal{T}_*$ is its uniform refinement, then $\mathcal{T}_*$ is Λ-admissible.*

Proof. A simple recursion argument on the global index of the hanging nodes of $\mathcal{T}$ shows that after refinement each such node either becomes a proper node or its global index is reduced by 1. At the same time, new nodes are created by the refinement, whose global index is at most 1 plus the maximal global index of the pre-existing nodes. In both cases, the maximal global index of $\mathcal{T}_*$ cannot exceed Λ. □

A simple consequence is the following result, which is useful for controlling the mesh size between consecutive refinements.

Corollary 3.31 (bound on the refinements). REFINE *with $b = 1$ never refines an element of a Λ-admissible partition $\mathcal{T}$ more than d times.*

Proof. REFINE gives the smallest Λ-admissible mesh $\mathcal{T}_*$ such that all the marked elements of $\mathcal{T}$ have been refined. Since the uniform refinement of $\mathcal{T}$ remains Λ-admissible, the minimality of $\mathcal{T}_*$ implies that no element of the marked set can be refined more than d times. □

We conclude by emphasizing that the polynomial interpolation and adaptive approximation theories of Sections 3.3 and 3.6 extend to non-conforming meshes with fixed level of incompatibility as well.

4. *A posteriori* error analysis

Numerical solutions to a boundary value problem serve to approximate its unknown exact solution. In such a context, it is of interest

- to quantify the error of the numerical solution,
- to gain information for *adapting* the discretization to the exact solution

in a computationally accessible manner. These are the two goals of an *a posteriori error analysis*, where the term *a posteriori* hints at the fact that the numerical solution itself can be involved. To achieve the two goals, the *a posteriori* analysis individuates so-called *error estimators* that, ideally, are computable, split into local contributions called *indicators*, and bound the error from above and below.

This section exemplifies such an analysis, considering the numerical solution of the boundary value problem (2.5), that is,

$$-\operatorname{div}(\boldsymbol{A}\nabla u) + cu = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

with Lagrange elements of arbitrary fixed order $n \geq 1$. Throughout this section we adopt the notation and assumptions of previous sections for this model setting. In particular, the *exact solution* $u \in H^1_0(\Omega)$ solves the variational problem (2.7) and, given a simplicial conforming mesh $\mathcal{T} \in \mathbb{T}$ of Ω and finite element space

$$\mathbb{V}_{\mathcal{T}} := \left\{ v \in \mathbb{S}^{n,0}_{\mathcal{T}} \mid v|_\Omega = 0 \right\} \subset H^1_0(\Omega),$$

the *Galerkin approximation* solves

$$u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}: \quad \mathcal{B}[u_{\mathcal{T}}, w] = \langle f, w \rangle \quad \text{for all } w \in \mathbb{V}_{\mathcal{T}}, \tag{4.1}$$

with the bilinear form $\mathcal{B}$ from (2.8).

We stress that the analysis will be conducted under the regularity assumptions

$$\boldsymbol{A} \in L^\infty(\Omega; \mathbb{R}^{d\times d}), \quad c \in L^\infty(\Omega), \quad f \in H^{-1}(\Omega) = H^1_0(\Omega)^*, \tag{4.2}$$

used in Section 2.4 to establish existence and uniqueness of the exact solution. This fact distinguishes the approach below, which builds on Kreuzer and Veeser (2019), from most other approaches requiring additional regularity; see e.g. Verfürth (2013). Notably, this difference not only allows for covering more examples but is also related to strengthening the relationship between error and estimator to a true equivalence on any admissible mesh $\mathcal{T} \in \mathbb{T}$.

It is useful to recall two differences between the forcing f and the coefficients $(\boldsymbol{A}, c)$. First, while the exact solution u depends linearly on the forcing f, it depends nonlinearly on the diffusion tensor $\boldsymbol{A}$ and the reaction coefficient c. To state the second difference, let $u \in H^1_0(\Omega)$ and note that the assumptions (4.2) on $(\boldsymbol{A}, c)$ imply the 'missing' $f \in H^{-1}(\Omega)$. On the other hand, the assumptions on (A, f) imply only $cu \in H^{-1}(\Omega)$, while the assumptions on (c, f) imply only $-\operatorname{div}(\boldsymbol{A}\nabla u) \in H^{-1}(\Omega)$. These conditions are weaker than the 'missing' $u \in H^1_0(\Omega)$, and are due to the multiplicative role of $(\boldsymbol{A}, c)$ in the differential equation.

In order to elucidate the new twists allowing for (4.2), this section is organized as follows. We start with steps of the *a posteriori* analysis that are common to the 'classical' and new approaches. We then illustrate the classical approach with the *standard residual estimator*, and afterwards develop the new approach resulting in a modification of the standard residual estimator, called the *modified residual estimator*. Finally, we conclude by adapting the new approach to other techniques of *a posteriori* error estimation and boundary conditions.

In what follows, the notation may or may not indicate the dependences of a given quantity. We shall balance readability and the importance of the dependence in the given context. For example, in (4.1), the discrete solution depends not only on the mesh $\mathcal{T}$ but also on the data Ω, $\boldsymbol{A}$, c and f in problem (2.5). We write $\mathcal{T}$ explicitly because of its more prominent role in the *a posteriori* analysis. Let $\mathcal{F} := \mathcal{F}_{\mathcal{T}}$ denote the set of all interior $(d-1)$-dimensional faces of $\mathcal{T}$. The letter C will be used for a generic constant, with possibly different values at each occurrence. If not stated otherwise, it may depend on the shape regularity coefficient σ from (3.9), the dimension d and the polynomial degree n in $\mathbb{V}_{\mathcal{T}}$.

4.1. Error, residual and localization of residual norm

This section starts the *a posteriori* analysis by establishing that a suitable norm of the so-called *residual*

- is equivalent to the error $\|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)}$, and
- admits a *localization* in the sense that it splits into suitable local contributions depending on accessible quantities, i.e. on data $\mathcal{D} = (\boldsymbol{A}, c, f)$ and the discrete solution $u_{\mathcal{T}}$.

We do not yet consider computability: this important aspect will be addressed in the following sections.

Replacing the exact solution u in the weak form (2.7) with its approximation $u_{\mathcal{T}}$, we define the *residual* $R_{\mathcal{T}} \in H^{-1}(\Omega)$:

$$\langle R_{\mathcal{T}}, w\rangle = \langle f, w\rangle - \mathcal{B}[u_{\mathcal{T}}, w] \quad \text{for all } w \in H_0^1(\Omega).$$

We thus have a quantity that depends only on data $\mathcal{D}$ and the approximate solution $u_{\mathcal{T}}$ and relates to the error function $u - u_{\mathcal{T}}$ as follows:

$$\langle R_{\mathcal{T}}, w\rangle = \mathcal{B}[u - u_{\mathcal{T}}, w] \quad \text{for all } w \in H_0^1(\Omega). \tag{4.3}$$

Continuity and coercivity of the bilinear form $\mathcal{B}$ then provide a quantitative relationship between error and residual.

Lemma 4.1 (error and residual). *The error of the approximation $u_{\mathcal{T}}$ is equivalent to the residual norm. More precisely,*

$$\frac{1}{\|B\|}\|R_{\mathcal{T}}\|_{H^{-1}(\Omega)} \le \|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)} \le \frac{1}{\alpha}\|R_{\mathcal{T}}\|_{H^{-1}(\Omega)},$$

where $\|\mathcal{B}\| \geq \alpha > 0$ *are, respectively, the continuity and coercivity constants of the bilinear form* $\mathcal{B}$.

Proof. The error–residual relationship (4.3) yields the lower bound,

$$\|R_{\mathcal{T}}\|_{H^{-1}(\Omega)} = \sup_{w \in H_0^1(\Omega)} \frac{\langle R_{\mathcal{T}}, w\rangle}{\|\nabla w\|_{L^2(\Omega)}} = \sup_{w \in H_0^1(\Omega)} \frac{\mathcal{B}[u - u_{\mathcal{T}}, w]}{\|\nabla w\|_{L^2(\Omega)}}$$
$$\leq \|\mathcal{B}\| \|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)},$$

while the choice $w = u - u_{\mathcal{T}}$ therein gives

$$\alpha \|\nabla(u - u_{\mathcal{T}})\|^2_{L^2(\Omega)} \leq \mathcal{B}[u - u_{\mathcal{T}}, u - u_{\mathcal{T}}] = \langle R_{\mathcal{T}}, u - u_{\mathcal{T}}\rangle$$
$$\leq \|R_{\mathcal{T}}\|_{H^{-1}(\Omega)} \|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)}$$

and thus the upper bound. □

Remark 4.2 (role of forcing vs. role of coefficients). In addition to the two differences between right-hand side f and coefficients $(\boldsymbol{A}, c)$ mentioned in the introduction of this section, a third one implicitly arises in the proof of Lemma 4.1: the coefficients defining the bilinear form $\mathcal{B}$ are fixed, while the right-hand side f is replaced by the residual $R_{\mathcal{T}}$ in (4.3), which varies with the mesh $\mathcal{T}$.

Remark 4.3 (local lower estimate for the error). The proof of Lemma 4.1 shows that the lower bound of the error hinges on the continuity of the bilinear form $\mathcal{B}$. Since the evaluation of $\mathcal{B}$ involves only local operators, one might expect that there are also local lower bounds. This, however, depends on the interplay of the underlying differential operator and the choice of the test space norm. Indeed, in the case of the Poisson problem, i.e. $\boldsymbol{A} = \boldsymbol{I}$, $c = 0$, and the test space norm $\|\nabla \cdot\|_{L^2(\Omega)}$, we easily see that

$$\|R_{\mathcal{T}}\|_{H^{-1}(\omega)} \leq \|\nabla(u - u_{\mathcal{T}})\|_{L^2(\omega)}$$

for any subdomain $\omega \subset \Omega$. This local lower bound, however, does not carry over to the general case with $c \neq 0$, as the error function itself is bounded by its gradient only through the global inequality in Lemma 2.2 (first Poincaré inequality). On the other hand, endowing the test space $H_0^1(\Omega)$ with the full H^1-norm $\|\cdot\|_{H^1(\Omega)}$ yields

$$\|R_{\mathcal{T}}\|_{(H^1(\omega))^*} \leq \max\{\alpha_1, \|c\|_{L^\infty(\Omega)}\} \|u - u_{\mathcal{T}}\|_{H^1(\omega)}$$

for any subdomain $\omega \subset \Omega$.

In line with Carstensen *et al.* (2014), we shall not invoke local lower bounds to derive convergence and rate-optimality for the error of AFEM, although they might appear useful or even crucial in other settings.

Remark 4.4 (constants in error–residual relationship). Given the norm measuring the error, i.e. the norm of the trial space, the choice of the test space norm

is important for the ensuing constants in the error–residual relationship; see e.g. Verfürth (2013, Sections 4.3, 4.6). To avoid related additional technicalities and difficulties, the test space is endowed with the straightforward norm $\|\nabla \cdot\|_{L^2(\Omega)}$.

Lemma 4.1 establishes the first goal that was set out at the beginning of this section. We now turn to the second one, that is, we split the residual norm $\|R_{\mathcal{T}}\|_{H^{-1}(\Omega)}$ into local contributions. Note that the nature of the dual norm $\|\cdot\|_{H^{-1}(\Omega)}$ makes this task less obvious than for integral norms as in the error $\|\nabla(u-u_{\mathcal{T}})\|_{L^2(\Omega)}$.

We start by recalling that the definition of the Galerkin approximation $u_{\mathcal{T}}$ implies that its residual is orthogonal to the discrete trial space $\mathbb{V}_{\mathcal{T}} = \mathbb{S}^{n,0}_{\mathcal{T}} \cap H^1_0(\Omega)$:

$$\langle R_{\mathcal{T}}, w\rangle = 0 \quad \text{for all } w \in \mathbb{V}_{\mathcal{T}}.$$

Let $\mathcal{V}$ denote the set of vertices of $\mathcal{T}$ and let $\phi_z \in \mathbb{S}^{1,0}_{\mathcal{T}}$ be the hat function with $\phi_z(y) = \delta_{yz}$ for all vertices $y \in \mathcal{V}$. In what follows, the *partial* orthogonality

$$\langle R_{\mathcal{T}}, \phi_z\rangle = 0 \quad \text{for all } z \in \mathcal{V} \tag{4.4}$$

will be crucial for splitting the non-local norm of the residual into local contributions. The latter ones will be formulated in terms of the supports of the hat functions, and thus to each vertex $z \in \mathcal{V}$ we associate the following subset and submesh:

$$\omega_z := \operatorname{supp} \phi_z = \bigcup_{T \in \mathcal{T}_z} T \quad \text{with} \quad \mathcal{T}_z := \{T \in \mathcal{T} \mid T \ni z\}. \tag{4.5}$$

These subsets, called *stars*, form a *subdomain covering* of Ω, that is, each interior $\mathring{\omega}_z$ is a domain and $\overline{\Omega} = \cup_{z\in\mathcal{V}} \omega_z$. The *overlapping index* $\operatorname{ess\,sup}_{x\in\Omega} \#\{z \in \mathcal{V} \mid \omega_z \ni x\}$ of this covering is bounded by $(d+1)$.

Lemma 4.5 (localization of H^{-1}-norm). *Let $\ell \in H^{-1}(\Omega)$ be an arbitrary linear functional on $H^1_0(\Omega)$.*

(i) *If $\langle \ell, \phi_z\rangle = 0$ for all interior vertices $z \in \mathcal{V} \cap \Omega$, then*

$$\|\ell\|^2_{H^{-1}(\Omega)} \le (d+1) C^2_{\mathrm{loc}} \sum_{z\in\mathcal{V}} \|\ell\|^2_{H^{-1}(\omega_z)},$$

where C_{loc} depends only on the shape regularity coefficient σ from (3.9) and d.

(ii) *For any subdomain covering $(\omega_i)_{i\in I}$ of Ω with finite overlapping index $C_{\mathrm{ovrl}} := \operatorname{ess\,sup}_{x\in\Omega} \#\{i \in I \mid \omega_i \ni x\}$, we have*

$$\sum_{i\in I} \|\ell\|^2_{H^{-1}(\omega_i)} \le C_{\mathrm{ovrl}} \|\ell\|^2_{H^{-1}(\Omega)}.$$

Theorem 3.5 of Blechta, Málek and Vohralík (2020) generalizes Lemma 4.5 to the W^{-1}_p-norm, $1 < p < \infty$. Lemma 4.66 below provides an alternative localization with different local norms.

Proof. 1 We start by showing statement (i). Thanks to the orthogonality of ℓ, we may write

$$\langle \ell, w\rangle = \left\langle \ell, w - \sum_{z\in\mathcal{V}} c_z\phi_z \right\rangle,$$

where any c_z is an arbitrary constant if $z \in \mathcal{V}\cap\Omega$ is an interior vertex and 0 if $z \in \mathcal{V}\cap\partial\Omega$ is a boundary vertex. Using the partition of unity $\sum_{z\in\mathcal{V}}\phi_z = 1$ on Ω, we split the new test function

$$w - \sum_{z\in\mathcal{V}} c_z\phi_z = \sum_{z\in\mathcal{V}}(w - c_z)\phi_z$$

into local contributions $(w - c_z)\phi_z \in H^1_0(\omega_z)$, $z \in \mathcal{V}$. The constant c_z allows us to counter the gradient generated by the cut-off with ϕ_z. Indeed, the product rule, $0 \le \phi_z \le 1$ and $|\nabla\phi_z| \le C(d)\sigma h_T^{-1}$ on an element $T \in \mathcal{T}$ lead to

$$\begin{aligned}
\|\nabla((w - c_z)\phi_z)\|_{L^2(\omega_z)} &\le \|\phi_z\nabla(w - c_z)\|_{L^2(\omega_z)} + \|(w - c_z)\nabla\phi_z\|_{L^2(\omega_z)}\\
&\le \|\phi_z\|_{L^\infty(\omega_z)}\|\nabla w\|_{L^2(\omega_z)} + \|\nabla\phi_z\|_{L^\infty(\omega_z)}\|w - c_z\|_{L^2(\omega_z)}\\
&\le \|\nabla w\|_{L^2(\omega_z)} + C(d)\sigma\left(\max_{T\subset\omega_z} h_T^{-1}\right)\|w - c_z\|_{L^2(\omega_z)}.
\end{aligned}$$

If we choose $c_z = \fint_{\omega_z} w$ for interior vertices $z \in \mathcal{V}\cap\Omega$, then Lemma 2.3 (second Poincaré inequality) on reference stars implies

$$\|w - c_z\|_{L^2(\omega_z)} \lesssim \operatorname{diam}\omega_z\|\nabla w\|_{L^2(\omega_z)}.$$

The same inequality follows for boundary vertices $z \in \mathcal{V}\cap\partial\Omega$ thanks to the fact that w vanishes on at least one face of $\partial\omega_z \cap \partial\Omega$. Combining this with $\operatorname{diam}\omega_z \lesssim h_T$ for $T \subset \omega_z$, we thus obtain, for all local contributions, the stability bound

$$\|\nabla((w - c_z)\phi_z)\|_{L^2(\omega_z)} \le C_{\mathrm{loc}}\|\nabla w\|_{L^2(\omega_z)}, \tag{4.6}$$

where the constant C_{loc} depends only on d and σ. Hence

$$\langle \ell, w\rangle = \left\langle \ell, w - \sum_{z\in\mathcal{V}} c_z\phi_z \right\rangle = \sum_{z\in\mathcal{V}}\langle \ell, (w - c_z)\phi_z\rangle$$

gives

$$\begin{aligned}
|\langle \ell, w\rangle| &\le C_{\mathrm{loc}}\sum_{z\in\mathcal{V}}\|\ell\|_{H^{-1}(\omega_z)}\|\nabla w\|_{L^2(\omega_z)}\\
&\le \sqrt{d+1}\,C_{\mathrm{loc}}\left(\sum_{z\in\mathcal{V}}\|\ell\|^2_{H^{-1}(\omega_z)}\right)^{1/2}\|\nabla w\|_{L^2(\Omega)}
\end{aligned}$$

and (i) is proved.

$\boxed{2}$ We verify statement (ii). For each index $i \in I$, define $v_i \in H_0^1(\omega_i) \subset H_0^1(\Omega)$ by

$$\int_{\omega_i} \nabla v_i \cdot \nabla w = \langle \ell, w \rangle \quad \text{for all } w \in H_0^1(\omega_i).$$

We obtain

$$\langle \ell, v_i \rangle = \|\nabla v_i\|^2_{L^2(\omega_i)} = \|\ell\|^2_{H^{-1}(\omega_z)}$$

by arguments similar to those in the proof of Lemma 4.1 (error and residual). The sum $v := \sum_{i \in I} v_i$ is in $H_0^1(\Omega)$ with

$$\begin{aligned}\|\nabla v\|^2_{L^2(\Omega)} &\le \int_\Omega \Big| \sum_{i \in I_x} \nabla v_i(x) \Big|^2 \,\mathrm{d}x \le \int_\Omega \#I_x \sum_{i \in I_x} |\nabla v_i(x)|^2 \,\mathrm{d}x \\ &\le C_{\mathrm{ovrl}} \sum_{i \in I} \|\nabla v_i\|^2_{L^2(\omega_i)} = C_{\mathrm{ovrl}} \sum_{i \in I} \|\ell\|^2_{H^{-1}(\omega_z)},\end{aligned}$$

where we denote the set of active indices in $x \in \Omega$ by $I_x := \{i \in I \mid \omega_i \ni x\}$. Inserting this in

$$\sum_{i \in I} \|\ell\|^2_{H^{-1}(\omega_z)} = \sum_{i \in I} \langle \ell, v_i \rangle = \langle \ell, v \rangle \le \|\ell\|_{H^{-1}(\omega_z)} \|\nabla v\|_{L^2(\Omega)}$$

establishes the desired inequality. □

Thanks to the partial orthogonality (4.4) and the properties of the star covering ω_z, $z \in \mathcal{V}$, we readily obtain the following statement.

Corollary 4.6 (star localization of residual norm). *The H^{-1}-norm of the residual can be split into local contributions on stars:*

$$\frac{1}{d+1} \sum_{z \in \mathcal{V}} \|R_{\mathcal{T}}\|^2_{H^{-1}(\omega_z)} \le \|R_{\mathcal{T}}\|^2_{H^{-1}(\Omega)} \le (d+1) C_{\mathrm{loc}} \sum_{z \in \mathcal{V}} \|R_{\mathcal{T}}\|^2_{H^{-1}(\omega_z)},$$

where C_{loc} depends only on d and the shape regularity coefficient σ.

The upper bound of the global residual norm in Corollary 4.6 employs the stars ω_z, $z \in \mathcal{V}$, as local domains. The next remark assesses this choice by discussing conceivable alternatives in terms of elements and domains of the type

$$\omega_F := \bigcup_{T \in \mathcal{T}_F} T \quad \text{with } \mathcal{T}_F := \{T \in \mathcal{T} \mid T \supset F\}, \tag{4.7}$$

where $F \in \mathcal{F}$ is an interior face of $\mathcal{T}$.

Remark 4.7 (star localization is minimal for $d \ge 2$). The use of stars in the localization of the global residual norm is a sort of minimal choice, except for the special case $d = 1$ where elements can be used.

- If $d = 1$, point values are defined for functions in $H^1(\Omega)$. This allows an upper bound with elements instead of stars as local domains. In fact, choosing

$c_z = w(z)$ for all interval endpoints, the function $\sum_{z\in\mathcal{V}} c_z\phi_z$ amounts to the Lagrange interpolant $I_{\mathcal{T}}w \in \mathbb{S}^{1,0}_{\mathcal{T}} \cap H^1_0(\Omega)$, and we have $(w - I_{\mathcal{T}}w)|_I \in H^1_0(I)$ with $\|\nabla(w - I_{\mathcal{T}}w)\|_{L^2(I)} \lesssim \|\nabla w\|_{L^2(I)}$ for any interval I of the mesh $\mathcal{T}$. Arguing as in the proof of Lemma 4.5(i) then gives

$$\|R\|^2_{H^{-1}(\Omega)} \lesssim \sum_{I\in\mathcal{T}} \|R\|^2_{H^{-1}(I)}.$$

- An upper bound where the stars are replaced by elements $T \in \mathcal{T}$ cannot hold in general because it does not account for face-supported residual contributions. For example, consider our setting with

$$d \geq 2, \quad \boldsymbol{A} = \boldsymbol{I}, \quad c = 0, \quad \langle f, w\rangle = \int_F qw, \ w \in H^1_0(\Omega),$$

  where $F \in \mathcal{F}$ and $q \neq 0$ is L^2-orthogonal to $\mathbb{P}_n(F)$. Then we have $u \neq 0 = u_{\mathcal{T}}$ and therefore $\|R_{\mathcal{T}}\|_{H^{-1}(\Omega)} > 0$ but $\|R_{\mathcal{T}}\|_{H^{-1}(T)} = 0$ for any $T \in \mathcal{T}$.

- An upper bound with pairs ω_F, $F \in \mathcal{F}$, instead of stars cannot hold in general. This can be shown by considering our setting with

$$d = 2, \quad \boldsymbol{A} = \boldsymbol{I}, \quad c = 0, \quad \langle f_\varepsilon, w\rangle = \frac{1}{\pi\varepsilon^2}\int_{B_\varepsilon(z)} w, \ w \in H^1_0(\Omega),$$

  where $z \in \mathcal{V}$ is a vertex of a suitable triangulation $\mathcal{T}$, for $\varepsilon \searrow 0$. The limiting right-hand side is the Dirac measure in z, which, formally, is not seen by any $\|\cdot\|_{H^{-1}(\omega_F)}$, $F \in \mathcal{F}$. We thus have $\|R_{\mathcal{T}}\|_{H^{-1}(\Omega)} \to \infty$ but $\sum_{F\in\mathcal{F}} \|R_{\mathcal{T}}\|_{H^{-1}(\omega_F)} \lesssim 1$; see Tantardini, Veeser and Verfürth (2024).

The bisection method for mesh refinement is element-oriented. It is therefore advantageous to dispose of an element-indexed reformulation of the localization in Corollary 4.6. For that purpose, we recall the notion of *patches*,

$$\omega_T := \bigcup_{\substack{T'\in\mathcal{T}\\ T'\cap T\neq\emptyset}} T', \tag{4.8}$$

and may use the following equivalence.

Lemma 4.8 (localization re-indexing). *For any functional $\ell \in H^{-1}(\Omega)$, we have*

$$\sum_{z\in\mathcal{V}} \|\ell\|^2_{H^{-1}(\omega_z)} \approx \sum_{T\in\mathcal{T}} \|\ell\|^2_{H^{-1}(\omega_T)},$$

where the hidden constants depend on d and the shape regularity coefficient σ.

Note that, in contrast to the localization itself, its re-indexing does not require any orthogonality such as $\langle \ell, \phi_z\rangle = 0$ for all $z \in \mathcal{V} \cap \Omega$.

Proof. For any vertex $z \in \mathcal{V}$, there is an element $T \in \mathcal{T}$ containing z. Then the inclusion $\omega_z \subset \omega_T$ yields the inequality $\|\ell\|_{H^{-1}(\omega_z)} \le \|\ell\|_{H^{-1}(\omega_T)}$. Hence

$$\sum_{z\in\mathcal{V}} \|\ell\|^2_{H^{-1}(\omega_z)} \le \sum_{T\in\mathcal{T}} \|\ell\|^2_{H^{-1}(\omega_T)}.$$

To show the converse inequality, let $T \in \mathcal{T}$ be any element and $w \in H^1_0(\omega_T)$. Given any vertex $z \in \mathcal{V} \cap \omega_T$, Lemma 2.2 (first Poincaré inequality) on ω_T implies the stability bound

$$\begin{aligned}\|\nabla(w\phi_z)\|_{L^2(\omega_z\cap\omega_T)} &\le \|\phi_z \nabla w\|_{L^2(\omega_z\cap\omega_T)} + \|w\nabla\phi_z\|_{L^2(\omega_z\cap\omega_T)}\\ &\le \|\nabla w\|_{L^2(\omega_z\cap\omega_T)} + C(d,\sigma) \max_{T\subset\omega_z\cap\omega_T} h_T^{-1}\, \|w\|_{L^2(\omega_T)}\\ &\lesssim \|\nabla w\|_{L^2(\omega_T)}.\end{aligned}$$

We thus derive

$$\begin{aligned}\langle \ell, w\rangle &= \sum_{z\in\mathcal{V}\cap\omega_T} \langle \ell, w\phi_z\rangle\\ &\le \sum_{z\in\mathcal{V}\cap\omega_T} \|\ell\|_{H^{-1}(\omega_z\cap\omega_T)} \|\nabla(w\phi_z)\|_{L^2(\omega_z\cap\omega_T)}\\ &\lesssim \Big(\sum_{z\in\mathcal{V}\cap\omega_T} \|\ell\|_{H^{-1}(\omega_z)}\Big) \|\nabla w\|_{L^2(\omega_T)}\end{aligned}$$

and, since $\#(\mathcal{V}\cap\omega_T)$ is bounded in terms of the shape regularity coefficient σ,

$$\|\ell\|^2_{H^{-1}(\omega_T)} \lesssim \sum_{z\in\mathcal{V}\cap\omega_T} \|\ell\|^2_{H^{-1}(\omega_z)}.$$

Summing over $T \in \mathcal{T}$, and taking into account that $\#\{T \in \mathcal{T} \mid \omega_T \ni z\}$ is again bounded in terms of σ, concludes the proof. □

4.2. *Standard residual estimator and its flaws*

Exploiting the results of Section 4.1, we derive an *a posteriori upper bound* of the error in terms of the standard residual estimator and discuss its flawed sharpness. This discussion will serve as the starting point for an improved *a posteriori* analysis in the following sections.

The *standard residual estimator* needs the additional regularity

$$f \in L^2(\Omega) \quad \text{and} \quad \boldsymbol{A} \in W^1_\infty(\Omega;\mathbb{R}^{d\times d}) \tag{4.9}$$

for the data in our model problem (2.5). Given the Galerkin approximation $u_\mathcal{T}$ from (4.1), it may be defined as follows (see e.g. Verfürth 2013):

$$\mathcal{E}^{\mathrm{std}}_\mathcal{T} := \mathcal{E}^{\mathrm{std}}_\mathcal{T}(u_\mathcal{T},\mathcal{D}) := \Big(\sum_{T\in\mathcal{T}} \mathcal{E}^{\mathrm{std}}_\mathcal{T}(u_\mathcal{T},\mathcal{D},T)^2\Big)^{1/2} \tag{4.10a}$$

with the *local indicators*

$$\mathcal{E}_{\mathcal{T}}^{\mathrm{std}}(u_{\mathcal{T}}, \mathcal{D}, T)^2 := h_T \|j(u_{\mathcal{T}})\|^2_{L^2(\partial T \setminus \partial\Omega)} + h_T^2 \|r(u_{\mathcal{T}})\|^2_{L^2(T)}, \tag{4.10b}$$

where

- the *scaling factor* $h_T = |T|^{1/d}$ measures the size of the element $T \in \mathcal{T}$,
- $j(v) = j_{\mathcal{T}}(v)$ is the *jump residual* given face-wise for $v \in \mathbb{V}_{\mathcal{T}}$ by

$$\begin{aligned} j(v)|_F &:= ([\![\boldsymbol{A}\nabla v]\!] \cdot \boldsymbol{n}_{T_1})|_F := ((\boldsymbol{A}\nabla v)|_{T_1} - (\boldsymbol{A}\nabla v)|_{T_2}) \cdot \boldsymbol{n}_{T_1} \\ &= (\boldsymbol{A}\nabla v)|_{T_1} \cdot \boldsymbol{n}_{T_1} + (\boldsymbol{A}\nabla v)|_{T_2} \cdot \boldsymbol{n}_{T_2}, \end{aligned}$$

  where $F \in \mathcal{F}$, $T_1, T_2 \in \mathcal{T}$ are such that $F = T_1 \cap T_2$, $\boldsymbol{n}_{T_i}$ denotes the outer normal of ∂T_i, $i = 1, 2$, and
- $r(v) = r_{\mathcal{T}}(v)$ is the *element residual*, a function given for $v \in \mathbb{V}_{\mathcal{T}}$ by

$$r(v)|_T := (f - cv + \operatorname{div}(\boldsymbol{A}\nabla v))|_T$$

  on any element $T \in \mathcal{T}$.

Note that the definition itself already uses the extra regularity (4.9). For notational simplicity, we shall write j and r instead of $j(u_{\mathcal{T}})$ and $r(u_{\mathcal{T}})$ for the rest of this section. Also, for any interior face $F \in \mathcal{F}$, we have $F = T_1 \cap T_2$ with $T_1, T_2 \in \mathcal{T}$. If $\boldsymbol{n}_{T_i}$ denotes the outer normal of ∂T_i, $i = 1, 2$, we set $\boldsymbol{n}_F = \boldsymbol{n}_{T_1}$. This particular choice of $\boldsymbol{n}_F$ is irrelevant as it does not affect the following definition of normal jump of any vector-valued field $\boldsymbol{g}$ with well-defined trace on F:

$$[\![\boldsymbol{g}]\!] \cdot \boldsymbol{n}_F := \boldsymbol{g}|_{T_1} \cdot \boldsymbol{n}_{T_1} + \boldsymbol{g}|_{T_2} \cdot \boldsymbol{n}_{T_2}.$$

Theorem 4.9 (upper bound with standard residual estimator). *Suppose the additional regularity* (4.9) *holds. Then the error is bounded by the standard residual estimator:*

$$\|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)} \lesssim \mathcal{E}_{\mathcal{T}}^{\mathrm{std}},$$

where the hidden constant depends on the coefficients $(\boldsymbol{A}, c)$*, the shape regularity coefficient* σ*, and* d*.*

Proof. As Lemma 4.1 (error and residual) and Corollary 4.6 (star localization of residual norm) imply

$$\|\nabla(u - u_{\mathcal{T}})\|^2_{L^2(\Omega)} \lesssim \|R_{\mathcal{T}}\|^2_{H^{-1}(\Omega)} \lesssim \sum_{z \in \mathcal{V}} \|R_{\mathcal{T}}\|^2_{H^{-1}(\omega_z)}$$

and $\#\{z \in \mathcal{V} \mid \omega_z \supset T\} = d + 1$, it suffices to establish

$$\|R_{\mathcal{T}}\|^2_{H^{-1}(\omega_z)} \lesssim \sum_{T \subset \omega_z} \mathcal{E}_{\mathcal{T}}^{\mathrm{std}}(u_{\mathcal{T}}, f, T)^2 \tag{4.11}$$

for any vertex $z \in \mathcal{V}$. To this end, let $w \in H_0^1(\omega_z)$. The extra regularity $f \in L^2(\Omega)$ and $\boldsymbol{A} \in W_\infty^1(\Omega; \mathbb{R}^{d\times d})$ allows for piecewise integration by parts, which leads to the following L^2-representation of the residual:

$$\langle R_{\mathcal{T}}, w\rangle = \sum_{F \ni z} \int_F jw + \sum_{T \ni z} \int_T rw.$$

In order to bound the right-hand side suitably, we use the scaled trace theorem

$$\|w\|_{L^2(F)}^2 \le \frac{|F|}{|T|}\|w\|_{L^2(T)}^2 + \frac{2}{d}\frac{|F|\operatorname{diam} T}{|T|}\|w\|_{L^2(T)}\|\nabla w\|_{L^2(T)} \tag{4.12}$$

for any face $F \subset \partial T$ (see e.g. Veeser and Verfürth 2009, Corollary 4.5), the inequality

$$\|w\|_{L^2(\omega_z)} \le \operatorname{diam} \omega_z \|\nabla w\|_{L^2(\omega_z)}$$

from Lemma 2.2 (first Poincaré inequality), and the two geometric relationships

$$\operatorname{diam} \omega_z \lesssim h_T \quad \text{whenever } T \subset \omega_z, \quad |F| \operatorname{diam} T \lesssim |T| \quad \text{for } F \subset \partial T.$$

We thus obtain

$$|\langle R_{\mathcal{T}}, w\rangle| \lesssim \left(\sum_{T \ni z} h_T \|r\|_{L^2(T)} + h_T^{1/2} \sum_{F \ni z, F \subseteq T} \|j\|_{L^2(F)}\right) \|\nabla w\|_{L^2(\omega_z)}.$$

As the number of faces and elements in the star ω_z is bounded in terms of the shape regularity coefficient σ, we arrive at the desired bound (4.11), and the proof is finished. □

Remark 4.10 (alternative derivation of upper bound). The upper bound for the standard residual estimator in Theorem 4.9 is often derived with a suitable interpolation operator, bypassing the localization of the H^{-1}-norm in Lemma 4.5. That approach is useful for the proof of Theorem 4.48 below and is presented therein. Here we opted for using the localization of the H^{-1}-norm in order to facilitate the comparison with the following subsections. The approach at hand is also convenient to keep the ensuing constants small; see Veeser and Verfürth (2009).

An important question is the sharpness of the upper bound in Theorem 4.9. The so-called *a posteriori lower bounds* provide some answer by trying to bound the estimator in terms of the error. For many estimators, however, there arise additional terms of an *oscillatory* nature. The following remark justifies the presence of such terms for the case at hand.

Remark 4.11 (non-asymptotic overestimation). The lower bound

$$\mathcal{E}_{\mathcal{T}}^{\mathrm{std}} \lesssim \|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)},$$

which would imply equivalence of error and estimator, *cannot hold* in general for the following reason.

Fix a mesh $\mathcal{T}$ and a functional $f \in H^{-1}(\Omega) \setminus L^2(\Omega)$ and consider a sequence $(f_n)_n$ of functions in $L^2(\Omega)$ with $\lim_{n\to\infty} \|f - f_n\|_{H^{-1}(\Omega)} = 0$. Then the sequences $(u_n)_n$ and $(u_{\mathcal{T},n})_n$ of exact and Galerkin solutions on a fixed mesh $\mathcal{T}$ remain bounded. The error sequence $\big(\|\nabla(u_n - u_{\mathcal{T},n})\|_{L^2(\Omega)}\big)_n$ is therefore also bounded, while the standard residual estimator $\mathcal{E}_{\mathcal{T}}^{\mathrm{std}}(u_{\mathcal{T},n}, f_n) \to \infty$ becomes unbounded. Note that in the special case $f = -\operatorname{div}(\boldsymbol{A}\nabla v) + cv$ with $v \in \mathbb{V}_{\mathcal{T}}$, we even have for the error $\lim_{n\to\infty} \|\nabla(u_n - u_{\mathcal{T},n})\|_{L^2(\Omega)} = 0$.

In other words, in certain cases, the standard residual estimator bounds almost 0 by almost ∞ and a lower bound has to involve an additional term that cannot be bounded by the error in general.

We shall define these additional terms with the help of the following local best approximations. Let K be an element or face of $\mathcal{T}$ and $m \in \mathbb{N}_0$ a polynomial degree. Given $v \in L^2(K)$, let $\Pi_K v := \Pi_K^m v$ denote the best approximation in $\mathbb{P}_m(K)$ with respect to the norm $\|\cdot\|_{L^2(K)}$. It is convenient to allow also for $m = -1$ with $\mathbb{P}_{-1}(K) = \{0\}$ and $\Pi_K^{(-1)} v = 0$. Writing $\mathcal{D} = (\boldsymbol{A}, c, f)$ for the data in problem (2.5), the (m_1, m_2)-oscillation for the standard residual estimator is then given by

$$\operatorname{osc}_{\mathcal{T}}^{\mathrm{std}}(u_{\mathcal{T}}, \mathcal{D})^2 := \sum_{T\in\mathcal{T}} \operatorname{osc}_{\mathcal{T}}^{\mathrm{std}}(u_{\mathcal{T}}, \mathcal{D}, T)^2, \tag{4.13a}$$

with the local indicators

$$\operatorname{osc}_{\mathcal{T}}^{\mathrm{std}}(u_{\mathcal{T}}, \mathcal{D}, T)^2 := h_T^2 \, \|r - \Pi_T^{m_2} r\|_{L^2(T)}^2 + h_T \sum_{F\subset\partial T\setminus\partial\Omega} \|j - \Pi_F^{m_1} j\|_{L^2(F)}^2. \tag{4.13b}$$

Proposition 4.12 (partial lower bound). *If $f \in L^2(\Omega)$ and $\boldsymbol{A} \in W^1_\infty(\Omega; \mathbb{R}^{d\times d})$, the standard residual estimator is bounded by error and oscillation:*

$$\mathcal{E}_{\mathcal{T}}^{\mathrm{std}} \lesssim \|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)} + \operatorname{osc}_{\mathcal{T}}^{\mathrm{std}}(u_{\mathcal{T}}, \mathcal{D}),$$

where the hidden constant depends on d, the coefficients $\boldsymbol{A}$ and c, the shape regularity coefficient σ as well as the oscillation degrees (m_1, m_2).

Proof. In light of Lemma 4.1 (error and residual) and Corollary 4.6 (star localization of residual norm), we may establish the claimed bound by bounding each indicator with a corresponding local residual norm. To this end, we shall consider here only the case of the oscillation degrees $(m_1, m_2) = (0, 0)$. The general case can be verified along the same lines with additional technicalities, and is treated in the proof of Lemma 4.28 below in a slightly different context.

$\boxed{1}$ We start by bounding an arbitrary element residual $h_T \|r\|_{L^2(T)}$, $T \in \mathcal{T}$, in terms of some local residual norm. To this end, we may try to invert the following consequence of Lemma 2.2 (first Poincaré inequality):

$$\|R_{\mathcal{T}}\|_{H^{-1}(T)} = \sup_{w\in H_0^1(T)} \frac{|\langle R_{\mathcal{T}}, w\rangle|}{\|\nabla w\|_{L^2(T)}} = \sup_{w\in H_0^1(T)} \frac{\int_T rw}{\|\nabla w\|_{L^2(T)}} \lesssim h_T \|r\|_{L^2(T)},$$

the residual norm of which avoids involving the jump residual. We thus actually ask for an equivalence of two different smoothness norms. Such an equivalence can hold only for special r, e.g. from a finite-dimensional space. Furthermore, writing $\|r\|^2_{L^2(T)} = \int_T r(r\chi_T)$ suggests the choice $w = r\chi_T$, which, however, is not admissible for the residual $R_{\mathcal{T}}$ as neither r nor the characteristic function χ_T belong to $H^1_0(\Omega)$. We shall overcome these issues by replacing r with its mean value $\Pi^0_T r$ and χ_T with the element bubble

$$\phi_T := (d+1)^{(d+1)} \prod_{z \in \mathcal{V} \cap T} \phi_z. \tag{4.14}$$

Thanks to $\int_T \phi_T = C_d|T|$ and the inverse estimate $\|\nabla w\|_{L^2(T)} \lesssim h_T^{-1}\|w\|_{L^2(T)}$ for $w = (\Pi^0_T r)\phi_T \in H^1_0(T) \cap \mathbb{P}_{d+1}(T)$, we derive

$$\begin{aligned}
\|\Pi^0_T r\|_{L^2(T)} &\lesssim \int_T (\Pi^0_T r) w \le \|(\Pi^0_T r)\chi_T\|_{H^{-1}(T)} \|\nabla w\|_{L^2(T)} \\
&\lesssim h_T^{-1} \|(\Pi^0_T r)\chi_T\|_{H^{-1}(T)} \|w\|_{L^2(T)} \\
&\le h_T^{-1} \|(\Pi^0_T r)\chi_T\|_{H^{-1}(T)} \|\Pi^0_T r\|_{L^2(T)},
\end{aligned}$$

whence

$$h_T \|\Pi^0_T r\|_{L^2(T)} \lesssim \|(\Pi^0_T r)\chi_T\|_{H^{-1}(T)}. \tag{4.15}$$

This implies the desired partial lower bound for the element residual by a perturbation argument and the inequality $\|r - \Pi^0_T r\|_{H^{-1}(T)} \lesssim h_T \|r - \Pi^0_T r\|_{L^2(T)}$, which follows from another application of Lemma 2.2 (first Poincaré inequality):

$$\begin{aligned}
h_T \|r\|_{L^2(T)} &\le h_T \|\Pi^0_T r\|_{L^2(T)} + h_T \|r - \Pi^0_T r\|_{L^2(T)} \\
&\lesssim \|(\Pi^0_T r)\chi_T\|_{H^{-1}(T)} + h_T \|r - \Pi^0_T r\|_{L^2(T)} \\
&\lesssim \|r\chi_T\|_{H^{-1}(T)} + h_T \|r - \Pi^0_T r\|_{L^2(T)} \\
&= \|R_{\mathcal{T}}\|_{H^{-1}(T)} + h_T \|r - \Pi^0_T r\|_{L^2(T)}.
\end{aligned} \tag{4.16}$$

$\boxed{2}$ We bound an arbitrary jump residual $\|j\|_{L^2(F)}$, $F \in \mathcal{F}$, in a similar manner. Note that here an interference of the element residual is unavoidable because the support of non-trivial test functions has non-empty interior. We thus may try to insert

$$\|R_{\mathcal{T}}\|_{H^{-1}(\omega_F)} = \sup_{w \in H^1_0(\omega_F)} \frac{\int_F jw + \int_{\omega_F} rw}{\|\nabla w\|_{L^2(\omega_F)}} \lesssim h_F^{1/2} \|j\|_{L^2(F)} + \sum_{T \subset \omega_F} h_T \|r\|_{L^2(T)},$$

where the scaled trace theorem (4.12) is also used. To this end, we write δ_F for the Dirac measure of the face F,

$$\phi_F := d^d \prod_{z \in \mathcal{V} \cap F} \phi_z \tag{4.17}$$

for the face bubble of F, and choose the test function $w = (\Pi^0_F j)\phi_F \in H^1_0(\omega_F)$.

Using in addition $\|w\|_{L^2(\omega_F)} \lesssim h_F^{1/2}\|w\|_{L^2(F)}$ and (4.15), we deduce

$$\begin{aligned}
\|\Pi_F^0 j\|_{L^2(F)}^2 &\lesssim \int_F (\Pi_F^0 j)w + \sum_{T\subset\omega_F}\int_T (\Pi_T^0 r)w - \sum_{T\subset\omega_F}\int_T (\Pi_T^0 r)w \\
&\le \Big\|(\Pi_F^0 j)\delta_F + \sum_{T\subset\omega_F}(\Pi_T^0 r)\chi_T\Big\|_{H^{-1}(\omega_F)}\|\nabla w\|_{L^2(\omega_F)} \\
&\quad + \sum_{T\subset\omega_F}\|\Pi_F^0 r\|_{L^2(T)}\|w\|_{L^2(T)} \\
&\lesssim \Big\|(\Pi_F^0 j)\delta_F + \sum_{T\subset\omega_F}(\Pi_T^0 r)\chi_T\Big\|_{H^{-1}(\omega_F)} h_F^{-1/2}\|\Pi_F^0 j\|_{L^2(F)},
\end{aligned}$$

whence

$$h_F^{1/2}\|\Pi_F^0 j\|_{L^2(F)}^2 \lesssim \Big\|(\Pi_F^0 j)\delta_F + \sum_{T\subset\omega_F}(\Pi_T^0 r)\chi_T\Big\|_{H^{-1}(\omega_F)}. \tag{4.18}$$

Passing to the proper jump residual j, we arrive at the partial lower bound for the jump residual:

$$\begin{aligned}
h_F^{1/2}\|\Pi_F^0 j\|_{L^2(F)} &\lesssim \|R_{\mathcal{T}}\|_{H^{-1}(\omega_F)} \\
&\quad + h_F^{1/2}\|j - \Pi_F^0 j\|_{L^2(F)} + \sum_{T\subset\omega_F} h_T\|r - \Pi_T^0 r\|_{L^2(T)}.
\end{aligned} \tag{4.19}$$

3 We square the bounds (4.16) and (4.19) from the previous steps and sum them, respectively, over all elements and faces to conclude the claimed partial lower bound with the help of Lemma 4.5(ii) (localization of H^{-1}-norm) and Lemma 4.1 (error and residual). □

The significance of Proposition 4.12 (partial lower bound) strongly depends on the choice of the polynomial degrees (m_1, m_2) in the oscillation from (4.13). The following two remarks address this important aspect.

Remark 4.13 (oscillation degrees: asymptotics). It is desirable that, under refinement, the oscillation in Proposition 4.12 (partial lower bound) converges to 0 at least as fast as the error. The maximal convergence order of the error under uniform refinement is $\|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)} = O(h^n)$ as $h \to 0$. In view of the scaling factors and derivative orders appearing in jump and element residual, we are thus led to require

$$m_1 \ge n - 1 \quad \text{and} \quad m_2 \ge n - 2.$$

One might hope that strict inequalities lead to higher order. Note, however, that since $\mathrm{osc}^{\mathrm{std}}$ involves in general both discrete solution $u_{\mathcal{T}}$ and data $\mathcal{D} = (\boldsymbol{A}, c, f)$, this will not be guaranteed without additional assumptions. Furthermore, increasing m_1 and m_2 entails bigger hidden constants in the lower bounds (4.15) and (4.18), as these bounds cannot hold for arbitrary L^2-functions. Consequently, a potentially

higher asymptotic speed of the oscillation $\mathrm{osc}^{\mathrm{std}}$ comes with a bigger constant in front of it and therefore with diminished non-asymptotic significance.

Remark 4.14 (oscillation degrees: data oscillation reduction). In the particular case of the Poisson equation, i.e. $\boldsymbol{A} = \boldsymbol{I}$ and $c = 0$, and linear elements, i.e. $n = 1$, the oscillation with the degrees $(m_1, m_2) = (0, 0)$ reduces to the *data oscillation*

$$\mathrm{osc}^{\mathrm{std}}_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D})^2 = \sum_{T \in \mathcal{T}} h_T^2 \|f - \Pi_T^0 f\|^2_{L^2(T)};$$

it depends only on the data, here the right-hand side f. Note also that here the regularity of f is determined by the regularity of the exact solution u.

For elements with degree $n \geq 2$, the choices $(m_1, m_2) = (n-1, n-2)$ ensure that for $F \in \mathcal{F}$,

$$\Pi_F^{n-1}([[\nabla u_{\mathcal{T}}]]|_F \cdot \boldsymbol{n}_F) = [[\nabla u_{\mathcal{T}}]]|_F \cdot \boldsymbol{n}_F \quad \text{and} \quad \Pi_T^{n-2}(\Delta u_{\mathcal{T}}|_T) = \Delta u_{\mathcal{T}}|_T, \tag{4.20}$$

and so, again, oscillation reduces to data oscillation in f:

$$\mathrm{osc}^{\mathrm{std}}_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D})^2 = \sum_{T \in \mathcal{T}} h_T^2 \|f - \Pi_T^{n-2} f\|^2_{L^2(T)}.$$

If we add a reaction term, i.e. we consider $\boldsymbol{A} = \boldsymbol{I}$ and $c = 1$, we can again obtain the reduction to data oscillation by increasing m_2 to n.

For a more general operator with piecewise polynomial coefficients

$$\boldsymbol{A} \in \left(\mathbb{S}_{\mathcal{T}}^{n_{\boldsymbol{A}},-1}\right)^{d\times d} \quad \text{and} \quad c \in \mathbb{S}_{\mathcal{T}}^{n_c,-1},$$

the choice

$$(m_1, m_2) = (n_A + n - 1, \max\{n_c + n, n_A + n - 2\}) \tag{4.21}$$

again reduces $\mathrm{osc}^{\mathrm{std}}$ to data oscillation in f.

Finally, for a general operator without piecewise polynomial coefficients $(\boldsymbol{A}, c)$, a reduction to data oscillation with piecewise polynomial best approximations as before is not possible. The argument in Remark 4.13 suggests approximating the general coefficients with piecewise polynomial coefficients satisfying

$$n_{\boldsymbol{A}} = n - 1 \quad \text{and} \quad n_c = n - 1.$$

As we shall see below in Section 4.8, the choice (4.21) with these values allows us to bound $\mathrm{osc}^{\mathrm{std}}$ in terms of $\|\nabla u_{\mathcal{T}}\|_{L^2(\Omega)}$, which is controlled by stability, and data oscillation terms involving f and the coefficients $\boldsymbol{A}$ and c. Note, however, that the nature of these data oscillation terms differs from the preceding reductions: for example, the regularity of the coefficients $\boldsymbol{A}$ and c is not determined by the exact solution u.

In light of Remark 4.13 (oscillation degrees: asymptotics), one might hope that the overestimation described in Remark 4.11 (non-asymptotic overestimation) disappears under refinement. This can be ensured under suitable regularity assumption but is not guaranteed in general, as the following remark reveals.

Remark 4.15 (asymptotic overestimation). Considering a variant of the standard residual estimator that allows for $f \in H^{-1}(\Omega)$ and adaptive refinement, Cohen, DeVore and Nochetto (2012, Section 6.4) give an example where the error converges asymptotically faster than the estimator; see also Kreuzer and Veeser (2021, Lemma 21).

Having recognized the above flaws of the standard residual estimator, let us conclude with an observation that will be the departure point of an improved analysis.

Corollary 4.16 (equivalence for discrete data). *Suppose all data $\mathcal{D} = (\boldsymbol{A}, f, c)$ of problem (2.5) are piecewise polynomial, that is, there are $n_{\boldsymbol{A}}, n_c, n_f \in \mathbb{N}_0$ such that*

$$\boldsymbol{A} \in \left(\mathbb{S}_{\mathcal{T}}^{n_{\boldsymbol{A}},-1}\right)^{d\times d}, \quad c \in \mathbb{S}_{\mathcal{T}}^{n_c,-1} \quad \text{and} \quad f \in \mathbb{S}_{\mathcal{T}}^{n_f,-1}.$$

Then error and standard residual estimator are equivalent:

$$\|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)} \approx \mathcal{E}_{\mathcal{T}}^{\mathrm{std}},$$

where the hidden constants depend only on d, the coefficients $\boldsymbol{A}$ and c, the shape regularity coefficient σ, and the degrees $n_{\boldsymbol{A}}$, n_c and n_f.

Proof. The upper bound follows from Theorem 4.9 (upper bound with standard residual estimator), while Proposition 4.12 (partial lower bound) with

$$(m_1, m_2) = (n_{\boldsymbol{A}} + n - 1, \max\{n_{\boldsymbol{A}} + n - 2, n_c + n, n_f\})$$

yields the lower bound. □

Motivated by the above discussion, one may define a variant of the standard residual estimator, characterized by a splitting into two different parts. More precisely, choosing (m_1, m_2) according to Remark 4.14 (oscillation degrees: data oscillation reduction), one may replace the local indicators in (4.10b) with

$$\mathcal{E}_{\mathcal{T}}^{\mathrm{std}}(u_{\mathcal{T}}, f, T)^2 := \eta_{\mathcal{T}}^{\mathrm{std}}(u_{\mathcal{T}}, T)^2 + \mathrm{osc}_{\mathcal{T}}^{\mathrm{std}}(u_{\mathcal{T}}, \mathcal{D}, T)^2, \tag{4.22a}$$

where the first part, the so-called *PDE indicator*, is given by

$$\eta_{\mathcal{T}}^{\mathrm{std}}(u_{\mathcal{T}}, T)^2 := h_T^2 \|\Pi_T^{m_2} r\|_{L^2(T)}^2 + h_T \sum_{F \subset \partial T \setminus \partial\Omega} \|\Pi_F^{m_1} j\|_{L^2(F)}^2, \tag{4.22b}$$

while the second part corresponds to the local oscillation from (4.13b); compare with Verfürth (2013, Theorems 1.5 and 4.7). In this way,

- the PDE indicators are computable (in terms of the Galerkin approximation $u_{\mathcal{T}}$ and the local projections),
- the oscillation indicators typically have to be approximated by numerical quadrature,
- both types of indicators are, in general, not dominated by the error.

4.3. *Discrete functionals and* a posteriori *error analysis*

This section introduces the notion of *discrete functionals* and individuates properties in their approximation that are useful in *a posteriori* error analysis. The realization of these properties distinguishes the subsequent approach, which is adapted from Kreuzer and Veeser (2021) and Kreuzer *et al.* (2024).

The notion of discrete functionals and its local counterparts are of interest for at least two reasons. The first one is that their H^{-1}-norm can be rather easily quantified, as we shall see in Corollary 4.30 below. This property is related to Corollary 4.16 (equivalence for discrete data), which can be read in the following way: the standard residual estimator is equivalent to the error whenever the residual is a discrete functional. The second reason lies in the observation that an important part of the residual, namely the application of the differential operator to a discrete function, is itself of discrete nature. This feature is partially captured by the following definition of discrete functionals with polynomial densities and is discussed in Remark 4.18.

Definition 4.17 (discrete functionals and meshed subdomains). For $m_1 \in \mathbb{N}_0$, $m_2 \in \mathbb{N}_0 \cup \{-1\}$, let $\mathbb{F}_{\mathcal{T}} := \mathbb{F}(\mathcal{T}) := \mathbb{F}_{m_1,m_2}(\mathcal{T})$ denote the subspace

$$\Big\{\ell \in H^{-1}(\Omega) \mid \text{for all } w \in H^1_0(\Omega),\ \langle \ell, w\rangle = \sum_{F\in\mathcal{F}} \int_F q_F w + \sum_{T\in\mathcal{T}} \int_T q_T w \\ \text{with fixed } q_F \in \mathbb{P}_{m_1}(F),\ q_T \in \mathbb{P}_{m_2}(T)\Big\}$$

of *discrete functionals*, i.e. functionals that are given by piecewise polynomial densities over elements and interior faces. We call (m_1, m_2) the *degrees of the discrete functionals*.

A set ω is a $\mathcal{T}$*-meshed subdomain* if it is a subdomain of Ω and it is triangulated by a submesh $\mathcal{T}_\omega \subset \mathcal{T}$, that is, we have $\overline{\omega} = \cup_{T\in\mathcal{T}_\omega} T$. A functional $\ell \in H^{-1}(\Omega)$ is then *discrete in the meshed subdomain* ω whenever $\ell|_{H^1_0(\omega)} \in \mathbb{F}(\mathcal{T}_\omega)$. Here the faces

$$\mathcal{F}_\omega := \{F \in \mathcal{F} \mid F \cap \omega \neq \emptyset,\ F \not\subset \partial\omega\}$$

involved in $\mathbb{F}(\mathcal{T}_\omega)$ are interior to ω; for example, the subspaces $\mathbb{F}(\{T\})$, $T \in \mathcal{T}$, do not involve any faces. In accordance with (4.5), we use the abbreviations $\mathcal{T}_z$ and $\mathcal{F}_z$ for $\mathcal{T}_{\omega_z}$ and $\mathcal{F}_{\omega_z}$.

Alternatively, the local space $\mathbb{F}(\mathcal{T}_\omega)$ can be obtained from the global space $\mathbb{F}(\mathcal{T})$ by restriction:

$$\mathbb{F}(\mathcal{T}_\omega) = \mathbb{F}(\mathcal{T})|_{H^1_0(\omega)} := \big\{\ell|_{H^1_0(\omega)} \mid \ell \in \mathbb{F}(\mathcal{T})\big\}. \tag{4.23}$$

Remark 4.18 (differential operator and discrete functionals). The image of the finite element space $\mathbb{V}_{\mathcal{T}}$ under the linear differential operator $-\operatorname{div}(\boldsymbol{A}\nabla\cdot) + c\cdot$ is again a finite-dimensional space. For differential operators with piecewise

polynomial coefficients $\boldsymbol{A}$ and c, the above notion captures this by the property that the application of such operators to discrete functions $v \in \mathbb{V}_{\mathcal{T}}$ yields discrete functionals. Indeed, if

$$\boldsymbol{A} \in \left(\mathbb{S}_{\mathcal{T}}^{n_{\boldsymbol{A}},-1}\right)^{d\times d} \quad \text{and} \quad c \in \mathbb{S}_{\mathcal{T}}^{n_c,-1}$$

with $n_{\boldsymbol{A}}, n_c \in \mathbb{N}_0$, piecewise integration by parts gives the representation

$$\int_\Omega \boldsymbol{A}\nabla v \cdot \nabla w + cvw = \sum_{F\in\mathcal{F}} \int_F [[\boldsymbol{A}\nabla v]] \cdot \boldsymbol{n}_F w + \sum_{T\in\mathcal{T}} \int_T (cv - \operatorname{div}(\boldsymbol{A}\nabla v))w, \tag{4.24}$$

where, for any interior face $F \in \mathcal{F}$ and any element $T \in \mathcal{T}$,

$$[[\boldsymbol{A}\nabla v]] \cdot \boldsymbol{n}_F \in \mathbb{P}_{m_1}(F), \quad cv - \operatorname{div}(\boldsymbol{A}\nabla v) \in \mathbb{P}_{m_2}(T)$$

with $m_1 = n_{\boldsymbol{A}} + n - 1$ and $m_2 = \max\{n_{\boldsymbol{A}} + n - 2, n_c + n\}$. Note, however, that not every functional in $\mathbb{F}_{m_1,m_2}(\mathcal{T})$ can be written in the form of (4.24). In fact, as the representation of a discrete functional is made up of L^2-scalar products on domains that are mutually disjoint or of different dimension, we have

$$\dim \mathbb{F}_{m_1,m_2}(\mathcal{T}) = \#\mathcal{F} \dim \mathbb{P}_{m_1} + \#\mathcal{T} \dim \mathbb{P}_{m_2}, \tag{4.25}$$

which is strictly greater than $\dim \mathbb{V}_{\mathcal{T}}$. This enlargement, which is implicitly used in the proof of Proposition 4.12 (partial lower bound), turns out to be convenient also in the constructive approximation of discrete functionals.

In view of the aforementioned properties of discrete functionals, we may split the residual into a discrete and a non-discrete part. Splitting the standard residual estimator in the alternative local indicators (4.22) is in a similar spirit. To see this, we introduce $\Pi_{\mathcal{T}}\ell \in H^{-1}(\Omega)$ given by

$$\langle \Pi_{\mathcal{T}}\ell, w\rangle := \sum_{F\in\mathcal{F}} \int_F \left(\Pi_F^{m_1} g\right) w + \sum_{T\in\mathcal{T}} \int_T \left(\Pi_T^{m_2} f\right) w, \quad w \in H_0^1(\Omega), \tag{4.26}$$

for all $\ell \in H^{-1}(\Omega)$ admitting the representation

$$\langle \ell, w\rangle = \sum_{F\in\mathcal{F}} \int_F gw + \sum_{T\in\mathcal{T}} \int_T fw, \quad w \in H_0^1(\Omega),$$

with suitable density functions g and f. Then the splitting of the alternative indicators (4.22) corresponds to writing

$$R_{\mathcal{T}} = \Pi_{\mathcal{T}} R_{\mathcal{T}} + (I - \Pi_{\mathcal{T}}) R_{\mathcal{T}}. \tag{4.27}$$

Moreover, Remark 4.14 (oscillation degrees: data oscillation reduction) discusses conditions for the identity

$$(I - \Pi_{\mathcal{T}}) R_{\mathcal{T}} = f - \Pi_{\mathcal{T}} f, \tag{4.28}$$

which follows from the property that $\Pi_{\mathcal{T}}$ reproduces the functionals in Remark 4.18 (differential operator and discrete functionals); compare with (4.20), which in terms of $\Pi_{\mathcal{T}}$ reads $\Pi_{\mathcal{T}}(\Delta u_{\mathcal{T}}) = \Delta u_{\mathcal{T}}$, where Δ is now the distributional Laplacian.

The fact that the definition of $\Pi_{\mathcal{T}}$ requires the extra regularity $f \in L^2(\Omega)$ and $\boldsymbol{A} \in W^1_\infty(\Omega; \mathbb{R}^{d\times d})$ not only excludes applications but, in light of Remark 4.11 (non-asymptotic overestimation), entails overestimation. To circumvent this flaw, we therefore aim to construct a new approximation operator $P_{\mathcal{T}}$ that is defined for all functionals $\ell \in H^{-1}(\Omega)$. Furthermore, we want this operator to be a projection onto $\mathbb{F}_{\mathcal{T}}$ so that the counterpart

$$(I - P_{\mathcal{T}})R_{\mathcal{T}} = f - P_{\mathcal{T}} f$$

of (4.28) holds under the same conditions.

To summarize, our plan is to develop a quasi-optimal *a posteriori* error analysis by constructing a locally computable linear projection

$$P_{\mathcal{T}} \colon H^{-1}(\Omega) \to \mathbb{F}_{\mathcal{T}} \subset H^{-1}(\Omega)$$

onto the discrete functionals that induces a splitting

$$R_{\mathcal{T}} = P_{\mathcal{T}} R_{\mathcal{T}} + (I - P_{\mathcal{T}})R_{\mathcal{T}} \tag{4.29}$$

of the residual into a *discretized residual* $P_{\mathcal{T}} R_{\mathcal{T}}$, which can be easily quantified, as well as an *oscillatory residual* $(I - P_{\mathcal{T}})R_{\mathcal{T}}$, which under the conditions of Remark 4.18 (differential operator and discrete functionals) reduces to an oscillation of the right-hand side f.

The proof of an upper bound of the error will then involve a triangle inequality applied to the right-hand side of (4.29). The following remark provides criteria to prevent overestimation in such a context, and is followed by a comparison of the two approaches represented by (4.27) and (4.29).

Remark 4.19 (avoiding overestimation). Overestimation can often be avoided by ensuring two relatively simple conditions. In order to discuss them informally, consider the model inequality

$$|\cdot| \le |\cdot|_1 + |\cdot|_2, \tag{4.30}$$

where $|\cdot|$, $|\cdot|_i$, $i = 1, 2$, are seminorms and denote the domain and kernel of $|\cdot|$, respectively, by $\operatorname{dom}|\cdot|$ and $\ker|\cdot|$, etc.

The first condition, the *kernel condition*, is that zero is not overestimated:

$$\ker|\cdot| \subset \ker|\cdot|_1 \cap \ker|\cdot|_2. \tag{4.31a}$$

The second condition, the *domain condition*, is that a finite value is never bounded by ∞, or in other words, still informal, if the evaluation of the left-hand side is (or can be uniquely defined to be) a finite value, the same holds for the right-hand side:

$$\operatorname{dom}|\cdot| \subset \operatorname{dom}|\cdot|_1 \cap \operatorname{dom}|\cdot|_2. \tag{4.31b}$$

Kreuzer *et al.* (2024) provide a precise version of the domain condition (4.31b),

showing that, given inequality (4.30), the two conditions (4.31) are also sufficient for equivalence, and discuss further applications of this viewpoint.

In order to illustrate the application of Remark 4.19, let us consider only the special case of the Poisson equation, i.e. $\boldsymbol{A} = \boldsymbol{I}$, $c = 0$, and linear elements, i.e. $n = 1$. We start with the upper bound in terms of the standard residual estimator in Theorem 4.9 and view it as a function of the right-hand side f. Then the domain condition is violated as the left-hand side is defined for any $f \in H^{-1}(\Omega)$, while the right-hand side is defined only for $f \in L^2(\Omega)$. Also, the kernel condition is not verified: the left-hand side vanishes whenever $f = -\Delta v$ for some $v \in \mathbb{V}_{\mathcal{T}}$, while the right-hand side vanishes only for $f = 0$. The splitting in the alternative local indicators (4.22) does not worsen this situation, that is, it does not add further instances in which kernel and domain condition are missed. Note, however, that the oscillation indicators alone are in conflict with the domain condition and therefore another PDE indicator cannot cure the overestimation. Finally, for the outlined approach, the splitting (4.29) and the required properties for the operator $P_{\mathcal{T}}$ ensure both kernel and domain condition.

4.4. *Testing discrete functionals*

The H^{-1}-projection $P_{\mathcal{T}}$ onto the discrete functionals $\mathbb{F}_{\mathcal{T}}$ will be defined by means of a Petrov–Galerkin-type approach. This section prepares its definition by individuating a suitable *test space* $\mathbb{V}^+_{\mathcal{T}}$. The key property of $\mathbb{V}^+_{\mathcal{T}}$ is that the dual pairing $\langle \cdot, \cdot \rangle$ in $H^{-1}(\Omega)$ is non-degenerate on the product $\mathbb{F}_{\mathcal{T}} \times \mathbb{V}^+_{\mathcal{T}}$. Doing so, the degrees (m_1, m_2) of the discrete functionals will be parameters that are omitted in the notation. The construction of the test space $\mathbb{V}^+_{\mathcal{T}}$ proceeds in two steps. First, we locally associate to the degrees of freedom in $\mathbb{F}_{\mathcal{T}}$ certain functions on Ω. For the degrees of freedom on the skeleton, this will involve a suitable *extension operator*. Second, we turn the ensuing functions into admissible test functions with the help of a *cut-off*.

The degrees of freedom in $\mathbb{F}_{\mathcal{T}}$ are given by density polynomials over element and faces. For an element $T \in \mathcal{T}$, if we extend such a density polynomial q_T by 0 off T, it is already a function on Ω. For a polynomial q_F associated with a face $F \in \mathcal{F}$, we employ the following extension operator E_F mapping a function v on F to a function on ω_F, the union of all elements T containing F.

Given such an element $T \subset \omega_F$, write $z_0, \dots, z_d$ for its vertices, z_d being the one opposite to F, let

$$b_F := \frac{1}{d} \sum_{i=0}^{d-1} z_i$$

denote the barycentre of F, and set

$$(E_F v)(x) := v\left(\phi_d(x) b_F + \sum_{i=0}^{d-1} \phi_{z_i}(x) z_i \right), \quad x \in T,$$

and extend by 0 off ω_F. Note that the definition of E_F is affine-invariant and does not depend on the enumeration of the vertices of F. The next lemma collects two useful properties of this extension operator.

Lemma 4.20 (extending from faces). *Let $F \in \mathcal{F}$ be a face. For any function $v \in L^2(F)$, we have*

$$\|E_F v\|_{L^2(\omega_F)} \lesssim h_F^{1/2} \|v\|_{L^2(F)},$$

where h_F stands for the diameter of F and the hidden constant depends only on d and the shape regularity coefficient σ. Furthermore, if v is a polynomial, then $E_F v$ is a continuous piecewise polynomial of the same degree.

Proof. $\boxed{1}$ In view of $(E_F v)^2 = E_F(v^2)$, we may show the inequality by verifying

$$\int_{\omega_F} E_F w \lesssim h_F \int_F w \tag{4.32}$$

for any positive function $w\colon F \to \mathbb{R}$, which amounts to L^1-stability. To this end, we shall use a standard argument involving the following reference situation, which slightly differs from the common one with

$$T_d = \left\{ x = (x_1, \dots, x_d) \in \mathbb{R}^d \mid 0 \le x_i \le 1,\ \sum_{i=1}^d x_i \le 1 \right\}$$

and $b_d := (d+1)^{-1}(1, \dots, 1) \in \mathbb{R}^d$. Let the reference face $\widehat{F} := T_{d-1} - b_{d-1} \subset \mathbb{R}^{d-1}$ be a translation of T_{d-1} and let the reference simplex $\widehat{T} \subset \mathbb{R}^d$ be the convex hull of $\widehat{F} \times \{0\}$ and the canonical basis vector $e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$. The barycentre of $\widehat{F} \times \{0\}$ is then the origin in $\mathbb{R}^d$ and the barycentric coordinate of the vertex e_d of $\widehat{T}$ is x_d. Fixing an element T with $T \subset \omega_F$, let $G_T : \widehat{T} \to T$ denote a bi-affine map sending vertices of $\widehat{F} \times \{0\}$ into vertices of F and e_d into the vertex of T opposite to F, and write $G_F : \widehat{F} \times \{0\} \to F$ for the restriction $G_T|_{\widehat{F}\times\{0\}}$. The pullbacks of $E_F w$ and w satisfy

$$G_T^*(E_F w)(x', x_d) = G_F^* w(x', 0)$$

for all $x = (x', x_d) \in \widehat{T} = \{y = (y', y_d) \in \widehat{F} \times \mathbb{R} \mid 0 \le y_d \le 1 - |y' + b_{d-1}|_1\}$, where $|z'|_1 = \sum_{i=1}^{d-1} |z_i'|$ stands for the ℓ_1-norm in $\mathbb{R}^{d-1}$. Consequently, the transformation rule, the fact that the Jacobians of G_T and G_F are constant, the Fubini theorem, $w \ge 0$, and $|\widehat{F}|/|\widehat{T}| = |T_{d-1}|/|T_d| = d$ yield

$$\begin{aligned}
\int_T E_F w &= \frac{|T|}{|\widehat{T}|} \int_{\widehat{T}} G_T^*(E_F w) = \frac{|T|}{|\widehat{T}|} \int_{\widehat{F}} \int_0^{1-|x'+b_{d-1}|_1} G_F^* w(x', 0)\, \mathrm{d}x_d\, \mathrm{d}x' \\
&\le \frac{|T|}{|\widehat{T}|} \int_{\widehat{F}} G_F^* w(\cdot, 0) = \frac{|T||\widehat{F}|}{|\widehat{T}||F|} \int_F w = d \frac{|T|}{|F|} \int_F w.
\end{aligned}$$

Since the hidden constant in $|T|/|F| \lesssim h_F$ depends only on the shape regularity coefficient σ, this implies the L^1-stability bound (4.32) and so also the claimed L_2-stability is proved.

$\boxed{2}$ The second statement for polynomial arguments of E_F is a direct consequence of its definition. □

In view of the above different treatment of elements and faces, we need two types of cut-off functions: one for elements denoted by ϕ_T and another for faces denoted by ϕ_F. Possible choices are the element and face bubbles from (4.14) and (4.17). Since other choices will be useful in Section 4.8 below, we shall henceforth rely only on the following properties.

Assumption 4.21 (abstract cut-off). The *cut-off functions* ϕ_T, $T \in \mathcal{T}$, and ϕ_F, $F \in \mathcal{F}$, satisfy

$$\operatorname{supp} \phi_T = T, \quad 0 \le \phi_T \le 1, \quad \operatorname{supp} \phi_F = \omega_F, \quad 0 \le \phi_F \le 1,$$

and act in an affine-equivalent manner on the element level: there exists a finite-dimensional linear space $\mathbb{S}^+ \subset L^\infty(T_d)$ of functions defined on the reference element T_d such that $G_T^* \phi_T$ does not depend on T, $G_F^* \phi_F$ does not depend on F, and

$$\begin{aligned} &\text{for all } T \in \mathcal{T} \text{ and } q \in \mathbb{P}_{m_2}(T), && G_T^*(q\phi_T) \in \mathbb{S}^+, \\ &\text{for all } F \in \mathcal{F},\ q \in \mathbb{P}_{m_1}(F) \text{ and } T \in \mathcal{T}, && G_T^*(((E_F q)\phi_F)|_T) \in \mathbb{S}^+, \end{aligned}$$

where G_T is a bi-affine map from the reference element T_d to the generic element T, and $G_T^*(v) = v \circ G_T$ denotes the pullback of a function $v \colon T \to \mathbb{R}$ via G_T.

In the case of the bubble functions (4.14) and (4.17), Assumption 4.21 holds with $\mathbb{S}^+ = \mathbb{P}_{\max\{m_1+d-1, m_2+d\}}(T_d)$ as the extension operators E_F, $F \in \mathcal{F}$, preserve the polynomial degree.

Lemma 4.22 (properties of cut-off). *If the cut-off functions ϕ_T, $T \in \mathcal{T}$, and ϕ_F, $F \in \mathcal{F}$, satisfy Assumption 4.21, then we have*

$$\|q\|_{L^2(T)} \lesssim \|q\phi_T^{1/2}\|_{L^2(T)} \quad \textit{and} \quad \|\nabla(q\phi_T)\|_{L^2(T)} \lesssim h_T^{-1}\|q\phi_T\|_{L^2(T)}$$

for all $q \in \mathbb{P}_{m_2}(T)$, as well as

$$\|q\|_{L^2(F)} \lesssim \|q\phi_F^{1/2}\|_{L^2(F)} \quad \textit{and} \quad \|\nabla((E_F q)\phi_F)\|_{L^2(\omega_F)} \lesssim h_F^{-1}\|(E_F q)\phi_F\|_{L^2(\omega_F)}$$

for all $q \in \mathbb{P}_{m_1}(F)$. The hidden constants depend only on d, the shape regularity coefficient σ, the degrees (m_1, m_2) of the discrete functionals, and the space $\mathbb{S}^+$.

Proof. $\boxed{1}$ To verify the first claimed inequality, we start by noting that, thanks to $\operatorname{supp} \phi_T = T$, we have $\phi_{T_d} := G_T^*(\phi_T) > 0$ in the interior of T_d. Hence $\|\cdot\|_{L^2(T_d)}$ and $\|\cdot\,\phi_{T_d}^{1/2}\|_{L^2(T_d)}$ are norms on $\mathbb{P}_{m_2}(T_d)$ and, thanks to $\dim \mathbb{P}_{m_2}(T_d) < \infty$, are

equivalent. A standard round trip to the reference element and $G_T^* q\, G_T^* \phi_T^{1/2} = G_T^*(q\phi_T^{1/2})$ thus yields

$$\begin{aligned}\|q\|_{L^2(T)} &\lesssim h_T^{d/2}\|G_T^* q\|_{L^2(T_d)} \lesssim h_T^{d/2}\|G_T^* q\, G_T^* \phi_T^{1/2}\|_{L^2(T_d)}\\ &\lesssim \|q\phi_T^{1/2}\|_{L^2(T)},\end{aligned}$$

and the first claimed inequality is established. The third one is proved along the same lines, but with a round trip to the reference face.

2 For the other claimed inequalities, note that $\|\nabla\cdot\|_{L^2(T_d)}$ and $\inf_{c\in\mathbb{R}}\|\cdot - c\|_{L^2(T_d)}$ are equivalent norms on the finite-dimensional quotient space $\mathbb{S}^+/\mathbb{R}$. Consequently, further round trips to the reference element give

$$\begin{aligned}\|\nabla(q\phi_T)\|_{L^2(T)} &\lesssim h_T^{-1+d/2}\|\nabla G_T^*(q\phi_T)\|_{L^2(T_d)} \lesssim h_T^{-1+d/2}\inf_{c\in\mathbb{R}}\|G_T^*(q\phi_T) - c\|_{L^2(T_d)}\\ &\lesssim h_T^{-1+d/2}\|G_T^*(q\phi_T)\|_{L^2(T_d)} \lesssim h_T^{-1}\|q\phi_T\|_{L^2(T)}\end{aligned}$$

and

$$\begin{aligned}\|\nabla\big(E_F(q)\phi_T\big)\|^2_{L^2(\omega_F)} &= \sum_{T\subset\omega_F}\|\nabla\big(E_F(q)\phi_T\big)\|^2_{L^2(T)}\\ &\lesssim h_F^{-1}\sum_{T\subset\omega_F}\|E_F(q)\phi_T\|^2_{L^2(T)} = h_F^{-1}\|E_F(q)\phi_T\|^2_{L^2(\omega_F)},\end{aligned}$$

and the proof is completed. □

These preparations lead to the following test space for discrete functionals.

Definition 4.23 (test space for discrete functionals). Using the cut-off functions from Assumption 4.21, we associate to the space $\mathbb{F}_{\mathcal{T}}$ of discrete functionals the following *test space*:

$$\begin{aligned}\mathbb{V}^+_{\mathcal{T}} := \mathbb{V}^+(\mathcal{T}) := \operatorname{span}\Big(&\{q_T\phi_T \mid q_T \in \mathbb{P}_{m_2}(T),\ T\in\mathcal{T}\}\\ &\bigcup\{E_F(q_F)\phi_F \mid q_F\in\mathbb{P}_{m_1}(F),\ F\in\mathcal{F}\}\Big).\end{aligned}$$

If ω is a subdomain of Ω meshed by $\mathcal{T}_\omega$, then $\mathbb{V}^+(\mathcal{T}_\omega)$ is the test space for $\mathbb{F}(\mathcal{T}_\omega)$.

Similarly as for the space $\mathbb{F}_{\mathcal{T}}$ of discrete functionals, the test space over a subdomain ω meshed by $\mathcal{T}_\omega$ can be obtained from the global test space, namely

$$\mathbb{V}^+(\mathcal{T}_\omega) = \mathbb{V}^+(\mathcal{T}) \cap H^1_0(\omega). \tag{4.33}$$

4.5. A projection onto discrete functionals

Having the test space $\mathbb{V}^+_{\mathcal{T}}$ from Definition 4.23 at our disposal, we are now ready to construct a H^{-1}-*projection* $P_{\mathcal{T}}$, as suggested in Section 4.3. As in the previous section, the degrees (m_1, m_2) of the discrete functionals in $\mathbb{F}_{\mathcal{T}}$ are hidden parameters.

Definition 4.24 (projection onto discrete functionals). Given the discrete functionals $\mathbb{F}_{\mathcal{T}}$ and the test space $\mathbb{V}^+_{\mathcal{T}}$, we define a *projection* $P_{\mathcal{T}}\colon H^{-1}(\Omega) \to \mathbb{F}_{\mathcal{T}}$ by

$$\langle P_{\mathcal{T}}\ell, w\rangle = \langle \ell, w\rangle \quad \text{for all } w \in \mathbb{V}^+_{\mathcal{T}}. \tag{4.34}$$

The well-posedness of this definition and algebraic properties of $P_{\mathcal{T}}$ are verified in the following Lemma 4.25. Moreover, a representation of $P_{\mathcal{T}}$ in the form of a quasi-interpolation operator is given in Corollary 4.61 below.

The *polynomial densities* of $P_{\mathcal{T}}\ell$ are denoted by $P_T\ell := P_{\mathcal{T},T}\ell$, $T \in \mathcal{T}$, and $P_F\ell := P_{\mathcal{T},F}$, $F \in \mathcal{F}$, so that

$$\langle P_{\mathcal{T}}\ell, w\rangle = \sum_{T\in\mathcal{T}} \int_T P_T\ell\, w + \sum_{F\in\mathcal{F}} \int_F P_F\ell\, w. \tag{4.35}$$

In the next lemma we show in particular that $P_{\mathcal{T}}$ is a local operator. In order to formulate this, we shall use *$\mathcal{T}$-meshed local subdomains*, i.e. $\mathcal{T}$-meshed subdomains ω for which there exists a mesh element $T \in \mathcal{T}$ with $\omega \subset \omega_T$. For the next lemma, addressing algebraic properties of the operator $P_{\mathcal{T}}$, recall the notation $\omega_F = \cup_{T\in\mathcal{T}_F} T$ from (4.7) for an interior face $F \in \mathcal{F}$.

Lemma 4.25 (algebraic properties). *The operator $P_{\mathcal{T}}$ is a local linear projection onto the subspace $\mathbb{F}_{\mathcal{T}}$ of discrete functionals. More precisely, for any local subdomain ω meshed by $\mathcal{T}$, there is a linear projection $P_\omega\colon H^{-1}(\omega) \to \mathbb{F}(\mathcal{T}_\omega)$ such that*

$$P_{\mathcal{T}}\ell|_{H^{-1}(\omega)} = P_\omega\big(\ell|_{H^1_0(\omega)}\big) \in \mathbb{F}(\mathcal{T}_\omega)$$

for all $\ell \in H^{-1}(\Omega)$.

Proof. $\boxed{1}$ We first show that the degrees of freedom in Definition 4.23 of $\mathbb{V}^+_{\mathcal{T}}$ are linearly independent. To this end, we fix an element $T \in \mathcal{T}$, let $F_1, \dots, F_l$, $l \le d+1$ denote its faces that are in $\mathbb{F}_{\mathcal{T}}$ and write $\phi_0 := \phi_T$, $\phi_i := \phi_{F_i}$ and $E_i := E_{F_i}$ for $i = 1, \dots, l$. We then claim that, for all $q_0 \in \mathbb{P}_{m_2}(T)\setminus\{0\}$ and all $q_i \in \mathbb{P}_{m_1}(F_i)\setminus\{0\}$, $i = 1, \dots, l$, we have

$$\alpha_0 q_0\phi_0 + \sum_{i=1}^{l} \alpha_i E_i(q_i)\phi_i = 0 \text{ in } T \quad \Rightarrow \quad \alpha_0 = \cdots = \alpha_l = 0. \tag{4.36}$$

In light of $\operatorname{supp}\phi_T = T$ and $\operatorname{supp}\phi_F = \omega_F$, we observe $\phi_0|_{\partial T} = 0$ and $\phi_i|_{\partial T\setminus F_i} = 0$ for $i = 1, \dots, l$. We thus evaluate the hypothesis of (4.36) first on the faces $F_1, \dots, F_l$ and then in the element T. This gives $\alpha_i = 0$ for $i = 0, \dots, l$ and (4.36) is verified.

$\boxed{2}$ Next, we discuss the well-posedness of (4.34). The linear independence (4.36) and (4.25) lead to

$$\dim \mathbb{V}^+_{\mathcal{T}} = \#\mathcal{T} \dim \mathbb{P}_{m_2} + \#\mathcal{F} \dim \mathbb{P}_{m_1} = \dim \mathbb{F}_{\mathcal{T}}. \tag{4.37}$$

Thus, it suffices to show the implication

$$\ell \in \mathbb{F}_{\mathcal{T}} \colon \langle \ell, w \rangle = 0 \quad \text{for all } w \in \mathbb{V}^+_{\mathcal{T}} \quad \Rightarrow \quad \ell = 0. \tag{4.38}$$

For that purpose, let $\ell \in \mathbb{F}_{\mathcal{T}}$ and let $q_T, T \in \mathcal{T}$ and $q_F, F \in \mathcal{F}$ be the determining polynomials. For any element $T \in \mathcal{T}$, the choice $w = q_T \phi_T \in H^1_0(T)$ implies

$$0 = \langle \ell, w \rangle = \int_T (q_T)^2 \phi_T, \quad \text{i.e. } q_T = 0.$$

Thus ℓ does not have contributions from elements. Regarding faces, any choice $w = (E_F q_F)\phi_F \in H^1_0(\omega_F)$, $F \in \mathcal{F}$, therefore gives

$$0 = \langle \ell, w \rangle = \int_F (q_F)^2 \phi_F, \quad \text{i.e. } q_F = 0,$$

and implication (4.38) is established. Combining (4.37) and (4.38), we have that the dual pairing in $H^{-1}(\Omega)$ is non-degenerate on $\mathbb{F}_{\mathcal{T}} \times \mathbb{V}^+_{\mathcal{T}}$, which in turn ensures that $P_{\mathcal{T}}$ is well-defined. The Petrov–Galerkin character of the definition (4.34) then ensures that $P_{\mathcal{T}}$ is a linear projection onto $\mathbb{F}_{\mathcal{T}}$.

3 It remains to show that $P_{\mathcal{T}}$ is a local operator. Given any local subdomain ω meshed by $\mathcal{T}_\omega$, we can apply the preceding proof to $\mathcal{T}_\omega$ instead of $\mathcal{T}$. This shows that the dual pairing in $H^{-1}(\omega)$ is non-degenerate on $\mathbb{F}(\mathcal{T}_\omega) \times \mathbb{V}^+(\mathcal{T}_\omega)$ and ensures a local projection operator $P_\omega \colon H^{-1}(\omega) \to \mathbb{F}(\mathcal{T}_\omega)$. Taking into account (4.23) and (4.33), we note that

$$P_{\mathcal{T}}|_{H^{-1}(\omega)} = P_\omega,$$

which completes the proof. □

The verification of (4.38), which amounts to a proof of uniqueness, suggests the following approach to computing $P_{\mathcal{T}}\ell$, $\ell \in H^{-1}(\Omega)$.

Remark 4.26 (local computation). Let $\ell \in H^{-1}(\Omega)$. Recalling (4.35), the polynomials $P_T\ell$, $T \in \mathcal{T}$, and $P_F\ell$, $F \in \mathcal{F}$ can be computed by solving first

$$\int_T P_T\ell \, q\phi_T = \langle \ell, q\phi_T \rangle \quad \text{for all } T \in \mathcal{T}, q \in \mathbb{P}_{m_2}(T), \tag{4.39}$$

and then

$$\int_F P_F\ell \, q\phi_F = \langle \ell, q\phi_F \rangle - \sum_{T \subset \omega_F} \int_T P_T\ell \, q\phi_F \quad \text{for all } F \in \mathcal{F}, q \in \mathbb{P}_{m_1}(F). \tag{4.40}$$

This amounts to two block diagonal linear systems with, respectively, $\#\mathcal{T}$ blocks of size $\dim \mathbb{P}_{m_2}$ and $\#\mathcal{F}$ blocks of size $\dim \mathbb{P}_{m_1}$. Each block and each corresponding right-hand side arises from local computations.

Remark 4.27 (star localization vs. locality of $P_{\mathcal{T}}$). Stars ω_z, $z \in \mathcal{V}$, are meshed local subdomains. Lemma 4.25 thus shows that, for any vertex $z \in \mathcal{V}$, there is a

linear projection $P_z\colon H^{-1}(\omega_z) \to \mathbb{F}(\mathcal{T}_z)$ such that

$$P_{\mathcal{T}}\ell|_{H^{-1}(\omega_z)} = P_z\big(\ell|_{H^1_0(\omega_z)}\big) \in \mathbb{F}(\mathcal{T}_z)$$

for all $\ell \in H^{-1}(\Omega)$. The stars also appear in the localizing upper bound of the global residual norm in Corollary 4.6. As they are minimal subdomains therein (see Remark 4.7), it may appear that a finer localization with smaller domains cannot be exploited in *a posteriori* analysis. Although this is true in the context of upper bounds, the increased locality of $P_{\mathcal{T}}$ is useful in the context of lower bounds; see the reduced lower bound (5.17), which follows from the interior vertex property introduced in Definition 4.50, and is crucial for deriving the contraction result (5.26).

We have already mentioned that we shall use $P_{\mathcal{T}}$ to split the residual. In light of the bounds for the residual norm in Corollary 4.6 (star localization of residual norm), this should be done in a locally stable manner. In order to formulate and employ the local stability properties of $P_{\mathcal{T}}$, the following notation is useful. Given a local subdomain ω meshed by $\mathcal{T}_\omega$, we define the $\mathbb{V}^+(\mathcal{T}_\omega)$*-discrete dual norm* by

$$\|\ell\|_{\mathbb{V}^+(\mathcal{T}_\omega)^*} := \sup_{w\in\mathbb{V}^+(\mathcal{T}_\omega),\,\|\nabla w\|_{L^2(\omega)}=1} \langle \ell, w\rangle \quad \ell \in H^{-1}(\omega). \tag{4.41}$$

In view of $\mathbb{V}^+(\mathcal{T}_\omega) \subset H^1_0(\omega)$, we have $\|\ell\|_{\mathbb{V}^+(\mathcal{T}_\omega)^*} \le \|\ell\|_{H^{-1}(\omega)}$ for all $\ell \in H^{-1}(\omega)$.

Lemma 4.28 (local H^{-1}-stability). *The projection $P_{\mathcal{T}}$ is locally H^{-1}-stable: for any local subdomain ω meshed by $\mathcal{T}_\omega$, we have*

$$\|P_{\mathcal{T}}\|_{\mathcal{L}(H^{-1}(\omega))} = \sup_{\ell\in\mathbb{F}(\mathcal{T}_\omega)} \frac{\|\ell\|_{H^{-1}(\omega)}}{\|\ell\|_{\mathbb{V}^+(\mathcal{T}_\omega)^*}} \le C_{\mathrm{lStb}},$$

where $C_{\mathrm{lStb}} = C_{\mathrm{lStb}}(d,\sigma,m_1,m_2)$ depends only on d, the shape regularity coefficient σ from (3.9), the degrees (m_1,m_2) of the discrete functionals and the space $\mathbb{S}^+$.

Proof. $\boxed{1}$ We start by verifying the '$\le$'-part of the claimed identity for the operator norm. The definition of the operator norm leads to

$$\|P_\omega\|_{\mathcal{L}(H^{-1}(\omega))} = \sup_{\ell\in H^{-1}(\omega)} \frac{\|P_\omega\ell\|_{H^{-1}(\omega)}}{\|\ell\|_{H^{-1}(\omega)}} \le \sup_{\ell\in H^{-1}(\omega)} \frac{\|P_\omega\ell\|_{H^{-1}(\omega)}}{\|\ell\|_{\mathbb{V}^+(\mathcal{T}_\omega)^*}}.$$

We now notice that $\|\ell\|_{\mathbb{V}^+(\mathcal{T}_\omega)^*} = \|P_\omega\ell\|_{\mathbb{V}^+(\mathcal{T}_\omega)^*}$ in view of (4.41) and (4.34). Hence

$$\|P_\omega\|_{\mathcal{L}(H^{-1}(\omega))} \le \sup_{\ell\in H^{-1}(\omega)} \frac{\|P_\omega\ell\|_{H^{-1}(\omega)}}{\|P_\omega\ell\|_{\mathbb{V}^+(\mathcal{T}_\omega)^*}} = \sup_{\ell\in\mathbb{F}(\mathcal{T}_\omega)} \frac{\|\ell\|_{H^{-1}(\omega)}}{\|\ell\|_{\mathbb{V}^+(\mathcal{T}_\omega)^*}},$$

because the projection P_ω is onto $\mathbb{F}(\mathcal{T}_\omega)$.

$\boxed{2}$ Next, we show that $\|P_{\mathcal{T}}\|_{\mathcal{L}(H^{-1}(\omega))}$ is uniformly bounded. Let $\ell \in \mathbb{F}(\mathcal{T}_\omega)$ be a discrete functional, namely

$$\langle \ell, w\rangle = \sum_{T\in\mathcal{T}_\omega}\int_T q_T w + \sum_{F\in\mathcal{F}_\omega}\int_F q_F w \quad \text{for all } w \in H^1_0(\omega).$$

We proceed in two steps that are quite similar to the classical standard residual estimates. Arguing as in (4.11), and using Lemma 2.2 (first Poincaré inequality) in the domain ω of diameter about h_T for $w \in H^1_0(\omega)$, we obtain

$$|\langle \ell, w\rangle| \lesssim \left(\sum_{T\in\mathcal{T}_\omega} h_T^2\|q_T\|^2_{L^2(T)} + \sum_{F\in\mathcal{F}_\omega} h_F\|q_F\|^2_{L^2(F)}\right)^{1/2}\|\nabla w\|_{L^2(\omega)},$$

whence

$$\|\ell\|^2_{H^{-1}(\omega)} \lesssim \sum_{T\in\mathcal{T}_\omega} h_T^2\|q_T\|^2_{L^2(T)} + \sum_{F\in\mathcal{F}_\omega} h_F\|q_F\|^2_{L^2(F)}. \tag{4.42}$$

Here we do not exploit the fact that ℓ is discrete in ω; this will be crucial in the second step, when we bound each term on the right-hand side, in a manner recalling the derivation of classical lower bounds. For any $T \in \mathcal{T}_\omega$, we write $\mathbb{V}^+(T)^*$ as shorthand for $\mathbb{V}^+(\{T\})^*$, and exploit Lemma 4.22 (properties of cut-off) to deduce

$$\begin{aligned}\|q_T\|^2_{L^2(T)} &\lesssim \int_T (q_T)^2\phi_T = \langle \ell, q_T\phi_T\rangle \le \|\ell\|_{\mathbb{V}^+(T)^*}\|\nabla(q_T\phi_T)\|_{L^2(T)}\\ &\lesssim \|\ell\|_{\mathbb{V}^+(T)^*}h_T^{-1}\|q_T\phi_T\|_{L^2(T)} \le \|\ell\|_{\mathbb{V}^+(T)^*}h_T^{-1}\|q_T\|_{L^2(T)},\end{aligned}$$

whence

$$h_T\|q_T\|_{L^2(T)} \lesssim \|\ell\|_{\mathbb{V}^+(T)^*}. \tag{4.43}$$

For an interior face $F \in \mathcal{F}_\omega$, we proceed similarly, also taking into account Lemma 4.20 and that $\mathbb{V}^+(T) \subset \mathbb{V}^+(\mathcal{T}_F)$ entails $\|\ell\|_{\mathbb{V}^+(T)^*} \le \|\ell\|_{\mathbb{V}^+(\mathcal{T}_F)}$ for $T \in \mathcal{T}_F$. We thus obtain

$$\begin{aligned}&\|q_F\|^2_{L^2(F)}\\ &\quad\lesssim \int_F (q_F)^2\phi_F = \langle \ell, (E_F q_F)\phi_F\rangle - \sum_{T\subset\omega_F}\int_T q_T(E_F q_F)\phi_F\\ &\quad\lesssim \|\ell\|_{\mathbb{V}^+(\mathcal{T}_F)^*}\|\nabla((E_F q_F)\phi_T)\|_{L^2(\omega_F)} + \sum_{T\subset\omega_F}\|q_T\|_{L^2(T)}\|(E_F q_F)\phi_T\|_{L^2(T)}\\ &\quad\lesssim \|\ell\|_{\mathbb{V}^+(\mathcal{T}_F)^*}h_T^{-1}\|(E_F q_F)\phi_T\|_{L^2(\omega_F)} \lesssim \|\ell\|_{\mathbb{V}^+(\mathcal{T}_F)^*}h_T^{-1/2}\|q_F\|_{L^2(F)},\end{aligned}$$

that is,

$$h_F^{1/2}\|q_F\|_{L^2(F)} \lesssim \|\ell\|_{\mathbb{V}^+(\mathcal{T}_F)^*}. \tag{4.44}$$

The number of elements and interior faces in the local subdomain ω is uniformly bounded by d and the shape regularity coefficient σ. Hence inequalities (4.43) and

(4.44) together with the inclusions $\mathbb{V}^+(T) \subset \mathbb{V}^+(\mathcal{T}_\omega)$ for $T \in \mathcal{T}_\omega$ and $\mathbb{V}^+(\mathcal{T}_F) \subset \mathbb{V}^+(\mathcal{T}_\omega)$ for $F \in \mathcal{F}_\omega$ imply

$$\sum_{T \in \mathcal{T}_\omega} h_T^2 \|q_T\|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}_\omega} h_F \|q_F\|_{L^2(F)}^2 \lesssim \|\ell\|_{\mathbb{V}^+(\mathcal{T}_\omega)^*}^2. \tag{4.45}$$

Combining (4.42) and (4.45) shows that the ratio $\|\ell\|_{H^{-1}(\omega)}/\|\ell\|_{\mathbb{V}^+(\mathcal{T}_\omega)^*}$ for $\ell \in \mathbb{F}(\mathcal{T}_\omega)$ is bounded by a universal constant depending on d, σ, m_1, m_2 and $\mathbb{S}^+$.

$\boxed{3}$ It remains to complete the proof of the claimed identity for the operator norm. To this end, we first introduce an operator $Q_\omega \colon H_0^1(\omega) \to \mathbb{V}^+(\mathcal{T}_\omega)$ by

$$\langle \ell, Q_\omega w \rangle = \langle \ell, w \rangle \quad \text{for all } \ell \in \mathbb{F}(\mathcal{T}_\omega).$$

Like the one for P_ω, this definition is well-posed because the pair $(\mathbb{F}(\mathcal{T}_\omega), \mathbb{V}^+(\mathcal{T}_\omega))$ is non-degenerate for the dual pairing of $H^{-1}(\omega)$; see (4.37) and (4.38) in the proof of Lemma 4.25 (algebraic properties). By the Petrov–Galerkin character of the definition, Q_ω is a linear projection onto $\mathbb{V}^+(\mathcal{T}_\omega)$. Given arbitrary $\ell \in H^{-1}(\omega)$ and $w \in H_0^1(\omega)$, the definitions of Q_ω and P_ω imply

$$\langle P_\omega \ell, w \rangle = \langle P_\omega \ell, Q_\omega w \rangle = \langle \ell, Q_\omega w \rangle,$$

that is, $Q_\omega = P_\omega^*$ is the (Hilbert) adjoint to P_ω. In other words, the adjoint P_ω^* is a projection onto $\mathbb{V}^+(\mathcal{T}_\omega)$. With this, we can prove the missing inequality. Let $\ell \in \mathbb{F}(\mathcal{T}_\omega)$ be discrete. In fact,

$$\langle \ell, w \rangle = \langle P_\omega \ell, w \rangle = \langle \ell, P_\omega^* w \rangle \quad \Rightarrow \quad \|\ell\|_{H^{-1}(\omega)} = \sup_{w \in H_0^1(\omega)} \frac{\langle \ell, P_\omega^* w \rangle}{\|\nabla w\|_{L^2(\omega)}}$$

leads to

$$\|\ell\|_{H^{-1}(\omega)} \le \|\ell\|_{\mathbb{V}^+(\mathcal{T}_\omega)^*} \|P_\omega^*\|_{\mathcal{L}(H_0^1(\omega))} = \|\ell\|_{\mathbb{V}^+(\mathcal{T}_\omega)^*} \|P_\omega\|_{\mathcal{L}(H^{-1}(\omega))}.$$

This concludes the proof. □

Remark 4.29 (failing global H^{-1}-stability). For Lebesgue norms, local stability of linear operators in terms of shape regularity entails that their respective global stability is uniform under mesh refinement. The fact that part (i) of Lemma 4.5 (localization of H^{-1}-norm) needs a condition to be true, may lead us to suspect that this implication might not be true in general for the H^{-1}-norm. This suspicion is confirmed by Example 4.63 below, where we show that $\|P_\mathcal{T}\|_{\mathcal{L}(H^{-1}(\Omega))}$ can tend to ∞ under mesh refinement.

The proof of Lemma 4.28 provides all non-trivial ingredients to allow the approximate computation of $\|\ell\|_{H^{-1}(\omega)}$ whenever $\ell \in H^{-1}(\Omega)$ is discrete in ω.

Corollary 4.30 (quantifying H^{-1}-norms of discrete functionals). *Let $\omega \subset \Omega$ be a local subdomain meshed by $\mathcal{T}_\omega$ and let $\ell \in H^{-1}(\Omega)$ be discrete in ω, given by the polynomials q_T for $T \in \mathcal{T}_\omega$ and q_F for $F \in \mathcal{F}_\omega$, where $F \in \mathcal{F}_\omega$ are the interior*

faces in ω. We then have

$$\|\ell\|^2_{H^{-1}(\omega)} \approx \sum_{T\in\mathcal{T}_\omega} h_T^2 \|q_T\|^2_{L^2(T)} + \sum_{F\in\mathcal{F}_\omega} h_F \|q_F\|^2_{L^2(F)},$$

where the hidden constants depend on d, the shape regularity coefficient σ, the degrees (m_1, m_2) of the discrete functionals, and the space $\mathbb{S}^+$ appearing in Assumption 4.21.

Proof. This is a consequence of (4.42) and (4.45), the latter requiring ℓ to be discrete, along with the fact that $\|\ell\|_{\mathbb{V}^+(\mathcal{T}_\omega)^*} \le \|\ell\|_{H^{-1}\omega)^*}$ for any $\ell \in H^{-1}(\omega)$. □

Corollary 4.31 (local near-best approximation). *The projection $P_\mathcal{T}$ yields local near-best approximations: for any functional $\ell \in H^{-1}(\Omega)$ and any local subdomain ω meshed by $\mathcal{T}_\omega$, we have*

$$\|\ell - P_\mathcal{T}\ell\|_{H^{-1}(\omega)} \le C_{\mathrm{lStb}} \inf_{\chi\in\mathbb{F}(\mathcal{T}_\omega)} \|\ell - \chi\|_{H^{-1}(\omega)},$$

where C_{lStb} is the constant of Lemma 4.28 (local H^{-1}-stability).

Proof. Fix a local subdomain ω meshed by $\mathcal{T}_\omega$ and let $\chi \in \mathbb{F}(\mathcal{T}_\omega)$ be arbitrary. Thanks to Lemma 4.25 (algebraic properties), we have $P_\omega\chi = \chi$ and

$$(\ell - P_\mathcal{T}\ell)|_{H^1_0(\omega)} = (I - P_\omega)\ell|_{H^1_0(\omega)} = (I - P_\omega)(\ell - \chi)|_{H^1_0(\omega)}.$$

As P_ω is a non-trivial projection on the Hilbert space $H^{-1}(\omega)$, Szyld (2006) ensures

$$\|I - P_\omega\|_{\mathcal{L}(H^{-1}(\omega))} = \|P_\omega\|_{\mathcal{L}(H^{-1}(\omega))} \le C_{\mathrm{lStb}}. \tag{4.46}$$

Hence

$$\|\ell - P_\mathcal{T}\ell\|_{H^{-1}(\omega)} \le C_{\mathrm{lStb}}\|\ell - \chi\|_{H^{-1}(\omega)}$$

concludes the proof because $\chi \in \mathbb{F}(\mathcal{T}_\omega)$ is arbitrary. □

We illustrate the approximation of possible parts of the residual with the projection $P_\mathcal{T}$ in a series of three remarks. For that purpose, the approximation quality is to be measured with a local $H^{-1}(\omega)$-norm, and it is instructive to compare with the operator $\Pi_\mathcal{T}$ from (4.26). Recall that the operator $\Pi_\mathcal{T}$ is used implicitly in the standard approach (see Section 4.2) to approximate the discrete functionals $\mathbb{F}_\mathcal{T}$.

Remark 4.32 (approximating functions). For functions, the local error with $P_\mathcal{T}$ is uniformly dominated by the one with $\Pi_\mathcal{T}$. More precisely, if $m_2 \ge 0$ and $\ell \in H^{-1}(\Omega)$ satisfies $\langle \ell, w\rangle = \int_\Omega fw$ where $f \in L^p(\Omega)$ with $p > 2d/(2+d)$, then Corollary 4.31 (local near-best approximation) and $\Pi_\mathcal{T} f = \sum_{T\in\mathcal{T}}(\Pi_T^{m_2} f)\chi_T \in \mathbb{F}(\mathcal{T}_\omega)$ imply, for any local meshed subdomain ω,

$$\|\ell - P_\mathcal{T}\ell\|_{H^{-1}(\omega)} \lesssim \|f - \Pi_\mathcal{T} f\|_{H^{-1}(\omega)}.$$

Observe that although ℓ is a function, $P_\mathcal{T}\ell$ is typically not a function. This property might look undesirable but it is crucial for an advantage of $P_\mathcal{T}$ over $\Pi_\mathcal{T}$ and closely

related to the fact that the opposite inequality does not hold; see Remark 4.34 about stability.

Furthermore, supposing $f \in L^2(\Omega)$ and combining the preceding inequality with Lemma 2.2 (first Poincaré inequality) gives

$$\|\ell - P_{\mathcal{T}}\ell\|^2_{H^{-1}(\omega_T)} \lesssim \sum_{T' \subset \omega_T} h_T^2 \|f - \Pi^{m_2}_{T'} f\|^2_{L^2(\omega_{T'})}, \tag{4.47}$$

which establishes that the local $P_{\mathcal{T}}$-oscillation of functions is uniformly dominated by its classical $\Pi_{\mathcal{T}}$-counterpart but not vice versa.

In Section 7.3.1 this case is considered in the context of adaptive approximation.

Remark 4.33 (approximating admissible functionals). For functionals allowing for the application of $\Pi_{\mathcal{T}}$, the local error with $P_{\mathcal{T}}$ is again uniformly dominated by the one with $\Pi_{\mathcal{T}}$. In view of the previous remark, let us consider only $\ell \in H^{-1}(\Omega)$ such that $\langle \ell, w\rangle = \int_\Sigma gw$, where $g \in L^p(\Sigma)$ with $\Sigma := \cup_{F\in\mathcal{F}} F$ and $p > 2(d-1)/d$. Note that we again have $\Pi_{\mathcal{T}}\ell \in \mathbb{F}(\mathcal{T}_\omega)$ as $\langle \Pi_{\mathcal{T}}\ell, w\rangle = \sum_{F\in\mathcal{F}} \int_F (\Pi^{m_1}_F g)\, w$ for all $w \in H^1_0(\Omega)$. Corollary 4.31 (local near-best approximation) thus ensures, for any local meshed subdomain ω,

$$\|\ell - P_{\mathcal{T}}\ell\|_{H^{-1}(\omega)} \lesssim \|\ell - \Pi_{\mathcal{T}}\ell\|_{H^{-1}(\omega)}.$$

Moreover, supposing $g \in L^2(\Sigma)$ and combining the scaled trace theorem (4.12) with Lemma 2.2 (first Poincaré inequality) yields

$$\|\ell - P_{\mathcal{T}}\ell\|^2_{H^{-1}(\omega_T)} \lesssim \sum_{F\subseteq\overline{\omega}_T, F\not\subseteq\partial\omega_T} h_F \|g - \Pi^{m_1}_F g\|^2_{L^2(F)}.$$

Also, this case will be revisited in the context of adaptive approximation, namely in Section 7.3.3.

Remark 4.34 (stability of approximation). The error with $P_{\mathcal{T}}$ is stable, while the one with $\Pi_{\mathcal{T}}$ is not. To see this by example, we restrict to $(m_1, m_2) = (0, 0)$, fix some interior face $F \in \mathcal{F}$ and, for $\varepsilon > 0$ sufficiently small, consider

$$\langle \ell_\varepsilon, w\rangle := \int_\Omega f_\varepsilon w = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_F w(y + s\boldsymbol{n}_F)\,\mathrm{d}y\,\mathrm{d}s, \quad w \in H^1_0(\Omega),$$

where $f_\varepsilon = (2\varepsilon)^{-1}\chi_{F_\varepsilon}$ is a multiple of the characteristic function of $F_\varepsilon := \{x + s\boldsymbol{n}_F \mid x \in F, -\varepsilon < s < \varepsilon\}$. As

$$\begin{aligned}\langle \ell_\varepsilon - \delta_F, w\rangle &= \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_F\int_0^s \partial_{\boldsymbol{n}_F} w(y + t\boldsymbol{n}_F)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \\ &\le \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_{F_\varepsilon} |\nabla w(x)|\,\mathrm{d}x\,\mathrm{d}s \le |F_\varepsilon|^{1/2}\|\nabla w\|_{L^2(\Omega)},\end{aligned}$$

the functions $(\ell_\varepsilon)_{\varepsilon>0}$ tend to the proper functional δ_F:

$$\ell_\varepsilon \to \delta_F \quad \text{in } H^{-1}(\Omega).$$

Combining the convergence with the local stability of $P_{\mathcal{T}}$ (see (4.46)) yields

$$\begin{aligned}|\|\ell_\varepsilon - P_{\mathcal{T}}\ell_\varepsilon\|_{H^{-1}(\omega_F)} - \|\delta_F - P_{\mathcal{T}}\delta_F\|_{H^{-1}(\omega_F)}| &\le \|(I - P_{\mathcal{T}})(\ell_\varepsilon - \delta_F)\|_{H^{-1}(\omega_F)} \\ &\lesssim \|\ell_\varepsilon - \delta_F\|_{H^{-1}(\omega_F)} \to 0,\end{aligned}$$

the stability of the error with $P_{\mathcal{T}}$. Furthermore, since $P_{\mathcal{T}}\delta_F = \delta_F$, the stability entails here $\|\ell_\varepsilon - P_{\mathcal{T}}\ell_\varepsilon\|_{H^{-1}(\omega_F)} \to 0$. For $\Pi_{\mathcal{T}}$, however, the approximation on the skeleton and in the volume are independent of each other. Hence, combining $\Pi_{\mathcal{T}}\delta_F = \delta_F$, which follows from (4.26), with $\lim_{\varepsilon\to 0}\Pi^0_T f_\varepsilon = |F|/(2|T|)$ for the two elements $T \in \mathcal{T}$ containing F, leads to

$$\|\delta_F - \Pi_{\mathcal{T}}\delta_F\|_{H^{-1}(\omega_F)} = 0 < \lim_{\varepsilon\to 0}\|\ell_\varepsilon - \Pi_{\mathcal{T}}\ell_\varepsilon\|_{H^{-1}(\omega_F)}.$$

Measuring the error in weighted L^2-norms instead of the H^{-1}-norm results in a more dramatic instability. Indeed, letting 1_K and 0_K denote the constant functions on a simplex K equal to 1 or 0, the two sides translate to

$$h_F^{1/2}\|1_F - \Pi^0_F 1_F\|_{L^2(F)} + \sum_{T\in\mathcal{T}_F} h_T\|0_T - \Pi^0_T 0_T\|_{L^2(T)} = 0$$

and

$$\lim_{\varepsilon\to 0}\left(h_F^{1/2}\|0_F - \Pi^0_F 0_F\|_{L^2(F)} + \sum_{T\in\mathcal{T}_F} h_T\|f_\varepsilon - \Pi^0_T f_\varepsilon\|_{L^2(T)}\right) = \infty.$$

Note that such a transformation of volume contributions into contributions on the skeleton may occur by perturbation in the right-hand side or, in the opposite direction, by an improvement of the Galerkin approximation thanks to refinement.

In view of this instability of $\Pi_{\mathcal{T}}$, the inequalities in the preceding Remarks 4.32 and 4.33 cannot be reversed – a fact that can also be inferred from Remark 4.19.

The above perturbations of δ_F are in the domain of $\Pi_{\mathcal{T}}$. For the functionals

$$\langle\widehat{\ell}_\varepsilon, w\rangle := \int_F w(y + \varepsilon\boldsymbol{n}_F)\,\mathrm{d}y, \quad w \in H^1_0(\Omega),$$

however, it is not clear how to directly apply $\Pi_{\mathcal{T}}$ for $\varepsilon \neq 0$. To the contrary, the approximations $P_{\mathcal{T}}\widehat{\ell}_\varepsilon$ are defined and stable around 0. Notably, $P_{\mathcal{T}}\widehat{\ell}_\varepsilon$ uses volume contributions to compensate for the displacement in the representation of the singular contribution.

4.6. *Discretized and oscillatory residual*

We now turn to the proper *a posteriori* analysis, that is, we shall derive *upper* and *lower bounds of the error*, implementing the following plan, which is motivated in Section 4.3. We use the projection $P_{\mathcal{T}}$ onto discrete functionals $\mathbb{F}_{\mathcal{T}}$ to split the residual into discretized and oscillatory parts. Then the quantification of the *oscillatory residual* is reduced to *data* oscillation through suitable choices of the

degrees (m_1, m_2) of the discrete functionals. The *discretized residual* can be quantified in various ways; see Sections 4.2 and 4.9 below.

We start by introducing indicators reflecting the stated splitting of the residual into discretized and oscillatory parts. They are vertex-indexed and, given $z \in \mathcal{V}$, defined by

$$\begin{aligned} &\mathcal{E}_{\mathcal{T}}^{\mathrm{abs}}(z)^2 := \eta_{\mathcal{T}}^{\mathrm{abs}}(z)^2 + \mathrm{osc}_{\mathcal{T}}(R_{\mathcal{T}}, z)^2 \quad \text{with} \\ &\eta_{\mathcal{T}}^{\mathrm{abs}}(z) := \|P_{\mathcal{T}} R_{\mathcal{T}}\|_{H^{-1}(\omega_z)} \quad \text{and} \quad \mathrm{osc}_{\mathcal{T}}(R_{\mathcal{T}}, z) := \|(I - P_{\mathcal{T}}) R_{\mathcal{T}}\|_{H^{-1}(\omega_z)}. \end{aligned} \tag{4.48}$$

Note that these quantities are not proper indicators: they still need to be quantified in a computable manner. By using 'abs' (shorthand for 'abstract'), we hint at the fact that $\eta_{\mathcal{T}}^{\mathrm{abs}}$ can be quantified by various approaches.

Lemma 4.35 (splitting of local residual norm). *For any vertex $z \in \mathcal{V}$, the local residual norm is equivalent to the abstract indicator from* (4.48)*:*

$$\frac{1}{\sqrt{2} C_{\mathrm{lStb}}} \mathcal{E}_{\mathcal{T}}^{\mathrm{abs}}(z) \le \|R_{\mathcal{T}}\|_{H^{-1}(\omega_z)} \le \sqrt{2} \mathcal{E}_{\mathcal{T}}^{\mathrm{abs}}(z),$$

where C_{lStb} is the constant of Lemma 4.28 (local H^{-1}-stability).

Proof. As announced, we use the linear projection $P_{\mathcal{T}}$ in order to split the residual into a discretized and an oscillatory part:

$$R_{\mathcal{T}} = P_{\mathcal{T}} R_{\mathcal{T}} + (I - P_{\mathcal{T}}) R_{\mathcal{T}}. \tag{4.49}$$

The upper bound of the local residual norm then readily follows from the triangle inequality:

$$\|R_{\mathcal{T}}\|_{H^{-1}(\omega_z)} \le \eta_{\mathcal{T}}^{\mathrm{abs}}(z) + \mathrm{osc}_{\mathcal{T}}(R_{\mathcal{T}}, z) \le \sqrt{2} \mathcal{E}_{\mathcal{T}}^{\mathrm{abs}}(z).$$

To show the lower bound, we exploit the local stability of $P_{\mathcal{T}}$ (see (4.46)) to obtain

$$\eta_{\mathcal{T}}^{\mathrm{abs}}(z) = \|P_{\mathcal{T}} R_{\mathcal{T}}\|_{H^{-1}(\omega_z)} \le C_{\mathrm{lStb}} \|R_{\mathcal{T}}\|_{H^{-1}(\omega_z)}$$

and

$$\mathrm{osc}_{\mathcal{T}}(R_{\mathcal{T}}, z) = \|(I - P_{\mathcal{T}}) R_{\mathcal{T}}\|_{H^{-1}(\omega_z)} \le C_{\mathrm{lStb}} \|R_{\mathcal{T}}\|_{H^{-1}(\omega_z)}.$$

Squaring both inequalities, summing them, and then taking the square-root finishes the proof. □

In many *a posteriori* analyses, this lemma is replaced by steps breaking a possible true equivalence between error and estimator. Therefore the following remark points out the key ingredients.

Remark 4.36 (ensuring proper equivalence). The fact that the projection $P_{\mathcal{T}}$ and so also $I - P_{\mathcal{T}}$ are linear and locally bounded operators precludes overestimation; see also Remark 4.19. Comparing with Section 4.2 and $\Pi_{\mathcal{T}}$ in (4.27), we see that the local stability in H^{-1} is crucial to that end and, in view of Remark 4.34, requires discrete functionals with contributions on the skeleton.

Next, we want to simplify the residual oscillations $\operatorname{osc}_{\mathcal{T}}(R_{\mathcal{T}}, z)$, $z \in \mathcal{V}$, in the spirit of Remark 4.14. This will be dependent on the coefficients $\boldsymbol{A}$ and c of the differential operator and involve the following 'polynomial degrees':

$$n_{\boldsymbol{A}} := \min\{k \in \mathbb{N}_0 \mid \boldsymbol{A} \in \left(\mathbb{S}^{k,-1}_{\mathcal{T}}\right)^{d\times d}\}, \tag{4.50a}$$

$$n_c := \min\{k \in \mathbb{N}_0 \cup \{-1\} \mid c \in \mathbb{S}^{k,-1}_{\mathcal{T}}\}, \tag{4.50b}$$

where we use the convention $\min \emptyset = \infty$. We shall say that the differential operator $-\operatorname{div}(\boldsymbol{A}\nabla\cdot) + c(\cdot)$ in (2.5) *has discrete coefficients* whenever $\max\{n_{\boldsymbol{A}}, n_c\} < \infty$; otherwise it has *non-discrete coefficients*.

Lemma 4.37 (data oscillation reduction for discrete coefficients). *If the coefficients A and c are discrete, the choices*

$$m_1 = n_{\boldsymbol{A}} + n - 1,$$

$$m_2 = \max\{n - 2 + n_{\boldsymbol{A}}, \widetilde{m}_c\} \quad \textit{with} \quad \widetilde{m}_c = \begin{cases} n + n_c, & \textit{if } c \neq 0 \\ 0, & \textit{otherwise} \end{cases}$$

ensure that the oscillatory residual reduces to data oscillation of the right-hand side:

$$(I - P_{\mathcal{T}})R_{\mathcal{T}} = f - P_{\mathcal{T}} f.$$

Proof. The choices for m_1 and m_2 yield, for any face $F \in \mathcal{F}$ and any element $T \in \mathcal{T}$,

$$[[\boldsymbol{A}\nabla u_{\mathcal{T}}]] \cdot \boldsymbol{n}_F \in \mathbb{P}_{m_1}(F) \quad \text{and} \quad \operatorname{div}(\boldsymbol{A}\nabla u_{\mathcal{T}})|_T \in \mathbb{P}_{m_2}(T).$$

Furthermore, if $c \neq 0$, we also have $cu_{\mathcal{T}}|_T \in \mathbb{P}_{m_2}(T)$, and the claimed identity follows from $-\operatorname{div}(\boldsymbol{A}\nabla u_{\mathcal{T}}) + cu_{\mathcal{T}} \in \mathbb{F}_{\mathcal{T}}$. □

Remark 4.38 (Poisson equation with linear elements). In the case of the Poisson equation with linear elements, the choices in Lemma 4.37 lead to $m_1 = 0$ and $m_2 = 0$. Alternatively, one may use $m_1 = 0$ and $m_2 = -1$ (recall we have set $\mathbb{P}_{-1}(T) = \{0\}$); see Diening, Kreuzer and Stevenson (2016) or Siebert and Veeser (2007). The choice here leads to an oscillation for which the standard oscillation indicators $h_T \|f - \Pi_T f\|_T$, $T \in \mathcal{T}$, can be used as a surrogate; see also Remark 4.43 about surrogates.

If one of the coefficients, $\boldsymbol{A}$ or c, is non-discrete, the range of the finite element space $\mathbb{V}_{\mathcal{T}}$ under the differential operator $-\operatorname{div}(\boldsymbol{A}\nabla\cdot) + c(\cdot)$ consists of functionals whose densities are not piecewise polynomial. Consequently, the oscillatory residual cannot be reduced to the oscillation $f - P_{\mathcal{T}} f$, or to any other oscillation of f involving discrete functionals with piecewise polynomial densities. The next result illustrates the idea of a non-perfect remedy, namely bounding the residual oscillation defined in (4.48) in terms of data oscillation and discrete stability. For

its formulation, we define global H^{-1}-oscillations by

$$\operatorname{osc}_{\mathcal{T}}(\ell)^2 := \sum_{z\in\mathcal{V}} \operatorname{osc}_{\mathcal{T}}(\ell, z)^2, \quad \ell \in H^{-1}(\Omega), \tag{4.51}$$

which, in contrast to $\|(I - P_{\mathcal{T}})\ell\|_{H^{-1}(\Omega)}$, is bounded in terms of ℓ; see Remark 4.29 (failing global H^{-1}-stability).

Lemma 4.39 (surrogate data oscillation reduction). *Let $m_{\boldsymbol{A}} := \min\{n_{\boldsymbol{A}}, n-1\}$ and $m_c := \min\{n_c, n-1\}$, and define m_1 and m_2 as in Lemma 4.37, but replacing $n_{\boldsymbol{A}}$ and n_c, respectively, with $m_{\boldsymbol{A}}$ and m_c. Given any approximations*

$$\widehat{\boldsymbol{A}} \in \big(\mathbb{S}_{\mathcal{T}}^{m_{\boldsymbol{A}},-1}\big)^{d\times d} \quad \text{and} \quad \widehat{c} \in \mathbb{S}_{\mathcal{T}}^{m_c,-1},$$

we then have, for all vertices $z \in \mathcal{V}$,

$$\begin{aligned}\operatorname{osc}_{\mathcal{T}}(R_{\mathcal{T}}, z) \le \operatorname{osc}_{\mathcal{T}}(f, z) &+ C_{\mathrm{lStb}} C(d,\sigma)\|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|_{L^\infty(\omega_z)}\|\nabla u_{\mathcal{T}}\|_{L^2(\omega_z)} \\ &+ C_{\mathrm{lStb}} C(d,\sigma)\|h(c - \widehat{c})\|_{L^\infty(\omega_z)}\|u_{\mathcal{T}}\|_{L^2(\omega_z)},\end{aligned}$$

and thus

$$\begin{aligned}\operatorname{osc}_{\mathcal{T}}(R_{\mathcal{T}})^2 &\le 3\operatorname{osc}_{\mathcal{T}}(f)^2 \\ &\quad + \frac{3(d+1)}{\alpha^2}\|f\|_{H^{-1}(\Omega)}^2\big(\|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|_{L^\infty(\Omega)}^2 + C_P^2 C(d,\sigma)\|h(c-\widehat{c})\|_{L^\infty(\Omega)}^2\big)\end{aligned}$$

where C_{lStb} is the constant from Lemma 4.28 (local H^{-1}-stability), h is the mesh size function defined by $h|_T = h_T$ for all $T \in \mathcal{T}$, α is the coercivity constant from (2.29), and C_P is the constant in Lemma 2.2 (first Poincaré inequality).

The bounds of Lemma 4.39 are obviously not convenient if $\boldsymbol{A}$ or c are not continuous. We therefore implement the underlying idea in Section 5.4 differently.

Proof. $\boxed{1}$ To verify the local bound, let $z \in \mathcal{V}$ be any vertex. By linearity of $P_{\mathcal{T}}$, we obtain

$$\begin{aligned}\operatorname{osc}_{\mathcal{T}}(R_{\mathcal{T}}, z) &\le \operatorname{osc}_{\mathcal{T}}(f, z) \\ &\quad + \|(I - P_{\mathcal{T}})(-\operatorname{div}(\boldsymbol{A}\nabla u_{\mathcal{T}}))\|_{H^{-1}(\omega_z)} + \|(I - P_{\mathcal{T}})(cu_{\mathcal{T}})\|_{H^{-1}(\omega_z)}\end{aligned}$$

and it remains to bound appropriately the two terms involving the coefficients $\boldsymbol{A}$ and c. As the definitions of m_1 and m_2 ensure $-\operatorname{div}(\widehat{\boldsymbol{A}}\nabla u_{\mathcal{T}}) \in \mathbb{F}_{\mathcal{T}}$, Corollary 4.31 (local near-best approximation), the scaled trace theorem (4.12) and Lemma 2.2 (first Poincaré inequality) give

$$\begin{aligned}\|(I - P_{\mathcal{T}})(-\operatorname{div}(\boldsymbol{A}\nabla u_{\mathcal{T}}))\|_{H^{-1}(\omega_z)} &\le C_{\mathrm{lStb}}\|-\operatorname{div}((\boldsymbol{A} - \widehat{\boldsymbol{A}})\nabla u_{\mathcal{T}})\|_{H^{-1}(\omega_z)} \\ &\le C C_{\mathrm{lStb}}\|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|_{L^\infty(\omega_z)}\|\nabla u_{\mathcal{T}}\|_{L^2(\omega_z)}.\end{aligned}$$

As $\widehat{c}u_{\mathcal{T}} \in \mathbb{F}_{\mathcal{T}}$ thanks to the definition of m_2, a similar argument again using

Lemma 2.2 (first Poincaré inequality) and $\operatorname{diam}\omega_z \le Ch_z$ on ω_z provides

$$\begin{aligned}\|(I-P_{\mathcal{T}})(cu_{\mathcal{T}})\|_{H^{-1}(\omega_z)} &\le C_{\mathrm{lStb}}\|(c-\widehat{c})u_{\mathcal{T}}\|_{H^{-1}(\omega_z)}\\ &\le CC_{\mathrm{lStb}}\|h(c-\widehat{c})\|_{L^\infty(\omega_z)}\|u_{\mathcal{T}}\|_{L^2(\omega_z)},\end{aligned}$$

and the local bound is verified.

$\boxed{2}$ To show the global bound, we square the local bound and sum it over all vertices $z \in \mathcal{V}$ to obtain

$$\begin{aligned}\operatorname{osc}_{\mathcal{T}}(R_{\mathcal{T}})^2 \le 3\sum_{z\in\mathcal{V}}\operatorname{osc}_{\mathcal{T}}(f,z)^2 &+ 3(d+1)C\|\boldsymbol{A}-\widehat{\boldsymbol{A}}\|^2_{L^\infty(\Omega)}\|\nabla u_{\mathcal{T}}\|^2_{L^2(\Omega)}\\ &+ 3(d+1)C\|h(c-\widehat{c})\|^2_{L^\infty(\Omega)}\|u_{\mathcal{T}}\|^2_{L^2(\Omega)}.\end{aligned}$$

Hence Lemma 2.2 (first Poincaré inequality) on Ω and discrete stability,

$$\|u_{\mathcal{T}}\|_{L^2(\Omega)} \le C_P\|\nabla u_{\mathcal{T}}\|_{L^2(\Omega)} \le \frac{C_P}{\alpha}\|f\|_{H^{-1}(\Omega)},$$

finish the proof. □

The following remarks set Lemma 4.35 (splitting of local residual norm) and the accompanying results Lemma 4.37 and Lemma 4.39 on the reduction to data oscillation in the context of adaptive algorithms.

Remark 4.40 (structure of splitting). Combining Lemma 4.35 (splitting of local residual norm) with Lemma 4.37 or Lemma 4.39 about reduction to data oscillation thus provides an abstract estimator with the following two global parts:

$$\eta^{\mathrm{abs}}_{\mathcal{T}}(u_{\mathcal{T}})^2 := \sum_{z\in\mathcal{V}}\eta^{\mathrm{abs}}_{\mathcal{T}}(u_{\mathcal{T}},z)^2$$

and, writing $\mathcal{D} = (\boldsymbol{A}, c, f)$ for the data of the partial differential equation,

$$\begin{aligned}&\operatorname{osc}^{\mathrm{abs}}_{\mathcal{T}}(\mathcal{D})^2 := \operatorname{osc}_{\mathcal{T}}(f)^2,\\ &\operatorname{osc}^{\mathrm{abs}}_{\mathcal{T}}(\mathcal{D})^2 := \operatorname{osc}_{\mathcal{T}}(f)^2 + C_1\max_{z\in\mathcal{V}}\|\boldsymbol{A}-\widehat{\boldsymbol{A}}\|^2_{L^\infty(\omega_z)} + C_2\max_{z\in\mathcal{V}}\|h(c-\widehat{c})\|^2_{L^\infty(\omega_z)},\end{aligned}$$

the latter provided $(\boldsymbol{A}, c)$ are not discrete. It is important to note the different nature of these two parts. The first part $\eta^{\mathrm{abs}}_{\mathcal{T}}(u_{\mathcal{T}})$, the abstract *PDE indicator*,

- is strictly related to the structure of the underlying PDE,
- involves only discrete functionals from $\mathbb{F}_z$, and
- the evaluation of its local indicators $\eta^{\mathrm{abs}}_{\mathcal{T}}(u_{\mathcal{T}}, z)$ requires the global computation of the discrete solution $u_{\mathcal{T}}$.

In contrast, the second part $\operatorname{osc}^{\mathrm{abs}}_{\mathcal{T}}(\mathcal{D})$, the *oscillation (indicator)*,

- depends only on the data $\mathcal{D}$ of the differential operator,
- involves non-discrete functionals, and

- the evaluation of its local indicators $\operatorname{osc}_{\mathcal{T}}(f, z)$, $\|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|_{L^\infty(\omega_z)}$, $\|h(c - \widehat{c})\|_{L^\infty(\omega_z)}$, $z \in \mathcal{V}$, is completely local.

The respective properties 'discrete nature' and 'local dependence' of the two parts are the key advantage over the whole local residual indicators $\|R_{\mathcal{T}}\|_{H^{-1}(\omega_z)}$, $z \in \mathcal{V}$, and will be instrumental to the algorithmic design in Sections 5 and 6 below.

Remark 4.41 (minimal regularity and regularizing $P_{\mathcal{T}}$). It is worth noting that the results in this section do not involve any regularity beyond (4.2) and that the projection $P_{\mathcal{T}}$ has a regularizing effect. In particular, we have

$$\operatorname{Im} P_{\mathcal{T}} = \mathbb{F}_{\mathcal{T}} \subset H^{-1/2-\varepsilon}(\Omega) \quad \text{for any small } \varepsilon > 0,$$

thanks to the trace theorem in fractional Sobolev spaces. As a consequence, most techniques for *a posteriori* error estimation can be directly applied to the discretized residual $P_{\mathcal{T}} R_{\mathcal{T}}$, without any special twisting and under natural regularity assumptions.

Remark 4.42 (reduction vs. surrogate reduction). The kernel condition of Remark 4.19 (avoiding overestimation) is not verified for the bounds in Lemma 4.39 (surrogate data oscillation reduction). These bounds may thus exhibit overestimation and cannot be reversed. If we use the right-hand side of an overestimating bound as a part of an estimator, we shall call that part a *surrogate*. This label marks a crucial difference between the cases represented by Lemma 4.39 (surrogate data oscillation reduction) and Lemma 4.37 (data oscillation reduction for discrete coefficients), which is free of any overestimation.

Remark 4.43 (surrogate data oscillation). Surrogates for data oscillation indicators can be useful for providing more direct access for computation. For example, if $f \in L^2(\Omega)$, the bound (4.47) by the classical $\Pi_{\mathcal{T}}$-oscillation can be approximated by numerical integration. In such a context, it is useful to take the following points into account.

- Computable surrogates, i.e. computable upper bounds, for data oscillation indicators are in general impossible. In fact, generic data from an infinite-dimensional space will not be completely seen by the finite information available at any stage of a computation; see also Kreuzer and Veeser (2021, Lemma 2 and Corollary 5) illustrating this fact for $\operatorname{osc}_{\mathcal{T}}(f)$ with the help of orthogonality. Hence computable surrogates will hinge on additional *a priori* information on the given data. We postpone a discussion of examples to Section 7.3.
- As a general rule, surrogates should be applied last. This avoids other parts of the estimators being affected by overestimation; see Remark 4.46 (modified vs. standard residual estimator) below.

4.7. Modified residual estimation

In view of the splitting into PDE and oscillation indicators and the discussion of the computability of the latter, it remains to quantify the abstract PDE indicators $\eta_{\mathcal{T}}^{\mathrm{abs}}(u_{\mathcal{T}}, z)$, $z \in \mathcal{V}$. To this end, we can employ Corollary 4.30 (quantifying H^{-1}-norms of discrete functionals), resulting in a modification of the standard residual estimator $\mathcal{E}_{\mathcal{T}}^{\mathrm{std}}(u_{\mathcal{T}}, \mathcal{D})$ from Section 4.2. Alternative quantifications by other techniques of *a posteriori* error estimation are discussed in Section 4.9 below. In so doing, for simplicity, we consider only the case given by the following assumption.

Assumption 4.44 (discrete coefficients and discrete functionals). Suppose that the coefficients $\boldsymbol{A}$ and c in (2.5) are discrete, and choose the degrees (m_1, m_2) of the discrete functionals in $\mathbb{F}_{\mathcal{T}}$ according to Lemma 4.37 (data oscillation reduction for discrete coefficients).

For non-discrete coefficients, we essentially have to invoke Lemma 4.39 (surrogate data oscillation reduction) instead of Lemma 4.37 in order to reduce to data oscillation.

We shall employ the bisection method in order to refine the mesh. Since this method is based upon the subdivision of elements, it is convenient to split the estimator into contributions associated with elements and not with vertices as in Section 4.6.

To define the modified residual estimator, we recall the representation (4.35) of the H^{-1}-projection $P_{\mathcal{T}}$, and we use Assumption 4.44 (discrete coefficients and discrete functionals) to set

$$\begin{aligned}
\mathcal{E}_{\mathcal{T}}^2 &:= \sum_{T \in \mathcal{T}} \mathcal{E}_{\mathcal{T}}(T)^2 \quad \text{with} \\
\mathcal{E}_{\mathcal{T}}(T)^2 &:= \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f, T)^2 := \eta_{\mathcal{T}}(u_{\mathcal{T}}, T)^2 + \mathrm{osc}_{\mathcal{T}}(f, T)^2, \\
\eta_{\mathcal{T}}(u_{\mathcal{T}}, T)^2 &:= h_T \sum_{F \subset \partial T \setminus \partial\Omega} \| [\![\boldsymbol{A} \nabla u_{\mathcal{T}}]\!] \cdot \boldsymbol{n}_F - P_F f \|_{L^2(F)}^2 \\
&\qquad + h_T^2 \| P_T f - c u_{\mathcal{T}} + \mathrm{div}(\boldsymbol{A} \nabla u_{\mathcal{T}}) \|_{L^2(T)}^2, \\
\mathrm{osc}_{\mathcal{T}}(f, T)^2 &:= \| f - P_{\mathcal{T}} f \|_{H^{-1}(\omega_T)}^2.
\end{aligned} \tag{4.52}$$

Clearly, this is a variant of the standard residual estimator in (4.10), where the main differences are given by the corrections $P_F f$, $F \in \mathcal{F}$, of the jump residual and the replacement of $f|_T$ by $P_T f$, $T \in \mathcal{T}$, in the PDE indicator. As shown by the following theorem and remarks, the modification leads to more accurate *a posteriori* bounds.

Theorem 4.45 (modified residual estimator). *Under Assumption 4.44, the modified residual estimator (4.52) is equivalent to the error: more precisely, we have*

$$C_L \mathcal{E}_{\mathcal{T}} \le \| \nabla (u - u_{\mathcal{T}}) \|_{L^2(\Omega)} \le C_U \mathcal{E}_{\mathcal{T}},$$

where the constants $C_U \geq C_L > 0$ depend only on the coefficients $(\boldsymbol{A}, c)$, the shape regularity coefficient σ from (3.9), the polynomial degree n, and d.

Proof. To derive the upper bound, we use those of Lemma 4.1 (error and residual), Corollary 4.6 (star localization of residual norm), Lemma 4.35 (splitting of local residual norm) and Corollary 4.30 (quantifying H^{-1}-norms of discrete functionals) with stars, and obtain

$$\begin{aligned}\|\nabla(u-u_{\mathcal{T}})\|^2_{L^2(\Omega)} &\lesssim \|R_{\mathcal{T}}\|^2_{H^{-1}(\Omega)} \lesssim \sum_{z\in\mathcal{V}} \|R_{\mathcal{T}}\|^2_{H^{-1}(\omega_z)} \\ &\lesssim \sum_{z\in\mathcal{V}} \mathcal{E}^{\mathrm{abs}}_{\mathcal{T}}(z)^2 = \sum_{z\in\mathcal{V}} \eta^{\mathrm{abs}}_{\mathcal{T}}(u_{\mathcal{T}},z)^2 + \sum_{z\in\mathcal{V}} \mathrm{osc}_{\mathcal{T}}(f,z)^2 \\ &\lesssim \sum_{z\in\mathcal{V}} \sum_{T\in\mathcal{T}_z} \eta_{\mathcal{T}}(u_{\mathcal{T}},T)^2 + \sum_{z\in\mathcal{V}} \mathrm{osc}_{\mathcal{T}}(f,z)^2,\end{aligned}$$

with $\mathcal{T}_z = \{T \in \mathcal{T} \mid T \ni z\}$. As a given mesh element appears in the star meshes $\mathcal{T}_z$ for at most $d+1$ vertices, we have

$$\sum_{z\in\mathcal{V}} \sum_{T\in\mathcal{T}_z} \eta_{\mathcal{T}}(u_{\mathcal{T}},T)^2 \leq (d+1) \sum_{T\in\mathcal{T}} \eta_{\mathcal{T}}(u_{\mathcal{T}},T)^2$$

for the first sum, and Lemma 4.8 (localization re-indexing) yields

$$\sum_{z\in\mathcal{V}} \mathrm{osc}_{\mathcal{T}}(f,z)^2 \lesssim \sum_{T\in\mathcal{T}} \mathrm{osc}_{\mathcal{T}}(f,T)^2$$

for the second sum. Inserting the last two inequalities in the previous one, we conclude the upper bound:

$$\|\nabla(u-u_{\mathcal{T}})\|^2_{L^2(\Omega)} \lesssim \sum_{T\in\mathcal{T}} \eta_{\mathcal{T}}(u_{\mathcal{T}},T)^2 + \sum_{T\in\mathcal{T}} \mathrm{osc}_{\mathcal{T}}(f,T)^2 = \sum_{T\in\mathcal{T}} \mathcal{E}_{\mathcal{T}}(T)^2.$$

To show the lower bound, fix a mesh element $T \in \mathcal{T}$. Applying the local lower bounds in Corollary 4.30, Lemma 4.28 (local H^{-1}-stability) and Lemma 4.1 on the local meshed subdomain $\widetilde{\omega}_T$ defined in (3.12) yields for the PDE indicator

$$\eta_{\mathcal{T}}(u_{\mathcal{T}},T) \lesssim \|P_{\mathcal{T}} R_{\mathcal{T}}\|_{H^{-1}(\widetilde{\omega}_T)} \lesssim \|R_{\mathcal{T}}\|_{H^{-1}(\widetilde{\omega}_T)}. \tag{4.53}$$

In the case of the oscillation indicator, we exploit $-cu_{\mathcal{T}} + \mathrm{div}(\boldsymbol{A}\nabla u_{\mathcal{T}}) \in \mathbb{F}_{\mathcal{T}}$ with the help of Lemma 4.25 (algebraic properties) and apply Lemma 4.28 on the local meshed subdomain ω_T:

$$\mathrm{osc}_{\mathcal{T}}(f,T) = \|f - P_{\mathcal{T}} f\|_{H^{-1}(\omega_T)} = \|(I-P_{\mathcal{T}})R_{\mathcal{T}}\|_{H^{-1}(\omega_T)} \lesssim \|R_{\mathcal{T}}\|_{H^{-1}(\omega_T)}.$$

Thanks to $\widetilde{\omega}_T \subset \omega_T$, combining the last two inequalities gives the desired local lower bound:

$$\begin{aligned}\mathcal{E}_{\mathcal{T}}(T)^2 &= \eta_{\mathcal{T}}(u_{\mathcal{T}},T)^2 + \mathrm{osc}_{\mathcal{T}}(f,T)^2 \\ &\lesssim \|R_{\mathcal{T}}\|_{H^{-1}(\widetilde{\omega}_T)} + \|R_{\mathcal{T}}\|_{H^{-1}(\omega_T)} \lesssim \|R_{\mathcal{T}}\|_{H^{-1}(\omega_T)}.\end{aligned} \tag{4.54}$$

As the number of patches ω_T, $T \in \mathcal{T}$, containing a given mesh element is uniformly bounded by d and the shape regularity coefficient σ, summing this bound over all mesh elements yields the global lower bound

$$\sum_{T \in \mathcal{T}} \mathcal{E}_{\mathcal{T}}(T)^2 \lesssim \sum_{T \in \mathcal{T}} \|R_{\mathcal{T}}\|^2_{H^{-1}(\omega_T)} \lesssim \|\nabla(u - u_{\mathcal{T}})\|^2_{L^2(\Omega)},$$

with the help of Lemma 4.5 (localization of H^{-1}-norm) and Lemma 4.1. Thus the equivalence of error and estimator is established. □

A detailed comparison of the modified residual estimator with the standard estimator is in order.

Remark 4.46 (modified vs. standard residual estimator). We compare the modified residual estimator (4.52) with the standard one given by (4.10a) and the local split indicators (4.22). As a common characterizing feature, both residual estimators use properly scaled L^2-norms of jump and element residual, ready for numerical integration. However, we observe the following differences.

- While the modified estimator $\mathcal{E}_{\mathcal{T}}$ is defined under the natural regularity assumptions (4.2), the standard estimator $\mathcal{E}^{\text{std}}_{\mathcal{T}}$ requires $\boldsymbol{A} \in W^1_\infty(\Omega; \mathbb{R}^{d\times d})$ and $f \in L^2(\Omega)$ in addition.
- While the modified estimator $\mathcal{E}_{\mathcal{T}}$ is truly equivalent to the error, the standard estimator $\mathcal{E}^{\text{std}}_{\mathcal{T}}$ may overestimate it, limited, however, by Proposition 4.12 (partial lower bound).

By the domain test in Remark 4.19 (avoiding overestimation), we know that these two points are interrelated. However, the kernel test is also at play in the overestimation. Indeed, revisiting the proof of Theorem 4.9 (upper bound with standard residual estimator), we can replace the scaled L^2-norms of the element residuals on a star ω_z with $\|r\|_{H^{-1}(\omega_z)}$, and the resulting vertex-oriented variant of the residual estimator with unsplit local indicators is defined for all $f \in H^{-1}(\Omega)$. Overestimation can, however, still occur non-asymptotically as well as asymptotically; see Cohen *et al.* (2012). Indeed, in the case of the Poisson equation and linear finite elements, the kernel test is obviously not satisfied. This shows that the splitting in jump and element residual is quite delicate and highlights the crucial role of the modifications of the standard residual estimator: not only do they allow for stability in line with Remark 4.34 (stability of approximation) but they also imply the kernel test.

To conclude this comparison, let us illustrate the second point of Remark 4.43 (surrogate data oscillation), namely that surrogates should be applied last. Using (4.47) in Remark 4.32 (approximating functions), in the modified residual estimator we may replace the H^{-1}-oscillation $\operatorname{osc}_{\mathcal{T}}(f)$ with the standard oscillation $\operatorname{osc}^{\text{std}}_{\mathcal{T}}(f)$, which can be readily approximated with numerical integration. In so doing, we first split the residual with $P_{\mathcal{T}}$ and then apply $\Pi_{\mathcal{T}}$ to obtain the surrogate. Note, however, that if we apply $\Pi_{\mathcal{T}}$ earlier to split the residual, the crucial modifications

will not appear and, therefore, the PDE indicator of the standard residual estimator also exhibits overestimation.

4.8. *Bounds for corrections and reduction of PDE estimator*

In the following sections we shall use the modified residual estimator $\mathcal{E}_{\mathcal{T}}$ from (4.52) in adaptive algorithms. In their convergence analyses, not only is its relationship with the error important, but also its relationship with the norm $\|\nabla(u_{\mathcal{T}_*} - u_{\mathcal{T}})\|_{L^2(\Omega)}$ of (possible) *corrections*, where $u_{\mathcal{T}_*}$ is the Galerkin approximation to u over some refinement $\mathcal{T}_*$ of $\mathcal{T}$. This section establishes corresponding upper and lower bounds, as well as related results about the *global PDE indicator*

$$\eta_{\mathcal{T}}(u_{\mathcal{T}}, f)^2 := \sum_{T \in \mathcal{T}} \eta_{\mathcal{T}}(u_{\mathcal{T}}, T)^2 \tag{4.55}$$

and the *global oscillation*

$$\operatorname{osc}_{\mathcal{T}}(f)^2 := \sum_{T \in \mathcal{T}} \operatorname{osc}_{\mathcal{T}}(f, T)^2. \tag{4.56}$$

When it is important to indicate that the oscillations are measured in H^{-1}, we use the notation

$$\operatorname{osc}_{\mathcal{T}}(f)_{-1} \quad \text{and} \quad \operatorname{osc}_{\mathcal{T}}(f, T)_{-1}.$$

Let $\mathcal{T}_*$ be a conforming mesh that is a *refinement* of $\mathcal{T}$, that is, for any element $T \in \mathcal{T}$, there exists a submesh $\mathcal{T}_{*,T}$ of $\mathcal{T}_*$ such that $T = \cup\{T_* \mid T_* \in \mathcal{T}_{*,T}\}$. The Galerkin approximation in $\mathbb{V}_{\mathcal{T}_*}$ is characterized by

$$u_{\mathcal{T}_*} \in \mathbb{V}_{\mathcal{T}_*}: \quad \mathcal{B}[u_{\mathcal{T}_*}, w] = \langle f, w\rangle \quad \text{for all } w \in \mathbb{V}_{\mathcal{T}_*}.$$

Hence the discrete solution $u_{\mathcal{T}}$ on the original mesh $\mathcal{T}$ is not only a Galerkin approximation to the exact solution u satisfying (2.7) but also to $u_{\mathcal{T}_*}$. The norm $\|\nabla(u_{\mathcal{T}_*} - u_{\mathcal{T}})\|_{L^2(\Omega)}$ of the correction therefore can be viewed as the error in approximating $u_{\mathcal{T}_*}$ on the mesh $\mathcal{T}$. This viewpoint suggests considering the variant

$$\langle R_{\mathcal{T}}, w\rangle = \mathcal{B}[u_{\mathcal{T}_*} - u_{\mathcal{T}}, w] \quad \text{for all } w \in \mathbb{V}_{\mathcal{T}_*} \tag{4.57}$$

of the error–residual identity (4.3) and introducing the discrete dual norm

$$\|R_{\mathcal{T}}\|_{(\mathbb{V}_{\mathcal{T}_*})^*} := \sup_{w \in \mathbb{V}_{\mathcal{T}_*}} \frac{\langle R_{\mathcal{T}}, w\rangle}{\|\nabla w\|_{L^2(\Omega)}} \tag{4.58}$$

of the residual as a counterpart of $\|R_{\mathcal{T}}\|_{H^{-1}(\Omega)}$. Arguing as in the proof of Lemma 4.1 (error and residual), we thus readily obtain the following quantitative relationship between the correction and the residual.

Lemma 4.47 (correction and residual). *If $\mathcal{T}_*$ is a refinement of the mesh $\mathcal{T}$, the norm of the correction $u_{\mathcal{T}_*} - u_{\mathcal{T}}$ is equivalent to the discrete residual norm. More*

precisely,

$$\frac{1}{\|\mathcal{B}\|}\|R_{\mathcal{T}}\|_{(\mathbb{V}_{\mathcal{T}_*})^*} \le \|\nabla(u_{\mathcal{T}_*} - u_{\mathcal{T}})\|_{L^2(\Omega)} \le \frac{1}{\alpha}\|R_{\mathcal{T}}\|_{(\mathbb{V}_{\mathcal{T}_*})^*},$$

where $\|\mathcal{B}\| \ge \alpha > 0$*, are, respectively, the continuity and coercivity constant of the bilinear form* $\mathcal{B}$*.*

We first exploit the upper bound in Lemma 4.47. As the inclusion $\mathbb{V}_{\mathcal{T}_*} \subset H^1_0(\Omega)$ implies

$$\|R_{\mathcal{T}}\|_{(\mathbb{V}_{\mathcal{T}_*})^*} \le \|R_{\mathcal{T}}\|_{H^{-1}(\Omega)}, \tag{4.59}$$

Theorem 4.45 (modified residual estimator) immediately yields the upper bound

$$\|\nabla(u_{\mathcal{T}_*} - u_{\mathcal{T}})\|_{L^2(\Omega)} \le C_U \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f). \tag{4.60}$$

This bound, however, appears not to be accurate in view of the use of (4.59). We will sharpen it by following the lines of its proof but exploiting the *full* orthogonality

$$\langle R_{\mathcal{T}}, w\rangle = 0 \quad \text{for all } w \in \mathbb{V}_{\mathcal{T}}, \tag{4.61}$$

with suitably tuned Scott–Zhang interpolation (Scott and Zhang 1990).

In order to prepare the use of this interpolation, let $\mathcal{N}$ denote the Lagrange nodes of order n of the mesh $\mathcal{T}$ and let $\mathcal{F}$ and $\mathcal{F}_*$, respectively, denote the $(d-1)$-dimensional faces of $\mathcal{T}$ and $\mathcal{T}_*$, including boundary faces. Given a node $z \in \mathcal{N}$, fix a face $F_z \in \mathcal{F}$ such that F_z contains z and the following conditions are met:

$$z \in \partial\Omega \quad \Rightarrow \quad F_z \subset \partial\Omega,$$
$$\{F \in \mathcal{F} \cap \mathcal{F}_* \mid F \ni z\} \neq \emptyset \quad \Rightarrow \quad F_z \in \mathcal{F}_*.$$

Furthermore, let ψ_z^* denote the polynomial in $\mathbb{P}_n(F_z)$ satisfying

$$\int_{F_z} \psi_z^* \psi_y = \delta_{yz} \quad \text{for all } y \in \mathcal{N},$$

where $\{\psi_y\}_{y\in\mathcal{N}}$ is the Lagrange basis of $\mathbb{S}^{n,0}_{\mathcal{T}}$, and define

$$I_{\mathcal{T}} w = \sum_{z\in\mathcal{N}} \left(\int_{F_z} \psi_z^* w\right)\psi_z. \tag{4.62}$$

The two conditions on the fixed face F_z then ensure, respectively,

$$w \in H^1_0(\Omega) \quad \Rightarrow \quad I_{\mathcal{T}} w \in \mathbb{V}_{\mathcal{T}}, \tag{4.63a}$$
$$w \in \mathbb{V}_{\mathcal{T}_*} \text{ and } T \in \mathcal{T} \cap \mathcal{T}_* \quad \Rightarrow \quad I_{\mathcal{T}} w = w \text{ on } T. \tag{4.63b}$$

In particular, if $w \in \mathbb{V}_{\mathcal{T}_*}$, its approximation $I_{\mathcal{T}} w \in \mathbb{V}_{\mathcal{T}}$ is an admissible test function and coincides with w whenever possible. Finally, $I_{\mathcal{T}}$ has the following stability and approximation properties, where the hidden constants depend only on d, n and the

shape regularity coefficient σ: for any element $T \in \mathcal{T}$ and any face $F \in \mathcal{F}$,

$$\|\nabla I_{\mathcal{T}} w\|_{L^2(T)} \lesssim \|\nabla w\|_{L^2(\omega_T)}, \tag{4.64a}$$

$$\|w - I_{\mathcal{T}} w\|_{L^2(T)} \lesssim h_T \|\nabla w\|_{L^2(\omega_T)}, \tag{4.64b}$$

$$\|w - I_{\mathcal{T}} w\|_{L^2(F)} \lesssim h_F^{1/2} \|\nabla w\|_{L^2(\omega_F)}. \tag{4.64c}$$

The sharpening of the simple upper bound (4.60) lies in the fact that only a part of the estimator in (4.52) will be invoked. To formulate this, we define

$$\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f, \widetilde{\mathcal{T}}) := \Big(\sum_{T \in \widetilde{\mathcal{T}}} \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f, T)^2 \Big)^{1/2}, \tag{4.65}$$

where $\widetilde{\mathcal{T}} \subset \mathcal{T}$ is a subset of elements in $\mathcal{T}$. In the same vein, we shall denote $\eta_{\mathcal{T}}(u_{\mathcal{T}}, f, \widetilde{\mathcal{T}})$ and $\operatorname{osc}_{\mathcal{T}}(f, \widetilde{\mathcal{T}})$.

Theorem 4.48 (upper bound for corrections). *Let Assumption 4.44 hold and let $\mathcal{T}_*$ be a refinement of the mesh $\mathcal{T}$. The correction $u_{\mathcal{T}_*} - u_{\mathcal{T}}$ is bounded in terms of the indicators of the refined elements $\mathcal{T} \setminus \mathcal{T}_*$:*

$$\|\nabla(u_{\mathcal{T}} - u_{\mathcal{T}_*})\|_{L^2(\Omega)} \le \widetilde{C}_U \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f, \mathcal{T} \setminus \mathcal{T}_*),$$

where the constant $\widetilde{C}_U > 0$ depends only on the dimension d, the coefficients $\boldsymbol{A}$ and c, the polynomial degree n, and the shape regularity coefficient σ from (3.9).

Proof. $\boxed{1}$ *Localization and splitting of the residual norm.* In light of Lemma 4.47, it suffices to bound the discrete residual norm $\|R_{\mathcal{T}}\|_{(\mathbb{V}_{\mathcal{T}_*})^*}$. Given $w \in \mathbb{V}_{\mathcal{T}_*}$, we prepare the localization of the residual by full orthogonality (4.61) and split it with help of the projection $P_{\mathcal{T}}$ on discrete functionals:

$$\begin{aligned} |\langle R_{\mathcal{T}}, w\rangle| &= |\langle R_{\mathcal{T}}, w - I_{\mathcal{T}} w\rangle| \\ &\le |\langle P_{\mathcal{T}} R_{\mathcal{T}}, w - I_{\mathcal{T}} w\rangle| + |\langle f - P_{\mathcal{T}} f, w - I_{\mathcal{T}} w\rangle|, \end{aligned}$$

where we used the identity $R - P_{\mathcal{T}} R = f - P_{\mathcal{T}} f$ in the last step. In light of

$$\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f, \mathcal{T} \setminus \mathcal{T}_*)^2 = \eta_{\mathcal{T}}(u_{\mathcal{T}}, f, \mathcal{T} \setminus \mathcal{T}_*)^2 + \operatorname{osc}_{\mathcal{T}}(f, \mathcal{T} \setminus \mathcal{T}_*)^2,$$

it remains to bound the two terms with discretized residual $P_{\mathcal{T}} R_{\mathcal{T}}$ and the oscillation of f appropriately.

$\boxed{2}$ *Bounding the discretized residual.* We adopt the notation (4.35) for the densities of $P_{\mathcal{T}}$, and exploit the piecewise nature of the discretized residual and the local invariance (4.63b) of $I_{\mathcal{T}}$ to deduce

$$\begin{aligned} &\langle P_{\mathcal{T}} R_{\mathcal{T}}, w - I_{\mathcal{T}} w\rangle \\ &\quad = \sum_{T \in \mathcal{T} \setminus \mathcal{T}_*} \Big(\int_T (P_T R_{\mathcal{T}})(w - I_{\mathcal{T}} w) + \frac{1}{2} \sum_{F \subset \partial T \setminus \partial\Omega} \int_F (P_F R_{\mathcal{T}})(w - I_{\mathcal{T}} w) \Big). \end{aligned}$$

Invoking the local approximation properties (4.64b) and (4.64c) of $I_{\mathcal{T}}$ leads to the desired bound for the discretized residual:

$$\begin{aligned}|\langle P_{\mathcal{T}} R_{\mathcal{T}}, w - I_{\mathcal{T}} w\rangle| &\lesssim \sum_{T \in \mathcal{T}\setminus\mathcal{T}_*} \eta_{\mathcal{T}}(u_{\mathcal{T}}, f, T)\, \|\nabla w\|_{L^2(\omega_T)} \\ &\lesssim \eta_{\mathcal{T}}\big(u_{\mathcal{T}}, f, \mathcal{T}\setminus\mathcal{T}_*\big)\, \|\nabla w\|_{L^2(\Omega)}.\end{aligned}$$

$\boxed{3}$ *Bounding the oscillation.* We need to split the oscillation into suitable local contributions and first proceed similarly to the proof of Lemma 4.5(i) (localization of H^{-1}-norm). Writing

$$w - I_{\mathcal{T}} w = \sum_{z\in\mathcal{V}} (w - I_{\mathcal{T}} w)\phi_z \quad \text{and} \quad \Omega_0 := \bigcup_{T\in\mathcal{T}\setminus\mathcal{T}_*} T,$$

we have $(w - I_{\mathcal{T}} w)\phi_z \in H^1_0(\omega_z \cap \Omega_0)$ thanks to (4.63b) and, for any $T \subset \omega_z \cap \Omega_0$,

$$\begin{aligned}\|\nabla((w - I_{\mathcal{T}} w)\phi_z)\|_{L^2(T)} &\le \|\phi_z \nabla(w - I_{\mathcal{T}} w)\|_{L^2(T)} + \|(w - I_{\mathcal{T}} w)\nabla\phi_z\|_{L^2(T)} \\ &\le \|\nabla(w - I_{\mathcal{T}} w)\|_{L^2(T)} + C(d)\sigma\|\nabla w\|_{L^2(\omega_T)} \\ &\lesssim \|\nabla w\|_{L^2(\omega_T)}\end{aligned}$$

by means of $0 \le \phi_z \le 1$, $|\nabla\phi_z| \le C(d)\sigma h_T^{-1}$, (4.64a) and (4.64b). Hence we get

$$\begin{aligned}&|\langle f - P_{\mathcal{T}} f, w - I_{\mathcal{T}} w\rangle| \\ &\quad\le \sum_{z\in\mathcal{V}} |\langle f - P_{\mathcal{T}} f, (w - I_{\mathcal{T}} w)\phi_z\rangle| \\ &\quad\le \sum_{z\in\mathcal{V}} \|f - P_{\mathcal{T}} f\|_{H^{-1}(\omega_z\cap\Omega_0)} \|\nabla((w - I_{\mathcal{T}} w)\phi_z)\|_{L^2(\omega_z\cap\Omega_0)} \\ &\quad\lesssim \sum_{z\in\mathcal{V}} \|f - P_{\mathcal{T}} f\|_{H^{-1}(\omega_z\cap\Omega_0)} \|\nabla w\|_{L^2(\cup_{T\subseteq\omega_z\cap\Omega_0}\omega_T)} \\ &\quad\lesssim \Big(\sum_{z\in\mathcal{V}} \|f - P_{\mathcal{T}} f\|^2_{H^{-1}(\omega_z\cap\Omega_0)}\Big)^{1/2} \|\nabla w\|_{L^2(\Omega)}.\end{aligned}$$

Since

$$\sum_{z\in\mathcal{V}} \|f - P_{\mathcal{T}} f\|^2_{H^{-1}(\omega_z\cap\Omega_0)} \le \sum_{T\in\mathcal{T}\setminus\mathcal{T}_*} \|f - P_{\mathcal{T}} f\|^2_{H^{-1}(\omega_T)},$$

the oscillation of f is therefore bounded by

$$|\langle f - P_{\mathcal{T}} f, w - I_{\mathcal{T}} w\rangle| \le \operatorname{osc}_{\mathcal{T}}(f, \mathcal{T}\setminus\mathcal{T}_*)\|\nabla w\|_{L^2(\Omega)}$$

and the proof is complete. □

Proposition 4.12 (partial lower bound) as well as Lemma 4.35 (splitting of local residual norm) illustrate that the test space $\mathbb{V}^+_{\mathcal{T}}$ is closely related to lower bounds for the error. This observation suggests establishing lower bounds for the correction

$\|\nabla(u_{\mathcal{T}} - u_{\mathcal{T}_*})\|_{L^2(\Omega)}$ by ensuring conditions such as

$$\mathbb{V}^+(\mathcal{T}_\omega) \subset \mathbb{V}(\mathcal{T}_*),$$

where ω is a $\mathcal{T}$-mesh subdomain. Inspecting the construction of $\mathbb{V}^+_{\mathcal{T}}$, we realize that such conditions can be achieved if $\max\{m_1, m_2\} \le n-1$ and the cut-off is implemented with hat functions of a virtual refinement of $\mathcal{T}$.

Lemma 4.49 (cut-off by refined hat functions). *Let $\mathcal{T}_+$ be the minimal bisection refinement of $\mathcal{T}$ such that the relative interior of each element $T \in \mathcal{T}$ and each face $F \in \mathcal{F}$ of the original mesh $\mathcal{T}$ contains at least one vertex from $\mathcal{T}_+$. Then there exist hat functions ϕ_T, $T \in \mathcal{T}$, and ϕ_F, $F \in \mathcal{F}$, in $\mathbb{S}^{1,0}(\mathcal{T}_+)$ satisfying Assumption 4.21 if* $\max\{m_1, m_2\} \le n-1$.

Proof. The details of the proof depend on bisection and we therefore restrict to the case $d = 2$; for $d > 2$, the following reference situation used to define the hat functions is replaced by several ones with 'tagged' reference simplices. Let $\widehat{T} = T_2$ be the reference element in $\mathbb{R}^2$ with the standard enumeration of its vertices $\widehat{z}_0 = 0$, $\widehat{z}_1 = e_1$ and $\widehat{z}_2 = e_2$. Furthermore, let $\widehat{\mathcal{T}}_+$ be the mesh obtained by applying five bisections so that vertices in the interiors of $\widehat{T}$ and of its faces are generated. Let $\widehat{\phi}_{\widehat{T}}$, $\widehat{\phi}_{F'}$, $F' \subset \widehat{T}$ denote the four hat functions in $\mathbb{S}^{1,0}(\widehat{\mathcal{T}}_+)$ associated with these generated vertices. Given an arbitrary element $T \in \mathcal{T}$, let H_T denote the bi-affine map $T \to \widehat{T}$ preserving the numbering of the vertices for bisection, and define the pullbacks

$$\phi_T := H_T^*(\widehat{\phi}_{\widehat{T}}), \quad \phi_F|_T := H_T^*(\widehat{\phi}_{H_T(F)}), \quad F \subset T,$$

and extend by 0 off T or ω_F. As the extension operators E_F preserve the polynomial degree (see Lemma 4.20 (extending from faces)), $\max\{m_1, m_2\} \le n-1$, and

$$\{H_T^{-1}(\widehat{T}_+) \mid \widehat{T}_+ \in \widehat{\mathcal{T}}_+\} = \{T_+ \in \mathcal{T}_+ \mid T_+ \subset T\},$$

the hat functions ϕ_T, $T \in \mathcal{T}$, and ϕ_F, $F \in \mathcal{F}$, then satisfy Assumption 4.21 with $G_T = H_T^{-1}$ and $\mathbb{S}^+ = \mathbb{S}^{n,0}(\widehat{\mathcal{T}}_+)$. □

Definition 4.50 (interior vertex property). A mesh element $T \in \mathcal{T}$ satisfies the *interior vertex property* with respect to $\mathcal{T}_* \ge \mathcal{T}$ whenever each interior face $F \subset \partial T \setminus \partial\Omega$ of T and each element in $\widetilde{\omega}_T$ (defined in (3.12)) have in their relative interiors at least one vertex from $\mathcal{T}_*$.

A set $\mathcal{M} \subset \mathcal{T}$ satisfies the interior vertex property with respect to a refinement $\mathcal{T}_* \ge \mathcal{T}$ if each element $T \in \mathcal{M}$ satisfies the interior vertex property.

The interior vertex property is valid upon enforcing a fixed number b of bisections ($b = 3, 6$ for $d = 2, 3$). An immediate consequence is the following lower bound for corrections.

Theorem 4.51 (lower bound for corrections). *Suppose $\boldsymbol{A}$ is piecewise constant over $\mathcal{T}$ and $c = 0$, define $P_{\mathcal{T}}$ with the help of the cut-off functions in Lemma 4.49,*

and let $\mathcal{M}$ denote the subset of elements in $\mathcal{T}$ satisfying the interior vertex property with respect to $\mathcal{T}_*$. Then

$$\sum_{T\in\mathcal{M}} \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f, T)^2 \le C\|\nabla(u_{\mathcal{T}} - u_{\mathcal{T}_*})\|^2_{L^2(\Omega)} + \sum_{T\in\mathcal{M}} \|f - P_{\mathcal{T}} f\|^2_{H^{-1}(\omega_T)},$$

where C depends on d, the shape regularity coefficient σ, the coefficients $(\boldsymbol{A}, c)$, and the polynomial degree n.

Proof. 1 We first show a local bound with the PDE indicator $\eta_{\mathcal{T}}(u_{\mathcal{T}}, T)$. In view of $\boldsymbol{A} \in \mathbb{S}^{0,-1}_{\mathcal{T}}$ and $c = 0$, we choose $m_1 = n - 1$ and $m_2 = n - 2$ as the degrees for the discrete functionals. We can thus apply Lemma 4.49 and construct $P_{\mathcal{T}}$ with the refined hat functions. Let $T \in \mathcal{M}$ and so, using the interior vertex property and the notation associated with $\widetilde{\omega}_T$ in (3.12), we deduce $\mathbb{V}^+(\widetilde{\mathcal{T}}_T) \subset \mathbb{V}(\mathcal{T}_*, T) := \mathbb{V}(\mathcal{T}_*) \cap H^1_0(\widetilde{\omega}_T)$, where $\widetilde{\mathcal{T}}_T = \{T \in \mathcal{T} \mid T \subseteq \widetilde{\omega}_T\}$. Combining this with inequality (4.45) and Definition 4.24 (projection onto discrete functionals), we conclude

$$\eta_{\mathcal{T}}(u_{\mathcal{T}}, T) \lesssim \|P_{\mathcal{T}} R_{\mathcal{T}}\|_{\mathbb{V}^+(\widetilde{\mathcal{T}}_T)^*} = \|R_{\mathcal{T}}\|_{\mathbb{V}^+(\widetilde{\mathcal{T}}_T)^*} \le \|R_{\mathcal{T}}\|_{\mathbb{V}(\mathcal{T}_*, T)^*}.$$

2 To collect the local bounds of the first step, we first show that, for any $\ell \in \mathbb{V}(\mathcal{T}_*)^*$,

$$\sum_{T\in\mathcal{T}} \|\ell\|^2_{\mathbb{V}(\mathcal{T}_*, T)^*} \le (d+2)\|\ell\|^2_{\mathbb{V}(\mathcal{T}_*)^*}.$$

To this end, we just repeat the proof of Lemma 4.5 (localization of H^{-1}-norm), replacing the spaces $H^1_0(\omega_i)$ and $H^1_0(\Omega)$, respectively, with $\mathbb{V}(\mathcal{T}_*, T)$ and $\mathbb{V}(\mathcal{T}_*)$. Hence, squaring and summing the bound of the first step as well as using Lemma 4.1 (error and residual) yield

$$\sum_{T\in\mathcal{M}} \eta_{\mathcal{T}}(u_{\mathcal{T}}, T)^2 \lesssim \sum_{T\in\mathcal{M}} \|R_{\mathcal{T}}\|^2_{\mathbb{V}(\mathcal{T}_*, T)^*} \lesssim \|\nabla(u_{\mathcal{T}_*} - u_{\mathcal{T}})\|^2_{L^2(\Omega)}. \tag{4.66}$$

3 We finally prove the claimed bound by simply inserting (4.66):

$$\begin{aligned}\sum_{T\in\mathcal{M}} \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f, T)^2 &= \sum_{T\in\mathcal{M}} \left(\eta_{\mathcal{T}}(u_{\mathcal{T}}, T)^2 + \operatorname{osc}_{\mathcal{T}}(f, T)^2\right) \\ &\le C\|\nabla(u_{\mathcal{T}_*} - u_{\mathcal{T}})\|^2_{L^2(\Omega)} + \sum_{T\in\mathcal{M}} \operatorname{osc}_{\mathcal{T}}(f, T)^2,\end{aligned}$$

and the proof is finished. □

Remark 4.52 (oscillation and correction). In general, by first fixing the finer mesh $\mathcal{T}_*$, it is impossible to bound oscillation indicators $\operatorname{osc}_{\mathcal{T}}(f, T)$ by some suitable correction. Indeed, these indicators can contain contributions to f and so to $R_{\mathcal{T}}$ of 'arbitrarily high frequency', while the correction can control only contributions of the residual $R_{\mathcal{T}}$ with frequencies representable over $\mathcal{T}_*$; see (4.58).

Monotonicity properties of the error $|u - u_{\mathcal{T}}|_{H^1_0(\Omega)} = \|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)}$ and the PDE error estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}}) = \eta_{\mathcal{T}}(u_{\mathcal{T}}, f)$ with respect to $\mathcal{T}$ would be useful but fail

to hold. To investigate this issue, we consider two admissible meshes $\mathcal{T}, \mathcal{T}_* \in \mathbb{T}$, the latter being a refinement of the former $\mathcal{T}_* \geq \mathcal{T}$, and a third admissible mesh $\widehat{\mathcal{T}} \leq \mathcal{T}$. We further assume that data $\mathcal{D} = (\boldsymbol{A}, c, f)$ is discrete over $\widehat{\mathcal{T}}$ in the sense that $\mathcal{D} \in \mathbb{D}_{\widehat{\mathcal{T}}}$, where

$$\mathbb{D}_{\widehat{\mathcal{T}}} := \left[\mathbb{S}^{n-1,-1}_{\widehat{\mathcal{T}}}\right]^{d\times d} \times \mathbb{S}^{n-1,-1}_{\widehat{\mathcal{T}}} \times \mathbb{F}_{\widehat{\mathcal{T}}}$$

and $\mathcal{D}$ does not change in the transition from $\mathcal{T}$ to $\mathcal{T}_*$ irrespective of the degree of local refinement; in particular, $f = P_{\widehat{\mathcal{T}}} f \in \mathbb{F}_{\widehat{\mathcal{T}}}$. We will later denote discrete data as $\widehat{\mathcal{D}} = (\widehat{\boldsymbol{A}}, \widehat{c}, \widehat{f})$ to distinguish it from exact data $\mathcal{D}$, and to study their discrepancy, but we prefer to keep the simple notation $\mathcal{D} = \widehat{\mathcal{D}}$ now because there is no reason for confusion. In particular, this implies that the bilinear form in (2.8) and the forcing function are the same for both Galerkin solutions $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$ and $u_{\mathcal{T}_*} \in \mathbb{V}_{\mathcal{T}_*}$, whence the energy errors are monotone according to (3.8),

$$|\!|\!| u - u_{\mathcal{T}_*} |\!|\!|_\Omega \leq |\!|\!| u - u_{\mathcal{T}} |\!|\!|_\Omega,$$

but not $|u - u_{\mathcal{T}}|_{H^1_0(\Omega)}$. Moreover, $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ is not monotone because the discrete solution $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$ changes with the mesh. It is thus useful to quantify the behaviour of $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ in terms of $\mathcal{T}$ and $u_{\mathcal{T}}$ following Cascón *et al.* (2008); see also Morin, Siebert and Veeser (2008). We do this next.

The first lemma exploits the structure of the PDE residual estimator, namely the presence of a positive power of the local mesh size, and expresses the *reduction* of $\eta_{\mathcal{T}_*}(v, f)$ relative to $\eta_{\mathcal{T}}(v, f)$ for fixed functions $v \in \mathbb{V}_{\mathcal{T}}$ and $f \in \mathbb{F}_{\mathcal{T}}$. This quantitative property is instrumental in studying convergence of AFEMs for coercive problems in Section 6 as well as discontinuous Galerkin methods in Section 9 and inf-sup stable problems in Section 10.

Lemma 4.53 (reduction property of the estimator). *If the elements of $\mathcal{M} \subset \mathcal{T}$ are bisected at least $b \geq 1$ times to refine $\mathcal{T}$ into $\mathcal{T}_*$, and $\lambda = 1 - 2^{-b/d}$, then*

$$\eta_{\mathcal{T}_*}(v, f, \mathcal{T}_*)^2 \leq \eta_{\mathcal{T}}(v, f, \mathcal{T})^2 - \lambda\, \eta_{\mathcal{T}}(v, f, \mathcal{M})^2 \quad \textit{for all } v \in \mathbb{V}_{\mathcal{T}},\ f \in \mathbb{F}_{\mathcal{T}}. \tag{4.67}$$

Proof. Given $T \in \mathcal{T}$, we rewrite (4.52) as follows:

$$\eta_{\mathcal{T}}(v, T)^2 = h_T\, j_{\mathcal{T}}(v, T)^2 + h_T^2\, r_{\mathcal{T}}(v, T)^2,$$

with $\eta_{\mathcal{T}}(v, T) = \eta_{\mathcal{T}}(v, f, T)$ and

$$j_{\mathcal{T}}(v, T)^2 = j_{\mathcal{T}}(v, f, T)^2 = \sum_{\substack{F \in \mathcal{F} \\ F \subset \partial T}} \|[\![\boldsymbol{A}\nabla v]\!] \cdot \boldsymbol{n}_F - P_{\mathcal{T}} f\|^2_{L^2(F)},$$

$$r_{\mathcal{T}}(v, T) = r_{\mathcal{T}}(v, f, T) = \|P_{\mathcal{T}} f + \operatorname{div}(\boldsymbol{A}\nabla v) - cv\|_{L^2(T)},$$

where $f = P_{\mathcal{T}} f = P_{\mathcal{T}_*} f \in \mathbb{F}_{\mathcal{T}}$ does not change from $\mathcal{T}$ to $\mathcal{T}_*$. We readily have

$$\sum_{T_* \in \mathcal{T}_*, T_* \subseteq T} \eta_{\mathcal{T}_*}(v, T_*)^2 \leq \eta_{\mathcal{T}}(v, T)^2,$$

because $h_{T_*} \le h_T$ for all $T_* \subset T$ and $T_* \in \mathcal{T}_*$. If, in addition, T is bisected at least b times, then any such T_* satisfies $h_{T_*} \le 2^{-b/d} h_T$, whence

$$\sum_{T_* \in \mathcal{T}_*, T_* \subseteq T} \eta_{\mathcal{T}_*}(v, T_*)^2 \le 2^{-b/d} \eta_{\mathcal{T}}(v, T)^2.$$

Therefore, adding over $T \in \mathcal{T}$, we obtain

$$\eta_{\mathcal{T}_*}(v)^2 = \sum_{T \in \mathcal{T}} \sum_{T_* \in \mathcal{T}_*, T_* \subset T} \eta_{\mathcal{T}_*}(v, T)^2 \le 2^{-b/d} \sum_{T \in \mathcal{M}} \eta_{\mathcal{T}}(v, T)^2 + \sum_{T \in \mathcal{T} \setminus \mathcal{M}} \eta_{\mathcal{T}}(v, T)^2,$$

which implies the assertion (4.67). □

The next result complements Lemma 4.53 in that it expresses the Lipschitz continuity of $\eta_{\mathcal{T}}(v, f)$ with respect to the argument $v \in \mathbb{V}_{\mathcal{T}}$ for fixed $\mathcal{T}$ and $f \in \mathbb{F}_{\mathcal{T}}$.

Lemma 4.54 (Lipschitz property of the estimator). *Let $\mathcal{T}$ and $f \in \mathbb{F}_{\mathcal{T}}$ be fixed. There exists a constant C_{Lip} proportional to $\|A\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)}$ such that*

$$|\eta_{\mathcal{T}}(v, f) - \eta_{\mathcal{T}}(w, f)| \le C_{\mathrm{Lip}} |v - w|_{H^1_0(\Omega)} \quad \textit{for all } v, w \in \mathbb{V}_{\mathcal{T}}. \tag{4.68}$$

Proof. Since $\eta_{\mathcal{T}}(v) = \eta_{\mathcal{T}}(v, f)$ is the ℓ^2-norm of the vector $(\eta_{\mathcal{T}}(v, T))_{T \in \mathcal{T}} \in \mathbb{R}^{\#\mathcal{T}}$, applying the triangle inequality gives

$$\begin{aligned} |\eta_{\mathcal{T}}(v) - \eta_{\mathcal{T}}(w)|^2 &\le \sum_{T \in \mathcal{T}} |\eta_{\mathcal{T}}(v, T) - \eta_{\mathcal{T}}(w, T)|^2 \\ &\le \sum_{T \in \mathcal{T}} h_T |j_{\mathcal{T}}(v, T) - j_{\mathcal{T}}(w, T)|^2 + h_T^2 |r_{\mathcal{T}}(v, T) - r_{\mathcal{T}}(w, T)|^2. \end{aligned}$$

We first consider the jump terms and again apply the triangle inequality followed by an inverse estimate to find that

$$\begin{aligned} |j_{\mathcal{T}}(v, T) - j_{\mathcal{T}}(w, T)|^2 &\le \sum_{\substack{F \in \mathcal{F} \\ F \subset \partial T}} \|[\![\boldsymbol{A} \nabla (v - w)]\!] \cdot \boldsymbol{n}_F\|^2_{L^2(F)} \\ &\lesssim h_T^{-1} \|\boldsymbol{A}\|^2_{L^\infty(\Omega)} \|\nabla (v - w)\|^2_{L^2(\omega_T)}, \end{aligned}$$

where ω_T is the patch of T. A similar reasoning for the element residuals yields

$$\begin{aligned} |r_{\mathcal{T}}(v, T) - r_{\mathcal{T}}(w, T)|^2 &\lesssim \|c(v - w)\|^2_{L^2(T)} + \|\operatorname{div}(\boldsymbol{A} \nabla (v - w))\|^2_{L^2(T)} \\ &\lesssim \|c\|^2_{L^\infty(\Omega)} \|(v - w)\|^2_{L^2(T)} + h_T^{-2} \|\boldsymbol{A}\|^2_{L^\infty(\Omega)} \|\nabla (v - w)\|^2_{L^2(T)}, \end{aligned}$$

because A is piecewise polynomial. Finally, adding over $T \in \mathcal{T}$ and applying Lemma 2.2 (first Poincaré inequality) concludes the proof. □

Since the estimator $\eta_{\mathcal{T}}(v, f)$ depends explicitly on $P_{\mathcal{T}} f$, and $P_{\mathcal{T}} f$ may change with $\mathcal{T}$, it is crucial to account for the variations of $\eta_{\mathcal{T}}(v, f)$ while keeping $\mathcal{T}$ and $v \in \mathbb{V}_{\mathcal{T}}$ fixed. This is the purpose of our next result.

Lemma 4.55 (estimator dependence on discrete forcing). *Let $\mathcal{T}$ and $v \in \mathbb{V}_{\mathcal{T}}$ be fixed. Then there exists a constant C_{Lip} such that*

$$|\eta_{\mathcal{T}}(v,f)-\eta_{\mathcal{T}}(v,g)| \le C_{\mathrm{Lip}}\Big(\sum_{T\in\mathcal{T}} \|f-g\|^2_{H^{-1}(\omega_T)}\Big)^{1/2} \quad \textit{for all } f,g \in \mathbb{F}_{\mathcal{T}}. \tag{4.69}$$

Proof. We proceed elementwise, as in Lemma 4.54, except that after applying the triangle inequality we end up with the weighted L^2-norms

$$h_T^2\|f-g\|^2_{L^2(T)} + h_T\|f-g\|^2_{L^2(\partial T)} \quad \text{for all } T \in \mathcal{T}.$$

Extending these norms to patches ω_T and its interior faces σ_T, and appealing to Corollary 4.30 (quantifying H^{-1}-norms of discrete functionals), we deduce

$$h_T^2\|f-g\|^2_{L^2(\omega_T)} + h_T\|f-g\|^2_{L^2(\sigma_T)} \approx \|f-g\|^2_{H^{-1}(\omega_T)}.$$

Adding over $T \in \mathcal{T}$ finishes the proof. □

In the subsequent applications of Lemma 4.54 the discrete coefficients $(\boldsymbol{A}, c)$ may change with the change of the supporting mesh $\widehat{\mathcal{T}}$, but they will always be uniformly bounded in $L^\infty(\Omega)$; hence the constant C_{Lip} is uniformly bounded as well. Upon combining Lemmas 4.53, 4.54 and 4.55, we obtain the following crucial property.

Proposition 4.56 (estimator reduction). *Given $\mathcal{T} \in \mathbb{T}$ and a subset $\mathcal{M} \subset \mathcal{T}$ of elements marked for refinement, let* REFINE *be the procedure discussed in Section 3.5 that bisects the elements of $\mathcal{M}$ at least b times, and let $\mathcal{T}_* = \mathsf{REFINE}(\mathcal{T},\mathcal{M})$ be the resulting conforming mesh. Let the coefficients $(\boldsymbol{A}, c)$ be discrete and fixed. Then, for $\lambda = 1-2^{-b/d}$, for all $v \in \mathbb{V}_{\mathcal{T}}$, $v_* \in \mathbb{V}_{\mathcal{T}_*}$, $f \in \mathbb{F}_{\mathcal{T}}$, $f_* \in \mathbb{F}_{\mathcal{T}_*}$, and any $\delta > 0$,*

$$\begin{aligned}\eta_{\mathcal{T}_*}(v_*,f_*,\mathcal{T}_*)^2 &\le (1+\delta)\big(\eta_{\mathcal{T}}(v,f,\mathcal{T})^2 - \lambda\,\eta_{\mathcal{T}}(v,f,\mathcal{M})^2\big)\\ &\quad + 2(1+\delta^{-1})\,C_{\mathrm{Lip}}^2\Big(|v_*-v|^2_{H^1_0(\Omega)} + \sum_{T_*\in\mathcal{T}_*}\|f_*-f\|^2_{H^{-1}(\omega_{T_*})}\Big),\end{aligned}$$

where C_{Lip} is the constant in Lemmas 4.54 and 4.55.

Proof. For any $\delta > 0$, write

$$\eta_{\mathcal{T}_*}(v_*,f_*,\mathcal{T}_*)^2 \le (1+\delta)\eta_{\mathcal{T}_*}(v,f,\mathcal{T}_*)^2 + (1+\delta^{-1})(\eta_{\mathcal{T}_*}(v_*,f_*,\mathcal{T}_*) - \eta_{\mathcal{T}_*}(v,f,\mathcal{T}_*))^2$$

and apply Lemma 4.53 to the first term and Lemmas 4.54 and 4.55 to the second term combined with a triangle inequality. □

We finish this section by investigating the behaviour of the global oscillation under refinement.

Lemma 4.57 (quasi-monotonicity of oscillation). *If $f \in H^{-1}(\Omega)$ and $\mathcal{T}, \mathcal{T}_* \in \mathbb{T}$ with $\mathcal{T}_* \ge \mathcal{T}$, then*

$$\operatorname{osc}_{\mathcal{T}_*}(f) \le C_{\mathrm{osc}}\operatorname{osc}_{\mathcal{T}}(f),$$

where C_{osc} depends only on the shape regularity coefficient σ and d.

Proof. Given $T \in \mathcal{T}$, let $T_* \in \mathcal{T}_*$ such that $T_* \subset T$. Since $\mathcal{T}_*$ is a refinement of $\mathcal{T}$, this implies that the patch $\omega_{\mathcal{T}_*}(T_*)$ in $\mathcal{T}_*$ around T_* is contained in the patch $\omega_{\mathcal{T}}(T)$ in $\mathcal{T}$ around T. Thanks to Lemma 4.31 (local near-best approximation), we derive

$$\mathrm{osc}_{\mathcal{T}_*}(f, T_*)^2 = \|(I - P_{\mathcal{T}_*})f\|^2_{H^{-1}(\omega_{\mathcal{T}_*}(T_*))} \le C^2_{\mathrm{lStb}}\|(I - P_{\mathcal{T}})f\|^2_{H^{-1}(\omega_{\mathcal{T}_*}(T_*))}$$

and therefore, with the help of (ii) of Lemma 4.5 (localization of H^{-1}-norm),

$$\sum_{T_* \subset T} \mathrm{osc}_{\mathcal{T}_*}(f, T_*)^2 \le C^2_{\mathrm{lStb}} C_{\mathrm{ovrl}} \|(I - P_{\mathcal{T}})f\|^2_{H^{-1}(\omega_{\mathcal{T}}(T))} = C^2_{\mathrm{lStb}} C_{\mathrm{ovrl}}\, \mathrm{osc}_{\mathcal{T}}(f, T)^2,$$

where C_{ovrl} is bounded in terms of the shape regularity coefficient σ and d. Hence summing over $T \in \mathcal{T}$ yields

$$\mathrm{osc}_{\mathcal{T}_*}(f)^2 = \sum_{T \in \mathcal{T}} \sum_{T_* \subset T} \mathrm{osc}_{\mathcal{T}_*}(f, T_*)^2 \le C^2_{\mathrm{lStb}} C_{\mathrm{ovrl}}\, \mathrm{osc}_{\mathcal{T}}(f)^2,$$

and the proof is finished. □

4.9. Alternative estimators

In Section 4.7 we used the H^{-1}-projection $P_{\mathcal{T}}$ to derive *a posteriori* bounds for the error in the spirit of the standard residual estimator. The goal of this section is to illustrate that the approach with $P_{\mathcal{T}}$ can also be combined with other techniques of *a posteriori* error estimation, generalizing and expanding the discussion in Kreuzer and Veeser (2021, Section 4) with the H^{-1}-projection $P_{\mathcal{T}}$.

Alternative techniques have been developed with the desire to reduce or even circumvent the fact that constants spoil the relationship between error and estimator. In the framework of the aforementioned approach, we shall see that the various techniques based upon

- local (discrete) problems,
- hierarchy,
- flux equilibration

amount to different ways of quantifying a local norm of the discretized residual $P_{\mathcal{T}} R_{\mathcal{T}}$. This observation is useful for comparing the techniques and for a common treatment in the following sections about adaptive algorithms.

As in Section 4.7 on modified residual estimation, we shall consider only the case given by Assumption 4.44 (discrete coefficients and discrete functionals). For the hidden constants in the results of this section, it is useful to keep in mind Remark 4.4 (constants in error–residual relationship).

Theorem 4.45 (modified residual estimator) analysed an element-indexed version of the residual estimator. For the sake of simplicity, we shall refrain here from such an element-indexed setting and remain in the vertex-indexed setting of the

abstract analysis of Section 4.6. In order to facilitate the comparison with the other estimators below, we offer the following vertex-indexed variant of Theorem 4.45. For $z \in \mathcal{V}$, we set $\mathcal{F}_z := \{F \in \mathcal{F} \mid z \in F\}$ and $\mathcal{T}_z := \{T \in \mathcal{T} \mid z \in T\}$. Given the Galerkin approximation $u_\mathcal{T}$ from (4.1), define the PDE indicator by

$$\begin{aligned} \eta_\mathcal{T}^{\mathrm{res}}(u_\mathcal{T}) &:= \sum_{z\in\mathcal{V}} \eta_\mathcal{T}^{\mathrm{res}}(u_\mathcal{T}, z)^2 \quad \text{with} \\ \eta_\mathcal{T}^{\mathrm{res}}(u_\mathcal{T}, z)^2 &:= \sum_{F\in\mathcal{F}_z} h_F \|P_F R_\mathcal{T}\|_{L^2(F)}^2 + \sum_{T\in\mathcal{T}_z} h_T^2 \|P_T R_\mathcal{T}\|_{L^2(T)}^2, \end{aligned} \tag{4.70a}$$

where $R_\mathcal{T} = f + \operatorname{div}(\boldsymbol{A}\nabla u_\mathcal{T}) - c u_\mathcal{T} \in H^{-1}(\Omega)$ is the residual and P_F, $F \in \mathcal{F}$ and P_T, $T \in \mathcal{T}$ yield the polynomial densities of $P_\mathcal{T}$; see Definition 4.24. The vertex-indexed modified residual estimator is then

$$\mathcal{E}_\mathcal{T}^{\mathrm{res}} := \mathcal{E}_\mathcal{T}^{\mathrm{res}}(u_\mathcal{T}, f)^2 := \eta_\mathcal{T}^{\mathrm{res}}(u_\mathcal{T})^2 + \operatorname{osc}_\mathcal{T}(f)^2. \tag{4.70b}$$

Theorem 4.58 (vertex-indexed modified residual estimator). *Suppose Assumption 4.44. The modified residual estimator* (4.70b) *is equivalent to the error:*

$$\frac{\min\{1, C_{L,\mathrm{res}}\}}{C_{\mathrm{lStb}}^\star C_d} \mathcal{E}_\mathcal{T}^{\mathrm{res}} \lesssim \|\nabla(u - u_\mathcal{T})\|_{L^2(\Omega)} \lesssim \max\{1, C_{U,\mathrm{res}}\} C_d C_{\mathrm{loc}} \mathcal{E}_\mathcal{T}^{\mathrm{res}},$$

while its PDE indicator (4.70a) *is locally equivalent to the discretized residual: for all vertices $z \in \mathcal{V}$,*

$$C_{L,\mathrm{res}} \eta_\mathcal{T}^{\mathrm{res}}(u_\mathcal{T}, z) \le \|P_\mathcal{T} R_\mathcal{T}\|_{H^{-1}(\omega_z)} \le C_{U,\mathrm{res}} \eta_\mathcal{T}^{\mathrm{res}}(u_\mathcal{T}, z).$$

Here, $C_{L,\mathrm{res}}$ and $C_{U,\mathrm{res}}$ are the hidden constants of Corollary 4.30 on stars, $C_{\mathrm{lStb}}^\star$ is the stability constant of $P_\mathcal{T}$ on stars from Lemma 4.28, C_{loc} is the constant from Corollary 4.6, $C_d = \sqrt{2(d+1)}$, and the hidden constants depend only on the error–residual relationship in Lemma 4.1.

Proof. Local equivalence is a reformulation of Corollary 4.30 (quantifying H^{-1}-norms of discrete functionals) on stars. The global bounds follow by combining local equivalence with Lemma 4.1 (error and residual), Corollary 4.6 (star localization of residual norm) and Lemma 4.35 (splitting of local residual norm). □

Adjoint projection

The projection $P_\mathcal{T}$ relates residual $R_\mathcal{T}$ and discretized residual $P_\mathcal{T} R_\mathcal{T}$. In order to exploit this relationship on the test space $H_0^1(\Omega)$, we shall need the adjoint $P_\mathcal{T}^*$ to the projection $P_\mathcal{T}$. Curiously, operators employed in this vein appeared first; see e.g. Morin, Nochetto and Siebert (2003) and Veeser (2002).

Given $w \in H_0^1(\Omega)$, the function $P_\mathcal{T}^* w$ can be directly defined by requiring

$$P_\mathcal{T}^* w \in \mathbb{V}_\mathcal{T}^+ : \quad \langle \ell, P_\mathcal{T}^* w \rangle = \langle \ell, w \rangle \quad \text{for all } \ell \in \mathbb{F}_\mathcal{T}. \tag{4.71}$$

This definition is well-posed thanks to Lemma 3.1 (discrete inf-sup condition)

and Lemma 4.25 (algebraic properties), especially (4.37) and (4.38). Clearly, $P^*_{\mathcal{T}}$ is a linear projection onto the finite-dimensional subspace $\mathbb{V}^+_{\mathcal{T}} \subset H^1_0(\Omega)$. A representation as interpolation operator will be derived in Corollary 4.61 below. Using both definitions of $P_{\mathcal{T}}$ and $P^*_{\mathcal{T}}$, we see that they are actually adjoint:

$$\langle P_{\mathcal{T}}\ell, w\rangle = \langle P_{\mathcal{T}}\ell, P^*_{\mathcal{T}}w\rangle = \langle \ell, P^*_{\mathcal{T}}w\rangle \quad \text{for all } \ell \in H^{-1}(\Omega),\ w \in H^1_0(\Omega). \tag{4.72}$$

Consequently, Lemmas 4.25 and 4.28 (local H^{-1}-stability) show that $P^*_{\mathcal{T}}$ is a local operator with

$$\|P^*_{\mathcal{T}}\|_{\mathcal{L}(H^1_0(\omega))} = \|P_{\mathcal{T}}\|_{\mathcal{L}(H^{-1}(\omega))} \le C_{\mathrm{lStb}}. \tag{4.73}$$

The choice $\ell = R_{\mathcal{T}}$ in (4.72) leads to

$$\langle P_{\mathcal{T}}R_{\mathcal{T}}, w\rangle = \langle P_{\mathcal{T}}R_{\mathcal{T}}, P^*_{\mathcal{T}}w\rangle = \langle R_{\mathcal{T}}, P^*_{\mathcal{T}}w\rangle \quad \text{for all } w \in H^1_0(\Omega),$$

where the two identities show that the discretized residual $P_{\mathcal{T}}R_{\mathcal{T}}$ can be analysed with discrete test functions in $\mathbb{V}^+_{\mathcal{T}}$ only; see the norm equivalence in Lemma 4.28. Restricting to discrete test functions in $\mathbb{V}^+_{\mathcal{T}} = \operatorname{Im} P^*_{\mathcal{T}}$, we find the definition of $P_{\mathcal{T}}$:

$$\langle P_{\mathcal{T}}R_{\mathcal{T}}, w\rangle = \langle R_{\mathcal{T}}, w\rangle \quad \text{for all } w \in \mathbb{V}^+_{\mathcal{T}}. \tag{4.74}$$

An estimator based upon local problems

Local dual norms can be quantified by solving local problems. Requiring computability of these solutions leads to *finite-dimensional* or *discrete* local problems. In other words, we lift the residual to local and finite-dimensional extensions of the finite element space. Starting with Babuška and Rheinboldt (1978), this idea was used to soften the impact of constants in the relationship between error and estimator; see Verfürth (2013, Remark 1.22) for more references.

Within the approach of Sections 4.1 and 4.6, we can use local discrete problems to quantify the local H^{-1}-norms of the discretized residual $P_{\mathcal{T}}R_{\mathcal{T}}$. In this manner, constants arise only due to the localization of the residual norm and to the splitting into discretized and oscillatory residual by the H^{-1}-projection $P_{\mathcal{T}}$.

We start by introducing the vertex-oriented PDE indicator. Given the Galerkin approximation $u_{\mathcal{T}}$ from (4.1), set

$$\eta^{\mathrm{lpb}}_{\mathcal{T}}(u_{\mathcal{T}}) := \sum_{z\in\mathcal{V}} \eta^{\mathrm{lpb}}_{\mathcal{T}}(u_{\mathcal{T}}, z)^2 \quad \text{with} \quad \eta^{\mathrm{lpb}}_{\mathcal{T}}(u_{\mathcal{T}}, z) := \|\nabla v_z\|_{L^2(\omega_z)}, \tag{4.75a}$$

where $v_z \in \mathbb{V}^+(\mathcal{T}_z)$ is the solution of the local problem

$$\int_{\omega_z} \nabla v_z \cdot \nabla w = \langle R_{\mathcal{T}}, w\rangle \quad \text{for all } w \in \mathbb{V}^+(\mathcal{T}_z). \tag{4.75b}$$

Note that this problem is discrete for $\dim \mathbb{V}^+(\mathcal{T}_z) < \infty$ and can therefore be solved up to machine precision. The resulting estimator is then

$$\mathcal{E}^{\mathrm{lpb}}_{\mathcal{T}} := \mathcal{E}^{\mathrm{lpb}}_{\mathcal{T}}(u_{\mathcal{T}}, f)^2 := \eta^{\mathrm{lpb}}_{\mathcal{T}}(u_{\mathcal{T}})^2 + \operatorname{osc}_{\mathcal{T}}(f)^2. \tag{4.75c}$$

Theorem 4.59 (estimator based on local problems). *Under Assumption 4.44 the estimator (4.75) based on local problems is equivalent to the error, while its PDE indicator is locally equivalent to the discretized residual with constant* 1 *in the lower bound, so that*

$$\frac{1}{C^\star_{\mathrm{lStb}} C_d} \mathcal{E}^{\mathrm{lpb}}_{\mathcal{T}} \lesssim \|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)} \lesssim C^\star_{\mathrm{lStb}} C_d C_{\mathrm{loc}} \mathcal{E}^{\mathrm{lpb}}_{\mathcal{T}},$$

and, for all vertices $z \in \mathcal{V}$,

$$\eta^{\mathrm{lpb}}_{\mathcal{T}}(u_{\mathcal{T}}, z) \le \|P_{\mathcal{T}} R_{\mathcal{T}}\|_{H^{-1}(\omega_z)} \le C^\star_{\mathrm{lStb}} \eta^{\mathrm{lpb}}_{\mathcal{T}}(u_{\mathcal{T}}, z)$$

where $C^\star_{\mathrm{lStb}}$ is the stability constant of $P_{\mathcal{T}}$ on stars from Lemma 4.28, C_{loc} from Corollary 4.6, $C_d = \sqrt{2(d+1)}$ and the hidden constants depend only on the error–residual relationship in Lemma 4.1.

Proof. It suffices to show the local equivalence for the PDE indicator; see Theorem 4.58 (vertex-indexed modified residual estimator) and note that $C^\star_{\mathrm{lStb}} \ge 1$. Let $z \in \mathcal{V}$ be any vertex. In view of (4.74), the definition of $v_z \in \mathbb{V}^+(\mathcal{T})$ readily implies

$$\eta^{\mathrm{lpb}}_{\mathcal{T}}(u_{\mathcal{T}}, z) = \|\nabla v_z\|_{L^2(\omega_z)} = \|P_{\mathcal{T}} R_{\mathcal{T}}\|_{\mathbb{V}^+(\mathcal{T})^*}.$$

Hence Lemma 4.28 on the local H^{-1}-stability of $P_{\mathcal{T}}$ yields the asserted local equivalence of PDE indicator and discretized residual $P_{\mathcal{T}} R_{\mathcal{T}}$. □

A stable biorthogonal system for $\mathbb{F}_{\mathcal{T}} \times \mathbb{V}^+_{\mathcal{T}}$

Stable biorthogonal systems induce linear bounded projections, which enjoy near-best approximation thanks to the Lebesgue lemma. Supposing Assumption 4.21 (abstract cut-off), we now outline the construction of such a system for the finite-dimensional product $\mathbb{F}_{\mathcal{T}} \times \mathbb{V}^+_{\mathcal{T}}$. The constructed system will induce both projections $P_{\mathcal{T}}$ and its adjoint $P^*_{\mathcal{T}}$. This generalizes the bi-orthogonal system in Kreuzer and Veeser (2021, Section 3.4) to arbitrary degrees of the discrete functionals and provides an alternative approach to $P_{\mathcal{T}}$, its local stability as well as its computation. Furthermore, we use it for devising a hierarchical estimator.

The construction is implemented in an affine equivalent manner and our first step consists in setting up a suitable *reference biorthogonal system*. Let $\widehat{T} := T_d$ be the reference element, let $\widehat{F} := T_{d-1} \times \{0\} \subset \widehat{T}$ be the reference face, and denote the polynomial degrees in $\mathbb{F}_{\mathcal{T}}$ by $m_1 \in \mathbb{N}_0$ and $m_2 \in \mathbb{N}_0$. Writing

$$K_1 := \dim \mathbb{P}_{m_1}(\widehat{F}), \quad K_2 := \dim \mathbb{P}_{m_2}(\widehat{T}),$$

assume that we are given orthonormal bases $q_{(\widehat{F},1)}, \dots, q_{(\widehat{F},K_1)} \in \mathbb{P}_{m_1}(\widehat{F})$ and $q_{(\widehat{T},1)}, \dots, q_{(\widehat{T},K_2)} \in \mathbb{P}_{m_2}(\widehat{T})$ in the sense that

$$\int_{\widehat{F}} q_{(\widehat{F},k)} q_{(\widehat{F},l)} \phi_{\widehat{F}} = \delta_{kl}, \quad \int_{\widehat{T}} q_{(\widehat{T},k)} q_{(\widehat{T},l)} \phi_{\widehat{T}} = \delta_{kl}. \tag{4.76}$$

for all admissible k, l, i.e. $k, l \in \{1, \dots, K_1\}$ or $\{1, \dots, K_2\}$ depending on the underlying domain. These bases induce the *reference functionals*

$$\widehat{\ell}_{(\widehat{F},k)}(w) := \int_{\widehat{F}} q_{(\widehat{F},k)} w, \quad \widehat{\ell}_{(\widehat{T},k)}(w) := \int_{\widehat{T}} q_{(\widehat{T},k)} w,$$

on $H^1(\widehat{T})$, which in turn span the reference space $\widehat{\mathbb{F}}$. In order to define complementary test functions, let $\widehat{E}$ be the extension operator (4.4) associated with the reference face $\widehat{F}$ adapted to the current situation with the only element $\widehat{T}$, and, given some $v \in L^2(\widehat{T})$, define $\widehat{Q}v \in \mathbb{P}_{m_2}$ by

$$\int_{\widehat{T}} q\big(\widehat{Q}v\big)\phi_{\widehat{T}} = \int_{\widehat{T}} qv \quad \text{for all } q \in \mathbb{P}_{m_2}. \tag{4.77}$$

We thus define the *reference test functions*

$$\begin{aligned} \widehat{w}_{(\widehat{T},k)} &:= q_{(\widehat{T},k)}\phi_{\widehat{T}}, \quad k = 1, \dots, K_2, \\ \widehat{w}_{(\widehat{F},k)} &:= \widehat{v}_{(\widehat{F},k)} - \big(\widehat{Q}\widehat{v}_{(\widehat{F},k)}\big)\phi_{\widehat{T}}, \quad k = 1, \dots, K_1, \end{aligned} \tag{4.78}$$

with $\widehat{v}_{(\widehat{F},k)} := (\widehat{E}q_{(\widehat{F},k)})\phi_{\widehat{F}}$. Note that $\widehat{w}_{(\widehat{F},k)} \neq 0$. Writing

$$\widehat{I} := \{(\widehat{F}, k) \mid k = 1, \dots, K_1\} \cup \{(\widehat{T}, k) \mid k = 1, \dots, K_2\},$$

we then have

$$\widehat{w}_i \in \widehat{\mathbb{V}}^+ := \{(\widehat{E}q_1)\phi_{\widehat{F}} + q_2\phi_{\widehat{T}} \mid q_1 \in \mathbb{P}_{m_1}, q_2 \in \mathbb{P}_{m_2}\} \quad \text{for all } i \in \widehat{I}$$

and the biorthogonality

$$\langle \widehat{\ell}_i, \widehat{w}_j \rangle = \delta_{ij} \quad \text{for all } i, j \in \widehat{I}, \tag{4.79}$$

thanks to (4.76) and (4.77). We thus dispose of a biorthogonal system in the reference product $\widehat{\mathbb{F}} \times \widehat{\mathbb{V}}^+$.

Using pullbacks with some minor tweaks, this reference biorthogonal system induces a global biorthogonal system. To this end, we employ bi-affine maps G_F, G_T, and $G_{(T,F)}$. Here, for example, given a pair $(T, F) \in \mathcal{T} \times \mathcal{F}$ with $F \subset T$, the map $G_{(T,F)}$ is bi-affine and sends vertices into vertices such that $G_{(T,F)}(\widehat{T}) = T$ and $G_{(T,F)}(\widehat{F}) = F$. The fact that these maps are only unique up to some renumbering of the vertices is irrelevant, as all objects in the reference situation on $(\widehat{T}, \widehat{F})$ are invariant under such renumberings. We denote the respective inverse maps of G_F, G_T and $G_{(T,F)}$ by H_F, H_T and $H_{(T,F)}$. Motivated by the transformation rule, we introduce the scaled pullbacks, for $F \in \mathcal{F}, T \in \mathcal{T}, k$ admissible,

$$q_{(F,k)} := \left(\frac{|\widehat{F}|}{|F|}\right)^{1/2} H_F^* q_{(\widehat{F},k)}, \quad q_{(T,k)} := \left(\frac{|\widehat{T}|}{|T|}\right)^{1/2} H_T^* q_{(\widehat{T},k)}, \tag{4.80}$$

of the reference orthonormal bases in (4.76). These lead to the basis

$$\ell_{(F,k)}(w) := \int_F q_{(F,k)} w, \quad \ell_{(T,k)}(w) := \int_T q_{(T,k)} w \tag{4.81}$$

of $\mathbb{F}_{\mathcal{T}}$, while the associated test functions are again given via pullbacks:

$$w_{(T,k)} := \left(\frac{|\widehat{T}|}{|T|}\right)^{1/2} H_T^* \widehat{w}_{(\widehat{T},k)}, \quad w_{(F,k)}|_T := \left(\frac{|\widehat{F}|}{|F|}\right)^{1/2} H_{(T,F)}^* \widehat{w}_{(\widehat{F},k)} \tag{4.82}$$

for all $T \in \mathcal{T}$, $F \in \mathcal{F}$ with $F \subset T$ and all admissible k. Note that $w_{(F,k)} \in H_0^1(\Omega)$. Finally, we introduce the index set

$$I := (\mathcal{F} \times \{1, \dots, K_1\}) \cup (\mathcal{T} \times \{1, \dots, K_2\}),$$

and observe that $w_i \in \mathbb{V}_{\mathcal{T}}^+$ for all $i \in I$.

Lemma 4.60 (biorthogonal system). *The pairs (ℓ_i, w_i), $i \in I$, provide a stable biorthogonal system of the product $\mathbb{F}_{\mathcal{T}} \times \mathbb{V}_{\mathcal{T}}^+$: indeed, $\langle \ell_i, w_j \rangle = \delta_{ij}$ for all $i, j \in I$ and, writing $I_z := \{(S,k) \in I \mid S \ni z\}$ for all $z \in \mathcal{V}$,*

$$\sum_{i \in I_z} \|\ell_i\|_{H^{-1}(\omega_z)} \|\nabla w_i\|_{L^2(\omega_z)} \le C_{\mathrm{bOS}}^\star,$$

where the constant $C_{\mathrm{bOS}}^\star$ depends only on d, m_1, m_2 and the shape regularity coefficient σ from (3.9).

Proof. 1 We first establish the biorthogonality. Thanks to the transformation rule, the scaled pullbacks (4.80) indeed form local orthonormal bases of $P_{m_1}(F)$, $F \in \mathcal{F}$, and $P_{m_2}(T)$, $T \in \mathcal{T}$:

$$\int_F q_{(F,k)} q_{(F,l)} \phi_F = \int_{\widehat{F}} q_{(\widehat{F},k)} q_{(\widehat{F},l)} \phi_{\widehat{F}} = \delta_{kl}, \quad \int_T q_{(T,k)} q_{(T,l)} \phi_T = \delta_{kl} \tag{4.83}$$

for all admissible k and l. This orthonormality, combined with the local supports of the pairs (ℓ_i, w_i), $i \in I$, shows the biorthogonality, except for the cases when $(T,F) \in \mathcal{T} \times \mathcal{F}$ with $F \subset T$ and k, l are admissible. Here the transformation rule and the definition of $\widehat{Q}$ imply

$$\begin{aligned}\langle \ell_{(T,k)}, w_{(F,l)} \rangle &= \int_T q_{(T,k)} w_{(F,l)} = \left(\frac{|T|\,|\widehat{F}|}{|\widehat{T}|\,|F|}\right)^{1/2} \int_{\widehat{T}} q_{(\widehat{T},k)} \widehat{w}_{(\widehat{F},l)} \\ &= \left(\frac{|T|\,|\widehat{F}|}{|\widehat{T}|\,|F|}\right)^{1/2} \int_{\widehat{T}} q_{(\widehat{T},k)} \big(\widehat{v}_{(\widehat{F},l)} - (\widehat{Q}\widehat{v}_{(\widehat{F},l)})\phi_{\widehat{T}}\big) = 0,\end{aligned}$$

and biorthogonality is verified.

2 It remains to show the stability bound for any vertex $z \in \mathcal{V}$. Given $i \in I_z$, we have either $i = (T,k)$ with $T \in \mathcal{T}$ or $i = (F,k)$ with $F \in \mathcal{F}$. On the one hand, the functional ℓ_i satisfies

$$\|\ell_{(T,k)}\|_{H^{-1}(\omega_z)} \lesssim h_T \quad \text{or} \quad \|\ell_{(F,k)}\|_{H^{-1}(\omega_z)} \lesssim h_F^{1/2}$$

since, by passing to the reference element and using orthonormality (4.83), (a variant of the) Poincaré inequality (2.2) and the trace inequality on $\widehat{T}$, we have

$$\begin{aligned}\langle \ell_{(T,k)}, w\rangle &= \int_T q_{(T,k)} w = \left(\frac{|T|}{|\widehat{T}|}\right)^{1/2} \int_{\widehat{T}} q_{(\widehat{T},k)} G_T^* w \\ &\le \left(\frac{|T|}{|\widehat{T}|}\right)^{1/2} \|q_{(\widehat{T},k)}\|_{L^2(\widehat{T})} \|G_T^* w\|_{L^2(\widehat{T})} \\ &\lesssim \left(\frac{|T|}{|\widehat{T}|}\right)^{1/2} \left(\int_{\widehat{T}} |q_{(\widehat{T},k)}|^2 \phi_{\widehat{T}}\right) \|\nabla(G_T^* w)\|_{L^2(\widehat{T})} \lesssim h_T \|\nabla w\|_{L^2(T)}\end{aligned}$$

or, with $T \in \mathcal{T}$ such that $T \supset F$,

$$\begin{aligned}\langle \ell_{(F,k)}, w\rangle &= \int_F q_{(F,k)} w = \left(\frac{|F|}{|\widehat{F}|}\right)^{1/2} \int_{\widehat{F}} q_{(\widehat{F},k)} G_{(T,F)}^* w \\ &\le \left(\frac{|F|}{|\widehat{F}|}\right)^{1/2} \|q_{(\widehat{F},k)}\|_{L^2(\widehat{F})} \|G_{(T,F)}^* w\|_{L^2(\widehat{F})} \\ &\lesssim \left(\frac{|F|}{|\widehat{F}|}\right)^{1/2} \left(\int_{\widehat{F}} |q_{(\widehat{F},k)}|^2 \phi_{\widehat{F}}\right) \|\nabla\big(G_{(T,F)}^* w\big)\|_{L^2(\widehat{T})} \\ &\lesssim \left(\frac{|F|\,|\widehat{T}|}{|\widehat{F}|\,|T|}\right)^{1/2} h_T \|\nabla w\|_{L^2(T)} \lesssim h_F^{1/2} \|\nabla w\|_{L^2(T)}.\end{aligned}$$

On the other hand, we obtain that the function w_i verifies

$$\|\nabla w_{(T,k)}\|_{L^2(T)} = \left(\frac{|\widehat{T}|}{|T|}\right)^{1/2} \|\nabla\big(H_T^* \widehat{w}_{(\widehat{T},k)}\big)\|_{L^2(T)} \lesssim h_T^{-1} \|\widehat{w}_{(\widehat{T},k)}\|_{L^2(\widehat{T})} \lesssim h_T^{-1}$$

or

$$\begin{aligned}\|\nabla w_{(F,k)}\|_{L^2(T)} &= \left(\frac{|\widehat{F}|}{|F|}\right)^{1/2} \|\nabla\big(H_{(T,F)}^* \widehat{w}_{(\widehat{T},k)}\big)\|_{L^2(T)} \\ &\lesssim h_T^{-1} \left(\frac{|T|}{|F|}\right)^{1/2} \|\widehat{w}_{(\widehat{T},k)}\|_{L^2(\widehat{T})} \lesssim h_F^{-1/2}.\end{aligned}$$

Using these inequalities to bound $\|\ell_i\|_{H^{-1}(\omega_z)} \|\nabla w_i\|_{L^2(\omega_z)}$, summing over all $i \in I_z$ then establishes the stability bound as the cardinality $\#I_z$ is uniformly bounded in terms of the shape regularity coefficient σ. □

Corollary 4.61 (projections as interpolation operators). *The biorthogonal system (ℓ_i, w_i), $i \in I$, induces the H^{-1}-projection $P_{\mathcal{T}}$ from Definition 4.24 and its adjoint $P_{\mathcal{T}}^*$. Indeed, we have*

$$P_{\mathcal{T}} \ell = \sum_{i \in I} \langle \ell, w_i\rangle \ell_i \quad \text{and} \quad P_{\mathcal{T}}^* w = \sum_{i \in I} \langle \ell_i, w\rangle w_i$$

for all $\ell \in H^{-1}(\Omega)$ and $w \in H_0^1(\Omega)$. The stability of the biorthogonal system then

provides an alternative proof of the H^{-1}-stability on stars of both projection $P_{\mathcal{T}}$ and $P^*_{\mathcal{T}}$, entailing $C^\star_{\mathrm{lStb}} \le C^\star_{\mathrm{bOS}}$.

Proof. 1 We only show the identity for $P_{\mathcal{T}}$; the one for $P^*_{\mathcal{T}}$ can be verified along the same lines. The biorthogonality in Lemma 4.60 readily implies

$$\left\langle \sum_{i\in I} \langle \ell, w_i\rangle \ell_i, w_j \right\rangle = \sum_{i\in I} \langle \ell, w_i\rangle \langle \ell_i, w_j\rangle = \langle \ell, w_j\rangle \quad \text{for all } j \in I.$$

As w_j, $j \in I$, is a basis of $\mathbb{V}^+_{\mathcal{T}}$, we conclude the claimed identity for $P_{\mathcal{T}}$.

2 To verify the stability statement, we again restrict ourselves to the case of the projection $P_{\mathcal{T}}$. Observe first that the proof of the stability of the biorthogonal system does invoke the local stability of $P_{\mathcal{T}}$. Thanks to the representation of $P_{\mathcal{T}}$ and the stability of the biorthogonal system in Lemma 4.60, we have

$$\langle P_{\mathcal{T}}\ell, w\rangle = \sum_{i\in I_z} \langle \ell, w_i\rangle \langle \ell_i, w\rangle \le C^\star_{\mathrm{bOS}} \|\ell\|_{H^{-1}(\omega_z)} \|\nabla w\|_{L^2(\omega_z)}$$

for any $w \in H^1_0(w_z)$, $z \in \mathcal{V}$, and the proof is finished. □

The following two remarks illustrate the practical and theoretical usefulness of the representation formulae.

Remark 4.62 (alternative computation of $P_{\mathcal{T}}$). A by-product of Corollary 4.61 is a way of computing $P_{\mathcal{T}}\ell$ for a given functional $\ell \in H^{-1}(\Omega)$ that 'diagonalizes' the approach in Remark 4.26. In fact, given reference orthonormal bases as in (4.76), we can compute the functionals ℓ_i, $i \in I$, and test functions w_i, $i \in I$, by means of the formulae (4.78), (4.80), (4.81) and (4.82), whence, evaluating $\langle \ell, w_i\rangle$, $i \in I$, everything in the representation of $P_{\mathcal{T}}\ell$ in Corollary 4.61 is at our disposal.

Example 4.63 (global instability of $P_{\mathcal{T}}$ and $P^*_{\mathcal{T}}$). While the projections $P_{\mathcal{T}}$ and $P^*_{\mathcal{T}}$ are locally stable, both may become globally unbounded under mesh refinement. To see this, recall (4.73), note that $\|P_{\mathcal{T}}\|_{\mathcal{L}(H^{-1}(\Omega))} = \|P^*_{\mathcal{T}}\|_{\mathcal{L}(H^1_0(\Omega))}$ and, following the spirit of an example in Tantardini *et al.* (2024), consider

$$w := \sum_{z\in\mathcal{V}\cap\Omega} \phi_z \in H^1_0(\Omega).$$

Then, for all *quasi-uniform* meshes $\mathcal{T}$ with shape regularity coefficient σ, there is a constant C depending on σ and quasi-uniformity such that

$$\|P^*_{\mathcal{T}}\|^2_{\mathcal{L}(H^1_0(\Omega))} \ge \frac{\|\nabla(P^*_{\mathcal{T}} w)\|^2_{L^2(\Omega)}}{\|\nabla w\|^2_{L^2(\Omega)}} \ge C \frac{\#\{T \in \mathcal{T} \mid T \cap \partial\Omega = \emptyset\}}{\#\{T \in \mathcal{T} \mid T \cap \partial\Omega \neq \emptyset\}}. \tag{4.84}$$

Obviously, the last term tends to ∞ under uniform refinement.

To prove (4.84), we proceed in several steps, mostly hiding constants depending on quasi-uniformity of $\mathcal{T}$ and, as usual, the shape regularity coefficient σ and d.

1 We first bound $\|\nabla w\|_{L^2(\Omega)}$ from above. Noting that

$$w = 1 \quad \text{on} \quad \bigcup_{T\cap\partial\Omega=\emptyset} T,$$

the bound $\|\nabla\phi_z\|_{L^\infty(T)} \lesssim h_T^{-1}$ readily implies

$$\|\nabla w\|^2_{L^2(\Omega)} = \sum_{T\cap\partial\Omega\neq\emptyset} \|\nabla w\|^2_{L^2(T)} \lesssim \#\{T\in\mathcal{T} \mid T\cap\partial\Omega\neq\emptyset\}\, h_{\mathcal{T}}^{d-2}, \tag{4.85}$$

where $h_{\mathcal{T}}$ stands for the mesh size of $\mathcal{T}$.

2 The lower bound for $\|\nabla(P^*_{\mathcal{T}} w)\|_{L^2(\Omega)}$ is more involved. We start by showing the following representation for any $T\in\mathcal{T}$ with $T\cap\partial\Omega=\emptyset$:

$$P^*_{\mathcal{T}} w|_T = H^*_T \widehat{v} \tag{4.86}$$

with the fixed function

$$\widehat{v} := \sum_{(\widehat{T},k)\in\widehat{I}} \left(\int_{\widehat{T}} q_{(\widehat{T},k)}\right)\widehat{w}_{(\widehat{T},k)} + \sum_{(F',k)} \left(\frac{|F'|}{|\widehat{F}|}\right)^{1/2}\left(\int_{\widehat{F}} q_{(\widehat{F},k)}\right)\widehat{w}_{(F',k)} \notin \mathbb{P}_0,$$

where the indices of the second sum vary according to $F'\subset\widehat{T}$, $k=1,\dots,K_1$ and $\widehat{w}_{(F',k)}$ is given by (4.82) with the transformation $H_{(\widehat{T},F')}$. Note first that, thanks to $w=1$ on T and (4.80), the coefficients in the expansion of $P^*_{\mathcal{T}} w|_T$ satisfy

$$\langle \ell_{(T,k)}, w\rangle = \int_T q_{(T,k)} = \left(\frac{|T|}{|\widehat{T}|}\right)^{1/2}\int_{\widehat{T}} q_{(\widehat{T},k)}$$

and, for any $F\subset T$,

$$\langle \ell_{(F,k)}, w\rangle = \int_F q_{(F,k)} = \left(\frac{|F|}{|\widehat{F}|}\right)^{1/2}\int_{\widehat{F}} q_{(\widehat{F},k)}.$$

Combining these identities with (4.82) yields the claimed identity (4.86), and it remains to verify $\widehat{v}\notin\mathbb{P}_0$. Suppose $\widehat{v}=c\in\mathbb{R}$. As a consequence, for any face $F'\subset\widehat{T}$ and $k\in\{1,\dots,K_1\}$, we have

$$c = \widehat{w}_{(F',k)} = |\widehat{F}|^{1/2}|F'|^{-1/2}\widehat{w}_{(\widehat{F},k)} = |\widehat{F}|^{1/2}|F'|^{-1/2}c.$$

As not all faces of the reference simplex have the same volume, this yields $c=0$. From (4.78) and (4.76), we infer that the coefficients in the definition of $\widehat{v}$ vanish. In particular, $\int_{\widehat{T}} q_{(\widehat{T},k)} = 0$ for all $k=1,\dots,K_2$ means $\widehat{Q}1=0$, where $\widehat{Q}$ is the operator given in (4.77). This, however, is a contradiction because the restriction of $\widehat{Q}$ to $\mathbb{P}_{m_2}$ is injective. Hence $\widehat{v}\notin\mathbb{P}_0$ is proved.

3 We are ready to show the bound for $\|\nabla(P^*_{\mathcal{T}} w)\|_{L^2(\Omega)}$. Given any element $T\in\mathcal{T}$ with $T\cap\partial\Omega=\emptyset$, we pass to the reference element to exploit the previous step, and obtain

$$\|\nabla(P^*_{\mathcal{T}} w)\|_{L^2(T)} = \|\nabla(H^*_T\widehat{v})\|_{L^2(T)} \gtrsim h_T^{d/2-1}\|\nabla\widehat{v}\|_{L^2(\widehat{T})}$$

with $\|\nabla \widehat{v}\|_{L^2(\widehat{T})} > 0$ independent of T. Consequently,

$$\|\nabla(P_{\mathcal{T}}^* w)\|_{L^2(\Omega)}^2 \geq \sum_{T \cap \partial\Omega = \emptyset} \|\nabla(P_{\mathcal{T}}^* w)\|_{L^2(T)}^2 \gtrsim \#\{T \in \mathcal{T} \mid T \cap \partial\Omega = \emptyset\}\, h_{\mathcal{T}}^{d-2},$$

because $\mathcal{T}$ is quasi-uniform. Combining this lower bound with the upper bound (4.85) of the first step, we conclude (4.84).

A hierarchical estimator

Like estimators based upon local problems, hierarchical estimators aim at softening the impact of constants in the lower bound, with the difference that they are explicit. While global higher-order extensions were used originally, Bornemann, Erdmann and Kornhuber (1996) use an extension tailored to the residual structure and derive an upper bound with indicators testing the residual with a basis of the extension. One may expect that such explicit indicators come at the price of increased constants in the upper bound. For the following example, this expectation is confirmed by the inequality $C_{\mathrm{lStb}}^\star \leq C_{\mathrm{bOS}}^\star$.

Given the Galerkin approximation $u_{\mathcal{T}}$ from (4.1), the hierarchical PDE indicator is defined by

$$\eta_{\mathcal{T}}^{\mathrm{hier}}(u_{\mathcal{T}}) := \sum_{z \in \mathcal{V}} \eta_{\mathcal{T}}^{\mathrm{hier}}(u_{\mathcal{T}}, z)^2 \quad \text{with} \quad \eta_{\mathcal{T}}^{\mathrm{hier}}(u_{\mathcal{T}}, z) := \max_{i \in I_z} \frac{|\langle R_{\mathcal{T}}, w_i \rangle|}{\|\nabla w_i\|_{L^2(\omega_z)}}, \tag{4.87a}$$

with I and I_z as in Lemma 4.60 (biorthogonal system). Note that the test functions w_i, $i \in I$, are available (see Remark 4.62), and therefore $\eta_{\mathcal{T}}^{\mathrm{hier}}(u_{\mathcal{T}})$ is explicit. The resulting estimator is then

$$\mathcal{E}_{\mathcal{T}}^{\mathrm{hier}} := \mathcal{E}_{\mathcal{T}}^{\mathrm{hier}}(u_{\mathcal{T}}, f)^2 := \eta_{\mathcal{T}}^{\mathrm{hier}}(u_{\mathcal{T}})^2 + \mathrm{osc}_{\mathcal{T}}(f)^2. \tag{4.87b}$$

Theorem 4.64 (hierarchical estimator). *Suppose the coefficients $\boldsymbol{A}$ and c are discrete. The hierarchical estimator (4.87) is equivalent to the error, while its PDE indicator is locally equivalent to the discretized residual with constant 1 in the lower bound, so that*

$$\frac{1}{C_{\mathrm{lStb}}^\star C_d} \mathcal{E}_{\mathcal{T}}^{\mathrm{hier}} \lesssim \|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)} \lesssim C_{\mathrm{bOS}}^\star C_d C_{\mathrm{loc}} \mathcal{E}_{\mathcal{T}}^{\mathrm{hier}},$$

and, for all vertices $z \in \mathcal{V}$,

$$\eta_{\mathcal{T}}^{\mathrm{hier}}(u_{\mathcal{T}}, z) \leq \|P_{\mathcal{T}} R_{\mathcal{T}}\|_{H^{-1}(\omega_z)} \leq C_{\mathrm{bOS}}^\star \eta_{\mathcal{T}}^{\mathrm{hier}}(u_{\mathcal{T}}, z),$$

where $C_{\mathrm{lStb}}^\star$ is the stability constant of $P_{\mathcal{T}}$ on stars from Lemma 4.5, C_{loc} from Corollary 4.6, $C_d = \sqrt{2(d+1)}$, and the hidden constants depend only on the error–residual relationship in Lemma 4.1.

Proof. It suffices to verify the local equivalence for the PDE indicator; see Theorem 4.58 (vertex-indexed modified residual estimator). Its lower bound simply

follows from (4.74): for all $i \in I_z$, we have

$$|\langle R_{\mathcal{T}}, w_i\rangle| = |\langle P_{\mathcal{T}} R_{\mathcal{T}}, w_i\rangle| \le \|P_{\mathcal{T}} R_{\mathcal{T}}\|_{H^{-1}(\omega_z)} \|\nabla w_i\|_{L^2(\omega_z)}.$$

To show its upper bound, let $w \in H^1_0(\omega_z)$ and, with the help of Corollary 4.61 (projections as interpolation operators) and Lemma 4.60 (biorthogonal system), we derive

$$\begin{aligned}\langle P_{\mathcal{T}} R_{\mathcal{T}}, w\rangle &= \sum_{i\in I_z} \langle R_{\mathcal{T}}, w_i\rangle \langle \ell_i, w\rangle \\ &\le \sum_{i\in I_z} \frac{\langle R_{\mathcal{T}}, w_i\rangle}{\|\nabla w_i\|_{L^2(\omega_z)}} \|\nabla w_i\|_{L^2(\omega_z)} \|\ell_i\|_{H^{-1}(\omega_z)} \|\nabla w\|_{L^2(\omega_z)} \\ &\le C^{\star}_{\mathrm{bOS}}\, \eta^{\mathrm{hier}}_{\mathcal{T}}(u_{\mathcal{T}}, z)\, \|\nabla w\|_{L^2(\omega_z)}\end{aligned}$$

and the local equivalence is established. □

Remark 4.65 (different test functions). The hierarchical estimator (4.87) does not generalize the one in Bornemann *et al.* (1996) as it uses slightly different test functions for edges. The given framework, however, applies to their variant too; see Kreuzer and Veeser (2021, Section 4.1).

Alternative localization and residual splitting

Lemma 4.5 (localization of H^{-1}-norm) is not well suited to reducing or avoiding constants in the upper bounds. The following modification, however, allows this.

To this end, we replace the local spaces $H^1_0(\omega_z)$, $z \in \mathcal{V}$, with

$$\mathbb{W}_z := \begin{cases} \{w \in H^1(\omega_z) \mid \int_{\omega_z} w = 0\}, & \text{if } z \in \mathcal{V} \cap \Omega, \\ \{w \in H^1(\omega_z) \mid w = 0 \text{ on } \partial\omega_z \cap \partial\Omega\}, & \text{if } z \in \mathcal{V} \cap \partial\Omega, \end{cases}$$

endow them with the norm $\|\nabla \cdot\|_{L^2(\omega_z)}$, and let $\mathbb{W}^*_z$ denote the respective dual spaces endowed in turn with

$$\|\ell\|_{\mathbb{W}^*_z} := \sup\big\{\langle \ell, w\rangle \mid w \in \mathbb{W}_z,\ \|\nabla w\|_{L^2(\omega_z)} \le 1\big\}. \tag{4.88}$$

Lemma 4.66 (alternative localization of H^{-1}-norm). *Let $\ell \in H^{-1}(\Omega)$ be any linear functional.*

(i) *If $\langle \ell, \phi_z\rangle = 0$ for all interior vertices $z \in \mathcal{V} \cap \Omega$, then*

$$\|\ell\|^2_{H^{-1}(\Omega)} \le (d+1) \sum_{z\in\mathcal{V}} \|\phi_z \ell\|^2_{\mathbb{W}^*_z}.$$

(ii) *We have*

$$\sum_{z\in\mathcal{V}} \|\phi_z \ell\|^2_{\mathbb{W}^*_z} \le (d+1) C^2_{\mathrm{loc}} \|\ell\|^2_{H^{-1}(\Omega)},$$

where C_{loc} is the constant in Lemma 4.5(i).

Proof. The proof is essentially a regrouping of the arguments in Lemma 4.5, where the constant C_{loc} in the stability bound (4.6) now arises in the proof of the lower bound from the following argument: we have

$$\|\phi_z \ell\|_{\mathbb{W}_z^*} \leq C_{\mathrm{loc}} \|\ell\|_{H^{-1}(\omega_z)}, \tag{4.89}$$

thanks to

$$\begin{aligned}\langle \phi_z \ell, w\rangle = \langle \ell, \phi_z w\rangle &\leq \|\ell\|_{H^{-1}(\omega_z)} \|\nabla(w\phi_z)\|_{L^2(\omega_z)} \\ &\leq C_{\mathrm{loc}} \|\ell\|_{H^{-1}(\omega_z)} \|\nabla w\|_{L^2(\omega_z)}\end{aligned}$$

for all $w \in \mathbb{W}_z$. □

The question arises whether the inequality (4.89) between the two local dual norms can be reversed. The following lemma reveals that this is only partially possible, covering discrete functionals as arguments.

Lemma 4.67 (partial equivalence for local dual norms). *If $z \in \mathcal{V} \cap \Omega$ is an interior vertex, the functional $\ell = \phi_z^{-1}$ satisfies*

$$\|\phi_z \ell\|_{\mathbb{W}_z^*} = 0 \quad \textit{and} \quad \|\ell\|_{H^{-1}(\omega_z)} > 0.$$

Furthermore, for any vertex $z \in \mathcal{V}$,

$$\|\ell\|_{H^{-1}(\omega_z)} \leq C_{\mathbb{F}} \|\phi_z \ell\|_{\mathbb{W}_z^*} \quad \textit{for all } \ell \in \mathbb{F}(\mathcal{T}_z),$$

where the constant $C_{\mathbb{F}}$ depends only on d, the shape regularity coefficient σ, and the degrees m_1 and m_2 of the discrete functionals.

Proof. $\boxed{1}$ We show the claims on the functional $\ell = \phi_z^{-1}$ for an interior vertex $z \in \mathcal{V} \cap \Omega$. By the definition of $\mathbb{W}_z$, we have, for all $w \in \mathbb{W}_z$,

$$\langle \phi_z \ell, w\rangle = \int_{\omega_z} w = 0,$$

whence $\phi_z \ell \in \mathbb{W}_z^*$ with $\|\phi_z \ell\|_{\mathbb{W}_z^*} = 0$.

To verify that $\ell = \phi_z^{-1} \in H^{-1}(\omega_z)$, we write $d_z := \mathrm{dist}(\cdot, \partial\omega_z)$ for the distance function of the star boundary and shall use the weighted Poincaré inequality

$$\|w d_z^{-1}\|_{L^2(\omega_z)} \lesssim \|\nabla w\|_{L^2(\omega_z)} \quad \text{for all } w \in H_0^1(\omega_z),$$

which follows from the Hardy inequality; see Sacchi and Veeser (2006, Lemma 3.6). Consequently, exploiting $d_z \leq \phi_z$ on ω_z as well, we obtain, for all $w \in H_0^1(\Omega)$,

$$\langle \ell, w\rangle = \int_{\omega_z} (\phi_z^{-1} d_z)(w d_z^{-1}) \leq |\omega_z|^{1/2} \|w d_z^{-1}\|_{L^2(\omega_z)} \lesssim |\omega_z|^{1/2} \|\nabla w\|_{L^2(\omega_z)}.$$

This and $\langle \ell, \phi_z\rangle = |\omega_z|$ ensure $\ell \in H^{-1}(\omega_z)$ with $\|\ell\|_{H^{-1}(\omega_z)} > 0$.

$\boxed{2}$ We start the proof of the asserted inequality by checking that $\|\cdot\|_{H^{-1}(\omega_z)}$ is a norm on $\mathbb{F}(\mathcal{T}_z)$. To this end, consider $q_F \in \mathbb{P}_{m_1}(F)$, $F \in \mathcal{F}_z$ and $q_T \in \mathbb{P}_{m_2}(T)$,

$T \in \mathcal{T}_z$ such that, for all $w \in H^1_0(\omega_z)$,

$$0 = \langle \ell, w \rangle := \sum_{F \in \mathcal{F}_z} \int_F q_F w + \sum_{T \in \mathcal{T}_z} \int_T q_T w.$$

We need to show $\ell = 0$. Testing with $w \in H^1_0(T)$, $T \in \mathcal{T}_z$, the fundamental lemma of the calculus of variations yields $q_T = 0$ for all $T \in \mathcal{T}_z$. Similarly, now testing with $w \in H^1_0(\omega_F)$, $F \in \mathcal{F}_z$, gives $q_F = 0$ for all $F \in \mathcal{T}_z$. Thus $\ell = 0$ holds.

3 Next, we check that $\|\phi_z \cdot\|_{\mathbb{W}_z^*}$ is also a norm on $\mathbb{F}(\mathcal{T}_z)$. This time, consider $q_F \in \mathbb{P}_{m_1}(F)$, $F \in \mathcal{F}_z$ and $q_T \in \mathbb{P}_{m_2}(T)$, $T \in \mathcal{T}_z$ such that, for all $w \in \mathbb{W}_z$,

$$0 = \langle \phi_z \ell, w \rangle := \sum_{F \in \mathcal{F}_z} \int_F \phi_z q_F w + \sum_{T \in \mathcal{T}_z} \int_T \phi_z q_T w,$$

and again, we need to conclude $\ell = 0$. If $z \in \mathcal{V} \cap \partial\Omega$ is a boundary node, we obtain $\ell = 0$ by the arguments of the previous step. We are thus left with the case $z \in \mathcal{V} \cap \Omega$ of interior nodes. Given $w \in H^1(\omega_z)$, we set $c_w := \fint_{\omega_z} w$ and $c_\ell = |\omega_z|^{-1} \langle \phi_z \ell, 1 \rangle$, and observe

$$0 = \langle \phi_z \ell, w - c_w \rangle = \langle \phi_z \ell - c_\ell, w - c_w \rangle = \langle \phi_z \ell - c_\ell, w \rangle.$$

Hence, testing with $w \in H^1_0(T)$, $T \in \mathcal{T}_z$, we deduce $\phi_z q_T = c_\ell$ on each $T \in \mathcal{T}_z$. However, this is only possible if $c_\ell = 0$ and $q_T = 0$ for all $T \in \mathcal{T}_z$. Therefore, testing with $w \in H^1_0(\omega_F)$, $F \in \mathcal{F}_z$, yields $q_F = 0$ for all $F \in \mathcal{F}_z$, and $\ell = 0$ is established in general.

4 To conclude the asserted inequality, note that $\mathbb{F}(\mathcal{T}_z)$ has finite dimension and, for fixed polynomial degrees m_1 and m_2, is invariant under continuous piecewise affine transformations. Furthermore, both norms scale in the same manner. We can therefore pass to reference stars and use the equivalence of norms in finite-dimensional spaces. Transforming the inequality back from the reference star then finishes the proof. □

The alternative localization entails that we need to adapt Lemma 4.35 (splitting of local residual norm). Relying on the local H^{-1}-stability of $P_{\mathcal{T}}$, Lemma 4.67 (partial equivalence for local dual norms) reveals that the adaptation has to be global.

Lemma 4.68 (alternative splitting). *Using the local norms $\|\cdot\|_{\mathbb{W}_z^*}$, $z \in \mathcal{V}$, the residual can be split into discretized and oscillatory residuals:*

$$\frac{1}{(C^\star_{\mathrm{lStb}})^2 C_d^2 C_{\mathrm{loc}}^2} \sum_{z \in \mathcal{V}} \big(\|\phi_z P_{\mathcal{T}} R_{\mathcal{T}}\|^2_{\mathbb{W}_z^*} + \|\phi_z (I - P_{\mathcal{T}}) R_{\mathcal{T}}\|^2_{\mathbb{W}_z^*} \big)$$
$$\leq \|R_{\mathcal{T}}\|^2_{H^{-1}(\Omega)} \leq C_d^2 \sum_{z \in \mathcal{V}} \big(\|\phi_z P_{\mathcal{T}} R_{\mathcal{T}}\|^2_{\mathbb{W}_z^*} + \|\phi_z (I - P_{\mathcal{T}}) R_{\mathcal{T}}\|^2_{\mathbb{W}_z^*} \big),$$

where $C^\star_{\mathrm{lStb}}$ *is the stability constant of* $P_\mathcal{T}$ *on stars from Lemma 4.28, and* $C_d = \sqrt{2(d+1)}$.

Proof. Combine the localization in Lemma 4.66 with the proof of Lemma 4.35, replacing the local norm $\|\cdot\|_{H^{-1}(\omega_z)}$ in most places, but apply (4.89) before using the local H^{-1}-stability of $P_\mathcal{T}$. □

An estimator based on flux equilibration

Estimators based on flux equilibration have been designed with the goal to obtain constant 1 in the upper bound. The principal obstruction that computation can access only a finite-dimensional part of infinite-dimensional objects such as the residual norm is overcome by means of the *Prager–Synge theorem*. Realizations of this approach can be found, for example, in Ainsworth (2010), Braess, Pillwein and Schöberl (2009), Ern, Smears and Vohralík (2017) and Luce and Wohlmuth (2004).

The definition of the PDE indicator needs some preparation. Let $d \in \{2,3\}$, as in the aforementioned works, and let $z \in \mathcal{V}$ be a vertex. Given the operator $\pi_z\colon \{\phi_z\ell \mid \ell \in H^{-1}(\Omega)\} \to \mathbb{W}_z^*$ defined by

$$\pi_z(\phi_z\ell) := \begin{cases} \phi_z\ell - \dfrac{\langle \ell, \phi_z\rangle}{|\omega_z|}, & \text{if } z \in \mathcal{V}\cap\Omega,\\ \phi_z\ell, & \text{if } z \in \mathcal{V}\cap\partial\Omega,\end{cases}$$

and

$$\gamma_z := \begin{cases} \partial\omega_z, & \text{if } z \in \mathcal{V}\cap\Omega,\\ \partial\omega_z \setminus \partial\Omega, & \text{if } z \in \mathcal{V}\cap\partial\Omega,\end{cases}$$

we introduce the local space $\mathbb{D}_z \neq \emptyset$,

$$\mathbb{D}_z := \{\boldsymbol{\xi} \in L^2(\omega_z;\mathbb{R}^d) \mid \operatorname{div}\boldsymbol{\xi} = \pi_z(\phi_z P_\mathcal{T} R_\mathcal{T}) \text{ and } \boldsymbol{\xi}\cdot\boldsymbol{n}_F = 0 \text{ on } F \text{ for all } F \subseteq \gamma_z\},$$

and its discretization

$$\mathbb{D}_z(\mathcal{T}) := \{\boldsymbol{\xi} \in \mathbb{D}_z \mid \boldsymbol{\xi} \in RTN_m(T) \text{ for all } T \in \mathcal{T}_z\},$$

with the Raviart–Thomas–Nédélec elements

$$RTN_m(T) = \{\boldsymbol{\xi}\colon T \to \mathbb{R}^d \mid \boldsymbol{\xi}(x) = \boldsymbol{q}(x) + q(x)x \text{ with } \boldsymbol{q} \in (\mathbb{P}_m)^d, q \in \mathbb{P}_m\}$$

of order $m := \max\{m_1, m_2\} + 1$. Given the Galerkin approximation $u_\mathcal{T}$ from (4.1), the PDE indicator is then given by

$$\eta^{\mathrm{feq}}_\mathcal{T}(u_\mathcal{T})^2 := \sum_{z\in\mathcal{V}} \eta^{\mathrm{feq}}_\mathcal{T}(u_\mathcal{T})^2 \quad \text{with} \quad \eta^{\mathrm{feq}}_\mathcal{T}(u_\mathcal{T}) := \min_{\boldsymbol{\xi}\in\mathbb{D}_z(\mathcal{T})} \|\boldsymbol{\xi}\|_{L^2(\omega_z)} \tag{4.90a}$$

and the total estimator by

$$\mathcal{E}^{\mathrm{feq}}_\mathcal{T} := \mathcal{E}^{\mathrm{feq}}_\mathcal{T}(u_\mathcal{T}, f)^2 := \eta^{\mathrm{feq}}_\mathcal{T}(u_\mathcal{T})^2 + \|\phi_z(f - P_\mathcal{T} f)\|^2_{\mathbb{W}_z^*}. \tag{4.90b}$$

Note that the local PDE indicators $\eta_{\mathcal{T}}^{\mathrm{feq}}(u_{\mathcal{T}}, z)$ are computable up to machine precision.

Theorem 4.69 (estimator based on flux equilibration). *Suppose that the coefficients* $\boldsymbol{A}$ *and* c *are discrete and that* $d \in \{2, 3\}$. *The estimator* (4.90) *based on flux equilibration is equivalent to the error, while its PDE indicator is locally equivalent to the discretized residual with constant* 1 *in the upper bound for the* $\|\cdot\|_{\mathbb{W}_z^*}$*-norm, so that*

$$\frac{C_{\mathbb{D}}}{C_{\mathrm{lStb}}^{\star} C_d C_{\mathrm{loc}}} \mathcal{E}_{\mathcal{T}}^{\mathrm{feq}} \lesssim \|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)} \lesssim C_d \mathcal{E}_{\mathcal{T}}^{\mathrm{feq}},$$

and, for all vertices $z \in \mathcal{V}$,

$$C_{\mathbb{D}} \eta_{\mathcal{T}}^{\mathrm{feq}}(u_{\mathcal{T}}, z) \leq \|\phi_z P_{\mathcal{T}} R_{\mathcal{T}}\|_{\mathbb{W}_z^*} \leq \eta_{\mathcal{T}}^{\mathrm{feq}}(u_{\mathcal{T}}, z)$$

as well as

$$\frac{C_{\mathbb{D}}}{C_{\mathrm{loc}}} \eta_{\mathcal{T}}^{\mathrm{feq}}(u_{\mathcal{T}}, z) \leq \|P_{\mathcal{T}} R_{\mathcal{T}}\|_{H^{-1}(\omega_z)} \leq C_{\mathbb{F}}\, \eta_{\mathcal{T}}^{\mathrm{feq}}(u_{\mathcal{T}}, z),$$

where $C_{\mathbb{D}}$ *depends on* d *and the shape regularity coefficient* σ, $C_{\mathrm{lStb}}^{\star}$ *is the stability constant of* $P_{\mathcal{T}}$ *on stars from Lemma 4.28,* C_{loc} *comes from Lemma 4.5,* $C_d = \sqrt{2(d+1)}$, $C_{\mathbb{F}}$ *comes from Lemma 4.67, and the hidden constants depend only on the error–residual relationship in Lemma 4.1.*

Proof. $\boxed{1}$ We start by verifying the local equivalence for the $\|\cdot\|_{\mathbb{W}_z^*}$-norm. Let $z \in \mathcal{V}$ be any vertex. The Prager–Synge theorem on the star ω_z implies

$$\|\phi_z P_{\mathcal{T}} R_{\mathcal{T}}\|_{\mathbb{W}_z^*} = \|\pi_z(\phi_z P_{\mathcal{T}} R_{\mathcal{T}})\|_{\mathbb{W}_z^*} = \min_{\boldsymbol{\xi} \in \mathbb{D}_z} \|\boldsymbol{\xi}\|_{L^2(\omega_z)};$$

see e.g. Verfürth (2013, Proposition 1.40). Hence the upper bound with constant 1 readily follows the inclusion $\mathbb{D}_z(\mathcal{T}) \subset \mathbb{D}_z$, while the lower bound is a consequence of the non-trivial inequality

$$C_{\mathbb{D}} \min_{\boldsymbol{\xi} \in \mathbb{D}_z(\mathcal{T})} \|\boldsymbol{\xi}\|_{L^2(\omega_z)} \leq \min_{\boldsymbol{\xi} \in \mathbb{D}_z} \|\boldsymbol{\xi}\|_{L^2(\omega_z)},$$

where $C_{\mathbb{D}}$ depends only on d and the shape regularity coefficient σ; see e.g. Braess *et al.* (2009, Theorem 7) and Ern *et al.* (2017, Theorem 1.1).

$\boxed{2}$ We verify the local equivalence for the $\|\cdot\|_{H^{-1}(\omega_z)}$-norm. On the one hand, combining the first equivalence with (4.89), we obtain

$$C_{\mathbb{D}} \eta_{\mathcal{T}}^{\mathrm{feq}}(u_{\mathcal{T}}, z) \leq \|\phi_z P_{\mathcal{T}} R_{\mathcal{T}}\|_{\mathbb{W}_z^*} \leq C_{\mathrm{loc}} \|P_{\mathcal{T}} R_{\mathcal{T}}\|_{H^{-1}(\omega_z)}.$$

On the other hand, using Lemma 4.67 instead of (4.89) yields

$$\|P_{\mathcal{T}} R_{\mathcal{T}}\|_{H^{-1}(\omega_z)} \leq C_{\mathbb{F}} \|\phi_z P_{\mathcal{T}} R_{\mathcal{T}}\|_{\mathbb{W}_z^*} \leq C_{\mathbb{F}} \eta_{\mathcal{T}}^{\mathrm{feq}}(u_{\mathcal{T}}, z),$$

and the equivalence for the $\|\cdot\|_{H^{-1}(\omega_z)}$ is verified, too.

[3] The global bounds follow by combining Lemmas 4.1 (error and residual), 4.68 (alternative splitting) and 4.37 (data oscillation reduction for discrete coefficients), as well as the first local equivalence. □

Remark 4.70 (improved upper bound). Applying the Prager–Synge theorem on Ω, we can improve the upper bound in Theorem 4.69 to

$$\|\nabla(u-u_{\mathcal{T}})\|_{L^2(\Omega)} \lesssim \|\boldsymbol{\xi}_\Omega\|_{L^2(\Omega)} + \sqrt{d+1}\Big(\sum_{z\in\mathcal{V}} \|\pi_z(\phi_z(P_{\mathcal{T}}f-f))\|^2_{\mathbb{W}^*_z}\Big)^{1/2}, \quad (4.91)$$

with $\boldsymbol{\xi}_\Omega := \sum_{z\in\mathcal{V}} \boldsymbol{\xi}_z$, where $\boldsymbol{\xi}_z := \arg\min_{\xi\in\mathbb{D}_z(\mathcal{T})} \|\boldsymbol{\xi}\|_{L^2(\omega_z)}$ are the minimizing vector fields associated with the PDE indicators, extended by 0 off ω_z.

To see this, we derive, thanks to the partial orthogonality (4.4) of the residual and Lemma 4.37 (data oscillation reduction for discrete coefficients),

$$\begin{aligned}\operatorname{div}\boldsymbol{\xi}_\Omega &= \sum_{z\in\mathcal{V}} \pi_z(\phi_z P_{\mathcal{T}} R_{\mathcal{T}}) = \sum_{z\in\mathcal{V}} \pi_z(\phi_z R_{\mathcal{T}}) + \sum_{z\in\mathcal{V}} \pi_z(\phi_z(P_{\mathcal{T}}R_{\mathcal{T}} - R_{\mathcal{T}}))\\ &= \sum_{z\in\mathcal{V}} \phi_z R_{\mathcal{T}} + \sum_{z\in\mathcal{V}} \pi_z(\phi_z(P_{\mathcal{T}}f - f)) = R_{\mathcal{T}} + \delta_{\mathcal{T}},\end{aligned}$$

with $\delta_{\mathcal{T}} := \sum_{z\in\mathcal{V}} \pi_z(\phi_z(P_{\mathcal{T}}f - f))$. Hence

$$\|\nabla(u-u_{\mathcal{T}})\|_{L^2(\Omega)} \lesssim \|R_{\mathcal{T}}\|_{H^{-1}(\Omega)} \le \|R_{\mathcal{T}} + \delta_{\mathcal{T}}\|_{H^{-1}(\Omega)} + \|\delta_{\mathcal{T}}\|_{H^{-1}(\Omega)},$$

inserting $\boldsymbol{\xi}_\Omega$ in the Prager–Synge theorem on Ω and Lemma 4.66 (alternative localization of H^{-1}-norm) establish the claimed bound.

In view of the bound (4.91), the alternative local PDE indicators $\|\phi_z^{1/2}\boldsymbol{\xi}_\Omega\|_{L^2(\omega_z)}$, $z \in \mathcal{V}$, may be used in an adaptive context. Note, however, that this alternative does not necessarily strengthen the link with the local residual, as the definition of $\boldsymbol{\xi}_\Omega$ suggests an increased overlapping in the lower bound.

4.10. Other boundary conditions

This section illustrates that the preceding analysis of homogeneous Dirichlet conditions can be adapted to other boundary conditions. In particular, we discuss

- *Robin and Neumann boundary conditions*, as an example for variationally formulated boundary conditions,
- the *pure Neumann problem*, with its global solvability constraint,
- *non-homogeneous Dirichlet boundary conditions*, formulated in an essential manner.

Mixed boundary conditions, suitably discretized, give rise to *a posteriori* error estimators combining in a straightforward manner the indicators of, for instance, the first and third of the above groups. We therefore omit further details of such a setting.

Robin and Neumann boundary conditions

The *Robin* bilinear form in (2.13) is coercive and continuous in $\mathbb{V} := H^1(\Omega)$ provided its coefficient $p \geq p_0$ on an open subset of $\partial\Omega$ for some constant $p_0 > 0$, according to the norm equivalence (2.31). Consequently, (2.12) admits a unique solution $u \in \mathbb{V}$. If $\mathbb{V}_{\mathcal{T}} = \mathbb{S}^{n,0}_{\mathcal{T}}$ is the subspace of $\mathbb{V}$ of continuous piecewise polynomial functions of degree $\leq n$, then the Galerkin counterpart of (4.1) reads

$$u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}: \quad \mathcal{B}[u_{\mathcal{T}}, v] = \ell(v) \quad \text{for all } v \in \mathbb{V}_{\mathcal{T}},$$

with $\ell = f + g\delta_{\partial\Omega} \in \mathbb{V}^*$; see (2.13). Its residual $R_{\mathcal{T}} \in \mathbb{V}^*$ is defined as

$$\langle R_{\mathcal{T}}, w\rangle := \ell(w) - \mathcal{B}[u_{\mathcal{T}}, w], \quad w \in \mathbb{V},$$

and $\|R_{\mathcal{T}}\|_{\mathbb{V}^*}$ is equivalent to the error $\|u - u_{\mathcal{T}}\|_{H^1(\Omega)}$ due to Lemma 4.1 (error and residual), whose proof easily extends to $\mathbb{V}$.

The global norm $\|R_{\mathcal{T}}\|_{\mathbb{V}^*}$ also localizes to all stars ω_z because Galerkin orthogonality $\langle R_{\mathcal{T}}, \phi_z\rangle = 0$ is now valid also for boundary vertices $z \in \mathcal{V} \cap \partial\Omega$. Indeed, the proof of Lemma 4.5 (localization of H^{-1}-norm) extends with minor modifications, where the local spaces for boundary vertices $z \in \mathcal{V} \cap \partial\Omega$ are now $\{v \in H^1(\omega_z) \mid v = 0 \text{ on } \partial\omega_z \setminus \partial\Omega\}$. Also, the proof of Lemma 4.66 (alternative localization of H^{-1}-norm) is easily modified, using the local space $\left\{v \in H^1(\omega_z) \mid \int_{\omega_z} v = 0\right\}$ at the boundary, too.

The next key step is the construction of a projection $P_{\mathcal{T}}\colon \mathbb{V}^* \to \mathbb{F}_{\mathcal{T}}$ that mimics the projection operator $P_{\mathcal{T}}$ of Section 4.4. For that purpose, the space of discrete functionals $\mathbb{F}_{\mathcal{T}}$ has to include boundary face Dirac masses $q_F\delta_F$ with densities $q_F \in \mathbb{P}_{m_1}(F)$ for $F \subset \partial\Omega$. Consequently, g can be approximated on $\partial\Omega$ similarly to the forcing f in Ω, while the coefficient p is at play like the coefficient c. Indeed, considering for simplicity only the case of discrete coefficients $(\boldsymbol{A}, c, p)$, the condition $m_1 \geq n_p + n$ arises in addition to those in Remark 4.14. With these caveats, the tools developed in Sections 4.3, 4.4 and 4.5 give rise to a suitably adapted projection $P_{\mathcal{T}}$ to split the residual $R_{\mathcal{T}}$ into a discretized residual $P_{\mathcal{T}}R_{\mathcal{T}}$ and an oscillatory residual $(f - P_{\mathcal{T}}f) + (g\delta_{\partial\Omega} - P_{\mathcal{T}}(g\delta_{\partial\Omega}))$. Here $g\delta_{\partial\Omega} - P_{\mathcal{T}}(g\delta_{\partial\Omega})$ is supported only on $\partial\Omega$ and therefore contributes only to the oscillation indicators based upon the aforementioned new local spaces for boundary stars. This modified oscillation $\operatorname{osc}^{\mathrm{Rob}}_{\mathcal{T}}(\mathcal{D})$ with $\mathcal{D} = (\boldsymbol{A}, c, p, f, g)$ can be combined with any of the presented PDE indicators, but we focus on residual estimation. In fact, the new discrete residual $P_{\mathcal{T}}R_{\mathcal{T}}$ leads to a definition of the PDE estimator $\eta^{\mathrm{Rob}}_{\mathcal{T}}(u_{\mathcal{T}})$ as in (4.52), but with additional contributions related to the boundary faces. Given any boundary face F of $\mathcal{T}$, such a contribution reads

$$h_F \|[\![\boldsymbol{A}\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!] \cdot \boldsymbol{n}_F + p\, u_{\mathcal{T}} - P_F g\|^2_{L^2(F)}$$

and measures the discretized Robin residual. Combining as usual the PDE estimator $\eta^{\mathrm{Rob}}_{\mathcal{T}}(u_{\mathcal{T}})$ and oscillation $\operatorname{osc}^{\mathrm{Rob}}_{\mathcal{T}}(\ell)$ yields the total estimator $\mathcal{E}^{\mathrm{Rob}}_{\mathcal{T}}(u_{\mathcal{T}}, \ell)$, whence

the following variant of Theorem 4.45 (modified residual estimator) follows: for discrete coefficients $(\boldsymbol{A}, c, p)$, the H^1-error and $\mathcal{E}_{\mathcal{T}}^{\mathrm{Rob}}(u_{\mathcal{T}}, \ell)$ are equivalent, that is,

$$C_L \mathcal{E}_{\mathcal{T}}^{\mathrm{Rob}}(u_{\mathcal{T}}, \ell) \le \|u - u_{\mathcal{T}}\|_{H^1(\Omega)} \le C_U \mathcal{E}_{\mathcal{T}}^{\mathrm{Rob}}(u_{\mathcal{T}}, \ell).$$

The estimates in Section 4.8 for corrections and estimator reduction extend as well.

Pure Neumann problem

Neumann conditions are already covered by the previous section, except for the case of the pure Neumann problem with $p = 0$ on $\partial\Omega$ in (2.12) requiring, as key novelty, the solvability constraint $\ell(1_\Omega) = 0$, that is, the right-hand side applied to the constant function equal to 1 gives 0. For such problems, unique exact and discrete solutions exist provided we choose $\mathbb{V}$ to be the subspace of $H^1(\Omega)$ of functions with zero mean value, and $\mathbb{V}_{\mathcal{T}}$ to be its natural finite element counterpart of degree n.

The residual $R_{\mathcal{T}}$ is defined on all $H^1(\Omega)$ and satisfies $\langle R_{\mathcal{T}}, 1_\Omega\rangle = 0$. Combining this fact with Lemma 2.3 (second Poincaré inequality) and $\inf_{c\in\mathbb{R}} \|v - c\|_{L^2(\Omega)} = \|w\|_{L^2(\Omega)}$ with $w = v - \fint_\Omega v \in \mathbb{V}$, we derive

$$\begin{aligned}\|R_{\mathcal{T}}\|_{\mathbb{V}^*} &= \sup_{w\in\mathbb{V}} \frac{\langle R_{\mathcal{T}}, w\rangle}{\|\nabla w\|_{L^2(\Omega)}} \approx \sup_{w\in\mathbb{V}} \frac{\langle R_{\mathcal{T}}, w\rangle}{\|w\|_{H^1(\Omega)}} \\ &= \sup_{v\in H^1(\Omega)} \frac{\langle R_{\mathcal{T}}, v\rangle}{\|v\|_{H^1(\Omega)}} = \|R_{\mathcal{T}}\|_{H^1(\Omega)^*}.\end{aligned}$$

Consequently, localizing $\|R_{\mathcal{T}}\|_{H^1(\Omega)^*}$ as in the previous section, we can derive *a posteriori* error estimators with suitable contributions from the boundary $\partial\Omega$.

However, the projection $P_{\mathcal{T}}$ from the previous section cannot be used to generate discrete data in some auxiliary problem because $\langle \ell, 1_\Omega\rangle = 0$ does not imply $\langle P_{\mathcal{T}}\ell, 1_\Omega\rangle = 0$ in general. Further, a simple modification like $P_{\mathcal{T}}\ell - \langle P_{\mathcal{T}}\ell, 1_\Omega\rangle\langle 1_\Omega, 1_\Omega\rangle^{-1} 1_\Omega$ with a global correction destroys the crucial local approximation properties.

To address this issue, we modify the projection $P_{\mathcal{T}}$ such that the new projection $\widetilde{P}_{\mathcal{T}}$ enforces locally $\langle \widetilde{P}_{\mathcal{T}}\ell, 1_\Omega\rangle = \langle \ell, 1_\Omega\rangle$ in the spirit of the construction of the Lagrange multiplier in Fierro and Veeser (2003). To this end, recall that $P_{\mathcal{T}}$ is now defined on $H^1(\Omega)$, and its range, the discrete functionals $\mathbb{F}(\mathcal{T})$, also includes boundary face Dirac masses, and that the first localization involves the local spaces $\mathbb{V}_z := \{v \in H^1(\omega_z) \mid v = 0 \text{ on } \partial\omega_z \setminus \partial\Omega\}$, $z \in \mathcal{V}$. Given $\ell \in H^1(\Omega)^*$, set

$$\widetilde{P}_{\mathcal{T}}\ell := \sum_{z\in\mathcal{V}} \phi_z \widetilde{P}_z \ell \quad \text{with} \quad \widetilde{P}_z\ell := P_{\mathcal{T}}\ell - \frac{\langle P_{\mathcal{T}}\ell - \ell, \phi_z\rangle}{\int_{\omega_z} \phi_z} 1_{\omega_z}. \tag{4.92}$$

Lemma 4.71 (new projection). *The operator* (4.92) *is linear, local, and satisfies*

$$\langle \widetilde{P}_z\ell, \phi_z\rangle = \langle \ell, \phi_z\rangle \quad \textit{for all } z \in \mathcal{V} \quad \textit{and} \quad \langle \widetilde{P}_{\mathcal{T}}\ell, 1\rangle = \langle \ell, 1\rangle.$$

Furthermore, $\widetilde{P}_z$ provides near-best approximation in $\mathbb{F}(\mathcal{T})|_{\mathbb{V}_z}$ and

$$\|\ell - \widetilde{P}_{\mathcal{T}}\ell\|^2_{H^1(\Omega)^*} \le C_{\mathrm{loc}} \sum_{z\in\mathcal{V}} \|\ell - \widetilde{P}_z\ell\|^2_{\mathbb{V}_z^*}.$$

Proof. $\boxed{1}$ We start with the algebraic properties. By the definition of $\widetilde{P}_z$, we have the local relationships $\langle \ell - \widetilde{P}_z\ell, \phi_z\rangle = 0$, i.e. $\langle \phi_z(\ell - \widetilde{P}_z\ell), 1\rangle = 0$ for all vertices $z \in \mathcal{V}$. Summing over all vertices immediately yields the global $\langle \ell - \widetilde{P}_{\mathcal{T}}\ell, 1\rangle = 0$.

$\boxed{2}$ To show that $\widetilde{P}_z$ is near-best approximating in $\mathbb{F}(\mathcal{T})|_{\mathbb{V}_z}$, we bound its error in terms of that of $P_{\mathcal{T}}$. The triangle inequality readily gives

$$\|\ell - \widetilde{P}_z\ell\|_{\mathbb{V}_z^*} \le \|\ell - P_{\mathcal{T}}\ell\|_{\mathbb{V}_z^*} + \left\| \frac{\langle P_{\mathcal{T}}\ell - \ell, \phi_z\rangle}{\int_\omega \phi_z} 1_{\omega_z} \right\|_{\mathbb{V}_z^*},$$

while a variant of Lemma 2.2 (first Poincaré inequality) and the properties of ϕ_z deliver

$$\begin{aligned}\left\| \frac{\langle P_{\mathcal{T}}\ell - \ell, \phi_z\rangle}{\int_\omega \phi_z} 1_{\omega_z} \right\|_{\mathbb{V}_z^*} &\lesssim |\omega_z|^{-1} h_z \|\langle P_{\mathcal{T}}\ell - \ell, \phi_z\rangle 1_{\omega_z}\|_{L^2(\omega_z)} \\ &\lesssim |\omega_z|^{-1/2} h_z \|\ell - P_{\mathcal{T}}\ell\|_{\mathbb{V}_z^*} \|\nabla\phi_z\|_{L^2(\omega_z)} \\ &\lesssim \|\ell - P_{\mathcal{T}}\ell\|_{\mathbb{V}_z^*}.\end{aligned}$$

Hence the error of $\widetilde{P}_z$ is dominated by that of $P_{\mathcal{T}}$,

$$\|\ell - \widetilde{P}_z\ell\|_{\mathbb{V}_z^*} \lesssim \|\ell - P_{\mathcal{T}}\ell\|_{\mathbb{V}_z^*}, \tag{4.93}$$

and the near-best approximation of $\widetilde{P}_z$ follows from Corollary 4.31 (local near-best approximation), adapted to the setting at hand.

$\boxed{3}$ It remains to prove the claimed inequality. Given $w \in H^1(\Omega)$, the definition of $\widetilde{P}_{\mathcal{T}}$ and the first step yield the following identity:

$$\begin{aligned}\langle \ell - \widetilde{P}_{\mathcal{T}}\ell, w\rangle &= \sum_{z\in\mathcal{V}} \langle \ell, w\phi_z\rangle - \langle \phi_z \widetilde{P}_z\ell, w\rangle \\ &= \sum_{z\in\mathcal{V}} \langle \ell - \widetilde{P}_z\ell, w\phi_z\rangle = \sum_{z\in\mathcal{V}} \langle \ell - \widetilde{P}_z\ell, (w - c_z)\phi_z\rangle,\end{aligned}$$

with $c_z = \fint_{\omega_z} w$. Proceeding as in Lemma 4.5 (localization of H^{-1}-norm) establishes the desired inequality and concludes the proof. □

The operator $\widetilde{P}_{\mathcal{T}}$ possesses additional enhanced global properties, which are not needed here. Lemma 4.71 and (4.93) allow us to solve auxiliary pure Neumann problems with discrete data $\widetilde{P}_{\mathcal{T}}\ell$, with the option of replacing the restrictions of $P_{\mathcal{T}}$ in the local indicators with $\widetilde{P}_z$.

Non-homogeneous Dirichlet boundary conditions
Let $\mathbb{V} := H^1(\Omega)$ and $\mathbb{V}_\mathcal{T} := \mathbb{S}^{n,0}_\mathcal{T}$ be the subspace of $\mathbb{V}$ of continuous piecewise polynomials of degree $\leq n$. Given Dirichlet boundary data $g \in H^{1/2}(\Gamma)$, where $\Gamma := \partial\Omega$ for simplicity, recall that $u \in \mathbb{V}(g) = \{v \in \mathbb{V} \mid v = g \text{ on } \Gamma\}$ satisfies (2.10). Let $g_\mathcal{T} \in \mathbb{S}^{n,0}_\mathcal{T}$ be a continuous finite element approximation of g on Γ and let $\mathbb{V}_\mathcal{T}(g_\mathcal{T})$ be the subspace of $\mathbb{V}_\mathcal{T}$ of discrete functions with trace $g_\mathcal{T}$. The Galerkin approximation of u satisfies

$$u_\mathcal{T} \in \mathbb{V}_\mathcal{T}(g_\mathcal{T}): \quad \mathcal{B}[u_\mathcal{T}, v] = \langle f, v\rangle \quad \text{for all } v \in \mathbb{V}_\mathcal{T}(0).$$

The error $e_\mathcal{T} = u - u_\mathcal{T}$ obviously satisfies Galerkin orthogonality

$$\mathcal{B}[e_\mathcal{T}, v] = 0 \quad \text{for all } v \in \mathbb{V}_\mathcal{T}(0),$$

but in general $e_\mathcal{T} = g - g_\mathcal{T} \neq 0$ on Γ. We follow Sacchi and Veeser (2006) to derive *a posteriori* bounds of $\|e_\mathcal{T}\|_{H^1(\Omega)}$ using minimal regularity $g \in H^{1/2}(\Gamma)$.

We start with an orthogonal decomposition of the error $e_\mathcal{T}$ arising from the two equations of the problem. Let $R_G = R_G(u_\mathcal{T}, f) \in H^{-1}(\Omega)$ be the Galerkin residual already introduced in Section 4.1, namely

$$\langle R_G, v\rangle = \langle f, v\rangle - \mathcal{B}[u_\mathcal{T}, v] \quad \text{for all } v \in \mathbb{V}(0) = H^1_0(\Omega),$$

and define the Galerkin error e_G as its representation in $H^1_0(\Omega)$:

$$e_G \in H^1_0(\Omega): \quad \mathcal{B}[e_G, v] = \langle R_G, v\rangle \quad \text{for all } v \in \mathbb{V}(0).$$

Furthermore, let $R_D = R_D(g) = g - g_\mathcal{T} \in H^{1/2}(\Gamma)$ be the *Dirichlet residual*, represented by the *Dirichlet error* e_D defined by

$$e_D \in \mathbb{V}(g - I_\mathcal{T} g): \quad \mathcal{B}[e_D, v] = 0 \quad \text{for all } v \in \mathbb{V}(0).$$

Then $e_\mathcal{T} = e_G + e_D$ and the orthogonality $\mathcal{B}[e_D, e_G] = 0$ yields

$$|\!|\!| e_\mathcal{T} |\!|\!|^2_\Omega = |\!|\!| e_G |\!|\!|^2_\Omega + |\!|\!| e_D |\!|\!|^2_\Omega,$$

while the derivation for homogeneous Dirichlet conditions readily provides

$$|\!|\!| e_G |\!|\!|_\Omega \approx \|\nabla e_G\|_{L^2(\Omega)} \approx \mathcal{E}_\mathcal{T}(u_\mathcal{T}, f),$$

where the Galerkin estimator $\mathcal{E}_\mathcal{T}(u_\mathcal{T}, f)$ is defined by (4.52), or any other estimator from Section 4.9. It thus remains to clarify whether $|\!|\!| e_\mathcal{T} |\!|\!|_\Omega$ is definite in the sense of $|\!|\!| e_\mathcal{T} |\!|\!|_\Omega = 0 \Rightarrow e_\mathcal{T} = 0$ and to derive suitable lower and upper bounds for $|\!|\!| e_D |\!|\!|_\Omega$.

To this end, we need to be more specific about the choice of $g_\mathcal{T}$. Let $g_\mathcal{T} = I_\mathcal{T} g$ be the Scott–Zhang quasi-interpolant of g, which is defined locally using boundary values of g exclusively (Scott and Zhang 1990, Brenner and Scott 2008) and satisfies

$$v \in \mathbb{V}_\mathcal{T}|_\Gamma \Rightarrow I_\mathcal{T} v = v \quad \text{on } \Gamma, \qquad \text{(invariance)} \tag{4.94a}$$

$$\|I_\mathcal{T} v\|_{L^2(\Gamma)} \lesssim \|v\|_{L^2(\Gamma)} \quad \text{for all } v \in \mathbb{V}. \qquad \text{(stability)} \tag{4.94b}$$

These two properties ensure a variant of the equivalence $\|\cdot\|_{H^1(\Omega)} \approx \|\nabla\cdot\|_{L^2(\Omega)}$ for functions with zero trace on Γ.

Lemma 4.72 (equivalence for vanishing discretized trace). *There exists a constant C depending only on the shape regularity of $\mathbb{T}$ and Ω such that*

$$\|v\|_{H^1(\Omega)} \le C\|\nabla v\|_{L^2(\Omega)} \quad \textit{for all } v \in \mathbb{V} \textit{ with } I_{\mathcal{T}} v = 0 \textit{ on } \Gamma.$$

Proof. Note that the core of the claimed inequality amounts to a variant of the first Poincaré inequality. In view of the norm equivalence (2.31), it suffices to prove that $\|v\|_{L^2(\Gamma)} \lesssim \|\nabla v\|_{L^2(\Omega)}$. Letting $\bar{v}_\Omega := \fint_\Omega v$, and using (4.94a) yields $v = v - I_{\mathcal{T}} v = (v - \bar{v}_\Omega) - I_{\mathcal{T}}(v - \bar{v}_\Omega)$ on Γ. Consequently, (4.94b) implies

$$\|v\|_{L^2(\Gamma)} \lesssim \|v - \bar{v}_\Omega\|_{L^2(\Gamma)} \lesssim \|v - \bar{v}_\Omega\|_{H^1(\Omega)} \lesssim \|\nabla v\|_{L^2(\Omega)}, \tag{4.95}$$

because of Lemma 2.4 (traces) and Lemma 2.3 (second Poincaré inequality). □

Observing $I_{\mathcal{T}} e_{\mathcal{T}} = I_{\mathcal{T}} g - I_{\mathcal{T}}^2 g = 0$ on Γ, we can apply Lemma 4.72 to get

$$\|e_{\mathcal{T}}\|_{H^1(\Omega)} \le C\|\nabla e_{\mathcal{T}}\|_{L^2(\Omega)} \le \frac{C}{\alpha_1}|\!|\!|e_{\mathcal{T}}|\!|\!|_\Omega \le \frac{C\max\{\alpha_2, \|c\|_{L^\infty(\Omega)}\}}{\alpha_1}\|e_{\mathcal{T}}\|_{H^1(\Omega)},$$

establishing in particular that $|\!|\!|e_{\mathcal{T}}|\!|\!|_\Omega$ is definite. In the same vein, we derive

$$|\!|\!|e_D|\!|\!|_\Omega \approx \|e_D\|_{H^1(\Omega)} \approx \|\nabla e_D\|_{L^2(\Omega)}$$

for the Dirichlet error.

With the intent to achieve directly computable bounds for the Dirichlet error, we next establish the equivalence $\|e_D\|_{H^1(\Omega)} \approx \|g - I_{\mathcal{T}} g\|_{H^{1/2}(\Gamma)}$, where the *intrinsic* $H^{1/2}$-norm combines the $L^2(\Gamma)$-norm with the seminorm

$$|v|^2_{H^{1/2}(\Gamma)} = \int_\Gamma \int_\Gamma \frac{|v(x) - v(y)|^2}{|x - y|^d}\,\mathrm{d}x\,\mathrm{d}y.$$

This equivalence follows with the help of the trace and extension theorems for $H^{1/2}(\Gamma)$; see e.g. Hackbusch (1992, Theorem 6.2.40). In fact, on the one hand, that trace theorem immediately gives $\|g - I_{\mathcal{T}} g\|_{H^{1/2}(\Gamma)} \lesssim \|e_D\|_{H^1(\Omega)}$. On the other hand, let $\chi \in H^1(\Omega)$ denote the extension of $g - I_{\mathcal{T}} g$ from Hackbusch (1992, Theorem 6.2.40). Then

$$\|e_D\|_{H^1(\Omega)} \lesssim |\!|\!|e_D|\!|\!|_\Omega \le |\!|\!|\chi|\!|\!|_\Omega \lesssim \|\chi\|_{H^1(\Omega)} \lesssim \|g - I_{\mathcal{T}} g\|_{H^{1/2}(\Gamma)},$$

where the second inequality is thanks to $\mathcal{B}[e_D, e_D - \chi] = 0$.

We are left with the issue that the $H^{1/2}(\Gamma)$-seminorm is *non-local*. To handle this delicate matter, we invoke its localization due to Faermann (2000, 2002), that is,

$$|v|^2_{H^{1/2}(\Gamma)} \le \sum_{F \in \mathcal{F}_\Gamma} \left(\int_F \int_{\omega_F} \frac{|v(x) - v(y)|^2}{|x - y|^d}\,\mathrm{d}x\,\mathrm{d}y + \frac{C}{h_F}\|v\|^2_{L^2(F)} \right),$$

where $F \in \mathcal{F}_\Gamma$ is a generic face of $\mathcal{T}$ lying on Γ and ω_F is the patch on Γ associated with F. The last term seems problematic. However, applied to $v = g - I_{\mathcal{T}} g$, we can

mimic the steps of (4.95) with local variants of (4.94), but using in the last step the second Poincaré inequality in $H^{1/2}$ (see e.g. Sacchi and Veeser 2006, Lemma 3.2):

$$\|v\|^2_{L^2(F)} = \|v - I_{\mathcal{T}} v\|^2_{L^2(F)} \lesssim \|v - \bar{v}_F\|^2_{L^2(\omega_F)} \lesssim h_F |v|^2_{H^{1/2}(\omega_F)},$$

where $\bar{v}_F$ is the mean value of v on ω_F. Note that this bound also means that the L^2-part in $\|g - I_{\mathcal{T}} g\|_{H^{1/2}(\Gamma)}$ is (locally) controlled by its seminorm. Altogether, this leads to defining the *Dirichlet oscillation* with the following local indicators:

$$\begin{aligned} \operatorname{osc}_{\mathcal{T}}(g)^2_{1/2} &:= \sum_{F \in \mathcal{F}_\Gamma} \operatorname{osc}_{\mathcal{T}}(g, F)^2_{1/2}, \\ \operatorname{osc}_{\mathcal{T}}(g, F)^2_{1/2} &:= \int_{\omega_F} \int_{\omega_F} \frac{|(g - I_{\mathcal{T}} g)(x) - (g - I_{\mathcal{T}} g)(y)|^2}{|x - y|^d} \, \mathrm{d}x \, \mathrm{d}y. \end{aligned} \tag{4.96}$$

We observe that $\operatorname{osc}_{\mathcal{T}}(g, F)$ is a double singular integral but computationally accessible, for instance, by using suitable quadrature provided g is continuous (Sacchi and Veeser 2006, Section 4.1).

Proposition 4.73 (Dirichlet oscillation). *There exist constants $D_1 \geq D_2 > 0$ depending on the shape regularity of $\mathbb{T}$ and geometry of Γ, such that*

$$D_2 \operatorname{osc}_{\mathcal{T}}(g)_{1/2} \leq \|\nabla e_D\|_{L^2(\Omega)} \leq D_1 \operatorname{osc}_{\mathcal{T}}(g)_{1/2}.$$

Proof. The preceding derivation verifies the upper bound. For the lower bound, note that for any $v \in H^1(\Omega)$ such that $v = g - I_{\mathcal{T}} g$ on Γ

$$\begin{aligned} \operatorname{osc}_{\mathcal{T}}(g)^2_{1/2} &= \sum_{F \in \mathcal{F}_\Gamma} \operatorname{osc}_{\mathcal{T}}(g, F)^2_{1/2} \\ &\leq \sum_{F \in \mathcal{F}_\Gamma} \int_{\omega_F} \int_{\Gamma} \frac{|v(x) - v(y)|^2}{|x - y|^d} \, \mathrm{d}x \, \mathrm{d}y \\ &\lesssim \int_{\Gamma} \int_{\Gamma} \frac{|v(x) - v(y)|^2}{|x - y|^d} \, \mathrm{d}x \, \mathrm{d}y = |v|^2_{H^{1/2}(\Gamma)}, \end{aligned}$$

because the patches ω_F, $F \in \mathcal{F}_\Gamma$, possess a uniform overlapping property due to shape regularity of $\mathbb{T}$. Applying this to $v = e_D$ finishes the proof. □

For suitable settings, local lower *a posteriori* estimates for the Dirichlet error e_D can be derived; see Sacchi and Veeser (2006, Theorem 3.2).

Combining the Dirichlet oscillation with some Galerkin estimator $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)$ by

$$\mathcal{E}^{\mathrm{Dir}}_{\mathcal{T}}(u_{\mathcal{T}}, f, g)^2 := \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)^2 + \operatorname{osc}_{\mathcal{T}}(g)^2_{1/2},$$

the preceding discussion is summarized by the following result.

Theorem 4.74 (estimators for general Dirichlet condition). *If Assumption 4.44 (discrete coefficients and discrete functionals) is valid, then there exist constants $C_L \leq C_U$ depending on $(\boldsymbol{A}, c)$, Ω, Γ, and the shape regularity of $\mathbb{T}$ such that*

$$C_L \mathcal{E}^{\mathrm{Dir}}_{\mathcal{T}}(u_{\mathcal{T}}, f, g) \leq \|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)} \leq C_U \mathcal{E}^{\mathrm{Dir}}_{\mathcal{T}}(u_{\mathcal{T}}, f, g).$$

5. Convergence of AFEM for coercive problems

In this section we consider the coercive problem (2.5) with the intent to design and analyse three AFEMs in increasing order of complexity and applicability, depending on properties of data $\mathcal{D}$. Our basic regularity assumption on data reads $\mathcal{D} = (\boldsymbol{A}, c, f) \in \mathbb{D}$, where

$$\mathbb{D} := L^\infty(\Omega; \mathbb{R}^{d\times d}) \times L^\infty(\Omega) \times H^{-1}(\Omega). \tag{5.1}$$

We approximate $\mathcal{D}$ with discrete data $\widehat{\mathcal{D}} = (\widehat{\boldsymbol{A}}, \widehat{c}, \widehat{f}) \in \mathbb{D}_{\widehat{\mathcal{T}}}$, where

$$\mathbb{D}_{\widehat{\mathcal{T}}} := \left[\mathbb{S}^{n-1,-1}_{\widehat{\mathcal{T}}}\right]^{d\times d} \times \mathbb{S}^{n-1,-1}_{\widehat{\mathcal{T}}} \times \mathbb{F}_{\widehat{\mathcal{T}}} \tag{5.2}$$

is subordinate to a partition $\widehat{\mathcal{T}} \in \mathbb{T}$. We will often assume that *data is discrete*, meaning precisely that $\mathcal{D} = \widehat{\mathcal{D}}$.

We start with the *one-step* AFEM, hereafter called GALERKIN, which is the standard SEMR loop

$$\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE}$$

introduced by Dörfler (1996) and further developed by Morin, Nochetto and Siebert (2000, 2002) and Cascón *et al.* (2008). This is the simplest algorithm in that it requires data $\mathcal{D} = (\boldsymbol{A}, c, f)$ to be discrete, but it is a building block for the other two methods. After reviewing a few crucial properties of error and estimator in Section 5.1, we fully discuss GALERKIN in Section 5.2.

The second algorithm is the *one-step* AFEM *with switch*, which still assumes the coefficients $(\boldsymbol{A}, c)$ to be discrete but allows for general forcing $f \in H^{-1}(\Omega)$. This is a new contribution of this survey that, similarly to Kreuzer *et al.* (2024), exploits the structure of the error estimator $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)$ of Section 4,

$$\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)^2 = \eta_{\mathcal{T}}(u_{\mathcal{T}})^2 + \operatorname{osc}_{\mathcal{T}}(f)_{-1}^2,$$

and its equivalence to the energy error. The PDE estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ relies on the discrete forcing $P_{\mathcal{T}} f \in \mathbb{F}_{\mathcal{T}}$ and is fully computable, whereas the data oscillation $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ encodes the infinite-dimensional nature of f and could be estimated in important cases of practical interest further discussed in Section 7.3. The quantity $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ measures the deviation of f from being discrete and may dictate the pre-asymptotic regime of AFEM. Therefore $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ must be handled separately from $\eta_{\mathcal{T}}(u_{\mathcal{T}})$; hence the name of the new method, hereafter called AFEM-SW. Assuming that $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ is computable, the module

$$[\widehat{\mathcal{T}}] = \mathsf{DATA}(\mathcal{T}, f, \tau)$$

deals with $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ whenever it is large relative to $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)$. In fact, it creates an admissible refinement $\widehat{\mathcal{T}}$ of the input mesh $\mathcal{T}$ such that $\operatorname{osc}_{\widehat{\mathcal{T}}}(f)_{-1}$ is below the desired tolerance τ, i.e. $\operatorname{osc}_{\widehat{\mathcal{T}}}(f)_{-1} \le \tau$. We explain the role of data oscillation for error analysis, design AFEM-SW and prove its linear convergence in Section 5.3.

The third algorithm deals with variable data $\mathcal{D}$ and various degrees of regularity of $\mathcal{D}$, and is able to handle discontinuous coefficients $(\boldsymbol{A}, c)$ not aligned with admissible meshes $\mathcal{T} \in \mathbb{T}$ emanating from $\mathcal{T}_0$. To handle the multiplicative structure of $(\boldsymbol{A}, c)$ in the model problem (2.5), we consider the following *two-step* AFEM.

Algorithm 5.1 (AFEM-TS). Given an initial mesh $\mathcal{T}_0$, an initial tolerance ε_0, and a parameter ω sufficiently small to be determined later, iterate

AFEM-TS$(\mathcal{T}_0, \varepsilon_0, \omega)$
$\quad k = 0$
$\quad [\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k] = \mathsf{DATA}(\mathcal{T}_k, \mathcal{D}, \omega\, \varepsilon_k)$
$\quad [\mathcal{T}_{k+1}, u_{k+1}] = \mathsf{GALERKIN}(\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k, \varepsilon_k)$
$\quad \varepsilon_{k+1} = \frac{1}{2}\varepsilon_k;\ k \leftarrow k+1$

This structure was first proposed by Stevenson (2008) and further explored by Bonito *et al.* (2013*b*), Cohen *et al.* (2012), Bonito *et al.* (2016), Bonito, Cascón, Morin and Nochetto (2013*a*) and Bonito and Devaud (2015). The three components of data $\mathcal{D} = (\boldsymbol{A}, c, f) \in \mathbb{D}$ are first approximated by discrete data $\widehat{\mathcal{D}} = (\widehat{\boldsymbol{A}}, \widehat{c}, \widehat{f}) \in \mathbb{D}_{\widehat{\mathcal{T}}}$, as defined in (5.1) and (5.2), within the module

$$[\widehat{\mathcal{T}}, \widehat{\mathcal{D}}] = \mathsf{DATA}(\mathcal{T}, \mathcal{D}, \tau)$$

to accuracy $\tau = \omega\varepsilon$ significantly smaller than ε. This is achieved by an algorithm similar to Algorithm 3.18 (greedy algorithm), which is fully discussed along with applications to $\mathcal{D}$ in Section 7. The resulting admissible refinement $\widehat{\mathcal{T}}$ of $\mathcal{T}$ and discrete data $\widehat{\mathcal{D}}$ over $\widehat{\mathcal{T}}$ are next taken by GALERKIN to reduce the PDE error to the desired tolerance ε, namely the module

$$[\mathcal{T}, u_{\mathcal{T}}] = \mathsf{GALERKIN}(\widehat{\mathcal{T}}, \widehat{\mathcal{D}}, \varepsilon)$$

constructs a refinement $\mathcal{T}$ of $\widehat{\mathcal{T}}$ with discrete data $\widehat{\mathcal{D}}$ over $\widehat{\mathcal{T}}$ such that $\eta_{\mathcal{T}}(u_{\mathcal{T}}) \leq \varepsilon$. We point out that if the data is discrete, i.e. $\mathcal{D} = \widehat{\mathcal{D}}$, then DATA is skipped and AFEM-TS reduces to GALERKIN. We tackle AFEM-TS in Section 5.4, where we prove a perturbation estimate with respect to $\mathcal{D}$ and next discuss convergence properties of AFEM-TS. We will extend this approach to discontinuous FEMs in Section 9 and to mixed FEMs for (2.5) as well as the Stokes system (2.14) in Section 10.

5.1. Properties of error and estimator

We follow Cascón, Kreuzer, Nochetto and Siebert (2008) and summarize some basic properties of GALERKIN that emanate from the symmetry of the differential operator (i.e. of $\boldsymbol{A}$) and features of the modules. In doing this, any explicit constant or hidden constant in $\lesssim$ will depend only on the uniform shape regularity of $\mathbb{T}$, the dimension d, the polynomial degree n and the (global) eigenvalues of $\boldsymbol{A}$, but not on a specific grid $\mathcal{T} \in \mathbb{T}$, unless explicitly stated.

We recall that the bilinear form $\mathcal{B}$ in (2.8) with continuous coefficients $(\boldsymbol{A}, c)$ is symmetric, coercive and continuous in the space $H^1_0(\Omega)$ (see (2.30)), namely $|\!|\!|v|\!|\!|_\Omega = \mathcal{B}[v,v]^{1/2}$ is a norm equivalent to $|v|_{H^1_0(\Omega)}$ with equivalence constants $0 < c_{\mathcal{B}} \le C_{\mathcal{B}}$

$$c_{\mathcal{B}}|v|^2_{H^1_0(\Omega)} \le |\!|\!|v|\!|\!|^2_\Omega \le C_{\mathcal{B}}|v|^2_{H^1_0(\Omega)} \quad \text{for all } v \in H^1_0(\Omega). \tag{5.3}$$

The module DATA approximates $(\boldsymbol{A}, c)$ over a mesh $\mathcal{T}$ by piecewise polynomial coefficients $(\widehat{\boldsymbol{A}}, \widehat{c})$ obeying side constraints so that the corresponding perturbed bilinear form $\widehat{B}$ still defines a uniform scalar product in $H^1_0(\Omega)$,

$$|\!|\!|v|\!|\!|_\Omega = \widehat{\mathcal{B}}[v,v]^{1/2} \quad \text{for all } v \in H^1_0(\Omega), \tag{5.4}$$

which satisfies (5.3) with constants $0 < c_{\widehat{\mathcal{B}}} \le C_{\widehat{\mathcal{B}}}$ independent of $\mathcal{T}$. We hope this slight abuse of notation will not create confusion because we will always refer to the energy norm in (5.4) when dealing with $\widehat{\mathcal{B}}$. We let $\widehat{u} = u(\widehat{\mathcal{D}}) \in H^1_0(\Omega)$ denote the solution of (2.7) with coefficients $(\widehat{\boldsymbol{A}}, \widehat{c})$ and forcing function either $\widehat{f} = f \in H^{-1}(\Omega)$ or its projection $\widehat{f} = P_{\mathcal{T}} f \in \mathbb{F}_{\mathcal{T}}$ defined in (4.35), namely

$$\widehat{\mathcal{B}}[\widehat{u}, v] = \langle \widehat{f}, v \rangle \quad \text{for all } v \in H^1_0(\Omega). \tag{5.5}$$

In what follows, we will often compare discrete functions on different meshes. Given $\mathcal{T} \in \mathbb{T}$, we let $\mathcal{T}_* \in \mathbb{T}$ denote an admissible refinement of $\mathcal{T}$, and write

$$\mathcal{T} \le \mathcal{T}_* \quad \Leftrightarrow \quad \mathbb{T}(\mathcal{T}) \subset \mathbb{T}(\mathcal{T}_*), \tag{5.6}$$

in the sense that the supporting tree of $\mathcal{T}$ is contained in the tree of $\mathcal{T}_*$. For any $\mathcal{T}_* \ge \mathcal{T}$, we have the following crucial property.

Lemma 5.2 (Pythagoras). *Let $\mathcal{T}_* \ge \mathcal{T} \ge \widehat{\mathcal{T}}$ and let $\widehat{u} \in H^1_0(\Omega)$ be the solution of (5.5) with discrete coefficients $(\widehat{\boldsymbol{A}}, \widehat{c})$ over $\widehat{\mathcal{T}}$. The corresponding Galerkin solutions $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$ and $u_{\mathcal{T}_*} \in \mathbb{V}_{\mathcal{T}_*}$ with coefficients $(\widehat{\boldsymbol{A}}, \widehat{c})$ and forcing $f \in H^{-1}(\Omega)$ satisfy the orthogonality property*

$$|\!|\!|\widehat{u} - v_{\mathcal{T}}|\!|\!|^2_\Omega = |\!|\!|\widehat{u} - u_{\mathcal{T}_*}|\!|\!|^2_\Omega + |\!|\!|u_{\mathcal{T}_*} - v_{\mathcal{T}}|\!|\!|^2_\Omega \quad \textit{for all } v_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}. \tag{5.7}$$

Proof. Exploit the nestedness property $\mathbb{V}_{\mathcal{T}} \subset \mathbb{V}_{\mathcal{T}_*}$ and the Galerkin orthogonality property $\widehat{\mathcal{B}}[\widehat{u} - u_{\mathcal{T}_*}, v_{\mathcal{T}} - u_{\mathcal{T}_*}] = 0$ in $\mathbb{V}_{\mathcal{T}_*}$ for the scalar product induced by $\widehat{\mathcal{B}}$. □

Property (5.7) is very restrictive: it relies on space nestedness and is valid exclusively for the energy norm. However, it is instrumental to the subsequent analysis in the energy norm or the equivalent norm $|\cdot|_{H^1_0(\Omega)}$, but it does not extend to other, perhaps more practical, norms such as the maximum norm. This is an important open problem and a serious limitation of this theory.

We recall that the residual *a posteriori* error analysis of Section 4 relies on the projection operator $P_{\mathcal{T}} \colon H^{-1}(\Omega) \to \mathbb{F}_{\mathcal{T}}$, with element and face components

$P_{\mathcal{T}} f|_T = P_T f$ for $T \in \mathcal{T}$ and $P_{\mathcal{T}} f|_F = P_F f$. The *full local error indicator*

$$\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f, T)^2 = \eta_{\mathcal{T}}(u_{\mathcal{T}}, T)^2 + \operatorname{osc}_{\mathcal{T}}(f, T)_{-1}^2$$

splits into a computable *PDE error indicator* with discrete coefficients $(\widehat{\boldsymbol{A}}, \widehat{c})$,

$$\eta_{\mathcal{T}}(v, T)^2 = h_T^2 \, \|r(v)\|_T^2 + h_T \, \|j(v)\|_{\partial T}^2 \quad \text{for all } T \in \mathcal{T}, \tag{5.8}$$

where the *interior* and *jump residuals* are given by

$$\begin{aligned} r(v)|_T &= P_T f + \operatorname{div}(\widehat{\boldsymbol{A}} \nabla v) - \widehat{c} v \quad \text{for all } T \in \mathcal{T}, \\ j(v)|_F &= [\widehat{\boldsymbol{A}} \nabla v] \cdot \boldsymbol{n}\,|_F - P_F f \qquad \text{for all } F \in \mathcal{F}, \end{aligned} \tag{5.9}$$

and $j(v)|_F = 0$ for boundary faces F, and *data oscillation*

$$\operatorname{osc}_{\mathcal{T}}(f, T)_{-1}^2 = \|f - P_{\mathcal{T}} f\|_{H^{-1}(\omega_T)}^2 \quad \text{for all } T \in \mathcal{T}, \tag{5.10}$$

where ω_T is the patch associated with T. The corresponding global quantities are

$$\begin{gathered} \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)^2 = \sum_{T \in \mathcal{T}} \mathcal{E}(u_{\mathcal{T}}, f, T)^2, \\ \eta_{\mathcal{T}}(u_{\mathcal{T}})^2 = \sum_{T \in \mathcal{T}} \eta_{\mathcal{T}}(u_{\mathcal{T}}, T)^2, \quad \operatorname{osc}_{\mathcal{T}}(f)_{-1}^2 = \sum_{T \in \mathcal{T}} \operatorname{osc}_{\mathcal{T}}(f, T)_{-1}^2, \end{gathered} \tag{5.11}$$

and have the following *a posteriori* error estimates proved in Theorem 4.45 (modified residual estimator) for the H_0^1-norm.

Proposition 5.3 (*a posteriori* error estimates). *Let $\widehat{u} \in H_0^1(\Omega)$ be the solution of (5.5) with discrete coefficients $(\widehat{\boldsymbol{A}}, \widehat{c})$ over $\mathcal{T} \in \mathbb{T}$ but general forcing $f \in H^{-1}(\Omega)$. Then there exist constants $0 < C_L \le C_U$, depending on the shape regularity of $\mathbb{T}$, such that the Galerkin solution $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$ satisfies*

$$C_L \, \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f) \le |\widehat{u} - u_{\mathcal{T}}|_{H_0^1(\Omega)} \le C_U \, \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f). \tag{5.12}$$

Moreover, if $|||\cdot|||_\Omega$ stands for the energy norm in (5.4) with equivalence constants $c_{\widehat{\mathcal{B}}} \le C_{\widehat{\mathcal{B}}}$ satisfying (5.3), then (5.12) yields

$$C_2 \, \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f) \le |||\widehat{u} - u_{\mathcal{T}}|||_\Omega \le C_1 \, \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f), \tag{5.13}$$

with $C_1 = \sqrt{C_{\widehat{\mathcal{B}}}} C_U$ and $C_2 = \sqrt{c_{\widehat{\mathcal{B}}}} C_L$.

There is a fundamental difference between (5.12) and earlier versions of *a posteriori* error estimates, going back to the seminal paper of Babuška and Miller (1987); see also Ainsworth and Oden (2000), Braess (2007), Nochetto *et al.* (2009) and Verfürth (2013). It is about the role of data oscillation $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$, which is now dominated by the error $|\widehat{u} - u_{\mathcal{T}}|_{H_0^1(\Omega)}$ and does not spoil the lower bound. This is due to the fact that $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ is evaluated in the natural space $H^{-1}(\Omega)$ and quantifies the discrepancy between f and a suitable projection $P_{\mathcal{T}} f$ which gives rise to a quasi-best local approximation of f. We refer to Nochetto *et al.* (2009) and Kreuzer and Veeser (2021) for a discussion of data oscillation.

Suppose now that we have two conforming meshes $\mathcal{T}, \mathcal{T}_* \in \mathbb{T}$ with $\mathcal{T}_* \geq \mathcal{T}$. Let

$$\mathcal{R} := \mathcal{R}_{\mathcal{T}\to\mathcal{T}_*} := \mathcal{T}\backslash\mathcal{T}_* \tag{5.14}$$

be the subset of refined elements of $\mathcal{T}$, namely those elements in $\mathcal{T}$ that are no longer in $\mathcal{T}_*$. We stress that the upper bound in (5.12) cannot be local due to the non-local nature of the error $|\widehat{u} - u_{\mathcal{T}}|_{H_0^1(\Omega)}$. However, in view of Theorem 4.48 (upper bound for corrections), the following remarkable local upper bound for Galerkin solutions $u_{\mathcal{T}} \in V_{\mathcal{T}}, u_{\mathcal{T}_*} \in V_{\mathcal{T}_*}$ holds:

$$|\!|\!| u_{\mathcal{T}} - u_{\mathcal{T}_*} |\!|\!|_\Omega \leq C_1 \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f, \mathcal{R}), \tag{5.15}$$

where for $\mathcal{S} \subset \mathcal{T}$,

$$\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f, \mathcal{S}) := \left(\sum_{T\in\mathcal{S}} \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f, T)^2 \right)^{1/2}$$

is the error estimator restricted to $\mathcal{S}$. Consequently, only the elements where $\mathcal{T}$ and $\mathcal{T}_*$ differ, namely the set $\mathcal{R}$, account for the discrepancy between $u_{\mathcal{T}}$ and $u_{\mathcal{T}_*}$. This turns out to be consistent with (5.13) because $\mathcal{T}$ has to be refined everywhere to get to $\widehat{u}$, whence $\mathcal{R} = \mathcal{T}$.

In contrast to the upper bound in (5.12), the corresponding lower bound is local according to Theorem 4.45 (modified residual estimator). This is due to the local nature of the PDE (2.5). However, when comparing $u_{\mathcal{T}}$ and $u_{\mathcal{T}_*}$, this bound is not valid unless the *interior vertex property* (given in Definition 4.50) is satisfied (Morin *et al.* 2000); in fact, we present a counterexample later in Example 5.7 taken from Morin *et al.* (2000).

The interior vertex property is valid upon enforcing a fixed number b of bisections ($b = 3, 6$ for $d = 2, 3$). An immediate consequence, proved in Theorem 4.51 (lower bound for corrections), is the *discrete* lower *a posteriori* bound for piecewise constant diffusion coefficient $\boldsymbol{A}$ and reaction coefficient $c = 0$ on $\mathcal{T}_0$,

$$C_{L,1}\, \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f, \mathcal{M}) \leq |\!|\!| u_{\mathcal{T}} - u_{\mathcal{T}_*} |\!|\!|_\Omega + C_{L,2}\, \mathrm{osc}_{\mathcal{T}}(f, \omega(\mathcal{M}))_{-1}, \tag{5.16}$$

where $\omega(\mathcal{M}) := \cup\{\omega_T \mid T \in \mathcal{M}\}$ is the union of all patches of elements in $\mathcal{M}$ and $\mathrm{osc}_{\mathcal{T}}(f, \omega(\mathcal{M}))_{-1}^2 = \sum_{T\in\omega(\mathcal{M})} \mathrm{osc}_{\mathcal{T}}(f, T)_{-1}^2$; we refer to Morin *et al.* (2000, 2002). We stress that if $f = P_{\mathcal{T}} f$ is discrete, then $\mathrm{osc}_{\mathcal{T}}(f)_{-1} = 0$ and (5.16) reduces to

$$C_2\, \eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{M}) \leq |\!|\!| u_{\mathcal{T}} - u_{\mathcal{T}_*} |\!|\!|_\Omega. \tag{5.17}$$

One serious difficulty in dealing with AFEM is that we have access to the energy error $|\!|\!| \widehat{u} - u_{\mathcal{T}} |\!|\!|_\Omega$, or equivalently to $|\widehat{u} - u_{\mathcal{T}}|_{H_0^1(\Omega)}$, only through the full error estimator $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)$. Lemma 5.2 (Pythagoras) implies monotonicity of the energy error with respect to $\mathcal{T}$, namely, for $\mathcal{T}_* \geq \mathcal{T}$,

$$|\!|\!| \widehat{u} - u_{\mathcal{T}_*} |\!|\!|_\Omega \leq |\!|\!| \widehat{u} - u_{\mathcal{T}} |\!|\!|_\Omega.$$

However, the PDE estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ fails to be monotone for fixed discrete coefficients $(\widehat{\boldsymbol{A}}, \widehat{c})$ because it depends on the discrete solution $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$ that changes with

the mesh. The following estimate, proved in Proposition 4.56 (estimator reduction), quantifies the deviation of $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ from monotonicity: there exists $\lambda > 0$ such that for any $\delta > 0$, $v \in \mathbb{V}_{\mathcal{T}}$ and $v_* \in \mathbb{V}_{\mathcal{T}_*}$,

$$\eta_{\mathcal{T}_*}(v_*, \mathcal{T}_*)^2 \le (1+\delta)\big(\eta_{\mathcal{T}}(v, \mathcal{T})^2 - \lambda\, \eta_{\mathcal{T}}(v, \mathcal{M})^2\big) + 2(1+\delta^{-1})\, C_{\mathrm{Lip}} \left(|\!|\!| v_* - v |\!|\!|_{\Omega}^2 + \sum_{T \in \mathcal{T}_*} \|P_{\mathcal{T}} f - P_{\mathcal{T}_*} f\|_{H^{-1}(\omega_T)}^2 \right),$$

where C_{Lip} depends on $(\boldsymbol{A}, c)$ and the shape regularity constant of $\mathbb{T}$. We refer to Cascón *et al.* (2008) and Morin *et al.* (2008) for the case $P_{\mathcal{T}} f = P_{\mathcal{T}_*} f = f \in L^2(\Omega)$.

5.2. *Convergence for discrete data: one-step* AFEM

We now present the four basic modules of GALERKIN, the one-step AFEM within Algorithm 5.1 (AFEM-TS), namely

$$\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE}, \tag{5.18}$$

discuss their main properties, and prove a contraction property between consecutive iterates of GALERKIN. According to Algorithm 5.1, given discrete data $\widehat{\mathcal{D}}$ over a conforming mesh $\widehat{\mathcal{T}}$, created by DATA, and a desired tolerance $\varepsilon > 0$, the module

$$[\mathcal{T}, u_{\mathcal{T}}] = \text{GALERKIN}(\widehat{\mathcal{T}}, \widehat{\mathcal{D}}, \varepsilon) \tag{5.19}$$

stops the loop (5.18) as soon as the error tolerance ε is reached, i.e. as soon as

$$\eta_{\mathcal{T}}(u_{\mathcal{T}}) \le \varepsilon. \tag{5.20}$$

Since the data never changes within GALERKIN and is always discrete, we assume in this section that $\mathcal{D} \in \mathbb{D}_{\mathcal{T}}$ and do not use the hat symbol to indicate quantities defined using the (discrete) data.

5.2.1. *Modules of* GALERKIN

Module SOLVE. If $\mathcal{T} \in \mathbb{T}$ is a conforming refinement of $\mathcal{T}_0$, and $\mathbb{V}_{\mathcal{T}}$ is the finite element space of C^0 piecewise polynomials of degree $\le n$, then

$$[u_{\mathcal{T}}] = \text{SOLVE}(\mathcal{T})$$

determines the Galerkin FEM solution *exactly*, namely without algebraic error,

$$u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}: \quad \mathcal{B}[u_{\mathcal{T}}, v] = \int_{\Omega} \nabla v \cdot \boldsymbol{A} \nabla u_{\mathcal{T}} + c v u = \langle f, v \rangle, \tag{5.21}$$

where $f \in H^{-1}(\Omega)$. However, if $f \in \mathbb{F}_{\mathcal{T}}$ is discrete as defined in (4.35), then

$$\langle f, v \rangle = \sum_{T \in \mathcal{T}} \int_T q_T v + \sum_{F \in \mathcal{F}} \int_F q_F v \quad \text{for all } v \in \mathbb{V}_{\mathcal{T}}.$$

The assumption of exact solvability is made for simplicity. The algebraic error committed in solving (5.21) by iterative solvers can be accommodated within the

forthcoming theory. We refer to Stevenson (2007) and Daniel and Vohralík (2023) for details about how to relate the algebraic and PDE errors.

Module ESTIMATE. Given a conforming mesh $\mathcal{T} \in \mathbb{T}$ and the Galerkin solution $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$, the output of

$$[\{\eta_{\mathcal{T}}(u_{\mathcal{T}}, T), \operatorname{osc}_{\mathcal{T}}(f, T)_{-1}\}_{T \in \mathcal{T}}] = \mathsf{ESTIMATE}(u_{\mathcal{T}}, \mathcal{T}, \mathcal{D})$$

gives the element error indicators $\eta_{\mathcal{T}}(u_{\mathcal{T}}, T)$ defined in (5.8) with the discrete data $\mathcal{D}$, namely

$$\eta_{\mathcal{T}}(u_{\mathcal{T}}, T)^2 = h_T^2 \|r(u_{\mathcal{T}})\|_T^2 + h_T \|j(u_{\mathcal{T}})\|_{\partial T}^2, \quad T \in \mathcal{T},$$

and element data oscillation $\operatorname{osc}_{\mathcal{T}}(f, T)_{-1}$ defined in (5.10), namely

$$\operatorname{osc}_{\mathcal{T}}(f, T)_{-1} = \|f - P_{\mathcal{T}} f\|_{H^{-1}(\omega_T)}.$$

We observe that for discrete forcing $f = P_{\mathcal{T}} f$, global data oscillation vanishes, that is,

$$\operatorname{osc}_{\mathcal{T}}(f)_{-1} = \|f - P_{\mathcal{T}} f\|_{H^{-1}(\Omega)} = 0; \tag{5.22}$$

this property is always valid within GALERKIN. In this case, the output of ESTIMATE reduces to just the PDE error indicators. Given $\mathcal{S} \subset \mathcal{T}$, we denote

$$\eta_{\mathcal{T}}(v, \mathcal{S})^2 := \sum_{T \in \mathcal{S}} \eta_{\mathcal{T}}(v, T)^2, \quad \eta_{\mathcal{T}}(v) = \eta_{\mathcal{T}}(v, \mathcal{T}), \quad v \in \mathbb{V}_{\mathcal{T}}.$$

Module MARK. Given $\mathcal{T} \in \mathbb{T}$, the Galerkin solution $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$, and element error indicators $\{\eta_{\mathcal{T}}(u_{\mathcal{T}}, T)\}_{T \in \mathcal{T}}$, the module MARK selects elements for refinement using *Dörfler marking* (or bulk chasing) (Dörfler 1996, Morin *et al.* 2000, Nochetto *et al.* 2009, Nochetto and Veeser 2012), that is, given a parameter $\theta \in (0, 1]$, the output $\mathcal{M}$ of

$$[\mathcal{M}] = \mathsf{MARK}(\{\eta_{\mathcal{T}}(u_{\mathcal{T}}, T)\}_{T \in \mathcal{T}}, \mathcal{T}, \theta)$$

satisfies

$$\eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{M}) \geq \theta\, \eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{T}). \tag{5.23}$$

This marking guarantees that $\mathcal{M}$ contains a substantial part of the total (or bulk) error, hence its name. The choice of $\mathcal{M}$ does not have to be minimal at this stage, that is, the marked elements $T \in \mathcal{M}$ do not necessarily have to be those with largest indicators.

Module REFINE. Let $b \in \mathbb{N}$ be the number of desired bisections per marked element. Given $\mathcal{T} \in \mathbb{T}$ and a subset $\mathcal{M}$ of marked elements, the output $\mathcal{T}_* \in \mathbb{T}$ of

$$[\mathcal{T}_*] = \mathsf{REFINE}(\mathcal{T}, \mathcal{M})$$

is the smallest admissible refinement $\mathcal{T}_*$ of $\mathcal{T}$ such that all elements of $\mathcal{M}$ are bisected at least b times. Therefore we have $h_{\mathcal{T}_*} \leq h_{\mathcal{T}}$ and the strict reduction

property

$$h_{\mathcal{T}_*}|_T \le 2^{-b/d} h_{\mathcal{T}}|_T \quad \text{for all } T \in \mathcal{M}, \tag{5.24}$$

where $h_{\mathcal{T}} \colon \Omega \to \mathbb{R}^+$ is a piecewise constant mesh size function that coincides with $h_T = |T|^{1/d}$ on every $T \in \mathcal{T}$. We finally let

$$\mathcal{R} := \mathcal{R}_{\mathcal{T} \to \mathcal{T}_*} := \mathcal{T} \backslash \mathcal{T}_*$$

be the subset of refined elements of $\mathcal{T}$ and note that $\mathcal{M} \subseteq \mathcal{R}$.

Concatenating these four modules, we get the standard SEMR one-step AFEM.

Algorithm 5.4 (GALERKIN). Let $\mathcal{T} \ge \mathcal{T}_0$ be a conforming refinement of a suitable initial mesh $\mathcal{T}_0$. Let data $\mathcal{D} = (\boldsymbol{A}, c, f) \in D_{\mathcal{T}}$ be discrete on $\mathcal{T}$ and let $\varepsilon > 0$ be a stopping tolerance. The following one-step AFEM creates a conforming refinement $\mathcal{T}_* \ge \mathcal{T}$ and Galerkin solution $u_{\mathcal{T}_*} \in \mathbb{V}_{\mathcal{T}_*}$ for data $\mathcal{D}$ such that $\eta_{\mathcal{T}_*}(u_{\mathcal{T}_*}) \le \varepsilon$:

$[\mathcal{T}_*, u_{\mathcal{T}_*}] = \mathsf{GALERKIN}(\mathcal{T}, \mathcal{D}, \varepsilon)$
  set $j = 0, \mathcal{T}_0 = \mathcal{T}$
  do
    $[u_j] = \mathsf{SOLVE}(\mathcal{T}_j)$
    $[\{\eta_j(u_j, T)\}_{T \in \mathcal{T}_j}] = \mathsf{ESTIMATE}(u_j, \mathcal{T}_j, \mathcal{D})$
    if $\eta_j(u_j) \le \varepsilon$
      return $\mathcal{T}_j, u_j$
    $[\mathcal{M}_j] = \mathsf{MARK}(\{\eta_j(u_j, T)\}_{T \in \mathcal{T}_j}, \mathcal{T}_j, \theta)$
    $[\mathcal{T}_{j+1}] = \mathsf{REFINE}(\mathcal{T}_j, \mathcal{M}_j)$
    $j \leftarrow j + 1$
  while true

5.2.2. *Contraction property of* GALERKIN

A key question to ask is what is (are) the quantity(ies) that GALERKIN may contract. In light of (5.7), an obvious candidate is the energy error $|||u - u_j|||_\Omega$, where $u_j \in \mathbb{V}_j = \mathbb{V}_{\mathcal{T}_j}$ solves the problem

$$\mathcal{B}[u_j, w] = \langle f, w \rangle \quad \text{for all } w \in \mathbb{V}_j. \tag{5.25}$$

We now show that this is in fact the case for discrete data $\mathcal{D} \in \mathbb{D}_{\mathcal{T}}$ provided the *discrete local estimate* (5.17) holds. The latter is a consequence of the *interior vertex property* of Definition 4.50 whenever $\boldsymbol{A}$ is piecewise constant, $c = 0$ in $\mathcal{T}$, and data oscillation vanishes, i.e. $\operatorname{osc}_{\mathcal{T}}(f)_{-1} = 0$ (Morin *et al.* 2000, 2002).

Lemma 5.5 (contraction property with discrete lower bound). *Let data $\mathcal{D} \in \mathbb{D}_{\mathcal{T}}$ be discrete and let $u = u(\mathcal{D}) \in H^1_0(\Omega)$ be the corresponding exact solution. If the subset $\mathcal{M}_j \subset \mathcal{T}_j$ of elements marked by* MARK *satisfies the discrete local estimate* (5.17) *with respect to $\mathcal{T}_{j+1} \ge \mathcal{T}_j$, then for*

$$\alpha := \left(1 - \left(\theta \frac{C_2}{C_1}\right)^2\right)^{1/2} < 1,$$

the Galerkin solutions $u_j \in \mathbb{V}_j, u_{j+1} \in \mathbb{V}_{j+1}$ *of* (5.25) *satisfy*

$$|||u - u_{j+1}|||_\Omega \le \alpha |||u - u_j|||_\Omega, \tag{5.26}$$

where $0 < \theta < 1$ *is the parameter in* (5.23) *and* $C_1 \ge C_2$ *are the constants in* (5.13) *and* (5.17) *respectively.*

Proof. For convenience, we use the notation

$$e_j = |||u - u_j|||_\Omega, \quad E_j = |||u_{j+1} - u_j|||_\Omega, \quad \eta_j = \eta_j(u_j, \mathcal{T}_j), \quad \eta_j(\mathcal{M}_j) = \eta_j(u_j, \mathcal{M}_j)$$

and recall that $\mathcal{E}_{\mathcal{T}_j}(u_j, f) = \eta_j$ because $\operatorname{osc}_{\mathcal{T}_j}(f)_{-1} = 0$. The key idea is to use the Pythagoras equality (5.7), namely $e_{j+1}^2 = e_j^2 - E_j^2$, and show that E_j is a significant portion of e_j. Since (5.17) implies

$$C_2 \eta_j(\mathcal{M}_j) \le E_j,$$

applying Dörfler marking (5.23) and the upper bound in (5.13), we deduce

$$E_j^2 \ge C_2^2 \theta^2 \eta_j^2 \ge \left(\theta \frac{C_2}{C_1}\right)^2 e_j^2.$$

This is the desired property of E_j and leads to (5.26). □

The contraction property (5.26) is very special and only valid for the energy norm. For the H_0^1-norm we have the following simple but useful consequence.

Corollary 5.6 (linear convergence). *If* $c_{\mathcal{B}} \le C_{\mathcal{B}}$ *are the constants in* (5.3)*, then*

$$|u - u_k|_{H_0^1(\Omega)} \le \sqrt{\frac{C_{\mathcal{B}}}{c_{\mathcal{B}}}}\, \alpha^{k-j}\, |u - u_j|_{H_0^1(\Omega)}, \quad k \ge j \ge 0.$$

We wonder whether or not the interior vertex property is necessary for (5.17), and thus for (5.26). We present an example, introduced by Morin *et al.* (2000, 2002) to justify such a property for constant data and $n = 1$.

Example 5.7 (lack of strict error monotonicity). Let $\Omega = (0,1)^2$, $\boldsymbol{A} = \boldsymbol{I}$, $c = 0$, $f = 1$ (constant data), and consider the sequences of meshes depicted in Figure 5.1. If ϕ_0 denotes the basis function associated with the only interior vertex of the initial mesh $\mathcal{T}_0$, then $u_0 = u_1 = \frac{1}{12}\phi_0$ and $u_2 \ne u_1$.

The mesh $\mathcal{T}_1 \ge \mathcal{T}_0$ is produced by a standard two-step bisection ($b = 2$) in two dimensions. Since $u_0 = u_1$, we conclude that the energy error does not change $|||u - u_0|||_\Omega = |||u - u_1|||_\Omega$, whence (5.17) fails, between two consecutive steps of GALERKIN for $b = d = 2$. This is no longer true provided an interior vertex in each marked element is created, because then Lemma 5.5 (contraction property with discrete lower bound) holds.

Circumventing the discrete lower bound. Enforcing (5.17) requires a minimal number b_* of bisections, say $b_* = 3, 6$ for $d = 2, 3$, to guarantee the interior vertex property. This can be quite taxing, especially for $d = 3$, and relies on the strong

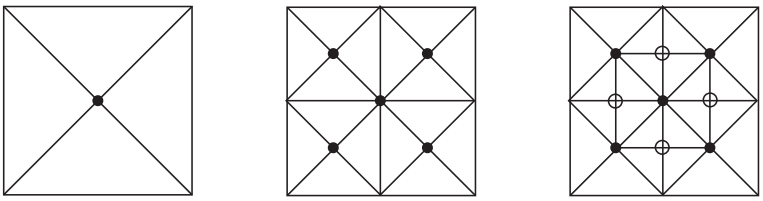


Figure 5.1. Grids $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ of Example 5.7. The mesh $\mathcal{T}_1$ has nodes in the middle of edges of $\mathcal{T}_0$, but only $\mathcal{T}_2$ has nodes in the interior of elements of $\mathcal{T}_0$. Hence $\mathcal{T}_2$ satisfies the interior vertex property of Definition 4.50 with respect to $\mathcal{T}_0$ whereas $\mathcal{T}_1$ does not.

assumption of $\boldsymbol{A}$ being piecewise constant and $c = 0$ on $\mathcal{T}$. It is clear from the preceding discussion that the energy error alone cannot be expected to contract between consecutive iterates. We explore next what quantity to monitor instead of the energy error in the analysis, with the aim of avoiding (5.17) and building a theory applicable to general discrete coefficients $(\boldsymbol{A}, c)$. This exploits the special structure of residual estimators and does not directly extend to non-residual estimators.

Heuristics. According to (5.7), the energy error is monotone $|||u - u_{j+1}|||_\Omega \leq |||u - u_j|||_\Omega$, but the previous example shows that strict inequality may fail. However, if $u_{j+1} = u_j$, estimate (4.67) reveals a strict estimator reduction $\eta_{j+1}(u_{j+1}) < \eta_j(u_j)$. We thus expect that, for a suitable scaling factor $\gamma > 0$, the so-called *quasi-error*

$$\zeta_j^2(u_j) := |||u - u_j|||_\Omega^2 + \gamma\, \eta_j^2(u_j) \tag{5.27}$$

may contract. This heuristic illustrates a distinct aspect of AFEM theory, the interplay between continuous quantities, such as the energy error $|||u - u_j|||_\Omega$, and discrete quantities, such as the estimator $\eta_j(u_j)$: neither one alone has the requisite properties to yield a contraction between consecutive adaptive steps. This result was originally proved by Cascón *et al.* (2008).

Theorem 5.8 (general contraction property). *Let $\mathcal{D} \in \mathbb{D}_\mathcal{T}$ be discrete data. Let $\theta \in (0, 1]$ be the Dörfler marking parameter, and let $\{\mathcal{T}_j, \mathbb{V}_j, u_j\}_{j=0}^\infty$ be a sequence of conforming meshes, finite element spaces and discrete solutions $u_j \in \mathbb{V}_j$ created by* GALERKIN *for the model problem (5.25). If $u = u(\mathcal{D}) \in H_0^1(\Omega)$ is the exact solution of (5.5), then there exist constants $\gamma > 0$ and $0 < \alpha < 1$, additionally depending on the number $b \geq 1$ of bisections and θ, such that for all $j \geq 0$*

$$|||u - u_{j+1}|||_\Omega^2 + \gamma\, \eta_{j+1}^2(u_{j+1}) \leq \alpha^2 \left(|||u - u_j|||_\Omega^2 + \gamma\, \eta_j^2(u_j) \right). \tag{5.28}$$

Proof. We split the proof into four steps and use the notation in Lemma 5.5 (contraction property with discrete lower bound).

1 The error orthogonality (5.7) reads

$$e_{j+1}^2 = e_j^2 - E_j^2. \tag{5.29}$$

Employing Proposition 4.56 (estimator reduction) with $\mathcal{T} = \mathcal{T}_j$, $\mathcal{T}_* = \mathcal{T}_{j+1}$, $v = u_j$, $v_* = u_{j+1}$ and $f = f_* \in \mathbb{F}_j$ gives

$$\eta_{j+1}^2 \le (1+\delta)\big(\eta_j^2 - \lambda\,\eta_j^2(\mathcal{M}_j)\big) + (1+\delta^{-1})\,C_{\mathrm{Lip}}^2\,E_j^2. \tag{5.30}$$

After multiplying (5.30) by $\gamma > 0$, to be determined later, we add (5.29) and (5.30) to obtain

$$e_{j+1}^2 + \gamma\,\eta_{j+1}^2 \le e_j^2 + \big(\gamma\,(1+\delta^{-1})\,C_{\mathrm{Lip}}^2 - 1\big)\,E_j^2 + \gamma\,(1+\delta)\,\big(\eta_j^2 - \lambda\,\eta_j^2(\mathcal{M}_j)\big).$$

2 We now choose the parameters δ, γ: let δ satisfy

$$(1+\delta)(1-\lambda\theta^2) = 1 - \frac{\lambda\theta^2}{2},$$

and let γ verify

$$\gamma\,(1+\delta^{-1})\,C_{\mathrm{Lip}}^2 = 1.$$

Note that this choice of γ yields

$$e_{j+1}^2 + \gamma\,\eta_{j+1}^2 \le e_j^2 + \gamma\,(1+\delta)\,\big(\eta_j^2 - \lambda\,\eta_j^2(\mathcal{M}_j)\big). \tag{5.31}$$

3 We next employ Dörfler marking (5.23), namely $\eta_j(\mathcal{M}_j) \ge \theta\eta_j$, to deduce

$$e_{j+1}^2 + \gamma\,\eta_{j+1}^2 \le e_j^2 + \gamma(1+\delta)(1-\lambda\theta^2)\eta_j^2.$$

This, in conjunction with the choice of δ, gives

$$e_{j+1}^2 + \gamma\,\eta_{j+1}^2 \le e_j^2 + \gamma\left(1 - \frac{\lambda\theta^2}{2}\right)\eta_j^2, \tag{5.32}$$

which we write as

$$e_{j+1}^2 + \gamma\,\eta_{j+1}^2 \le e_j^2 - \frac{\gamma\lambda\theta^2}{4}\eta_j^2 + \gamma\left(1 - \frac{\lambda\theta^2}{4}\right)\eta_j^2.$$

4 Finally, the upper bound in (5.13), namely $e_j \le C_1\,\eta_j$, implies that

$$e_{j+1}^2 + \gamma\,\eta_{j+1}^2 \le \left(1 - \frac{\gamma\lambda\theta^2}{4C_1^2}\right)e_j^2 + \gamma\left(1 - \frac{\lambda\theta^2}{4}\right)\eta_j^2.$$

This in turn leads to

$$e_{j+1}^2 + \gamma\,\eta_{j+1}^2 \le \alpha^2\big(e_j^2 + \gamma\,\eta_j^2\big),$$

with

$$\alpha^2 := \max\left\{1 - \frac{\gamma\lambda\theta^2}{4C_1^2}, 1 - \frac{\lambda\theta^2}{4}\right\} < 1,$$

and thus concludes the proof of the theorem. □

Remark 5.9 (basic ingredients). This proof solely uses Dörfler marking (5.23), the Pythagoras identity (5.7), the *a posteriori* upper bound in (5.13), and Proposition 4.56 (estimator reduction). The proof altogether circumvents use of the lower bound in (5.13) and the discrete lower bound (5.17).

The contraction property (5.28) is valid for a suitable combination of the energy norm $|||u - u_j|||_\Omega$ and the PDE estimator $\eta_j(u_j)$. We cannot expect this type of result for the underlying space norm $|u - u_j|_{H_0^1(\Omega)}$. We instead have the following statement, whose structure reflects the possible stagnation of $|u - u_j|_{H_0^1(\Omega)}$ during the refinement process, as documented in Example 5.7.

Corollary 5.10 (linear convergence of error). *If the assumptions of Theorem 5.8 are valid, and* $0 < \alpha < 1, \gamma > 0$ *are the constants in* (5.28)*, then*

$$|u - u_k|_{H_0^1(\Omega)} \le C_* \alpha^{k-j} |u - u_j|_{H_0^1(\Omega)} \quad \textit{for all } k \ge j \ge 0, \tag{5.33}$$

with

$$C_* = \left(\frac{C_{\mathcal{B}}}{c_{\mathcal{B}}} \left(1 + \frac{\gamma}{C_2^2} \right) \right)^{1/2} > 1$$

and constants $C_{\mathcal{B}} \ge c_{\mathcal{B}} > 0$ *and* $C_2 > 0$ *given in* (5.3) *and* (5.13) *respectively.*

Proof. Simply concatenate (5.3), (5.28) and (5.13) to obtain

$$\begin{aligned} c_{\mathcal{B}} |u - u_k|^2_{H_0^1(\Omega)} &\le |||u - u_k|||^2_\Omega + \gamma\, \eta_k(u_k)^2 \\ &\le \alpha^{2(k-j)} \left(|||u - u_j|||^2_\Omega + \gamma\, \eta_j(u_j)^2 \right) \\ &\le \alpha^{2(k-j)} \left(C_{\mathcal{B}} \left(1 + \frac{\gamma}{C_2^2} \right) \right) |u - u_j|^2_{H_0^1(\Omega)}. \end{aligned}$$

This implies (5.33) and concludes the proof. □

We stress that, in contrast to (5.28), (5.33) relies on the lower bound in (5.13). This is not the case if we express linear convergence in terms of the PDE estimator. The proof is similar to the preceding one and is omitted.

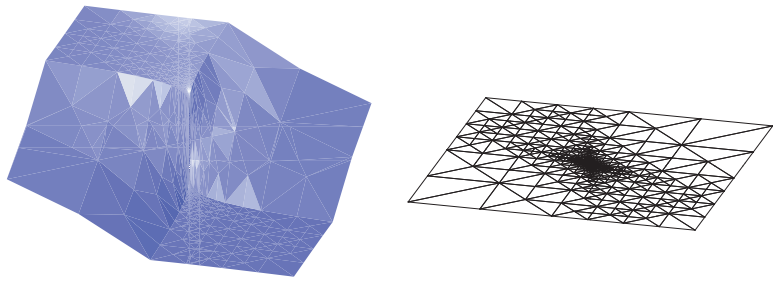
Corollary 5.11 (linear convergence of estimator). *If the assumptions of Theorem 5.8 are valid, and* $0 < \alpha < 1, \gamma > 0$ *are the constants in* (5.28)*, then*

$$\eta_k(u_k) \le C_\# \alpha^{k-j} \eta_j(u_j) \quad \textit{for all } k \ge j \ge 0, \tag{5.34}$$

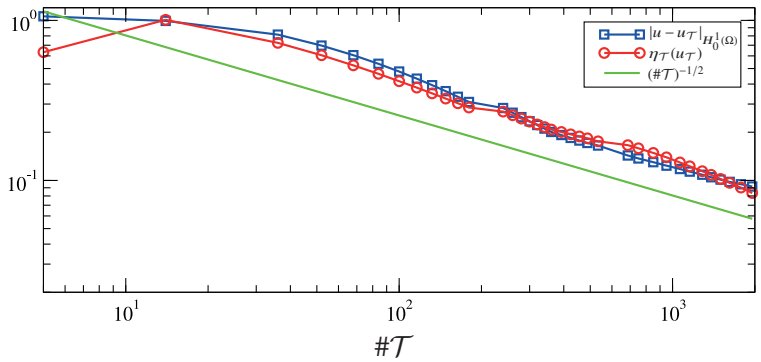
with $C_\# = (1 + C_1^2/\gamma)^{1/2} > 1$ *and* C_1 *given in* (5.13).

Remark 5.12 (stopping). In view of (5.34), (5.12), we realize that GALERKIN requires $j \le J$ iterations until the stopping criterion $\eta_j \le \varepsilon$ is satisfied and delivers the error $|u - u_j|_{H_0^1(\Omega)} \le C_U \varepsilon$, where

$$J \le 1 + \frac{\log(\varepsilon/(C_\# \eta_0))}{\log \alpha}.$$



(a)



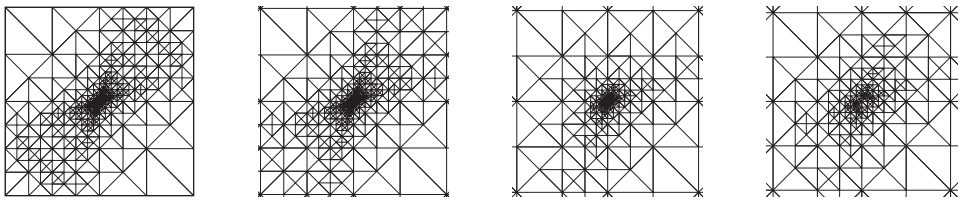
(b)

Figure 5.2. Discontinuous coefficients in a checkerboard pattern: (a) graph of the discrete solution u, which is $u \approx r^{0.1}$, and underlying strongly graded grid $\mathcal{T}$ towards the origin (notice the steep gradient of u at the origin); (b) estimate and true error in terms of $\#\mathcal{T}$ (the optimal decay for piecewise linear elements in two dimensions is indicated by the green line with slope $-1/2$).

5.2.3. *Discontinuous coefficients: Kellogg's example*

We examine a simple yet quite demanding example with piecewise constant coefficients in a checkerboard pattern for $d = 2$ due to Kellogg (1974/75), and used by Morin *et al.* (2000, 2002) as a benchmark for GALERKIN. We consider $\Omega = (-1, 1)^2$, $\boldsymbol{A} = a_1\boldsymbol{I}$ in the first and third quadrants, and $\boldsymbol{A} = a_2\boldsymbol{I}$ in the second and fourth quadrants. This checkerboard pattern is the worst for the regularity of the solution u at the origin. For $f = c = 0$, a function of the form $u(r, \theta) = r^\gamma \mu(\theta)$ in polar coordinates solves (2.5) with non-vanishing Dirichlet condition for suitable $0 < \gamma < 2$ and μ (Morin *et al.* 2000, 2002, Nochetto *et al.* 2009). We choose $\gamma = 0.1$, which leads to $u \in H^s(\Omega)$ for $1 \le s < 1.1$ and piecewise in W_p^2 for some $p > 1$. This corresponds to diffusion coefficients $a_1 \approx 161.44$ and $a_2 = 1$, which can be computed via Newton's method; the closer γ is to 0, the larger is the ratio a_1/a_2. The solution u and a sample mesh are depicted in Figure 5.2(a).

Figure 5.2(b) documents the optimal performance of GALERKIN: both the energy error and estimator exhibit optimal decay $(\#\mathcal{T})^{-1/2}$ in terms of the cardinality $\#\mathcal{T}$



(a) (b) (c) (d)

Figure 5.3. Discontinuous coefficients in a checkerboard pattern: (a) final grid $\mathcal{T}$ highly graded towards the origin with cardinality $\#\mathcal{T} \approx 2000$; (b) zoom to $(-10^{-3}, 10^{-3})^2$; (c) zoom to $(-10^{-6}, 10^{-6})^2$; (d) zoom to $(-10^{-9}, 10^{-9})^2$. For a similar resolution, a uniform grid $\mathcal{T}$ would require cardinality $\#\mathcal{T} \approx 10^{20}$.

of the underlying mesh $\mathcal{T}$ for piecewise linear finite elements. On the other hand, Figure 5.3 displays a strongly graded mesh $\mathcal{T}$ towards the origin generated by GALERKIN using bisection, and three zooms which reveal a self-similar structure. It is worth stressing that the mesh size is of order 10^{-10} at the origin and that $\#\mathcal{T} \approx 2 \times 10^3$, whereas to reach a similar resolution with a uniform mesh $\mathcal{T}$ we would need $\#\mathcal{T} \approx 10^{20}$. This example clearly reveals that adaptivity can restore optimal performance even with modest computational resources.

Classical FEMs with quasi-uniform meshes $\mathcal{T}$ require regularity $u \in H^2(\Omega)$ to deliver an optimal convergence rate $(\#\mathcal{T})^{-1/2}$ with polynomial degree $n = 1$. Since $u \notin H^s(\Omega)$ for any $s > 1.1$, this is not possible for the example above. However, the problem is not quite the lack of second derivatives, but rather the fact that they are not square integrable. In fact, the function u is in W^2_p for $p > 1$ in each quadrant, and so over the initial mesh $\mathcal{T}_0$, namely $u \in W^2_p(\Omega; \mathcal{T}_0)$. The computational rate of convergence $(\#\mathcal{T})^{-1/2}$ is consistent with Corollary 3.20. We will prove that GALERKIN delivers this rate in Section 6.

5.3. *Data oscillation: one-step* AFEM *with switch*

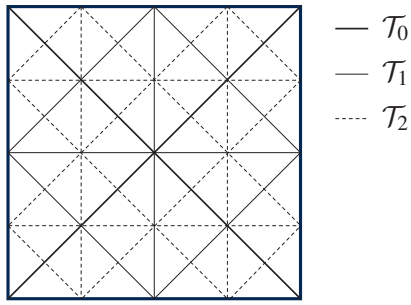
In Section 5.2 we assumed that the full data $\mathcal{D} = (\boldsymbol{A}, c, f) \in \mathbb{D}_{\mathcal{T}}$ is discrete, and in particular $f = P_{\mathcal{T}} f \in \mathbb{F}_{\mathcal{T}}$. The finite-dimensional nature of $\widehat{\mathcal{D}}$ allowed us to develop a rather simple theory of convergence for GALERKIN, the one-step AFEM, that hinges exclusively on the PDE local error indicator $\eta_{\mathcal{T}}(u_{\mathcal{T}}, T)$ defined in (5.8). We now keep $(\boldsymbol{A}, c)$ discrete, whence the elliptic operator in (2.5) includes the Laplacian, but explore the role of a general forcing $f \neq P_{\mathcal{T}} f$. Therefore, in contrast to (5.22), we now investigate the effect of data oscillation (5.11),

$$\operatorname{osc}_{\mathcal{T}}(f)^2_{-1} = \sum_{T \in \mathcal{T}} \|f - P_{\mathcal{T}} f\|^2_{H^{-1}(\omega_T)}$$

for any $\mathcal{T} \in \mathbb{T}$, and present a linear convergence theory. We recall from Theorem 4.45 (modified residual estimator) that the total error estimator $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)^2 =$

1	−1	1	−1	1	−1	1	−1
−1	1	−1	1	−1	1	−1	1
1	−1	1	−1	1	−1	1	−1
−1	1	−1	1	−1	1	−1	1
1	−1	1	−1	1	−1	1	−1
−1	1	−1	1	−1	1	−1	1
1	−1	1	−1	1	−1	1	−1
−1	1	−1	1	−1	1	−1	1

(a)



(b)

Figure 5.4. Representation of the checkerboard function f of Example 5.13 for $m = 3$ (a), and grids $\mathcal{T}_k$ for $k = 0, 1, 2$ (b).

$\eta_{\mathcal{T}}(u_{\mathcal{T}})^2 + \operatorname{osc}_{\mathcal{T}}(f)^2_{-1}$ is equivalent to the H^1-error, namely

$$C_L \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f) \le \|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)} \le C_U \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f). \tag{5.35}$$

As in the previous section, to simplify notation we do not use the hat symbol to indicate quantities defined with the discrete data $(\boldsymbol{A}, c)$.

5.3.1. Role of data oscillation

At first sight, it might seem that Example 5.7 (lack of strict error monotonicity) is very special and can only occur at the beginning of the refinement process. We now show that this situation can happen at any stage and that even an interior vertex property may not guarantee error or data oscillation decrease.

Example 5.13 (interior vertex). Let the polynomial degree be $n = 1$, fix $m \in \mathbb{N}$ and consider (5.21) with $\boldsymbol{A} = \boldsymbol{I}$ the identity matrix, $c = 0$, $\Omega = (0, 1)^2$ and checkerboard f given by the following expression and depicted in Figure 5.4(a):

$$f(x) = \begin{cases} 1, & \text{if } x \in (i\,2^{-m}, (i+1)\,2^{-m}) \times (j\,2^{-m}, (j+1)\,2^{-m}) \text{ and } i + j \text{ odd}, \\ -1, & \text{otherwise.} \end{cases}$$

We start with the same mesh $\mathcal{T}_0$ with four elements as in Example 5.7, and construct recursively grids $\mathcal{T}_{k+1} \in \mathbb{T}$, $k \ge 0$, as a conforming refinement of $\mathcal{T}_k \in \mathbb{T}$ via two newest-vertex bisections of every triangle of $\mathcal{T}_k$; see Figure 5.4(b). Since f is L^2-orthogonal to every piecewise linear basis function of the space $\mathbb{V}_{\mathcal{T}_k} = \mathbb{S}^{1,0}_{\mathcal{T}_k}$ for $0 \le k \le m - 1$, we deduce that $u_{\mathcal{T}_k} = 0$ and the energy error does not change

$$|||u - u_{\mathcal{T}_k}|||_\Omega = |||u - u_{\mathcal{T}_0}|||_\Omega, \quad 0 \le k \le m - 1. \tag{5.36}$$

We see that this procedure creates three interior vertices in every triangle of $\mathcal{T}_k$ after two refinement steps, namely in $\mathcal{T}_{k+2}$ as long as $k + 2 \le m$. Since the error does not change, we conclude that the interior vertex property is necessary for error

reduction but is not sufficient in the presence of data oscillation $\operatorname{osc}_{\mathcal{T}_k}(f)_{-1} \neq 0$. We conclude that

Data oscillation $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ *is not generally of higher order than the error, especially in the early stages of the adaptive process.* (5.37)

On the other hand, for $k = m$ the discrete solution $u_{\mathcal{T}_m}$ no longer vanishes globally, but is still zero along the lines where f changes sign due to the symmetry of the problem, and the same happens with $u_{\mathcal{T}_{m+1}}$. Therefore the behaviour of $u_{\mathcal{T}m}$ and $u_{\mathcal{T}_{m+1}}$ in a fixed square, where f is constant, is exactly the same as in Example 5.7. This implies that $u_{\mathcal{T}_m} = u_{\mathcal{T}_{m+1}}$, and illustrates that the rather special situation of Example 5.7 can occur at any stage of the refinement process.

Example 5.14 (vanishing of $P_{\mathcal{T}} f$ for $n = 1$). Since $P_{\mathcal{T}} f$ is constructed locally upon testing f against cubic and quadratic bubbles (see Remark 4.26 (local computation)), and f of Example 5.13 is highly oscillatory, we realize that $P_{\mathcal{T}_k} f$ is rather small relative to f in $H^{-1}(\Omega)$, but it is not zero. This is due to the lack of complete symmetry of the checkerboard pattern and the triangular grid. Suppose that each square of Figure 5.4, where $f = \pm 1$, is further split across the diagonals into four triangles, and that f is assigned the alternating values ± 1 and ∓ 1 in each triangle depending on whether f was originally 1 or -1 in that square; this configuration is displayed in Figure 5.5. Suppose further that the coefficients $(\boldsymbol{A}, c)$ of the operator (2.5) are piecewise constant, as happens for the Laplacian, the polynomial degree is $n = 1$, and the definition (4.39) of P_T over a triangle $T \in \mathcal{T}$ uses $q \in \mathbb{P}_0$ rather than $\mathbb{P}_1$. In light of (4.39) and (4.40), symmetry yields, for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$,

$$\int_T f \phi_T = 0 \quad \Rightarrow \quad P_T f = 0, \qquad \int_F f \phi_F = 0 \quad \Rightarrow \quad P_F f = 0, \tag{5.38}$$

whence $P_{\mathcal{T}} f = 0$. Since also $u_{\mathcal{T}} = 0$ because f is orthogonal to all basis functions of $\mathbb{V}_{\mathcal{T}}$, we deduce $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f) = 0$, and all the information about the error $|\!|\!| u - u_{\mathcal{T}} |\!|\!|_\Omega \neq 0$ resides in the data oscillation $\operatorname{osc}_{\mathcal{T}}(f)_{-1} \neq 0$. Moreover, the fact that $P_{\mathcal{T}} f = 0$ for several iterations reveals the important property that $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ may not change upon refinement because

$$\operatorname{osc}_{\mathcal{T}}(f)_{-1}^2 = \sum_{T \in \mathcal{T}} \|f\|^2_{H^{-1}(\omega_T)}. \tag{5.39}$$

Since $|\!|\!| u - u_{\mathcal{T}} |\!|\!|_\Omega \approx \operatorname{osc}_{\mathcal{T}}(f)_{-1}$, according to (4.45), special care must be exercised to reduce data oscillation when it dominates. This justifies the structure of Algorithm 5.16 (AFEM-SW) below.

Example 5.15 (vanishing of $P_{\mathcal{T}} f$ for $n > 1$). Given $n \geq 1$ a polynomial degree and $\mathcal{T}_k$, $k = 1, \ldots, m$, uniform refinements of $\mathcal{T}_0$, there are finitely many conditions to verify for $f \in H^{-1}(\Omega)$ to be orthogonal to $\mathbb{V}_{\mathcal{T}_k}$ and to $\mathbb{F}_{\mathcal{T}_k}$. Since $\dim H^{-1}(\Omega) = \dim L^2(\Omega) = \infty$, there are infinitely many loads $f \in H^{-1}(\Omega)$ as well as in $L^2(\Omega)$ that yield $u_{\mathcal{T}_k} = P_{\mathcal{T}_k} f = 0$, which implies (5.36). Moreover, $\eta_{\mathcal{T}_k}(u_k) = 0$ and $\mathcal{E}_{\mathcal{T}_k}(u_{\mathcal{T}_k}) = \operatorname{osc}_{\mathcal{T}_k}(f)_{-1}$ satisfies (5.39). One explicit example is as follows.

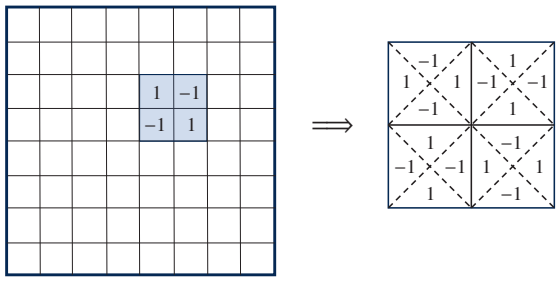


Figure 5.5. Refinement of the shaded area according to the process described in Example 5.14.

Given an initial mesh $\mathcal{T}_0$, suppose that f consists of line Dirac masses supported on the skeleton of $\mathcal{T}_0$ with densities g_F on $F \in \mathcal{F}_{\mathcal{T}_0}$ made of piecewise polynomials of degree $2n+1$. We further assume that the g_F are orthogonal to $\mathbb{P}_{2n}$ over F as well as over all sub-faces obtained from $m \geq 1$ uniform refinements of $\mathcal{T}_0$; see Figure 5.4. In such a situation, (5.38) applies and $u_{\mathcal{T}_k} = P_{\mathcal{T}_k} f = 0$, whence (5.36) and (5.39) are valid for $0 \leq k \leq m$.

These three examples reveal the following crucial and novel feature about the interplay of the energy error $|||u - u_{\mathcal{T}}|||_\Omega$ and data oscillation $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$:

Data oscillation $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ *may be responsible for the energy error* $|||u - u_{\mathcal{T}}|||_\Omega$ *to stagnate, even with the interior vertex property, and may entirely dominate it relative to the error estimator* $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)$ *over many mesh refinements unless it is reduced.* (5.40)

5.3.2. Reducing data oscillation

The PDE error estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ in (5.8) is fully discrete and thus computable. In contrast, the computation, or rather estimation, of $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ hinges on *a priori* knowledge of f and cannot be assessed in general. Assuming that the local indicators introduced in Lemma 4.8 (localization re-indexing),

$$\operatorname{osc}_{\mathcal{T}}(f, T)_{-1} = \|f - P_{\mathcal{T}} f\|_{H^{-1}(\omega_T)}, \quad T \in \mathcal{T}, \tag{5.41}$$

are computable without further regularity than $f \in H^{-1}(\Omega)$, it is natural to think of *tree approximation* as the algorithm of choice to reduce $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ (Binev and DeVore 2004, Binev, Fierro and Veeser 2023, Binev 2018). However, this optimal algorithm is not readily applicable because of the lack of a suitable sub-additivity property.

On the other hand, greedy algorithms, such as that in Section 3.6 (constructive approximation), do not work under minimal regularity. In Section 7.3 we present practical examples of rough f for which $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ can be replaced by a larger computable surrogate estimator $\widetilde{\operatorname{osc}}_{\mathcal{T}}(f)_{-1}$. The latter splits into element contributions

and is amenable to a greedy strategy. Since this is specialized and technical, we prefer to postpone the full discussion to Section 7.3 and now assume the existence of a module DATA with the following property: given a tolerance $\tau > 0$ and a conforming mesh $\mathcal{T} \in \mathbb{T}$, DATA constructs a conforming refinement $\mathcal{T}_* \in \mathbb{T}$,

$$[\mathcal{T}_*] = \mathsf{DATA}(\mathcal{T}, f, \tau),$$

such that $\operatorname{osc}_{\mathcal{T}_*}(f)_{-1} \leq \tau$. The complexity of DATA depends on the decay rate of the best approximation error $\min_{\mathcal{T}\in\mathbb{T}_N} \widetilde{\operatorname{osc}}_{\mathcal{T}}(f)_{-1}$ of f with N degrees of freedom. We address this important issue in Section 7.3 for each example separately.

5.3.3. *Linear convergence*

The following algorithm, AFEM-SW, a one-step AFEM with switch, is a minor, but essential, modification of GALERKIN in that the call to the modules MARK and REFINE is conditional on the size of $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ relative to $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)$. This structure is consistent with the heuristic discussion by Cascón *et al.* (2008, Section 6) to avoid separate marking. A similar algorithm is being developed in Kreuzer *et al.* (2024).

Algorithm 5.16 (AFEM-SW). Let $\mathcal{T}_0$ be a suitable initial mesh, let the coefficients $(\boldsymbol{A}, c)$ be discrete over $\mathcal{T}_0$, and let $\varepsilon > 0$ be a stopping tolerance. Given parameters $0 < \theta, \omega, \xi < 1$, AFEM-SW iterates the following loop until $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f) \leq \varepsilon$:

$[\mathcal{T}, u_{\mathcal{T}}] = \mathsf{AFEM\text{-}SW}(\mathcal{T}_0, \mathcal{D}, \varepsilon)$
  set $j = 0$
  do
    $[u_{\mathcal{T}_j}] = \mathsf{SOLVE}(\mathcal{T}_j)$
    $[\eta_{\mathcal{T}_j}(u_{\mathcal{T}_j}), \operatorname{osc}_{\mathcal{T}_j}(f)_{-1}] = \mathsf{ESTIMATE}(u_{\mathcal{T}_j}, \mathcal{T}_j, \mathcal{D})$
    if $\mathcal{E}_{\mathcal{T}_j}(u_{\mathcal{T}_j}, f) \leq \varepsilon$
      return $\mathcal{T}_j, u_{\mathcal{T}_j}$
    else if $\operatorname{osc}_{\mathcal{T}_j}(f)_{-1} \leq \sigma_j := \omega\mathcal{E}_{\mathcal{T}_j}(u_{\mathcal{T}_j}, f)$
      $[\mathcal{M}_j] = \mathsf{MARK}(\{\eta_{\mathcal{T}_j}(u_{\mathcal{T}_j}, T)\}_{T\in\mathcal{T}_j}, \mathcal{T}_j, \theta)$
      $[\mathcal{T}_{j+1}] = \mathsf{REFINE}(\mathcal{T}_j, \mathcal{M}_j)$
    else
      $[\mathcal{T}_{j+1}] = \mathsf{DATA}(\mathcal{T}_j, f, \xi\sigma_j)$
    $j \leftarrow j + 1$
  while true

Note that SOLVE computes the Galerkin approximation using the exact right-hand side $f \in H^{-1}(\Omega)$ (not necessarily in $\mathbb{F}_{\mathcal{T}_j}$), thereby preserving the Galerkin orthogonality property. Moreover, ESTIMATE is now responsible for computing the PDE estimator

$$\eta_{\mathcal{T}_j}(u_{\mathcal{T}_j}) = \eta_{\mathcal{T}_j}(u_{\mathcal{T}_j}, f, \mathcal{T}_j)$$

using $P_{\mathcal{T}_j} f \in \mathbb{F}_{\mathcal{T}_j}$, as well as data oscillation $\operatorname{osc}_{\mathcal{T}_j}(f)_{-1}$, which together give

$$\mathcal{E}_{\mathcal{T}_j}(u_{\mathcal{T}_j}, f) = \left(\eta_{\mathcal{T}_j}(u_{\mathcal{T}_j})^2 + \operatorname{osc}_{\mathcal{T}_j}(f)_{-1}^2\right)^{1/2},$$

and MARK consists of Dörfler marking (5.23) with parameter θ.

We proceed as in Section 5.2.2 to prove linear convergence of AFEM-SW. We first show a contraction property for the *quasi-error*, which instead of (5.27) reads

$$\zeta_{\mathcal{T}_j}(u_{\mathcal{T}_j}, f)^2 := |\!|\!| u - u_{\mathcal{T}_j} |\!|\!|_\Omega^2 + \gamma \eta_{\mathcal{T}_j}(u_{\mathcal{T}_j})^2 + \operatorname{osc}_{\mathcal{T}_j}(f)_{-1}^2, \tag{5.42}$$

where $u = u(\boldsymbol{A}, c, f)$ is the Galerkin solution with $(\boldsymbol{A}, c)$ discrete but f exact and the scaling parameter satisfies $0 < \gamma \le 1$.

Theorem 5.17 (contraction property of AFEM-SW). *Let $(\boldsymbol{A}, c)$ be discrete coefficients over $\mathcal{T}_0$ and let $f \in H^{-1}(\Omega)$. Let $\theta \in (0, 1]$ be the Dörfler parameter and let $(\mathcal{T}_j, \mathbb{V}_j, u_j)$ be the sequence of conforming meshes $\mathcal{T}_j$, finite element spaces $\mathbb{V}_j$, and Galerkin solutions $u_j \in \mathbb{V}_j$ produced by* AFEM-SW. *There exist parameters $0 < \omega_0 < 1$ sufficiently small and $0 < \gamma \le 1$ and $0 < \alpha < 1$ such that for any $\omega \le \omega_0$ and $\xi \le 1/2$, the quasi-error $\zeta_{\mathcal{T}_j}$ in (5.42) contracts*

$$\zeta_{\mathcal{T}_{j+1}}(u_{\mathcal{T}_{j+1}}, f) \le \alpha \zeta_{\mathcal{T}_j}(u_{\mathcal{T}_j}, f), \quad j \ge 0. \tag{5.43}$$

Proof. We argue as in Theorem 5.8 (general contraction property) upon distinguishing the two possible cases within Algorithm 5.16. But first we must account for a crucial difference: the discrete forcing function $P_{\mathcal{T}_j} f$ used in the definition of the estimator $\mathcal{E}_{\mathcal{T}_j}(u_{\mathcal{T}_j}, f)$ changes in each iteration. We use the same notation as in Theorem 5.8 along with $\operatorname{osc}_j := \operatorname{osc}_{\mathcal{T}_j}(f)_{-1}$, $\mathcal{E}_j^2 := \eta_j^2 + \operatorname{osc}_j^2$ and $P_j := P_{\mathcal{T}_j}$.

$\boxed{1}$ *Estimator reduction property.* In view of Proposition 4.56 (estimator reduction), we need to estimate the discrepancy between discrete forcing functions

$$\sum_{T \in \mathcal{T}_{j+1}} \|P_{j+1} f - P_j f\|_{H^{-1}(\omega_T)}^2 \le 2 \sum_{T \in \mathcal{T}_{j+1}} \big(\|f - P_{j+1} f\|_{H^{-1}(\omega_T)}^2 + \|f - P_j f\|_{H^{-1}(\omega_T)}^2 \big).$$

For the first term we recall Lemma 4.57 (quasi-monotonicity of oscillation) to write

$$\sum_{T \in \mathcal{T}_{j+1}} \|f - P_{j+1} f\|_{H^{-1}(\omega_T)}^2 = \operatorname{osc}_{j+1}^2 \le C_{\mathrm{osc}}^2 \operatorname{osc}_j^2 .$$

For the second term, instead, we combine the projection property $P_{j+1}(P_j f) = P_j f$ with Lemma 4.5 (localization of H^{-1}-norm) and Corollary 4.31 (local near-best approximation), and the fact that $\mathcal{T}_{j+1}$ is a refinement of $\mathcal{T}_j$, to see that

$$\begin{aligned} \sum_{T' \subset \omega_T} \|f - P_{j+1}(P_j f)\|_{H^{-1}(\omega_{T'})}^2 &\le C_{\mathrm{lStb}}^2 \sum_{T' \subset \omega_T} \|f - P_j f\|_{H^{-1}(\omega_{T'})}^2 \\ &\le C_{\mathrm{lStb}}^2 C_{\mathrm{ovrl}}^2 \operatorname{osc}_{\mathcal{T}_j}(f, \omega_T)_{-1}^2 \quad \text{for all } T \in \mathcal{T}_j. \end{aligned}$$

Adding over T and recalling Proposition 4.56, we end up with

$$\begin{aligned} \eta_{\mathcal{T}_{j+1}}(u_{j+1}, f, \mathcal{T}_{j+1})^2 \le{}& (1+\delta)\big(\eta_{\mathcal{T}_j}(u_j, f, \mathcal{T}_j)^2 - \lambda\, \eta_{\mathcal{T}_j}(u_j, f, \mathcal{M}_j)^2\big) \\ &+ (1+\delta^{-1})\, C_{\mathrm{Lip}}^2 \big(|u_j - u_{j+1}|_{H_0^1(\Omega)}^2 + \operatorname{osc}_j^2 \big), \end{aligned} \tag{5.44}$$

for a constant C_{Lip} large enough to absorb all preceding constants, and any $\delta > 0$.

$\boxed{2}$ *Case* $\operatorname{osc}_j \le \omega \mathcal{E}_j$. We first observe that $\eta_j^2 \ge (1-\omega^2)\mathcal{E}_j^2$ and $\operatorname{osc}_j^2 \le \omega^2(1-\omega^2)^{-1}\eta_j^2$. We then proceed as in Theorem 5.8 with the quantity $e_j^2 + \gamma\eta_j^2$, and observe that the choices of δ and γ,

$$\delta \le -1 + \frac{1-\lambda\theta^2/2}{1-\lambda\theta^2} = \frac{\lambda\theta^2}{2(1-\lambda\theta^2)}, \quad \gamma \le \frac{\delta}{4C_{\mathrm{Lip}}^2} \le \frac{1}{2(1+\delta^{-1})C_{\mathrm{Lip}}^2}, \tag{5.45}$$

imply $\gamma(1+\delta^{-1})C_{\mathrm{Lip}}^2 \le 1/2$. This, together with (5.7) and (5.44), leads to

$$e_{j+1}^2 + \gamma\eta_{j+1}^2 \le e_j^2 + \gamma\left(1 - \frac{\lambda\theta^2}{2}\right)\eta_j^2 + \frac{1}{2}\operatorname{osc}_j^2;$$

compare with (5.32). We invoke the upper bound in (5.13) to write

$$\eta_j^2 \ge (1-\omega^2)\mathcal{E}_j^2 \ge (1-\omega^2)\frac{e_j^2}{C_1^2} \ge \frac{e_j^2}{2C_1^2}$$

provided $\omega^2 \le 1/2$, whence

$$e_{j+1}^2 + \gamma\eta_{j+1}^2 \le \left(1 - \frac{\gamma\lambda\theta^2}{8C_1^2}\right)e_j^2 + \gamma\left(1 - \frac{\lambda\theta^2}{8}\right)\eta_j^2 - \frac{\gamma\lambda\theta^2}{8}\eta_j^2 + \frac{1}{2}\operatorname{osc}_j^2.$$

We next consider the data oscillation, for which we invoke Lemma 4.57 (quasi-monotonicity of oscillation):

$$\operatorname{osc}_{j+1} \le C_{\mathrm{osc}}\operatorname{osc}_j, \quad \operatorname{osc}_j^2 \le C_{\mathrm{osc}}^2\frac{\omega^2}{1-\omega^2}\eta_j^2 \le 2C_{\mathrm{osc}}^2\omega^2\eta_j^2.$$

Adding the two preceding inequalities yields

$$\begin{aligned}\zeta_{j+1}^2 = e_{j+1}^2 + \gamma\eta_{j+1}^2 + \operatorname{osc}_{j+1}^2 \le{}& \left(1 - \frac{\gamma\lambda\theta^2}{8C_1^2}\right)e_j^2 \\ &+ \left(1 - \frac{\lambda\theta^2}{8}\right)\left(\gamma\eta_j^2 + \operatorname{osc}_j^2\right) \\ &+ \left[-\gamma\frac{\lambda\theta^2}{8} + 2\left(C_{\mathrm{osc}}^2 - 1 + \frac{\lambda\theta^2}{8} + \frac{1}{2}\right)\omega^2\right]\eta_j^2.\end{aligned}$$

We drop the term $-1/2 + \lambda\theta^2/8 \le 0$ and let $\gamma = \delta/(4C_{\mathrm{Lip}}^2)$, which is consistent with (5.45). We seek conditions on ω that make the factor of η_j^2 non-positive. Imposing

$$\omega^2 \le \frac{\gamma\lambda\theta^2}{16C_{\mathrm{osc}}^2} = \frac{\lambda\theta^2}{64C_{\mathrm{osc}}^2C_{\mathrm{Lip}}^2}\delta \tag{5.46}$$

yields

$$\zeta_{j+1}^2 \le \alpha_1^2\zeta_j^2,$$

with

$$\alpha_1^2 := \max\left\{1 - \frac{\delta\lambda\theta^2}{32C_1^2C_{\mathrm{Lip}}^2}, 1 - \frac{\lambda\theta^2}{8}\right\} < 1.$$

$\boxed{3}$ *Case* $\operatorname{osc}_j > \omega\mathcal{E}_j$. The module DATA with input parameter $\xi \le 1/2$ gives

$$\operatorname{osc}_{j+1} \le \xi\omega\mathcal{E}_j < \xi \operatorname{osc}_j .$$

We now exploit the contraction of osc_j to compensate the moderate increase of η_j^2 and presence of osc_j^2, both governed by (5.44). In fact, $\gamma(1+\delta^{-1})C_{\mathrm{Lip}}^2 \le 1/2$ yields

$$e_{j+1}^2 + \gamma\eta_{j+1}^2 \le e_j^2 + \gamma(1+\delta)\eta_j^2 + \frac{1}{2}\operatorname{osc}_j^2 .$$

We add $\operatorname{osc}_{j+1}^2$ to both sides and rewrite the right-hand side to arrive at

$$\begin{aligned}\zeta_{j+1}^2 = e_{j+1}^2 + \gamma\eta_{j+1}^2 + \operatorname{osc}_{j+1}^2 \le e_j^2 &- \frac{1-2\xi^2}{8}\operatorname{osc}_j^2 \\ &+ (1-\delta)\gamma\eta_j^2 + \left(\frac{1+2\xi^2}{4} + \frac{1}{2}\right)\operatorname{osc}_j^2 \\ &+ 2\delta\gamma\eta_j^2 - \frac{1-2\xi^2}{8}\operatorname{osc}_j^2 .\end{aligned}$$

Our next task is to find conditions on ω for the last line to be non-positive. To this end, we resort to the upper bound $\eta_j^2 < \omega^{-2}\operatorname{osc}_j^2$ and $\xi \le 1/2$ to obtain

$$2\delta\gamma\eta_j^2 - \frac{1-2\xi^2}{8}\operatorname{osc}_j^2 < \left(\frac{2\delta\gamma}{\omega^2} - \frac{1}{16}\right)\operatorname{osc}_j^2 \le 0$$

provided we impose the relation

$$\omega^2 \ge 32\delta\gamma = \frac{8}{C_{\mathrm{Lip}}^2}\delta^2. \tag{5.47}$$

We next use the upper bound $e_j \le C_1\mathcal{E}_j \le C_1\omega^{-1}\operatorname{osc}_j$ to write

$$e_j^2 - \frac{1-2\xi^2}{8}\operatorname{osc}_j^2 \le \left(1 - \frac{\omega^2}{16C_1^2}\right)e_j^2,$$

whence we end up with

$$\zeta_{j+1}^2 \le \alpha_2^2\zeta_j^2$$

provided we define

$$\alpha_2^2 := \max\left\{1 - \frac{\omega^2}{16C_1^2}, 1-\delta, \frac{3+2\xi^2}{4}\right\} < 1.$$

$\boxed{4}$ *Choosing the parameters.* We see that the asserted estimate (5.43) is valid with $\alpha = \max\{\alpha_1, \alpha_2\} < 1$ provided the constraints (5.46) and (5.47) are compatible, that is,

$$\frac{8}{C_{\mathrm{Lip}}}\delta^2 \le \omega^2 \le \frac{\lambda\theta^2}{64C_{\mathrm{osc}}^2 C_{\mathrm{Lip}}^2}\delta.$$

We choose

$$\delta_0 = \frac{\lambda\theta^2}{512C_{\mathrm{osc}}^2 C_{\mathrm{Lip}}} \quad \text{and} \quad \omega_0 = \frac{\lambda\theta^2}{128C_{\mathrm{osc}}^2 C_{\mathrm{Lip}}\sqrt{2C_{\mathrm{Lip}}}}.$$

Then, for all $\omega \le \omega_0$, there exists $\delta \le \delta_0$ that satisfies the previous inequalities as well as $\gamma = \delta/(4C_{\mathrm{Lip}}^2) \le 1$, perhaps upon reducing δ_0. This completes the proof of Theorem 5.43. □

Note that we could replace the conditional $\mathrm{osc}_{\mathcal{T}_j}(f)_{-1} \le \omega\,\mathcal{E}_{\mathcal{T}_j}(u_{\mathcal{T}_j}, f)$ with $\mathrm{osc}_{\mathcal{T}_j}(f)_{-1} \le \omega\,\eta_{\mathcal{T}_j}(u_{\mathcal{T}_j})$, but the tolerance τ of DATA cannot be

$$\tau = \xi\,\omega\,\eta_{\mathcal{T}_j}(u_{\mathcal{T}_j})$$

because the algorithm might not terminate when $\eta_{\mathcal{T}_j}(u_{\mathcal{T}_j}) = 0$; see e.g. Examples 5.13–5.15. In fact, the tolerance $\tau = \xi\omega\mathcal{E}_{\mathcal{T}_j}(u_{\mathcal{T}_j}, f)$ is *dynamic* and relative to $\mathcal{E}_{\mathcal{T}_j}(u_{\mathcal{T}_j}, f)$. This avoids separate marking, which was shown by Cascón *et al.* (2008, Section 6) to give non-optimal convergence rates. In contrast, we will prove in Section 6 that Algorithm 5.16 is rate-optimal.

It turns out that Theorem 5.17 yields linear convergence of error and estimator.

Corollary 5.18 (linear convergence of error). *For* $0 < \alpha < 1$ *and* $0 < \omega \le \omega_0$, $\xi \le 1/2$ *as in Theorem 5.17, and* $C_* = \left(1 + C_2^{-1}\right)^{1/2}$ *with* C_2 *as in* (5.13), *we have*

$$|u - u_{\mathcal{T}_k}|_{H_0^1(\Omega)} \le C_*\,\alpha^{k-j}|u - u_{\mathcal{T}_j}|_{H_0^1(\Omega)} \quad \textit{for all } k \ge j \ge 0.$$

Proof. We again use the same notation as in Lemma 5.5 and Theorem 5.17. In view of the definition (5.42) of quasi-error $\zeta_j := \zeta_{\mathcal{T}_j}(u_{\mathcal{T}_j}, f)$, we thus have $e_j \le \zeta_j$ and

$$\zeta_j^2 \le e_j^2 + \eta_j^2 + \mathrm{osc}_j^2 \le \left(1 + C_2^{-1}\right)e_j^2$$

because $C_2\mathcal{E}_j \le e_j$ from (5.13). This implies

$$e_j \le \zeta_j \le C_* e_j \quad \text{for all } j \ge 0,$$

and invoking Theorem 5.17 (contraction property for AFEM-SW),

$$e_k^2 \le \zeta_k^2 \le \alpha^{2(k-j)}\zeta_j^2 \le \alpha^{2(k-j)}C_*^2 e_j$$

gives the desired estimate. □

We stress that Corollary 5.18 relies on the lower bound in (5.13) whereas Corollary 5.19 uses only the upper bound. Its proof is similar and is thus omitted.

Corollary 5.19 (linear convergence of estimator). *For* $0 < \alpha < 1$ *and* $0 < \omega \le \omega_0$, $\xi \le 1/2$ *as in Theorem 5.17, and* $C_\# = \left(\left(1 + C_1^2\right)\gamma^{-1}\right)^{1/2}$ *with* C_1 *as in* (5.13), *we have*

$$\mathcal{E}_{\mathcal{T}_k}(u_{\mathcal{T}_k}, f) \le C_\# \, \alpha^{k-j} \, \mathcal{E}_{\mathcal{T}_j}(u_{\mathcal{T}_j}, f) \quad \textit{for all } k \ge j \ge 0.$$

5.4. *Convergence for general data: two-step* AFEM

We now remove the restriction of Sections 5.2 and 5.3 to discrete data and allow for general data $\mathcal{D} = (\boldsymbol{A}, c, f) \in \mathbb{D}$ as defined in (5.2). The current goal is to study Algorithm 5.1 (AFEM-TS), which concatenates the modules DATA and GALERKIN. We start with the study of continuous dependence with respect to data $\mathcal{D}$. We next discuss the approximation of $\mathcal{D}$ within the module DATA, the computational cost of GALERKIN, and eventually the convergence of Algorithm 5.1.

5.4.1. *Perturbation theory*

We start with a brief discussion of data perturbation. Given constants $0 < \alpha_1 \le \alpha_2$ and $0 \le c_1 \le c_2$, we define the constrained spaces for the diffusion and reaction coefficients by

$$M(\alpha_1, \alpha_2) := \left\{\boldsymbol{A} \in L^\infty\left(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}\right) \mid 0 < \alpha_1 \le \lambda_j(\boldsymbol{A}(x)) \le \alpha_2 \right.$$
$$\left. \text{for a.e. } x \in \Omega,\ 1 \le j \le d\right\}, \tag{5.48}$$

where $\lambda_j(\boldsymbol{A}(x))$ denotes the jth eigenvalue of $\boldsymbol{A}$ at $x \in \Omega$ and

$$R(c_1, c_2) := \{c \in L^\infty(\Omega) \mid c_1 \le c(x) \le c_2 \text{ for a.e. } x \in \Omega\}. \tag{5.49}$$

The coefficients $(\boldsymbol{A}, c)$ are assumed to satisfy the structural assumption

$$\boldsymbol{A} \in M(\alpha_1, \alpha_2), \quad c \in R(c_1, c_2); \tag{5.50}$$

see (2.6). This guarantees coercivity and continuity of the bilinear form $\mathcal{B}$ in (2.8), and thus unique solvability of (2.7).

Regarding the discrete coefficients, $(\widehat{\boldsymbol{A}}, \widehat{c})$ will ultimately be piecewise polynomials in a grid $\widehat{\mathcal{T}} \in \mathbb{T}$. The side constraints in (5.48) and (5.49) are generally violated by any linear projection onto piecewise polynomials of degree $n - 1 \ge 1$, e.g. the L^2-projection, and require a nonlinear correction maintaining high-order accuracy. This is a crucial but delicate matter addressed later in Section 7.2. For the moment, we simply assume that the discrete coefficients $(\widehat{\boldsymbol{A}}, \widehat{c})$ satisfy

$$\widehat{\boldsymbol{A}} \in M(\widehat{\alpha}_1, \widehat{\alpha}_2), \quad \widehat{c} \in R(\widehat{c}_1, \widehat{c}_2), \tag{5.51}$$

with

$$\frac{\alpha_1}{2} \le \widehat{\alpha}_1 \le \widehat{\alpha}_2 \le C_{\mathrm{ctr}}\alpha_2, \quad -\frac{\alpha_1}{4C_P^2} \le \widehat{c}_1 \le \widehat{c}_2 \le C_{\mathrm{ctr}}(\alpha_1 + c_2), \tag{5.52}$$

where $C_P > 0$ is the Poincaré constant in (2.2) and $C_{\mathrm{ctr}} \ge 1$ is a constant; see (7.21)

and (7.23). This implies coercivity and continuity of the perturbed bilinear form

$$\widehat{\mathcal{B}}[v,w] := \int_\Omega \nabla v \cdot \widehat{\boldsymbol{A}} \nabla w + \widehat{c} v w, \quad \text{for all } v, w \in H_0^1(\Omega), \tag{5.53}$$

because for all $v, w \in H_0^1(\Omega)$

$$\widehat{\mathcal{B}}[v,v] \geq \widehat{\alpha}_1 \int_\Omega |\nabla v|^2 - \frac{\alpha_1}{4C_P^2} \int_\Omega |v|^2 \geq \frac{\alpha_1}{4} |v|^2_{H_0^1(\Omega)}$$

and

$$|\widehat{\mathcal{B}}[v,w]| \leq \int_\Omega \widehat{\alpha}_2 |\nabla v|\,|\nabla w| + \widehat{c}_2 |v|\,|w| \leq \left(\widehat{\alpha}_2 + \widehat{c}_2 C_P^2\right) |v|_{H_0^1(\Omega)} |w|_{H_0^1(\Omega)}.$$

Therefore the energy norm $|\!|\!| v |\!|\!|_\Omega^2 = \widehat{\mathcal{B}}[v,v]$ is equivalent to the H_0^1-seminorm

$$c_{\widehat{\mathcal{B}}} |v|^2_{H_0^1(\Omega)} \leq |\!|\!| v |\!|\!|_\Omega^2 \leq C_{\widehat{\mathcal{B}}} |v|^2_{H_0^1(\Omega)}, \tag{5.54}$$

where $c_{\widehat{\mathcal{B}}} = \alpha_1/4$ and $C_{\widehat{\mathcal{B}}} = \widehat{\alpha}_2 + \widehat{c}_2 C_P^2$. Hence the Lax–Milgram theorem guarantees the existence of a unique solution $\widehat{u} = u(\widehat{\mathcal{D}}) \in H_0^1(\Omega)$ of the perturbed problem (5.5) defined using the discrete data $\widehat{\mathcal{D}} = (\widehat{\boldsymbol{A}}, \widehat{c}, \widehat{f})$.

We now quantify the effect of perturbing data from $\mathcal{D}$ to $\widehat{\mathcal{D}}$ in the space

$$\widehat{D}(\Omega) := L^r(\Omega; \mathbb{R}^{d\times d}) \times W_q^{-s}(\Omega) \times H^{-1}(\Omega), \tag{5.55}$$

where $2 \leq r \leq \infty$ and $0 \leq s \leq 1$, $d/(2-s) < q \leq \infty$; $W_q^{-s}(\Omega)$ is the dual of $W_{q^*}^s(\Omega)$ with $q^* = q/(q-1)$. The use of $r = \infty$ for $\boldsymbol{A}$ entails the further assumption

$$\boldsymbol{A} \text{ is piecewise uniformly continuous over a generic mesh } \mathcal{T} \in \mathbb{T}, \tag{5.56}$$

which turns out to be rather restrictive but customary in the theory of AFEM. Our present approach allows for $r < \infty$ and thus for discontinuous coefficients $(\boldsymbol{A}, c)$ not aligned with $\mathcal{T}$, which is important in practice. However, it requires the following slightly stronger regularity property of the solution $u \in H_0^1(\Omega)$ of (2.7):

$$\|\nabla u\|_{L^p(\Omega)} \leq C_p \|f\|_{W_p^{-1}(\Omega)}, \quad 2 < p \leq p_0. \tag{5.57}$$

We refer to Lemma 2.13 (W_p^1-regularity), which shows the existence of $C_p > 0$ and $p_0 > 2$ that depend only on $\Omega, \alpha_1, \alpha_2$ and c_2.

Lemma 5.20 (continuous dependence on data). *Let $\mathcal{D} = (\boldsymbol{A}, c, f) \in \mathbb{D}$ be such that $\boldsymbol{A} \in M(\alpha_1, \alpha_2)$ and $c \in R(c_1, c_2)$. Let $\widehat{\mathcal{D}} = (\widehat{\boldsymbol{A}}, \widehat{c}, \widehat{f}) \in \mathbb{D}$ be an approximation of $\mathcal{D}$ such that $\widehat{\boldsymbol{A}} \in M(\widehat{\alpha}_1, \widehat{\alpha}_2)$ and $\widehat{c} \in R(\widehat{c}_1, \widehat{c}_2)$. Let $2 \leq r \leq \infty, 2 \leq r_* = 2r/(r-2) \leq p_0$ be such that $f \in W_{r_*}^{-1}(\Omega)$. If $u = u(\mathcal{D}), \widehat{u} = u(\widehat{\mathcal{D}}) \in H_0^1(\Omega)$ are the solutions of (2.7) and (5.5) with data $\mathcal{D}, \widehat{\mathcal{D}}$, respectively, and u satisfies (5.57) with $p = r_*$ for $r < \infty$, then for any $0 \leq s \leq 1$ and $d/(2-s) < q \leq \infty$ we have*

$$\|\nabla(u - \widehat{u})\|_{L^2(\Omega)} \leq C(\mathcal{D}, \Omega)\, \|\mathcal{D} - \widehat{\mathcal{D}}\|_{\widehat{D}(\Omega)}, \tag{5.58}$$

where the constant $C(\mathcal{D},\Omega)$ depends on $\mathcal{D}$, Ω, p_0, q and s, and blows up as $q \to d/(2-s)$ for $d=2$ while it remains bounded for $d>2$.

Proof. Subtracting the weak formulations (2.7) for u and (5.5) for $\widehat{u}$, and reordering, we easily obtain for any $v \in H^1_0(\Omega)$

$$\int_\Omega \nabla v \cdot \widehat{\boldsymbol{A}} \nabla(u-\widehat{u}) + \widehat{c} v(u-\widehat{u}) = \int_\Omega \nabla v \cdot (\widehat{\boldsymbol{A}} - \boldsymbol{A})\nabla u + (\widehat{c}-c)vu + \langle f - \widehat{f}, v\rangle.$$

We choose $v = u - \widehat{u} \in H^1_0(\Omega)$ and invoke (5.54) to deduce

$$c_{\widehat{\mathcal{B}}} \|\nabla v\|^2_{L^2(\Omega)} \le \int_\Omega \nabla v \cdot (\widehat{\boldsymbol{A}} - \boldsymbol{A})\nabla u + (\widehat{c}-c)vu + \langle f - \widehat{f}, v\rangle.$$

We estimate each term separately, starting with the first and last terms

$$\int_\Omega \nabla v \cdot (\widehat{\boldsymbol{A}} - \boldsymbol{A})\nabla u \le \|\widehat{\boldsymbol{A}} - \boldsymbol{A}\|_{L^r(\Omega)} \|\nabla u\|_{L^{r^*}(\Omega)} \|\nabla v\|_{L^2(\Omega)} \tag{5.59}$$

with $2 \le r_* = 2r/(r-2) \le p_0$, as well as

$$\langle f - \widehat{f}, v\rangle \le \|f - \widehat{f}\|_{H^{-1}(\Omega)} \|\nabla v\|_{H^1(\Omega)}.$$

For the reaction term, which is more delicate, we invoke the duality pairing $W^s_{q'}$–W^{-s}_q for any $0 \le s \le 1$ and $q' = q/(q-1) \ge 1$, to obtain

$$\int_\Omega (\widehat{c}-c)vu \le \|\widehat{c}-c\|_{W^{-s}_q(\Omega)} |vu|_{W^s_{q'}(\Omega)}.$$

We now estimate $|vu|_{W^s_{q'}(\Omega)} \lesssim |vu|_{W^1_{p'}(\Omega)}$, where $1/p' = \min\{1, (1-s)/d + 1/q'\}$ guarantees that $W^1_{p'}(\Omega) \subset W^s_{q'}(\Omega)$ (Leoni 2009, Theorem 14.32). Recalling that $q > d/(2-s)$, we deduce

$$\frac{1-s}{d} + \frac{1}{q'} = \frac{1-s}{d} + 1 - \frac{1}{q} > \frac{d-1}{d} \ge \frac{1}{2},$$

whence $1/p' > 1/2$, and there exists $t < \infty$ satisfying $1/t + 1/2 = 1/p'$ and

$$|vu|_{W^s_{q'}(\Omega)} \lesssim \|\nabla v\|_{L^2(\Omega)} \|u\|_{L^t(\Omega)} + \|v\|_{L^t(\Omega)} \|\nabla u\|_{L^2(\Omega)}.$$

Using the definition of p', we obtain the explicit expression $t = \max\{2, t_0\}$, where

$$t_0 = \frac{2dq}{q(2(1-s)+d) - 2d}.$$

Moreover, for the Sobolev embedding $H^1(\Omega) \hookrightarrow L^t(\Omega)$, we require

$$1 - d\left(\frac{1}{2} - \frac{1}{t}\right) > 0 \quad \Rightarrow \quad q > \frac{d}{2-s},$$

which is our assumption on q. Therefore, as $q \to d/(2-s)$, we see that $t_0 \to 2d/(d-2)$, and the limit is infinite for $d=2$ but finite and larger than 2 for

$d > 2$. Sobolev embedding together with the first Poincaré inequality (2.2) gives the estimate

$$\|vu\|_{W^s_{q'}(\Omega)} \leq C(\Omega, t)\|\nabla v\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)},$$

where $C(\Omega, t)$ is proportional to t for $d = 2$.

We finally observe that the factors $\|\nabla u\|_{L^{r_*}(\Omega)}$ and $\|\nabla u\|_{L^2(\Omega)}$ appear in the estimates of the coefficients $\boldsymbol{A}$ and c, thereby reflecting the multiplicative nature of these terms. Since $2 \leq r_* \leq p_0$, they can be further bounded in terms of $\|f\|_{W^{-1}_{r_*}(\Omega)}$ according to (5.57). This in conjunction with the preceding estimates yields the assertion (5.58). □

A natural and rather popular choice of parameters (r, q, s) in Lemma 5.20 (continuous dependence on data) is $r = q = \infty$ and $s = 0$, but this would prevent the coefficients $(\boldsymbol{A}, c)$ from being discontinuous within elements; see (5.56). We will explore this matter further in Section 7 (data approximation).

Remark 5.21 (L^2-approximation of $\boldsymbol{A}$). It is appealing to estimate the distortion $\boldsymbol{A} - \widehat{\boldsymbol{A}}$ in $L^2(\Omega)$ rather than in $L^r(\Omega)$ because it is a simpler norm to deal with. Since $\|\boldsymbol{A}\|_{L^\infty(\Omega)} \leq \alpha_2$, $\|\widehat{\boldsymbol{A}}\|_{L^\infty(\Omega)} \leq \widehat{\alpha}_2$ and $2 \leq r \leq \infty$, we deduce

$$\|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|_{L^r(\Omega)} \leq \|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|^{1-2/r}_{L^\infty(\Omega)}\|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|^{2/r}_{L^2(\Omega)} \lesssim \|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|^{2/r}_{L^2(\Omega)}.$$

However, this may be sub-optimal in general. One important situation where this is sharp corresponds to $\boldsymbol{A}$ being piecewise constant with jump discontinuities across a Lipschitz hypersurface γ and $\widehat{\boldsymbol{A}} = \boldsymbol{A}$ on every element $T \in \mathcal{T}$ not intersecting γ. In that case, the equivalence

$$\|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|_{L^p(\Omega)} \approx |\{x \in \Omega \mid \boldsymbol{A}(x) \neq \widehat{\boldsymbol{A}}(x)\}|^{1/p}$$

is valid for $1 \leq p \leq \infty$, whence

$$\|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|_{L^r(\Omega)} \approx \|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|^{2/r}_{L^2(\Omega)}.$$

5.4.2. *Approximation of $\mathcal{D}$: module* DATA

In this section we briefly discuss the structure of DATA, which is the module of Algorithm 5.1 (AFEM-TS) responsible for data approximation.

Henceforth we will no longer rely on the Banach space $\widehat{D}(\Omega)$ defined in (5.55) and used in Lemma 5.20 (continuous dependence on data). Instead we restrict the error notion to the following stronger Banach space:

$$D(\Omega) := L^r(\Omega; \mathbb{R}^{d\times d}) \times L^q(\Omega) \times H^{-1}(\Omega), \tag{5.60}$$

where $q = 2$ for $d < 4$ or $q > d/2$ for $d \geq 4$; we justify the choice of q below. Let $\mathbb{D}$ and $\mathbb{D}_{\mathcal{T}}$ be the spaces defined in (5.1) and (5.2) for a conforming mesh $\mathcal{T} \in \mathbb{T}$. Given $\mathcal{D} = (\boldsymbol{A}, c, f) \in \mathbb{D}$, let $\delta_{\mathcal{T}}(\mathcal{D})$ be the best approximation error of $\mathcal{D}$ within

$\mathbb{D}_{\mathcal{T}}$ measured in the space $D(\Omega)$, namely

$$\delta_{\mathcal{T}}(\mathcal{D}) := \inf_{\mathcal{D}_{\mathcal{T}} \in \mathbb{D}_{\mathcal{T}}} \|\mathcal{D} - \mathcal{D}_{\mathcal{T}}\|_{D(\Omega)}. \tag{5.61}$$

This quantity characterizes the approximation quality of $\mathbb{D}_{\mathcal{T}}$, thereby having theoretical value. Since $\delta_{\mathcal{T}}(\mathcal{D})$ is hard to access in view of the norms involved in the definition of $D(\Omega)$, the module DATA computes the surrogate quantity

$$\operatorname{osc}_{\mathcal{T}}(\mathcal{D}) := \|\mathcal{D} - \widehat{\mathcal{D}}\|_{D(\Omega)}$$

for some approximation $\widehat{\mathcal{D}} \in \mathbb{D}_{\mathcal{T}}$ to be specified below.

Assumption 5.22 (properties of DATA). Given a conforming mesh $\mathcal{T} \in \mathbb{T}$ and a tolerance $\tau > 0$, the call

$$[\widehat{\mathcal{T}}, \widehat{\mathcal{D}}] = \mathsf{DATA}(\mathcal{T}, \mathcal{D}, \tau) \tag{5.62}$$

creates an admissible refinement $\widehat{\mathcal{T}}$ of $\mathcal{T}$ and discrete data $\widehat{\mathcal{D}} = \mathcal{D}_{\widehat{\mathcal{T}}} \in \mathbb{D}_{\widehat{\mathcal{T}}}$ such that for a constant C_{data},

$$\operatorname{osc}_{\widehat{\mathcal{T}}}(\mathcal{D}) := \|\mathcal{D} - \widehat{\mathcal{D}}\|_{D(\Omega)} \le C_{\text{data}}\tau, \tag{5.63}$$

as well as the structural conditions (5.51), are achieved in a finite number of iterations that depends on the regularity of $\mathcal{D}$, and such that

$$\operatorname{osc}_{\widehat{\mathcal{T}}}(\mathcal{D}) \le \Lambda_{\text{data}}\, \delta_{\widehat{\mathcal{T}}}(\mathcal{D}), \tag{5.64}$$

with $\Lambda_{\text{data}} \ge 1$ depending only on the shape regularity of $\mathbb{T}$, the polynomial degree n and the Lebesgue exponents in the space $D(\Omega)$.

In view of Lemma 5.20 (continuous dependence on data), there exists a constant $C_D > 0$ depending on $\mathcal{D}, \Omega$, and the shape regularity of $\mathbb{T}$, such that the exact solutions $u = u(\mathcal{D})$ and $\widehat{u} = u(\widehat{\mathcal{D}})$ of (2.5) and (5.5), corresponding to data $\mathcal{D}$ and $\widehat{\mathcal{D}}$ respectively, satisfy the error estimate

$$|u - \widehat{u}|_{H_0^1(\Omega)} \le C_D \tau. \tag{5.65}$$

A brief discussion follows about computing $\operatorname{osc}_{\widehat{\mathcal{T}}}(\mathcal{D})$, where $\widehat{\mathcal{T}}$ remains fixed and is replaced by $\mathcal{T}$ to simplify the notation. Specific details are given later in Assumptions 6.10 and 6.11 of Section 6.10 and especially in Section 7.

Approximating the coefficients. We now construct approximations $(\widehat{\boldsymbol{A}}, \widehat{c})$ using local L^2-projections, and emphasize that this does not enforce the side constraints in the structural assumption (5.51). In Section 7 we propose a nonlinear correction satisfying the side constraints without sacrificing the accuracy.

Given $T \in \mathcal{T}$, and $v \in L^p(T)$ with $1 \le p \le \infty$, we let $\Pi_T v := \Pi_T^{n-1} v$ denote the L^2-projection of v onto the space $\mathbb{P}_{n-1}$ of polynomials of degree $\le n-1$, namely

$$\Pi_T v \in \mathbb{P}_{n-1}: \qquad \int_T \Pi_T v\, w = \int_T v\, w \quad \text{for all } w \in \mathbb{P}_{n-1}. \tag{5.66}$$

Lemma 5.23 (L^p-stability of Π_T). *For every $1 \le p \le \infty$ and $v \in L^p(T)$, there exists a constant C depending on p, n and the shape regularity of $\mathbb{T}$ such that*

$$\|\Pi_T v\|_{L^p(T)} \le C\|v\|_{L^p(T)} \quad \text{for all } T \in \mathcal{T}. \tag{5.67}$$

Proof. It is trivial to see that $\|\Pi_T v\|_{L^2(T)} \le \|v\|_{L^2(T)}$. Let $2 < p \le \infty$ and combine an inverse estimate with a Hölder inequality to write

$$\|\Pi_T v\|_{L^p(T)} \le C h_T^{d/p-d/2}\|\Pi_T v\|_{L^2(T)} \le C h_T^{d/p-d/2}\|v\|_{L^2(T)} \le C\|v\|_{L^p(T)}.$$

For $1 \le p < 2$ we proceed by duality. Let $\varphi \in L^q(T)$ with $q = p/(p-1)$. Then

$$\int_T \Pi_T v\varphi = \int_T v\Pi_T\varphi \le \|v\|_{L^p(T)}\|\Pi_T\varphi\|_{L^q(T)} \le C\|v\|_{L^p(T)}\|\varphi\|_{L^q(T)},$$

which implies (5.67) and concludes the proof. □

We immediately have the following simple consequence of Lemma 5.23.

Corollary 5.24 (best approximation of Π_T). *For every $1 \le p \le \infty$ and $v \in L^p(T)$, there exists a constant $C_{BA} \ge 1$ depending on p, n and the shape regularity of $\mathbb{T}$ such that*

$$\|v - \Pi_T v\|_{L^p(T)} \le C_{BA} \inf_{w \in \mathbb{P}_{n-1}} \|v - w\|_{L^p(T)}. \tag{5.68}$$

Proof. We combine the invariance of Π_T on $\mathbb{P}_{n-1}$, i.e. $\Pi_T w = w$ for $w \in \mathbb{P}_{n-1}$, with (5.67) to see that

$$\|v - \Pi_T v\|_{L^p(T)} = \|(v - w) - \Pi_T(v - w)\|_{L^p(T)} \le C\|v - w\|_{L^p(T)}.$$

This implies (5.68) as asserted. □

The L^2-projection is easily computable because it entails solving the linear system (5.66). However, this flexibility comes at the expense of a best approximation constant $C_{BA} > 1$ in (5.68) for $p \ne 2$. The best L^p-approximation of v in T is also computable, because it boils down to a convex minimization problem, and would result in $C_{BA} = 1$. This excellent property is superseded by the simplicity of (5.66), which makes $\Pi_T v$ the approximation of choice.

Corollary 5.25 (quasi-monotonicity of Π_T). *Let $\mathcal{T}, \mathcal{T}_* \in \mathbb{T}$ be such that $\mathcal{T} \le \mathcal{T}_*$, and let $T \in \mathcal{T}, T_* \in \mathcal{T}_*$ satisfy $T_* \subset T$. If C_{BA} is the constant in (5.68), then*

$$\|v - \Pi_{T_*} v\|_{L^p(T_*)} \le C_{BA}\|v - \Pi_T v\|_{L^p(T)} \tag{5.69}$$

for all $1 \le p \le \infty$, and $C_{BA} = 1$ for $p = 2$.

Proof. Simply use (5.68) to write

$$\|v - \Pi_{T_*} v\|_{L^p(T_*)} \le C_{BA}\|v - \Pi_T v\|_{L^p(T_*)} \le C_{BA}\|v - \Pi_T v\|_{L^p(T)}.$$

This is the desired bound. □

We are now ready to define the discontinuous $\mathbb{P}_{n-1}$-approximation $\widehat{v}$ of $v \in L^p(\Omega)$. Inequality (5.69) with $C_{BA} > 1$ is fine for most instances except Lemma 7.5 below. Therefore we introduce a nonlinear modification of the obvious choice $\widehat{v}$ for $T \in \mathcal{T}$, namely $\widehat{v} = \Pi_T v$. We give a recursive (and computable) definition as follows: if $T \in \mathcal{T}_0$, then $\widehat{v}|_T := \Pi_T v$; if $T \in \mathcal{T}$, let $\widehat{v}|_{P(T)} \in \mathbb{P}_{n-1}$ be the approximation of v in the parent element $P(T)$ of T, and set

$$\widehat{v}|_T := \begin{cases} \Pi_T v, & \text{if } \|v - \Pi_T v\|_{L^p(T)} \le \|v - \widehat{v}|_{P(T)}\|_{L^p(T)}, \\ \widehat{v}|_{P(T)}, & \text{if } \|v - \Pi_T v\|_{L^p(T)} > \|v - \widehat{v}|_{P(T)}\|_{L^p(T)}. \end{cases} \tag{5.70}$$

We then define

$$\operatorname{osc}_{\mathcal{T}}(v, T)_p := \|v - \widehat{v}\|_{L^p(T)} \quad \text{for all } T \in \mathcal{T}. \tag{5.71}$$

Since the chain of elements emanating from $\mathcal{T}_0$ and culminating with T is unique, the notion $\operatorname{osc}_{\mathcal{T}}(v, T)_p$ is well-defined and independent of $\mathcal{T}$. The following result is an immediate consequence of (5.70).

Lemma 5.26 (monotonicity of oscillation). *For all $1 \le p \le \infty$, $\mathcal{T}, \mathcal{T}_* \in \mathbb{T}$ with $\mathcal{T} \le \mathcal{T}_*$, and $T_* \in \mathcal{T}_*, T \in \mathcal{T}$ so that $T_* \subset T$, we have*

$$\operatorname{osc}_{\mathcal{T}_*}(v, T_*)_p \le \operatorname{osc}_{\mathcal{T}}(v, T)_p. \tag{5.72}$$

Consequently, for any $n \ge 1$ and $T \in \mathcal{T}$, let

$$\widehat{\boldsymbol{A}} \in \left[\mathbb{S}_{\mathcal{T}}^{n-1,-1}\right]^{d\times d}, \quad \widehat{c} \in \mathbb{S}_{\mathcal{T}}^{n-1,-1}$$

be defined locally via (5.70), and let the surrogate element error indicators of $(\boldsymbol{A}, c)$ be given by

$$\operatorname{osc}_{\mathcal{T}}(\boldsymbol{A}, T)_r := \|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|_{L^r(T)}, \quad \operatorname{osc}_{\mathcal{T}}(c, T)_q := \|c - \widehat{c}\|_{L^q(T)}, \tag{5.73}$$

for some $2 \le r \le \infty$ and $d/2 < q \le \infty$ according to (5.58) for $s = 0$. The simplest choice $q = 2$ yields $\widehat{c}_T = \Pi_T c$ in (5.70), but requires the restriction $d < 4$, which is fine in practice.

For $n = 1$ the situation is a bit special on two counts. First, $\Pi_T v$ reduces to mean values of v, namely

$$\Pi_T \boldsymbol{A} := \frac{1}{|T|}\int_T \boldsymbol{A}, \quad \Pi_T c := \frac{1}{|T|}\int_T c \quad \text{for all } T \in \mathcal{T}. \tag{5.74}$$

for $\boldsymbol{A} \in M(\alpha_1, \alpha_2)$, $c \in R(c_1, c_2)$ defined in (5.50). Hence $\widehat{\boldsymbol{A}} \in M(\widehat{\alpha}_1, \widehat{\alpha}_2)$ with $\widehat{\alpha}_1 = \alpha_1, \widehat{\alpha}_2 = \alpha_2$ and $\widehat{c} \in R(\widehat{c}_1, \widehat{c}_2)$ with $\widehat{c}_1 = c_1, \widehat{c}_2 = c_2$, that is, the L^2-projections (5.74) on piecewise constants over $\mathcal{T}$ as well as $\widehat{\boldsymbol{A}}$ and $\widehat{c}$ satisfy the side conditions in (5.51) without changing the original range of parameters. In addition, instead of (5.73), we can exploit *superconvergence* in $W_q^{-1}(\Omega)$ with $q > d/(2-s) = d$ in (5.58). In fact we utilize the orthogonality of Π_T in conjunction with (5.68) and

(3.16), to obtain, for an arbitrary function $w \in W^1_{q^*}(\Omega)$ and $q^* = q/(q-1)$,

$$\int_T (c - \Pi_T c)w = \int_T (c - \Pi_T c)(w - \Pi_T w) \lesssim h_T^t \|c - \Pi_T c\|_{L^r(T)} |w|_{W^1_{q^*}(\omega_T)},$$

where $t = 1 - d/q^* + d/r^* = 1 + d/q - d/r > 0$ and $r^* = r/(r-1)$. We consider two cases: $r = 2, \infty$. If $r = 2$ and $s = 1$, then $q > d$ results in $0 < t < 2 - d/2$ and entails the restriction $d < 4$. This implies $\|c - \widehat{c}\|_{W^{-1}_q(\Omega)} \lesssim \operatorname{osc}_{\mathcal{T}}(c)_2$, where

$$\operatorname{osc}_{\mathcal{T}}(c, T)_2 := h_T^t \|c - \Pi_T c\|_{L^2(T)}. \tag{5.75}$$

If $r = \infty$ and $s = 1$, then $q = \infty$ yields $t = 1$ and $\|c - \widehat{c}\|_{W^{-1}_q(\Omega)} \lesssim \operatorname{osc}_{\mathcal{T}}(c)_\infty$, where

$$\operatorname{osc}_{\mathcal{T}}(c, T)_\infty := h_T \|c - \Pi_T c\|_{L^\infty(T)}. \tag{5.76}$$

Approximating the load. Dealing with $f \in H^{-1}(\Omega)$ is trickier for several reasons. First, the norm in $H^{-1}(\Omega)$ is non-local, so its localization is non-obvious. We recall the definition (4.52) of local oscillation $\operatorname{osc}_{\mathcal{T}}(f, T)_{-1}$ for $T \in \mathcal{T}$ and Corollary 4.31 (local near-best approximation), to deduce

$$\operatorname{osc}_{\mathcal{T}}(f, T)_{-1} := \|f - P_{\mathcal{T}} f\|_{H^{-1}(\omega_T)} \le C_{\mathrm{lStb}} \inf_{\chi \in \mathbb{R}_{\mathcal{T}_{\omega_T}}} \|f - \chi\|_{H^{-1}(\omega_T)}, \tag{5.77}$$

where C_{lStb} is the constant in Lemma 4.28 (local H^{-1}-stability); equivalently, $\operatorname{osc}_{\mathcal{T}}(f, T)_{-1}$ delivers a near-best approximation of f in $H^{-1}(\omega_T)$. The second issue at stake is that without further assumptions on f, it is not possible to evaluate or bound the left-hand side of (5.77). In Section 7 we will consider several classes of loads amenable to computation and yet relevant in practice.

A popular variant of this approach for $f \in L^2(\Omega)$ replaces χ in (5.77) with the L^2-projection $\Pi_{\mathcal{T}}$ onto discontinuous piecewise polynomials of degree $n-1$, and sets $\widehat{f} = \Pi_{\mathcal{T}} f$. This leads to the standard local weighted L^2-element error indicator

$$\widetilde{\operatorname{osc}}_{\mathcal{T}}(f, T)_{-1} := h_T \|f - \widehat{f}\|_{L^2(T)} \quad \text{for all } T \in \mathcal{T}. \tag{5.78}$$

Data error estimators. They are the following quantities for the coefficients $(\boldsymbol{A}, c)$:

$$\begin{aligned} \operatorname{osc}_{\mathcal{T}}(\boldsymbol{A})_r &:= \left(\sum_{T \in \mathcal{T}} \operatorname{osc}_{\mathcal{T}}(\boldsymbol{A}, T)_r^r \right)^{1/r}, \\ \operatorname{osc}_{\mathcal{T}}(c)_q &:= \left(\sum_{T \in \mathcal{T}} \operatorname{osc}_{\mathcal{T}}(c, T)_q^q \right)^{1/q}, \end{aligned} \tag{5.79}$$

which accumulate in ℓ^r and ℓ^q for $2 \le r \le \infty$ and $d/2 < q$; recall that $q = 2$ is an admissible choice provided $d < 4$. In contrast, the global error estimator for f,

$$\operatorname{osc}_{\mathcal{T}}(f)_{-1} := \left(\sum_{T \in \mathcal{T}} \operatorname{osc}_{\mathcal{T}}(f, T)_{-1}^2 \right)^{1/2}, \tag{5.80}$$

accumulates in ℓ^2. The total data error estimator satisfies (5.64) and reads

$$\operatorname{osc}_{\mathcal{T}}(\mathcal{D}) := \operatorname{osc}_{\mathcal{T}}(\boldsymbol{A})_r + \operatorname{osc}_{\mathcal{T}}(c)_q + \operatorname{osc}_{\mathcal{T}}(f)_{-1}. \tag{5.81}$$

The module DATA. This module reduces the oscillation of data $\mathcal{D} = (\boldsymbol{A}, c, f)$ sequentially. It consists of a linear approximation followed by a nonlinear correction.

Given a coefficient $v = \boldsymbol{A}, c$, a mesh $\mathcal{T} \in \mathbb{T}$, a tolerance τ, an accumulation index $1 \le p \le \infty$, and a number of bisections $b \ge 1$ per marked element, the call

$$[\widetilde{\mathcal{T}}, \widetilde{v}] = \mathsf{GREEDY}(v, \mathcal{T}, \tau, p, b)$$

returns a conforming refinement $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ and a piecewise polynomial approximation $\widetilde{v}$ of v over $\widetilde{\mathcal{T}}$ such that the oscillation computed with $v - \widetilde{v}$ satisfies

$$\operatorname{osc}_{\widetilde{\mathcal{T}}}(v)_p \le \tau.$$

For the load function f, since the computation of $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ is impossible without further assumptions on f, we will consider three surrogate estimators $\widetilde{\operatorname{osc}}_{\mathcal{T}}(f)_{-1}$ in Section 7.3 that also accumulate in ℓ^p such that, for all $\mathcal{T} \in \mathbb{T}$,

$$\operatorname{osc}_{\mathcal{T}}(f)_{-1} \le C_{\text{data}} \widetilde{\operatorname{osc}}_{\mathcal{T}}(f)_{-1},$$

where $C_{\text{data}} \ge 1$. GREEDY applied to the surrogate estimator constructs $\widetilde{\mathcal{T}} \ge \mathcal{T}$ satisfying

$$\widetilde{\operatorname{osc}}_{\widetilde{\mathcal{T}}}(f)_{-1} \le \tau \quad \Rightarrow \quad \operatorname{osc}_{\widetilde{\mathcal{T}}}(f)_{-1} \le C_{\text{data}}\tau. \tag{5.82}$$

In all cases, the routine GREEDY is similar to that in Algorithm 3.18 (greedy algorithm) with several important distinctions: it accumulates the local error indicators in the ℓ^p-norm and starts from any mesh $\mathcal{T} \ge \mathcal{T}_0$ to save computational work.

Finally, the structure of the module DATA is as follows: it concatenates GREEDY with CONSTRAINT-A and CONSTRAINT-c in order to satisfy Assumption 5.22 (properties of DATA). The routine GREEDY deals with pure approximation without constraints: called with tolerance $\tau/3$, it sequentially reduces the oscillation for $\boldsymbol{A}, c, f$ with the most recent updated mesh to reduce their errors so that

$$\operatorname{osc}_{\widehat{\mathcal{T}}}(\boldsymbol{A})_r \le \tau/3, \quad \operatorname{osc}_{\widehat{\mathcal{T}}}(c)_q \le \tau/3, \quad \widetilde{\operatorname{osc}}_{\widehat{\mathcal{T}}}(f)_{-1} \le \tau/3$$

on a conforming refinement $\widehat{\mathcal{T}} \ge \mathcal{T}$. This is discussed in detail in Section 7.1.

From (5.82) we get $\operatorname{osc}_{\widehat{\mathcal{T}}}(f)_{-1} \le C_{\text{data}}\tau/3$. On the other hand, the resulting coefficients $(\widetilde{\boldsymbol{A}}, \widetilde{c})$ most likely do not satisfy the constraints (5.51) for $n > 1$. This requires a further nonlinear correction

$$[\widehat{\boldsymbol{A}}] = \mathsf{CONSTRAINT\text{-}A}(\widehat{\mathcal{T}}, \widetilde{\boldsymbol{A}}), \quad [\widehat{c}] = \mathsf{CONSTRAINT\text{-}c}(\widehat{\mathcal{T}}, \widetilde{c}),$$

that enforces (5.51) on the same grid $\widehat{\mathcal{T}}$ without compromising the accuracy gain produced by GREEDY: there exists a constant ≥ 1, still denoted by C_{data} for simplicity, such that

$$\operatorname{osc}_{\widehat{\mathcal{T}}}(\widehat{\boldsymbol{A}})_r \le C_{\text{data}}\tau/3, \quad \operatorname{osc}_{\widehat{\mathcal{T}}}(\widehat{c})_q \le C_{\text{data}}\tau/3 \quad \Rightarrow \quad \operatorname{osc}_{\widehat{\mathcal{T}}}(\mathcal{D}) \le C_{\text{data}}\tau.$$

For instance, for a fixed parameter $L \geq 2$, we get $\widehat{\alpha}_1 = \frac{1}{2}\alpha_1$ and $\widehat{\alpha}_2 = (1+4L)(\alpha_2/2)$ for the parameters in (5.51). We give details in Sections 7.2, 7.3 and 7.4.

The optimality properties of DATA hinge on the performance of GREEDY and the regularity of $\mathcal{D}$. Since this is not necessary for the present convergence assessment, we discuss it later in Section 7.

5.4.3. *Computational cost of* GALERKIN

The output pair $(\widehat{\mathcal{T}}, \widehat{\mathcal{D}})$ of DATA is next taken by GALERKIN, the one-step AFEM of Algorithm 5.4 in Section 5.2.1, to run an inner loop of the form (5.18) with fixed discrete data $\widehat{\mathcal{D}}$ and initial mesh $\widehat{\mathcal{T}}$. The call (5.19) of GALERKIN stops as soon as the error tolerance ε is reached, which takes a finite number of iterations because GALERKIN is a contraction between consecutive iterates, and creates the next mesh-solution pair $(\mathcal{T}, u_{\mathcal{T}})$. It is worth noticing that, in the absence of this stopping test, the Galerkin solution $u_{\mathcal{T}}$ would converge to the solution $\widehat{u} = u(\widehat{\mathcal{D}})$ of (5.5), which is not the desired solution $u = u(\mathcal{D})$ of (2.5).

We stress that, in view of (5.63) and (5.65), the relative resolution of the modules DATA and GALERKIN is critical for the discrepancy between the exact and perturbed solutions u and $\widehat{u}$. This is ultimately responsible for the performance of AFEM-TS and is studied in Section 6.

We now investigate the number of iterations within GALERKIN, which dictate its computational cost. We point out that at iteration $k-1 \geq 0$ of AFEM-TS, the output $(\mathcal{T}_k, u_k)$ of GALERKIN, and thus of AFEM-TS, satisfies

$$\eta_k(u_k) = \eta_{\mathcal{T}_k}(u_k) \leq \varepsilon_{k-1} \quad \Rightarrow \quad |u_k - \widehat{u}_{k-1}|_{H^1_0(\Omega)} \leq C_U \varepsilon_{k-1} \tag{5.83}$$

according to (5.12). We recall that $\widehat{u}_{k-1} = \widehat{u}_{k-1}(\widehat{\mathcal{D}}_{k-1}) \in H^1_0(\Omega)$ is the exact solution with discrete data $\widehat{\mathcal{D}}_{k-1}$, and that $\mathcal{E}_{\mathcal{T}_k}(u_k, f)$ is defined with discrete data $\widehat{\mathcal{D}}_{k-1}$ and satisfies $\mathcal{E}_{\mathcal{T}_k}(u_k, f) = \eta_{\mathcal{T}_k}(u_k)$ because data oscillation $\mathrm{osc}_{\mathcal{T}_k}(f)_{-1} = 0$. The next iteration k of AFEM-TS calls DATA, which in turn refines the mesh $\mathcal{T}_k$ to $\widehat{\mathcal{T}}_k$ and updates the data approximation from $\widehat{\mathcal{D}}_{k-1}$ to $\widehat{\mathcal{D}}_k$ over $\widehat{\mathcal{T}}_k$. The pair $(\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k)$ determines the first Galerkin solution $u_{k,0} \in \mathbb{V}_{k,0} = \mathbb{V}_{\widehat{\mathcal{T}}_k}$ of GALERKIN and corresponding estimator $\eta_{k,0}(u_{k,0})$ with $\mathcal{T}_{k,0} = \widehat{\mathcal{T}}_k$, which must satisfy

$$\eta_{k,0}(u_{k,0}) > \varepsilon_k \tag{5.84}$$

for GALERKIN to be executed. The reduction of $\eta_{\mathcal{T}_{k,j}}(u_{k,j})$ for $j \geq 1$ dictates the number of iterations of GALERKIN. We examine this next.

Proposition 5.27 (computational cost of GALERKIN). *If the assumptions of Theorem 5.8 are valid, then for any $k \in \mathbb{N}$, the number of subiterations J_k inside a call to* GALERKIN *at iteration k of* AFEM-TS *is bounded independently of k.*

Proof. The jth error $e_{k,j} := |\widehat{u}_k - u_{k,j}|_{H^1_0(\Omega)}$ within GALERKIN converges linearly in view of Corollary 5.10 (linear convergence of error) because the discrete data $\widehat{\mathcal{D}}_k$

is fixed in these inner iterations. Exploiting the lower bound $C_L\eta_{k,j}(u_{k,j}) \le e_{k,j}$ stated in (5.12), we thus deduce

$$\eta_{k,j}(u_{k,j}) \le C_L^{-1} e_{k,j} \le C_L^{-1} C_* \alpha^{j-i} e_{k,i}, \quad j \ge i \ge 0,$$

whence $\eta_{k,j}(u_{k,j}) \le C_\# \alpha^j e_{k,0}$ with $C_\# := C_L^{-1} C_*$. The number of iterations of GALERKIN depends on the size of $\eta_{k,0}(u_{k,0})$ relative to ε_k. We assume that $\eta_{k,0}(u_{k,0}) > \varepsilon_k$ according to (5.84). We first prove that $\eta_{k,0}(u_{k,0}) \lesssim \varepsilon_k$, and next argue that J_k is bounded uniformly in k. We proceed in two steps.

$\boxed{1}$ *Bound on* $|\widehat{u}_k - u_{k,0}|_{H_0^1(\Omega)}$. Since $u_k \in \mathbb{V}_k \subset \mathbb{V}_{k,0} = \mathbb{V}_{\widehat{\mathcal{T}}_k}$, and the Galerkin solution $u_{k,0} \in \mathbb{V}_{k,0}$ minimizes the error $|||u_{k,0} - \widehat{u}_k|||_\Omega$ in $\mathbb{V}_{k,0}$, relative to the energy norm induced by the bilinear form $\widehat{\mathcal{B}}$ with discrete data $\widehat{\mathcal{D}}_k$, we deduce

$$|||u_{k,0} - \widehat{u}_k|||_\Omega \le |||u_k - \widehat{u}_k|||_\Omega \le \sqrt{C_{\widehat{\mathcal{B}}}}\big(|u_k - \widehat{u}_{k-1}|_{H_0^1(\Omega)} + |\widehat{u}_{k-1} - \widehat{u}_k|_{H_0^1(\Omega)}\big),$$

where the last inequality uses (5.3) for $\widehat{\mathcal{B}}$. Invoking the *a posteriori* upper bound (5.13) and the termination condition of GALERKIN at step $k-1$, we obtain

$$|u_k - \widehat{u}_{k-1}|_{H_0^1(\Omega)} \le C_U \mathcal{E}_{\mathcal{T}_k}(u_k, f) = C_U \eta_k(u_k) \le C_U \varepsilon_{k-1} = 2C_U \varepsilon_k.$$

On the other hand, using (5.65) with $\tau = \omega\varepsilon_k$ and $0 < \omega \le 1$, we arrive at

$$|u - \widehat{u}_k|_{H_0^1(\Omega)} \le C_1 \varepsilon_k,$$

with $C_1 = \omega C_D$. The triangle inequality thus yields

$$|\widehat{u}_{k-1} - \widehat{u}_k|_{H_0^1(\Omega)} \le |u - \widehat{u}_{k-1}|_{H_0^1(\Omega)} + |u - \widehat{u}_k|_{H_0^1(\Omega)} \le C_1(\varepsilon_{k-1} + \varepsilon_k) = 3C_1\varepsilon_k,$$

whence

$$e_{k,0} = |u_{k,0} - \widehat{u}_k|_{H_0^1(\Omega)} \le \sqrt{\frac{C_{\widehat{\mathcal{B}}}}{c_{\widehat{\mathcal{B}}}}}(2C_U + 3C_1)\varepsilon_k =: C_2\varepsilon_k.$$

$\boxed{2}$ *Bound on* J_k. We observe that GALERKIN stops once $\eta_{k,j}(u_{k,j}) \le \varepsilon_k$. Since the smallest such j is J_k, we see that

$$\varepsilon_k < \eta_{k,J_k-1}(u_{k,J_k-1}) \le C_\# \alpha^{J_k-1} e_{k,0} \le C_\# C_2 \varepsilon_k \alpha^{J_k-1}.$$

This implies the asserted bound

$$J_k \le 1 + \frac{\log(C_\# C_2)}{\log \alpha^{-1}}$$

uniform in k. □

5.4.4. *Realization of* AFEM-TS

We now make the two-step AFEM algorithm precise.

Algorithm 5.28 (AFEM-TS). Given an initial tolerance $\varepsilon_0 > 0$, a target tolerance $\texttt{tol}$ and initial mesh $\mathcal{T}_0$, as well as a safety parameter $\omega \in (0, 1]$, AFEM consists of the two-step algorithm:

$[\mathcal{T}, u_{\mathcal{T}}] = \mathsf{AFEM\text{-}TS}(\mathcal{T}_0, \varepsilon_0, \omega, \texttt{tol})$
  set $k = 0$ and do
    $[\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k] = \mathsf{DATA}(\mathcal{T}_k, \mathcal{D}, \omega\, \varepsilon_k)$
    $[\mathcal{T}_{k+1}, u_{k+1}] = \mathsf{GALERKIN}(\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k, \varepsilon_k)$
    $\varepsilon_{k+1} = \frac{1}{2}\varepsilon_k$
    $k \leftarrow k + 1$
  while $\varepsilon_{k-1} > \texttt{tol}$
  return $\mathcal{T}_k, u_k$

Proposition 5.29 (convergence of AFEM-TS). *For each $k \geq 0$, the modules* DATA *and* GALERKIN *converge in a finite number of iterations, the latter independent of k. Moreover, there exists a constant C_* depending on $\mathcal{T}_0, \Omega, d, n$, the Lebesgue exponents r, q in $D(\Omega)$, the parameters $\alpha_1, \alpha_2, c_1, c_2$ in (5.48) and (5.49), and the shape regularity constant of $\mathbb{T}$, such that the output of the $(k + 1)$th iteration $[\mathcal{T}_{k+1}, u_{k+1}] = \mathsf{GALERKIN}(\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k, \varepsilon_k)$ satisfies $|u - u_{k+1}|_{H^1_0(\Omega)} \leq C_* \varepsilon_k$ for all $k \geq 0$. Therefore* AFEM-TS *stops after*

$$K < 2 + \frac{\log(\varepsilon_0/\texttt{tol})}{\log 2}$$

iterations and delivers

$$|u - u_K|_{H^1_0(\Omega)} \leq C_* \, \texttt{tol}\,.$$

Proof. In view of Assumption 5.22 (properties of DATA), the module DATA iterates a finite number of steps to reach tolerance $\tau = \omega\varepsilon_k$ for every $k \geq 0$. Moreover, the number of iterations of GALERKIN is independent of k due to Proposition 5.27 (computational cost of GALERKIN), whence we deduce that each loop of AFEM-TS requires finite iterations. Thus, the output u_{k+1} of the $(k + 1)$th loop satisfies

$$|u - u_{k+1}|_{H^1_0(\Omega)} \leq |u - \widehat{u}_k|_{H^1_0(\Omega)} + |\widehat{u}_k - u_{k+1}|_{H^1_0(\Omega)} \leq (\omega C_D + C_U)\varepsilon_k = C_* \varepsilon_k,$$

according to (5.65) with $\tau \leq \omega\varepsilon_k$ and (5.83) for all $k \geq 0$. Finally, AFEM-TS terminates after K loops, where K satisfies $\frac{1}{2}\,\texttt{tol} < \varepsilon_{K-1} \leq \texttt{tol}$, and the asserted estimate holds. □

This elementary proof gives no insight into whether the H^1_0-error decays optimally in terms of degrees of freedom. We assess this fundamental question in Sections 6 and 7, but investigate it computationally in Section 5.4.5.

A two-step algorithm similar to AFEM-TS was first proposed by Stevenson (2008), and further explored by Bonito *et al.* (2013*b*) and Cohen *et al.* (2012). Note that other quantities, such as the number of degrees of freedom, could be employed to stop AFEM-TS instead. It is also worth realizing that the structure of

the algorithm is independent of the size of tolerance `tol`. In this vein, a user could take $\varepsilon_0 = \texttt{tol}$, provided `tol` is affordable by the computational resources at hand. With such a choice, the modules DATA and GALERKIN run only once, in sequence: data are approximated to the desired accuracy in one shot, then fed to the PDE solver which produces the approximate solution. Since the quasi-optimality theory in Section 6 would also hold for this choice of ε_0, one might wonder why we do not use this simpler strategy. We stress that iterating over ε_k has the following advantages.

- *Restarts.* Dynamical shrinking of `tol`, for instance to account for the user decision to improve the accuracy, does not entail a restart of AFEM-TS but rather a continuation from the previous computed solution. In this sense, the resulting iteration would be similar to the proposed structure of AFEM-TS.
- *Computational resources.* AFEM-TS allows for 'balanced investment' of computational resources between the modules DATA and GALERKIN. If the stopping criterion, either accuracy or number of degrees of freedom, is unrealistic for the problem at hand, AFEM-TS would still produce a discrete solution with equilibrated data and solution errors.
- *Nonlinear problems.* The interleaving approach of AFEM-TS appears to be better suited to treating nonlinear problems for which data $\mathcal{D}$ may depend on the solution. Therefore a call to GALERKIN, and corresponding solution update, must precede a call to DATA.
- *Iterative solvers.* If an efficient iterative solver is adopted within SOLVE, then the previous discrete solution of GALERKIN could be taken as initial iterate, thereby making SOLVE fast because $\epsilon_{k+1}/\epsilon_k = 1/2$. If instead we compute with DATA alone until the fixed tolerance `tol` is reached, then GALERKIN would work directly on fine meshes, which are not adapted to the geometric domain singularities, and without a good initial guess. This would lead to fewer but heavier iterations of GALERKIN, which is detrimental from a linear algebra perspective.

5.4.5. Computational assessment of AFEM-TS

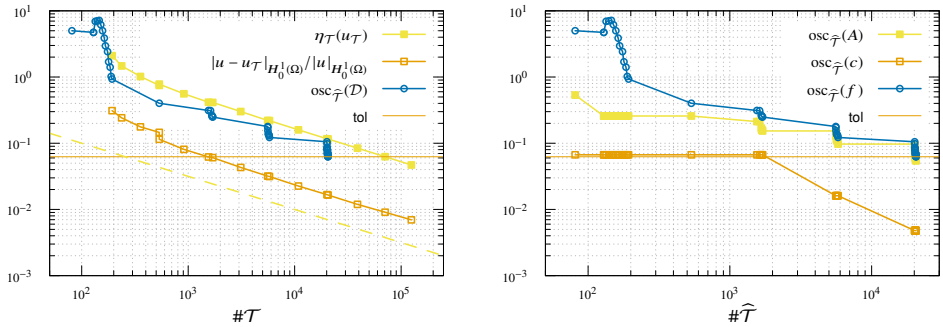
In this section we explore computationally the relative performance of GALERKIN and DATA, for the two-step AFEM, and elucidate the behaviour of data and coefficient oscillations within DATA. Our observations motivate the rigorous study of Section 6, which provides theoretical support to our experiments. The numerical computations are made with the help of Funken, Praetorius and Wissgott (2011).

We consider problem (2.5) in the L-shaped domain $\Omega = (-1,1)^2 \setminus ([0,1] \times [-1,0])$, with diffusion term $\boldsymbol{A} = a\boldsymbol{I}$, where

$$a(x,y) = 1 + \exp(-50((x+0.5)^2 + (y+0.5)^2)) + \exp(-50((x+0.5)^2 + (y-0.5)^2))$$

and reaction term

$$c(x,y) = 1 + \exp(-50((x+0.5)^2 + y^2)) + \exp(-50(x^2 + (y-0.5)^2));$$



(a) (b)

Figure 5.6. (a) Estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}})$, data error $\mathrm{osc}_{\widehat{\mathcal{T}}}(\mathcal{D})$, and relative H^1-error obtained with the algorithm AFEM performing $b = 3$ bisections per marked element. The optimal decay is indicated by the dashed line with slope -0.5. (b) Diffusion error $\mathrm{osc}_{\widehat{\mathcal{T}}}(\boldsymbol{A})$, reaction error $\mathrm{osc}_{\widehat{\mathcal{T}}}(c)$, load error $\mathrm{osc}_{\widehat{\mathcal{T}}}(f)_{-1}$, obtained with the algorithm AFEM.

note that the Gaussians in the definition of a and c have the same intensity but are located in different places within Ω. The load term f and the Dirichlet boundary conditions are chosen in accordance with the analytical solution

$$u(x, y) = r^{2/3}\sin(2\alpha/3) + \exp(-1000((x-0.5)^2 + (y-0.5)^2)),$$

where (r, α) are the polar coordinates around the origin. Notice that the exact solution u is singular at the re-entrant corner: it belongs to the Sobolev spaces $H(\Omega)^{5/3-\varepsilon}$ with $\varepsilon > 0$ and $W_p^2(\Omega)$ with $p > 1$. It also exhibits a rapid transition of order $10^{-3/2}$ around the point $(0.5, 0.5)$ due to the presence of a very narrow Gaussian. The Gaussians are meant to test the performance of the module DATA, while in addition the corner singularity of the solution tests the execution of the module GALERKIN.

We utilize the following parameters in the numerical test:

$$\theta = 0.5, \quad \omega = 1, \quad \mathtt{tol} = 2^{-4}, \quad h_0 = 0.125, \quad \varepsilon_0 = 1.$$

Notice that the number of iterations of the algorithm AFEM is $K = \log_2(\epsilon_0/\mathtt{tol}) = 4$. We compute the relative H^1-error between the exact solution u and the FEM solution $u_{\mathcal{T}}$ and notice that its decay rate is $(\#\mathcal{T})^{-1/2}$ in Figure 5.6(a). This rate is consistent with that of the PDE estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ and data estimator $\mathrm{osc}_{\mathcal{T}}(\mathcal{D})$. In Figure 5.6(b) we display the component of the data error $\mathrm{osc}_{\widehat{\mathcal{T}}}(\boldsymbol{A})$, $\mathrm{osc}_{\widehat{\mathcal{T}}}(c)$, $\mathrm{osc}_{\widehat{\mathcal{T}}}(f)_{-1}$ defined in (5.79) and (5.80) with local contributions defined in (5.73) for $\boldsymbol{A}$ with $r = \infty$, in (5.76) for c with $t = 1$ and (5.78) for f. Recall that at each iteration k, DATA circles through $\mathrm{osc}_{\widehat{\mathcal{T}}}(\boldsymbol{A})$, $\mathrm{osc}_{\widehat{\mathcal{T}}}(c)$, and $\mathrm{osc}_{\widehat{\mathcal{T}}}(f)_{-1}$, reducing each of these oscillations to $\frac{1}{3}$ of the iteration tolerance $\varepsilon_k = 2^{-k}$. The presence of the weight h^t in $\mathrm{osc}_{\widehat{\mathcal{T}}}(c)$ considerably reduces the influence of the approximation of c,

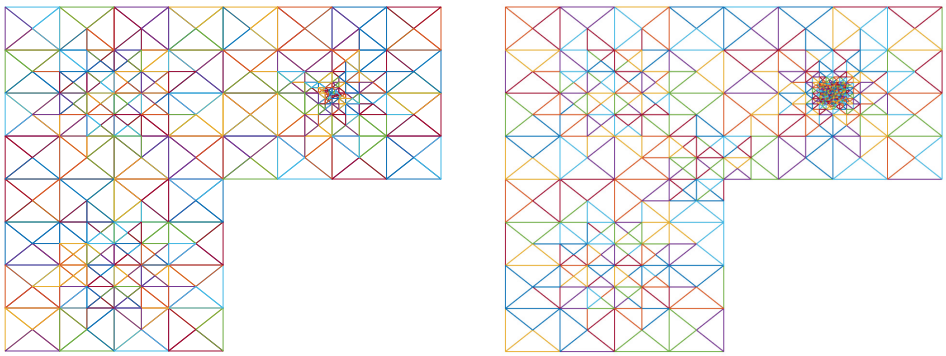
Table 5.1. Number of marked elements to reduce the data and Galerkin errors at each iteration $k = 1, 2, 3, 4$ of AFEM-TS when using $b = 1$ and $b = 3$ refinements per marked element. Regardless of the value used for b, the reduction of the Galerkin error is driving most of the refinements followed by the error in the approximation of the diffusion coefficient $\boldsymbol{A}$. The approximation of f is subordinate to the approximation of u and $\boldsymbol{A}$ arising earlier in the adaptive loop, and thus does not generate any refinement except during the first iteration, when the Galerkin error has not yet been tackled by the algorithm. The approximation of c is below the final tolerance from the start and does not generate any refinement.

k	$\mathrm{osc}_{\widehat{\mathcal{T}}}(\boldsymbol{A})$		$\mathrm{osc}_{\widehat{\mathcal{T}}}(c)$		$\mathrm{osc}_{\widehat{\mathcal{T}}}(f)_{-1}$		$\eta_{\mathcal{T}}(u_{\mathcal{T}})$	
	$b=1$	$b=3$	$b=1$	$b=3$	$b=1$	$b=3$	$b=1$	$b=3$
1	32	16	0	0	26	13	363	308
2	16	16	0	0	0	0	1 636	1 138
3	120	43	0	0	0	0	7 447	4 227
4	123	62	0	0	0	0	42 792	15 268
5	82	138	0	0	0	0	144 345	102 350

which is below threshold from the start and thus never generates any refinement (see Table 5.1). The local oscillation for f also includes a weight vanishing as $h \to 0$ but $\mathrm{osc}_{\widehat{\mathcal{T}}}(f)_{-1}$ is above the desired tolerance, which would in principle generate refinements. However, since at each iteration DATA considers $\mathrm{osc}_{\widehat{\mathcal{T}}}(\boldsymbol{A})$ first and the regions refined to reduce $\mathrm{osc}_{\widehat{\mathcal{T}}}(f)_{-1}$ are included in the regions needed to be refined to reduce $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ and $\mathrm{osc}_{\widehat{\mathcal{T}}}(\boldsymbol{A})$, the GREEDY routine applied to f does not refine any element except during the first iteration, when the Galerkin error has not yet been reduced by the algorithm. Overall, the reduction of the Galerkin error is driving most of the refinements. The number of marked elements to reduce the approximation errors of $\boldsymbol{A}$, c, f and the residual estimator are reported in Table 5.1 along with those when $b = 1$ refinement is used per marked element. In Figure 5.7 we provide the resulting meshes after the first iteration of DATA and GALERKIN.

5.5. *Convergence for other boundary conditions*

First we consider the variational problem (2.13) with *Robin* boundary condition. We approximate data $\mathcal{D} = (\boldsymbol{A}, c, p, f, g)$ by piecewise polynomials $\widehat{\mathcal{D}} = (\widehat{\boldsymbol{A}}, \widehat{c}, \widehat{p}, \widehat{f}, \widehat{g})$, The only difference with respect to (5.2) is that the new functions (p, g) are approximated on $\partial\Omega$ by discontinuous polynomials $(\widehat{p}, \widehat{g})$ of degree $n-1$ and $2n-1$. The projection operator $P_{\mathcal{T}}$ approximates $g\delta_{\partial\Omega}$ by $\widehat{g}\delta_{\partial\Omega} = P_{\mathcal{T}}(g\delta_{\partial\Omega})$ without component in the bulk because $g\delta_{\partial\Omega}$ is a line Dirac mass aligned with the mesh. Discrete functions $(\widehat{p}, \widehat{g})$ must be produced by DATA, subject to a sign constraint on p. The



(a) (b)

Figure 5.7. Resulting meshes after the first iteration of DATA (a) and after the first iteration of GALERKIN (b). DATA marked 29 elements for refinement while GALERKIN marked 308 elements. Refer to Table 5.1 for more details.

approximate bilinear form $\widehat{\mathcal{B}}$ and linear functional $\widehat{\ell}$ read

$$\widehat{\mathcal{B}}[w,v] := \int_\Omega \nabla v \cdot \widehat{\boldsymbol{A}} \nabla w + \widehat{c} v w + \int_{\partial\Omega} \widehat{p} v w, \quad \widehat{\ell}(v) := \langle \widehat{f}, v \rangle + \int_{\partial\Omega} \widehat{g}\, v. \tag{5.85}$$

The *a posteriori* error estimates of Section 4 extend to this pair $(\widehat{\mathcal{B}}, \widehat{\ell})$. The algorithms GALERKIN, AFEM-SW and AFEM-TS are similar to those above and possess a similar supporting convergence theory. The *Neumann* boundary condition is a particular case with $p = 0$. We do not pursue this any further.

However, the *pure Neumann* boundary condition is special because of the global compatibility condition $\widehat{\ell}(1) = \langle \widehat{\ell}, 1 \rangle = 0$. In Section 4.10 we introduce a new projection operator $\widetilde{P}_{\mathcal{T}}$, a modification of $P_{\mathcal{T}}$, with the requisite properties of local approximation and global compatibility $\langle \widetilde{P}_{\mathcal{T}} \ell, 1 \rangle = 0$ provided $\ell \in H^1(\Omega)^*$ satisfies $\langle \ell, 1 \rangle = 0$. We thus set $\widetilde{\ell} = \widetilde{P}_{\mathcal{T}} \ell$ to solve the Galerkin problems and use $\widetilde{P}_z$ in the local indicators. We do not explore this matter further.

For a *non-homogeneous Dirichlet* boundary data $g \in H^{1/2}(\partial\Omega)$, DATA must produce a continuous piecewise polynomial approximation $\widehat{g}$ of degree n, thereby consistent with the Galerkin solution $u_{\mathcal{T}}$. The Dirichlet oscillation $\operatorname{osc}_{\mathcal{T}}(g)_{1/2}$ is defined in (4.96) and is locally computable. Data oscillation now becomes

$$\operatorname{osc}_{\mathcal{T}}(\ell) = \operatorname{osc}_{\mathcal{T}}(f)_{-1} + \operatorname{osc}_{\mathcal{T}}(g)_{1/2},$$

and added to the PDE estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ for $g = 0$ gives a full estimator equivalent to the error, according to Theorem 4.74 (estimators for general Dirichlet conditions). With these minor modifications, the convergence theory for GALERKIN, AFEM-SW and AFEM-TS extends to this case. We do not provide any further details.

5.6. *Convergence for alternative estimators*

We have so far developed a convergence theory for the residual estimator $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)$. The purpose of this section is to extend this theory to the three alternative estimators discussed in Section 4.9, namely

- $\mathcal{E}_{\mathcal{T}}^{\mathrm{lpb}}(u_{\mathcal{T}}, f)^2 = \eta_{\mathcal{T}}^{\mathrm{lpb}}(u_{\mathcal{T}})^2 + \mathrm{osc}_{\mathcal{T}}(f)_{-1}^2$: estimator based on *local problems*,
- $\mathcal{E}_{\mathcal{T}}^{\mathrm{hier}}(u_{\mathcal{T}}, f)^2 = \eta_{\mathcal{T}}^{\mathrm{hier}}(u_{\mathcal{T}})^2 + \mathrm{osc}_{\mathcal{T}}(f)_{-1}^2$: *hierarchical* estimator,
- $\mathcal{E}_{\mathcal{T}}^{\mathrm{feq}}(u_{\mathcal{T}}, f)^2 = \eta_{\mathcal{T}}^{\mathrm{feq}}(u_{\mathcal{T}})^2 + \mathrm{osc}_{\mathcal{T}}(f)_{-1}^2$: estimator based on *flux equilibration*.

They are all computed on stars ω_z with $z \in \mathcal{V}$ and possess a similar structure. The first term is the PDE estimator, from now on called $\zeta_{\mathcal{T}}(u_{\mathcal{T}})$ to refer to any of them, and is locally equivalent to the discrete residual $P_{\mathcal{T}} R_{\mathcal{T}}$

$$\zeta_{\mathcal{T}}(u_{\mathcal{T}}, z) \approx \|P_{\mathcal{T}} R_{\mathcal{T}}\|_{H^{-1}(\omega_z)} \quad \text{for all } z \in \mathcal{V}; \tag{5.86}$$

see Theorems 4.59, 4.64 and 4.69. In fact they are all different mechanisms to extract information from $P_{\mathcal{T}} R_{\mathcal{T}}$. Since the vertex-indexed residual PDE indicator $\eta_{\mathcal{T}}(u_{\mathcal{T}}, z) := \eta_{\mathcal{T}}^{\mathrm{res}}(u_{\mathcal{T}}, z)$, defined in (4.70a), is also proved to be equivalent to $\|P_{\mathcal{T}} R_{\mathcal{T}}\|_{H^{-1}(\omega_z)}$ in Theorem 4.58 (vertex-indexed modified residual estimator), we deduce the existence of two equivalence constants $C_L^{\mathrm{eq}} \le C_U^{\mathrm{eq}}$ such that

$$C_L^{\mathrm{eq}} \eta_{\mathcal{T}}(u_{\mathcal{T}}, z) \le \zeta_{\mathcal{T}}(u_{\mathcal{T}}, z) \le C_U^{\mathrm{eq}} \eta_{\mathcal{T}}(u_{\mathcal{T}}, z) \quad \text{for all } z \in \mathcal{V}. \tag{5.87}$$

Following Kreuzer and Siebert (2011), we will exploit this property to prove convergence of AFEM driven by $\zeta_{\mathcal{T}}(u_{\mathcal{T}})$. An obstruction to a direct convergence theory is that our preceding results rely heavily on Lemma 4.53 (reduction property of the estimator), which is not necessarily valid for any of the alternative estimators. We refer to Cascón and Nochetto (2012), who present a direct approach based on the local lower bound for discrete solutions of Theorem 4.51 (lower bound for corrections). The latter is guaranteed by Definition 4.50 (interior vertex property) for operators with coefficients $\boldsymbol{A}$ piecewise constant and $c = 0$, and any polynomial degree $n \ge 1$, but we do not know its validity for more general coefficients $(\boldsymbol{A}, c)$.

The key for convergence is imposing a Dörfler marking. We say that a set of vertices $\mathcal{M}_{\mathcal{V}}$ satisfies a Dörfler property with parameter $\theta \le 1$ if

$$\zeta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{M}_{\mathcal{V}})^2 := \sum_{z \in \mathcal{M}_{\mathcal{V}}} \zeta_{\mathcal{T}}(u_{\mathcal{T}}, z)^2 \ge \theta^2 \sum_{z \in \mathcal{V}} \zeta_{\mathcal{T}}(u_{\mathcal{T}}, z)^2 =: \zeta_{\mathcal{T}}(u_{\mathcal{T}})^2. \tag{5.88}$$

Let $\mathcal{M}$ be the collection of elements contained in the stars ω_z with $z \in \mathcal{M}_{\mathcal{V}}$. Then MARK marks all elements in $\mathcal{M}$, and REFINE bisects them $b \ge 1$ times. This gives rise to a star-driven GALERKIN procedure.

Lemma 5.30 (Dörfler property). *If the set of vertices $\mathcal{M}_{\mathcal{V}}$ satisfies a Dörfler property with parameter θ for $\zeta_{\mathcal{T}}(u_{\mathcal{T}})$, then $\mathcal{M}$ satisfies a Dörfler property with parameter $\overline{\theta} = (C_L^{\mathrm{eq}}/C_U^{\mathrm{eq}})\theta$ for $\eta_{\mathcal{T}}(u_{\mathcal{T}})$.*

Proof. Simply use (5.87) to derive (5.88) for $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ with parameter $\overline{\theta}$. □

Hence, star-driven procedures for $\zeta_{\mathcal{T}}(u_{\mathcal{T}})$ lead to the corresponding counterparts for $\eta_{\mathcal{T}}(u_{\mathcal{T}})$. It turns out that algorithms GALERKIN, AFEM-SW and AFEM-TS can be reformulated for vertex-indexed indicators $\{\eta_{\mathcal{T}}(u_{\mathcal{T}}, z)\}_{z\in\mathcal{V}}$ as defined in (4.70a), without changing their essential properties. We may thus wonder about them driven by $\{\zeta_{\mathcal{T}}(u_{\mathcal{T}}, z)\}_{z\in\mathcal{V}}$ instead. Since these algorithms hinge on the Dörfler property (5.88), Lemma 5.30 gives rise to similar convergence properties for $\zeta_{\mathcal{T}}(u_{\mathcal{T}})$-driven algorithms provided (5.88) is enforced. We state this next without proof.

Corollary 5.31 (convergence of GALERKIN). *If the coefficients* $(\boldsymbol{A}, c, f) \in \mathbb{D}_{\mathcal{T}}$, *then there exist* $0 < \overline{\alpha} < 1$ *and* $C_*, C_\# > 0$ *such that the solution–estimator pairs* $(u_j, \zeta_j(u_j))$ *of* GALERKIN *converge linearly, namely, for all* $k \geq j \geq 0$,

$$|u - u_k|_{H_0^1(\Omega)} \leq C_* \,\overline{\alpha}^{k-j} |u - u_j|_{H_0^1(\Omega)}, \quad \zeta_k(u_k) \leq C_\# \,\overline{\alpha}^{k-j} \zeta_j(u_j).$$

Corollary 5.32 (convergence of AFEM-SW). *If the coefficients* $(\boldsymbol{A}, c)$ *are discrete and* $f \in H^{-1}(\Omega)$, *then for* $0 < \omega \leq \omega_0$, $\xi \leq \frac{1}{2}$ *as in Theorem 5.17, there exist* $0 < \overline{\alpha} < 1$ *and* $C_*, C_\# > 0$ *such that the solution–estimator pairs* $(u_j, \mathcal{E}_j(u_j, f))$ *of* AFEM-SW, *where* $\mathcal{E}_j(u_j, f)^2 = \zeta_j(u_j)^2 + \operatorname{osc}_j(f)^2_{-1}$, *converge linearly: for all* $k \geq j \geq 0$,

$$|u - u_k|_{H_0^1(\Omega)} \leq C_* \,\overline{\alpha}^{k-j} |u - u_j|_{H_0^1(\Omega)}, \quad \mathcal{E}_k(u_k, f) \leq C_\# \,\overline{\alpha}^{k-j} \mathcal{E}_j(u_j, f).$$

Both GALERKIN and AFEM-SW converge under restrictions on $\mathcal{D} = (\boldsymbol{A}, c, f)$. For arbitrary data $\mathcal{D}$, AFEM-TS concatenates GALERKIN and DATA, the latter being unrelated to $\zeta_{\mathcal{T}}(u_{\mathcal{T}})$. Therefore Corollary 5.31 and Proposition 5.27 (computational cost of GALERKIN) yield the following extension of Proposition 5.29 (convergence of AFEM-TS).

Corollary 5.33 (convergence of AFEM-TS). *The algorithm* AFEM-TS *driven by* $\zeta_{\mathcal{T}}(u_{\mathcal{T}})$ *stops after*

$$K < 2 + \frac{\log(\varepsilon_0/\texttt{tol})}{\log 2}$$

iterations and delivers the error

$$|u - u_K|_{H_0^1(\Omega)} \leq C_* \,\texttt{tol}\,.$$

The number of iterations of GALERKIN *is bounded uniformly for all outer loops.*

6. Convergence rates of AFEM for coercive problems

The ultimate goal of AFEM is to produce a *quasi-best* approximation $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$ to the solution $u \in \mathbb{V}$ of (2.7) with error measured in $\mathbb{V} = H_0^1(\Omega)$. The performance of AFEM is measured by the size of the error $|u - u_{\mathcal{T}}|_{H_0^1(\Omega)}$ relative to the cardinality $\#\mathcal{T}$ of $\mathcal{T}$. The latter usually reflects the total computational cost of implementing AFEM. As a benchmark, it is useful to compare the performance of AFEM with

the best approximation of $u \in \mathbb{V}$ and $\mathcal{D} = (\boldsymbol{A}, c, f) \in \mathbb{D}$, provided we have full knowledge of them. This is the main purpose of this section.

Under suitable assumptions on the solution u and data $\mathcal{D}$, we prove the existence of constants $C(u, \mathcal{D}) > 0$ and $s \in (0, n/d]$ such that

$$|u - u_{\mathcal{T}_k}|_{H^1_0(\Omega)} \leq C(u, \mathcal{D})\,(\#\mathcal{T}_k)^{-s}, \tag{6.1}$$

provided s is the best decay rate with meshes in $\mathbb{T}$ with a comparable number of degrees of freedom. The upper bound n/d of s is dictated by the best decay rate with polynomials of degree $n \geq 1$ in dimension d unless u is degenerate (for instance, u belongs to a finite element space $\mathbb{V}_{\mathcal{T}}$ with $\mathcal{T} \in \mathbb{T}$). The dependence on $\mathcal{D}$ of the constant $C(u, \mathcal{D})$ accounts for the multiplicative structure of the interaction between the coefficients $(\boldsymbol{A}, c)$ and u, and cannot be avoided in general.

A crucial insight for the simplest scenario, the Laplacian and piecewise constant forcing f, is due to Stevenson (2007). It has been extended to operators with variable coefficients by Cascón *et al.* (2008) and later expressed in terms of the estimator by Carstensen *et al.* (2014). It reads as follows:

If a marking strategy reduces the PDE estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}})$ to a fraction of its current value, then the refined set of elements $\mathcal{R}$ inherits an error indicator $\eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{R})$ comparable to $\eta_{\mathcal{T}}(u_{\mathcal{T}})$, hence a Dörfler marking. (6.2)

This allows us to compare meshes produced by AFEM with optimal meshes and to conclude a quasi-optimal error decay. To this end, in Section 6.1 we introduce approximation classes for functions in $\mathbb{V}$ and $\mathbb{D}$, tailored to the decomposition of Ω into conforming refinements of an initial conforming partition $\mathcal{T}_0$, the root of $\mathbb{T}$. We will assume that $u = u(\mathcal{D}) \in \mathbb{V}$ and $\mathcal{D} = (\boldsymbol{A}, c, f) \in \mathbb{D}$ belong to these classes which, however, are not characterized in terms of regularity of u and $\mathcal{D}$. In Section 6.2, we investigate the approximability properties of perturbations $\widehat{u} = u(\widehat{\mathcal{D}})$ of the exact solution u, namely exact solutions of (5.5) with perturbed data $\widehat{\mathcal{D}}$. Next, in Section 6.3, we consider a conforming refinement $\mathcal{T}_* \in \mathbb{T}$ of a partition $\mathcal{T} \in \mathbb{T}$, and give conditions under which an optimal Dörfler marking property holds. We first apply this in Section 6.4 to study and derive rate-optimality of GALERKIN and AFEM-SW, the one-step AFEMs. We then combine the quasi-optimal performances of GALERKIN and DATA to prove rate-optimality of the two-step AFEM in Section 6.5. We conclude in Section 6.8 upon bridging the gap between appproximation and regularity classes. In particular, we give sufficient conditions for functions in Besov, Sobolev and Lipschitz spaces to belong to the approximation classes.

6.1. Nonlinear approximation classes

In Section 6.1.1 we discuss approximation classes for functions in $\mathbb{V}$, which are applicable to the solution u of (2.7). In Section 6.1.2 we turn our attention to approximation classes for functions in $\mathbb{D}$, which are in turn applicable to data $\mathcal{D}$.

We refer to DeVore (1998), as well as DeVore and Lorentz (1993) and Binev *et al.* (2002), for a discussion within nonlinear approximation theory.

6.1.1. Nonlinear approximation classes for functions in $\mathbb{V}$

For any $N \in \mathbb{N}$, $N \geq \#\mathcal{T}_0$, we define the following collection of partitions within $\mathbb{T}$:

$$\mathbb{T}_N = \{\mathcal{T} \mid \mathcal{T} \in \mathbb{T} \text{ satisfies } \#\mathcal{T} \leq N\}.$$

This is the set of *conforming* meshes generated from $\mathcal{T}_0$ with at most $N - \#\mathcal{T}_0$ bisections. Given $v \in \mathbb{V}$, we let $\sigma_N(v)$ be the smallest approximation H_0^1-error incurred on v with *continuous* piecewise polynomial functions of degree $\leq n$ over meshes $\mathbb{T}_N$:

$$\sigma_N(v) := \inf_{\mathcal{T} \in \mathbb{T}_N} \inf_{v_\mathcal{T} \in \mathbb{V}_\mathcal{T}} |v - v_\mathcal{T}|_{H_0^1(\Omega)}. \tag{6.3}$$

This is a theoretical measure of performance, in that finding a mesh $\mathcal{T} \in \mathbb{T}_N$ that realizes $\sigma_N(v)$ has exponential complexity. Proving a bound $|v - v_\mathcal{T}|_{1,\Omega} \leq C_1 \sigma_{C_2 N}(v)$ for $\mathcal{T} \in \mathbb{T}_N$ with $C_2 \leq 1 \leq C_1$ independent of N, the so-called *instance optimality*, is rather difficult and beyond the scope of this survey. In fact, a function $v \in \mathbb{V}_\mathcal{T}$ with $\mathcal{T} \in \mathbb{T}_N$ could be the solution of our model problem (2.7), because we allow forcing $f \in H^{-1}(\Omega)$. Hence we see that $\sigma_N(v) = 0$, and AFEM should then capture v exactly on a finer mesh $\mathcal{T} \in \mathbb{T}_{C_2^{-1} N}$. We refer to Diening *et al.* (2016) for a proof of instance optimality for a forcing $f \in L^2(\Omega)$ and the Laplace operator, namely for coefficients $\boldsymbol{A} = \boldsymbol{I}$ and $c = 0$.

We will instead be able to prove that the error $|v - v_\mathcal{T}|_{H_0^1(\Omega)}$ for the Galerkin solution $v_\mathcal{T}$ for $\mathcal{T} \in \mathbb{T}_N$ decays in terms of N with the same rate N^{-s} as $\sigma_N(v)$; we thus say that AFEM is *rate-optimal*. We first note that for $v \in H^{n+1}(\Omega)$ and $\mathcal{T} \in \mathbb{T}_N$ quasi-uniform, we expect to have

$$\inf_{v_\mathcal{T} \in \mathbb{V}_\mathcal{T}} |v - v_\mathcal{T}|_{H_0^1(\Omega)} \lesssim N^{-n/d} |v|_{H^{n+1}(\Omega)} \tag{6.4}$$

because the global mesh size h and N satisfy $h \approx N^{-1/d}$. This error estimate within the linear Sobolev scale provides the largest possible decay rate $-n/d$.

Definition 6.1 (approximation class of u). Given $0 < s \leq n/d$, the class $\mathbb{A}_s := \mathbb{A}_s(H_0^1(\Omega); \mathcal{T}_0)$, relative to the partition $\mathcal{T}_0$ and approximation in the H_0^1-norm by continuous piecewise polynomials of degree $\leq n$ on the forest $\mathbb{T}$ emanating from $\mathcal{T}_0$, is the set of functions $v \in \mathbb{V} = H_0^1(\Omega)$ such that

$$|v|_{\mathbb{A}_s} := \sup_{N \geq \#\mathcal{T}_0} (N^s \sigma_N(v)) < \infty, \tag{6.5a}$$

whence

$$\sigma_N(v) \leq |v|_{\mathbb{A}_s} N^{-s} \quad \text{for all } N \geq \#\mathcal{T}_0. \tag{6.5b}$$

We also write $\mathbb{A}_s = \mathbb{A}_s^0$ to emphasize continuity of the discrete functions in $\mathbb{V}_\mathcal{T} = \mathbb{S}_\mathcal{T}^{n,0} \cap \mathbb{V}$ with $\mathcal{T} \in \mathbb{T}$. The quantity $|v|_{\mathbb{A}_s}$ is a quasi seminorm in $\mathbb{A}_s$, which is not

a linear space but rather a nonlinear class of functions. Notice that as s increases, the cost of membership to be in $\mathbb{A}_s$ increases, namely $\mathbb{A}_{s_1} \subset \mathbb{A}_{s_2}$ for $s_1 \geq s_2$.

We may as well consider approximating $v \in \mathbb{V}$ with *discontinuous* piecewise polynomials $\mathbb{S}_{\mathcal{T}}^{n,-1}$ of degree $\leq n$, which is a richer space than $\mathbb{S}_{\mathcal{T}}^{n,0}$. We can likewise define the corresponding modulus of approximation

$$\sigma_N^{(-1)}(v) := \inf_{\mathcal{T} \in \mathbb{T}_N} \inf_{v_{\mathcal{T}} \in \mathbb{S}_{\mathcal{T}}^{n,-1}} |v - v_{\mathcal{T}}|_{H_0^1(\Omega;\mathcal{T})} \tag{6.6}$$

and approximation class $\mathbb{A}_s^{-1} := \mathbb{A}_s^{-1}(H_0^1(\Omega); \mathcal{T}_0)$ of functions $v \in H_0^1(\Omega)$ such that

$$|v|_{\mathbb{A}_s^{-1}} := \sup_{N \geq \#\mathcal{T}_0} \left(N^s \sigma_N^{(-1)}(v)\right) < \infty \quad \Rightarrow \quad \sigma_N^{(-1)}(v) \leq |v|_{\mathbb{A}_s^{-1}} N^{-s}. \tag{6.7}$$

It is obvious that $\sigma_N^{(-1)}(v) \leq \sigma_N(v)$ for all $v \in H_0^1(\Omega)$ because $\mathbb{S}_{\mathcal{T}}^{n,0} \subset \mathbb{S}_{\mathcal{T}}^{n,-1}$. However, we have the following equivalence result taken from Veeser (2016). The original proof, although more complicated and for a different notion of error relevant to discontinuous Galerkin approximations, can be traced back to Bonito and Nochetto (2010, Proposition 5.2); see Proposition 9.4.

Proposition 6.2 (equivalence of classes). *Assume that all stars of meshes $\mathcal{T} \in \mathbb{T}$ are $(d-1)$-face-connected. Then, there exists a constant C_{dG} that depends on the shape regularity of $\mathbb{T}$, the dimension d and the polynomial degree $n \geq 1$, such that*

$$\sigma_N(v) \leq C_{\mathrm{dG}}\, \sigma_N^{(-1)}(v) \quad \textit{for all } v \in H_0^1(\Omega), \quad N \geq \#\mathcal{T}_0.$$

Moreover, the approximation classes coincide, i.e. $\mathbb{A}_s^0 = \mathbb{A}_s^{-1}$.

Proof. We simply resort to (3.19) of Proposition 3.9 (approximation of gradients), namely, for $v \in H_0^1(\Omega)$,

$$1 \leq \frac{\min_{w \in \mathbb{S}_{\mathcal{T}}^{n,0}} |v - w|_{H_0^1(\Omega)}}{\min_{w \in \mathbb{S}_{\mathcal{T}}^{n,-1}} |v - w|_{H_0^1(\Omega;\mathcal{T})}} \leq C_{\mathrm{dG}},$$

and use the definitions (6.3) and (6.6). This completes the proof. □

In the rest of the paper we will make the following approximability assumption.

Assumption 6.3 (approximability of u). The exact solution $u \in H_0^1(\Omega)$ of problem (2.5) belongs to the approximation class $\mathbb{A}_s(H_0^1(\Omega); \mathcal{T}_0)$ with $s = s_u \in (0, n/d]$.

The following condition (6.8) is simpler to handle in practice than (6.5).

Lemma 6.4 (membership of $\mathbb{A}_s$). *Let $v \in \mathbb{A}_s$ and $\varepsilon_0 := \inf_{v_{\mathcal{T}_0} \in \mathbb{V}_{\mathcal{T}_0}} |v - v_{\mathcal{T}_0}|_{H_0^1(\Omega)}$. Then, for all $0 < \varepsilon \leq \varepsilon_0$, there exist $\mathcal{T}_\varepsilon \in \mathbb{T}$ and $v_\varepsilon \in \mathbb{V}_{\mathcal{T}_\varepsilon}$ such that*

$$|v - v_\varepsilon|_{H_0^1(\Omega)} \leq \varepsilon, \quad \#\mathcal{T}_\varepsilon \leq 1 + |v|_{\mathbb{A}_s}^{1/s} \varepsilon^{-1/s}. \tag{6.8}$$

Proof. Given $0 < \varepsilon \le \varepsilon_0$, let $\mathcal{T}_\varepsilon \in \mathbb{T}$ be a conforming refinement of $\mathcal{T}_0$ with minimal cardinality and $v_\varepsilon \in \mathbb{V}_{\mathcal{T}_\varepsilon}$ such that

$$|v - v_\varepsilon|_{H_0^1(\Omega)} \le \varepsilon.$$

Therefore, if $\varepsilon < \varepsilon_0$, we deduce from the minimal property of $\mathcal{T}_\varepsilon$ that

$$\inf_{v_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}} |v - v_{\mathcal{T}}|_{H_0^1(\Omega)} > \varepsilon \quad \text{for all } \mathcal{T} \in \mathbb{T} \quad \text{such that} \quad \#\mathcal{T} \le \#\mathcal{T}_\varepsilon - 1.$$

If $N := \#\mathcal{T}_\varepsilon - 1$, definition (6.5) implies

$$\varepsilon < \inf_{\mathcal{T} \in \mathbb{T}_N} \inf_{v_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}} |v - v_{\mathcal{T}}|_{H_0^1(\Omega)} = \sigma_N(v) \le |v|_{\mathbb{A}_s} N^{-s},$$

whence

$$\#\mathcal{T}_\varepsilon = 1 + N \le 1 + |v|_{\mathbb{A}_s}^{1/s} \varepsilon^{-1/s},$$

as asserted in (6.8). On the other hand, if $\varepsilon = \varepsilon_0$ we see that

$$\varepsilon_0 \le |v|_{\mathbb{A}_s} (\#\mathcal{T}_0)^{-s} \quad \Rightarrow \quad \#\mathcal{T}_0 \le |v|_{\mathbb{A}_s}^{1/s} \varepsilon_0^{-1/s} < 1 + |v|_{\mathbb{A}_s}^{1/s} \varepsilon_0^{-1/s}.$$

This completes the proof. □

Remark 6.5. If $d = 2$ and $n = 1$, then Corollary 3.20 (optimal H^1-convergence rate) shows that $W_p^2(\Omega) \subset \mathbb{A}^{1/2}$ for $p > 1$. The space $W_p^2(\Omega)$ is much larger than $H^2(\Omega)$, fits within the nonlinear Sobolev scale, and delivers the same decay rate as (6.4). We will investigate the connection between approximation classes $\mathbb{A}_s$ and regularity classes in any dimension d and for any polynomial degree $n \ge 1$ later in Section 6.8.

6.1.2. Nonlinear approximation classes for data in $\mathbb{D}$

Given data $\mathcal{D} = (\boldsymbol{A}, c, f) \in \mathbb{D}$ and a mesh $\mathcal{T} \in \mathbb{T}$, we consider the best approximation of $\mathcal{D}$ by discrete (piecewise polynomial) data $\widehat{\mathcal{D}} = (\widehat{\boldsymbol{A}}, \widehat{c}, \widehat{f}) \in \mathbb{D}_{\mathcal{T}}$, where $\mathbb{D}$ and $\mathbb{D}_{\mathcal{T}}$ are defined in (5.1) and (5.2). We measure the error in the space $D(\Omega)$ defined in (5.60) with $q = 2$ for $d < 4$ or $q > d/2$ for $d \ge 4$. We now discuss the best approximation errors for the components of data in $D(\Omega)$, which are used to define the corresponding approximation classes. For the coefficients $(\boldsymbol{A}, c)$, they are characterized by the quantities

$$\delta_{\mathcal{T}}(\boldsymbol{A})_r := \inf_{\widehat{\boldsymbol{A}} \in [\mathbb{S}_{\mathcal{T}}^{n-1,-1}]^{d\times d}} \|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|_{L^r(\Omega)}, \quad \delta_{\mathcal{T}}(c)_q := \inf_{\widehat{c} \in \mathbb{S}_{\mathcal{T}}^{n-1,-1}} \|c - \widehat{c}\|_{L^q(\Omega)} \tag{6.9}$$

for $r, q \in [2, \infty]$ as above. Note that $\widehat{\boldsymbol{A}}$ and $\widehat{c}$ in (6.9) are unconstrained in the sense that they do not necessarily satisfy the structural assumption (5.51) and are thus not suited to the perturbed problem (5.5). We define the best constrained

approximation errors for $\boldsymbol{A} \in M(\alpha_1, \alpha_2)$ and $c \in R(c_1, c_2)$ by

$$\begin{aligned} \widetilde{\delta}_{\mathcal{T}}(\boldsymbol{A})_r &:= \inf_{\widehat{\boldsymbol{A}} \in [\mathbb{S}_{\mathcal{T}}^{n-1,-1}]^{d\times d} \cap M(\widehat{\alpha}_1, \widehat{\alpha}_2)} \|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|_{L^r(\Omega)}, \\ \widetilde{\delta}_{\mathcal{T}}(c)_q &:= \inf_{\widehat{c} \in \mathbb{S}_{\mathcal{T}}^{n-1,-1} \cap R(\widehat{c}_1, \widehat{c}_2)} \|c - \widehat{c}\|_{L^q(\Omega)}, \end{aligned} \tag{6.10}$$

where in view of (5.52)

$$\widehat{\alpha}_1 = \frac{\alpha_1}{2}, \quad \widehat{\alpha}_2 = C_{\mathrm{ctr}}\alpha_2, \quad \widehat{c}_1 = -\frac{\alpha_1}{4C_P^2}, \quad \widehat{c}_2 = C_{\mathrm{ctr}}(\alpha_1 + c_1). \tag{6.11}$$

We mention in anticipation that in Section 7.4 we prove the equivalences

$$\delta_{\mathcal{T}}(\boldsymbol{A})_r \le \widetilde{\delta}_{\mathcal{T}}(\boldsymbol{A})_r \le C_{\mathrm{data}}\delta_{\mathcal{T}}(\boldsymbol{A})_r, \quad \delta_{\mathcal{T}}(c)_q \le \widetilde{\delta}_{\mathcal{T}}(c)_q \le C_{\mathrm{data}}\delta_{\mathcal{T}}(c)_q \tag{6.12}$$

for all $\boldsymbol{A} \in M(\alpha_1, \alpha_2)$ and $c \in R(c_1, c_2)$; see Remarks 7.13 and 7.17. For the load function f, the definition (4.56) of $\mathrm{osc}_{\mathcal{T}}(f)_{-1}$ suggests considering

$$\delta_{\mathcal{T}}(f)_{-1} := \left(\sum_{T \in \mathcal{T}} \inf_{\widehat{f} \in \mathbb{F}_{\mathcal{T}_{\omega_T}}} \|f - \widehat{f}\|^2_{H^{-1}(\omega_T)} \right)^{1/2}.$$

All these best approximation errors are hard to evaluate and are thus replaced by the computable oscillations defined in (5.79) and (5.80) in practice. We recall that they rely on the local L^2-projection operator $\Pi_{\mathcal{T}}$ for $(\boldsymbol{A}, c)$ and the local H^{-1}-projection operator $P_{\mathcal{T}}$ for f to compute linear approximations $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ to a desired accuracy. These projections are later modified nonlinearly to give rise to $\widehat{\mathcal{D}}$ satisfying the side constraints (5.51) without compromising accuracy. We recall that the DATA module is assumed to construct approximations so that

$$\mathrm{osc}_{\mathcal{T}}(\boldsymbol{A})_r \le \Lambda_{\mathrm{data}}\delta_{\mathcal{T}}(\boldsymbol{A})_r, \quad \mathrm{osc}_{\mathcal{T}}(c)_q \le \Lambda_{\mathrm{data}}\delta_{\mathcal{T}}(c)_q, \quad \mathrm{osc}_{\mathcal{T}}(f)_{-1} \le \Lambda_{\mathrm{data}}\delta_{\mathcal{T}}(f)_{-1}$$

with a mesh independent constant Λ_{data}; see Assumption 5.22. In Section 5.4.2 we discuss practical realizations of DATA.

For the purpose of assessing the cardinality of AFEM, we do not need the specific form of $\widehat{\mathcal{D}}$ but rather the decay of the best approximation errors in terms of degrees of freedom. Therefore we postpone to Section 7 the construction of $\widehat{\mathcal{D}}$ for $n \ge 1$, and to Section 6.8 the discussion of regularity properties of $\mathcal{D}$ that guarantee membership of the following approximation classes.

Definition 6.6 (approximation classes of $\boldsymbol{A}$). For $0 < \alpha_1 \le \alpha_2$, $2 \le r \le \infty$, let $\mathbb{M}_s := \mathbb{M}_s(L^r(\Omega; \mathbb{R}^{d\times d}); \mathcal{T}_0)$ be the set of matrix-valued functions $\boldsymbol{A} \in M(\alpha_1, \alpha_2)$ satisfying

$$|\boldsymbol{A}|_{\mathbb{M}_s} := \sup_{N \ge \#\mathcal{T}_0} \left(N^s \inf_{\mathcal{T} \in \mathbb{T}_N} \widetilde{\delta}_{\mathcal{T}}(\boldsymbol{A})_r \right) < \infty \Rightarrow \inf_{\mathcal{T} \in \mathbb{T}_N} \widetilde{\delta}_{\mathcal{T}}(\boldsymbol{A})_r \le |\boldsymbol{A}|_{\mathbb{M}_s} N^{-s}. \tag{6.13}$$

Definition 6.7 (approximation classes of c). The class $\mathbb{C}_s := \mathbb{C}_s(L^q(\Omega); \mathcal{T}_0)$ is the set of functions $c \in R(c_1, c_2)$ such that

$$|c|_{\mathbb{C}_s} := \sup_{N \geq \#\mathcal{T}_0} \Big(N^s \inf_{\mathcal{T} \in \mathbb{T}_N} \widetilde{\delta}_{\mathcal{T}}(c)_q \Big) < \infty \;\Rightarrow\; \inf_{\mathcal{T} \in \mathbb{T}_N} \widetilde{\delta}_{\mathcal{T}}(c)_q \leq |c|_{\mathbb{C}_s} N^{-s}. \tag{6.14}$$

Definition 6.8 (approximation classes of f). The class $\mathbb{F}_s := \mathbb{F}_s(H^{-1}(\Omega); \mathcal{T}_0)$ is the set of functions $f \in H^{-1}(\Omega)$ such that

$$|f|_{\mathbb{F}_s} := \sup_{N \geq \#\mathcal{T}_0} \Big(N^s \inf_{\mathcal{T} \in \mathbb{T}_N} \delta_{\mathcal{T}}(f)_{-1} \Big) < \infty \;\Rightarrow\; \inf_{\mathcal{T} \in \mathbb{T}_N} \delta_{\mathcal{T}}(f)_{-1} \leq |f|_{\mathbb{F}_s} N^{-s}. \tag{6.15}$$

Since the polynomial degree of discrete coefficients $(\widehat{\boldsymbol{A}}, \widehat{c})$ in definition (5.2) is $n-1$, we expect decay rates $s_A, s_c \leq n/d$ according to nonlinear approximation theory. The specific values of (s_A, s_c) depend on the regularity of $(\boldsymbol{A}, c)$, a delicate topic that we further investigate in Sections 6.8 and 7. However, because u and $\mathcal{D} = (\boldsymbol{A}, c, f)$ satisfy the elliptic problem (2.5), the above approximation classes are somewhat related. We now quantify this statement.

Lemma 6.9 (relation between approximation classes). *Let $2 \leq r, q \leq \infty$ be such that $d/2 < q$. If $u \in \mathbb{A}_{s_u}(H^1_0(\Omega); \mathcal{T}_0)$, $\boldsymbol{A} \in \mathbb{M}_{s_A}(L^r(\Omega; \mathbb{R}^{d\times d}); \mathcal{T}_0)$ and $c \in \mathbb{C}_{s_c}(L^q(\Omega); \mathcal{T}_0)$, with $0 < s_u, s_A, s_c \leq n/d$, then $f \in \mathbb{F}_{s_f}(H^{-1}(\Omega); \mathcal{T}_0)$ and*

$$|f|_{\mathbb{F}_{s_f}} \leq C\big(|u|_{\mathbb{A}_{s_u}} + |\boldsymbol{A}|_{\mathbb{M}_{s_A}} + |c|_{\mathbb{C}_{s_c}}\big), \quad s_f = \min\{s_u, s_A, s_c\}, \tag{6.16}$$

where the constant $C > 0$ depends on $\|u\|_{W^1_p(\Omega)}$, $p = 2r/(r-2)$ and $\alpha_1, \alpha_2, c_1, c_2$. In particular, if $(\boldsymbol{A}, c)$ are discrete in $\mathcal{T}_0$, then

$$|f|_{\mathbb{F}_{s_f}} \leq C|u|_{\mathbb{A}_{s_u}}, \quad s_f = s_u. \tag{6.17}$$

Proof. Let $L[u] := -\operatorname{div}(\boldsymbol{A}\nabla u) + cu$ be the operator in (2.5) and note that $f = L[u] \in H^{-1}(\Omega)$ can be approximated by $\widehat{f} = -\operatorname{div}(\widehat{\boldsymbol{A}}\nabla\widehat{v}) + \widehat{c}\widehat{v} \in \mathbb{F}_{\mathcal{T}}$, where the discrete space $\mathbb{F}_{\mathcal{T}}$ is given in Definition 4.17 and $\widehat{v} \in \mathbb{S}^{n,0}_{\mathcal{T}}$, $\widehat{\boldsymbol{A}} \in (\mathbb{S}^{n-1,-1}_{\mathcal{T}})^{d\times d}$, $\widehat{c} \in \mathbb{S}^{n-1,-1}_{\mathcal{T}}$. Let us now express $f - \widehat{f}$ as follows:

$$f - \widehat{f} = -\operatorname{div}((\boldsymbol{A} - \widehat{\boldsymbol{A}})\nabla u) + (c - \widehat{c})u - \operatorname{div}(\widehat{\boldsymbol{A}}\nabla(u - \widehat{v})) + \widehat{c}(u - \widehat{v}),$$

and recall that we have to estimate $\|f - \widehat{f}\|_{H^{-1}(\omega_T)}$ for every $T \in \mathcal{T}$, rather than a global norm in $H^{-1}(\Omega)$, to get an upper bound on $\delta_{\mathcal{T}}(f)_{-1}$. Therefore we proceed as in the proof of Lemma 5.20 (continuous dependence on data), to obtain

$$\begin{aligned}\sum_{T \in \mathcal{T}} \|f - \widehat{f}\|^2_{H^{-1}(\omega_T)} \lesssim{}& \|\nabla u\|^2_{L^p(\Omega)} \|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|^2_{L^r(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \|c - \widehat{c}\|^2_{L^q(\Omega)} \\ &+ \|\widehat{\boldsymbol{A}}\|^2_{L^\infty(\Omega)} \|\nabla(u - \widehat{v})\|^2_{L^2(\Omega)} + \|\widehat{c}\|^2_{L^\infty(\Omega)} \|u - \widehat{v}\|^2_{L^2(\Omega)},\end{aligned}$$

where $p = 2r/(r-1)$ and $\|\nabla u\|_{L^p(\Omega)} < \infty$ according to (2.41) and $d/2 < q \leq \infty$.

Note that thanks to (6.11), $\|\widehat{A}\|_{L^\infty(\Omega)} \le \widehat{\alpha}_2 = C_{\mathrm{ctr}}\alpha_2$ and $\|\widehat{c}\|_{L^\infty(\Omega)} \le \widehat{c}_2 = C_{\mathrm{ctr}}(\alpha_1 + c_2)$. Moreover, since $\widehat{v}, \widehat{\boldsymbol{A}}, \widehat{c}$ can be chosen separately, invoking (6.5), (6.13), (6.14) and (6.15), we realize that

$$\begin{aligned}\inf_{\mathcal{T}\in\mathbb{T}_N} \delta_{\mathcal{T}}(f)_{-1} &\lesssim \inf_{\mathcal{T}\in\mathbb{T}_N} \inf_{\widehat{v}\in\mathbb{S}^{n,0}_{\mathcal{T}}} \|\nabla(u-\widehat{v})\|_{L^2(\Omega)} \\ &\quad + \inf_{\mathcal{T}\in\mathbb{T}_N} \inf_{\widehat{\boldsymbol{A}}\in[\mathbb{S}^{n-1,-1}_{\mathcal{T}}]^{d\times d}} \|\boldsymbol{A}-\widehat{\boldsymbol{A}}\|_{L^r(\Omega)} \\ &\quad + \inf_{\mathcal{T}\in\mathbb{T}_N} \inf_{\widehat{c}\in\mathbb{S}^{n-1,-1}_{\mathcal{T}}} \|c-\widehat{c}\|_{L^q(\Omega)} \\ &\le |u|_{\mathbb{A}_{s_u}} N^{-s_u} + |\boldsymbol{A}|_{\mathbb{M}_{s_A}} N^{-s_A} + |c|_{\mathbb{C}_{s_c}} N^{-s_c}\end{aligned}$$

gives (6.16) with $s_f = \min\{s_u, s_A, s_c\}$; (6.17) is a trivial consequence. □

Estimate (6.17) will be useful later in Theorem 6.20 (rate-optimality of one-step AFEMs). It is important to realize that the multiplicative structure between solution u and coefficients $(\boldsymbol{A}, c)$ is hidden in the constants C in (6.16) and (6.17). Moreover, these estimates are possible due to the fact that the space $H^{-1}(\Omega)$ is the range of the linear operator $L\colon H^1_0(\Omega) \to H^{-1}(\Omega)$ and that the discrete functions in $\mathbb{F}_{\mathcal{T}}$ are images by L of functions in $\mathbb{V}_{\mathcal{T}}$. This would not be true for L^2-weighted surrogates of $\delta_{\mathcal{T}}(f)_{-1}$ that typically overestimate the error in $H^{-1}(\Omega)$.

Assumption 6.10 (approximability of data). There exist $s_A, s_c, s_f \in (0, n/d]$ such that data $\mathcal{D} = (\boldsymbol{A}, c, f) \in \mathbb{D}$ satisfies $\boldsymbol{A} \in \mathbb{M}_{s_A}$, $c \in \mathbb{C}_{s_c}$, $f \in \mathbb{F}_{s_f}$.

We recall that if $\mathrm{osc}_{\mathcal{T}}(\mathcal{D}) = \|\mathcal{D} - \widehat{\mathcal{D}}\|_{D(\Omega)} > C_{\mathrm{data}}\tau$ over a conforming refinement $\mathcal{T} \in \mathbb{T}$ of $\mathcal{T}_0$, then the call

$$[\widehat{\mathcal{T}}, \widehat{\mathcal{D}}] = \mathsf{DATA}(\mathcal{T}, \mathcal{D}, \tau)$$

produces a conforming refinement $\widehat{\mathcal{T}}$ of $\mathcal{T}$ and approximate data $\widehat{\mathcal{D}} = (\widehat{\boldsymbol{A}}, \widehat{c}, \widehat{f}) \in \mathbb{D}_{\widehat{\mathcal{T}}}$ over $\widehat{\mathcal{T}}$ that satisfies $\mathrm{osc}_{\widehat{\mathcal{T}}}(\mathcal{D}) \le \Lambda_{\mathrm{data}}\delta_{\widehat{\mathcal{T}}}(\mathcal{D})$, and for $r, q \in [2, \infty]$,

$$\mathrm{osc}_{\widehat{\mathcal{T}}}(\mathcal{D}) = \mathrm{osc}_{\widehat{\mathcal{T}}}(\boldsymbol{A})_r + \mathrm{osc}_{\widehat{\mathcal{T}}}(c)_q + \mathrm{osc}_{\widehat{\mathcal{T}}}(f)_{-1} \le C_{\mathrm{data}}\tau,$$

as well as the constraints $\widehat{\boldsymbol{A}} \in M(\widehat{\alpha}_1, \widehat{\alpha}_2)$ and $\widehat{c} \in R(\widehat{c}_1, \widehat{c}_2)$ defined in (5.51). We will show in Section 7 that the routine responsible for reducing oscillations, namely GREEDY, exhibits optimal performance in the sense that the cardinalities $N_{\mathcal{T}}(\boldsymbol{A}), N_{\mathcal{T}}(c), N_{\mathcal{T}}(f)$ of the sets of elements necessary to reduce the individual oscillations of $(\boldsymbol{A}, c, f)$ below the threshold $C_{\mathrm{data}}(\tau/3)$ starting from any $\mathcal{T} \ge \mathcal{T}_0$ satisfy

$$N_{\mathcal{T}}(\boldsymbol{A}) \lesssim |\boldsymbol{A}|^{1/s_A}_{\mathbb{M}_{s_A}} \tau^{-1/s_A}, \quad N_{\mathcal{T}}(c) \lesssim |c|^{1/s_c}_{\mathbb{C}_{s_c}} \tau^{-1/s_c}, \quad N_{\mathcal{T}}(f) \lesssim |f|^{1/s_f}_{\mathbb{F}_{s_f}} \tau^{-1/s_f}. \tag{6.18}$$

Therefore the cost of one call to DATA can be quantified by the total number $N_{\mathcal{T}}(\mathcal{D})$

of elements marked, which obeys the relation

$$\begin{aligned} N_{\mathcal{T}}(\mathcal{D}) &= N_{\mathcal{T}}(\boldsymbol{A}) + N_{\mathcal{T}}(c) + N_{\mathcal{T}}(f) \\ &\lesssim |\boldsymbol{A}|_{\mathbb{M}_{s_A}}^{1/s_A} \tau^{-1/s_A} + |c|_{\mathbb{C}_{s_c}}^{1/s_c} \tau^{-1/s_c} + |f|_{\mathbb{F}_{s_f}}^{1/s_f} \tau^{-1/s_f} \\ &\leq |\mathcal{D}|_{\mathbb{A}_{\mathcal{D}}}^{1/s_{\mathcal{D}}} \tau^{-1/s_{\mathcal{D}}}, \end{aligned}$$

with

$$s_{\mathcal{D}} := \min\{s_A, s_c, s_f\}, \quad |\mathcal{D}|_{\mathbb{A}_{\mathcal{D}}} := \left(|\boldsymbol{A}|_{\mathbb{M}_{s_A}}^{1/s_A} + |c|_{\mathbb{C}_{s_c}}^{1/s_c} + |f|_{\mathbb{F}_{s_f}}^{1/s_f} \right)^{s_{\mathcal{D}}}. \tag{6.19}$$

It is thus natural to make the following assumption on DATA.

Assumption 6.11 (quasi-optimality of DATA). The call $[\widehat{\mathcal{T}}, \widehat{\mathcal{D}}] = \mathsf{DATA}(\mathcal{T}, \mathcal{D}, \tau)$, from an arbitrary conforming refinement $\mathcal{T}$ of $\mathcal{T}_0$ with tolerance τ, marks the number of elements $N_{\mathcal{T}}(\mathcal{D})$ to produce an approximation $\widehat{\mathcal{D}}$ of $\mathcal{D}$ over $\widehat{\mathcal{T}}$ so that

$$\operatorname{osc}_{\widehat{\mathcal{T}}}(\mathcal{D}) = \|\mathcal{D} - \widehat{\mathcal{D}}\|_{D(\Omega)} \leq C_{\text{data}}\tau, \quad N_{\mathcal{T}}(\mathcal{D}) \lesssim |\mathcal{D}|_{\mathbb{A}_{\mathcal{D}}}^{1/s_{\mathcal{D}}} \tau^{-1/s_{\mathcal{D}}}. \tag{6.20}$$

6.2. ε-approximation of order s

Inspection of the structure of algorithm AFEM-TS (Algorithm 5.28) reveals that the approximate data $\mathcal{D}_k$ is fixed inside GALERKIN. Therefore the performance of GALERKIN is dictated by the regularity of the exact solution $\widehat{u}_k = \widehat{u}_k(\mathcal{D}_k) \in H_0^1(\Omega)$ with data $\mathcal{D}_k$, rather than the exact solution $u = u(\mathcal{D})$ with data $\mathcal{D}$. We know that $u \in \mathbb{A}_s$, and wonder what regularity is inherited by $\widehat{u}_k$. This leads to the following concept introduced in Bonito *et al.* (2013*b*, Definition 3.1, Lemma 3.2).

Definition 6.12 (ε-approximation of order s). Given $u \in \mathbb{A}_s(H_0^1(\Omega); \mathcal{T}_0)$ and $\varepsilon > 0$, a function $v \in H_0^1(\Omega)$ is said to be an ε-approximation of order s to u if $|u - v|_{H_0^1(\Omega)} \leq \varepsilon$ and there exists a constant $C > 0$ independent of ε, u and v such that for all $\delta \geq \varepsilon$ there exists $N \geq \#\mathcal{T}_0$ satisfying

$$\sigma_N(v) \leq \delta, \quad N \leq 1 + C|u|_{\mathbb{A}_s}^{1/s} \delta^{-1/s}. \tag{6.21}$$

Lemma 6.13 (ε-approximation of u of order s). *Let $u \in \mathbb{A}_s(H_0^1(\Omega); \mathcal{T}_0)$ and $v \in H_0^1(\Omega)$ satisfy $|u - v|_{H_0^1(\Omega)} \leq \varepsilon$ for some $0 < \varepsilon \leq \varepsilon_0$ with ε_0 defined in Lemma 6.4. Then v is a 2ε-approximation of order s to u.*

Proof. Let $\delta \geq 2\varepsilon$. By definition (6.3) of $\sigma_N(v)$, it suffices to invoke the triangle inequality to realize that

$$\sigma_N(v) \leq |u - v|_{H_0^1(\Omega)} + \sigma_N(u) \leq \frac{\delta}{2} + \sigma_N(u).$$

Since $u \in \mathbb{A}_s(H_0^1(\Omega))$, in view of Lemma 6.4, there exist $N \geq \#\mathcal{T}_0$ and $\mathcal{T} \in \mathbb{T}_N$:

$$\sigma_N(u) \leq \frac{\delta}{2}, \quad N \leq 1 + |u|_{\mathbb{A}_s}^{1/s} \left(\frac{\delta}{2}\right)^{-1/s}.$$

The estimate (6.21) thus follows with constant $C = 2^{1/s}$. □

This is a simple but crucial result for studying AFEM-TS. It says that any function v that is ε-close to a function $u \in \mathbb{A}_s(X; \mathcal{T}_0)$ in the norm of the space X defining the approximation class $\mathbb{A}_s(X; \mathcal{T}_0)$ can be approximated with a decay rate similar to u in X for as long as the desired accuracy does not exceed ε. In other words, the approximability of u is inherited by v up to scale ε. However, beyond the scale ε, the approximability of v may differ from that of u. Note that neither the definition (6.3) of $\sigma_N(v)$ nor Lemma 6.13 require $X = H_0^1(\Omega)$.

6.3. *Properties of Dörfler marking*

We follow the ideas of Stevenson (2007), Cascón *et al.* (2008) and Carstensen *et al.* (2014) to explore the insight (6.2) about Dörfler marking. Hereafter, we recall (5.6) and consider two admissible partitions $\mathcal{T}, \mathcal{T}_* \in \mathbb{T}$ such that $\mathcal{T} \leq \mathcal{T}_*$, that is, the latter is a refinement of the former obtained by applying (newest-vertex) bisection to some of the elements of $\mathcal{T}$.

In what follows, we let $u = \widehat{u} \in H_0^1(\Omega)$ be the exact solution with discrete coefficients $(\widehat{A}, \widehat{c})$ over a fixed mesh $\widehat{\mathcal{T}} \leq \mathcal{T}$ and forcing function f that may or may not be discrete. We rewrite the *a posteriori* error estimates (5.12),

$$C_L\, \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f) \leq |\widehat{u} - u_{\mathcal{T}}|_{H_0^1(\Omega)} \leq C_U\, \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f), \tag{6.22}$$

where the total estimator $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)$ consists of the PDE estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}}, f)$ and the oscillation $\operatorname{osc}_{\mathcal{T}}(f)_{-1}$ and reads, according to (5.11),

$$\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)^2 = \eta_{\mathcal{T}}(u_{\mathcal{T}}, f)^2 + \operatorname{osc}_{\mathcal{T}}(f)_{-1}^2.$$

We also recall that when $f = P_{\mathcal{T}} f \in \mathbb{F}_{\mathcal{T}}$ is discrete, $\operatorname{osc}_{\mathcal{T}}(f)_{-1} = 0$ and $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)$ reduces to $\eta_{\mathcal{T}}(u_{\mathcal{T}}, f)$, and that $P_{\mathcal{T}} f$ is used within $\eta_{\mathcal{T}}(u_{\mathcal{T}}, f)$ rather than f. The global nature of the elliptic boundary value problem (2.5) prevents upper *a posteriori* energy error estimates such as (6.22) between the continuous and discrete solution from being local. Remarkably, the situation for two Galerkin solutions $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$ and $u_{\mathcal{T}_*} \in \mathbb{V}_{\mathcal{T}_*}$ is different, as stated in Theorem 4.48 (upper bound for corrections):

$$|u_{\mathcal{T}_*} - u_{\mathcal{T}}|_{H_0^1(\Omega)} \leq \widetilde{C}_U\, \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f, \mathcal{R}), \tag{6.23}$$

where $\mathcal{R} = \mathcal{T} \backslash \mathcal{T}_*$ is the refined set defined in (5.14). It thus turns out that $|u_{\mathcal{T}_*} - u_{\mathcal{T}}|_{H_0^1(\Omega)}$ is controlled by the estimator $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f, \mathcal{R})$ on the set of elements $\mathcal{R}$ where the meshes differ. This crucial observation goes back to Stevenson (2007); see also Cascón *et al.* (2008) and Nochetto *et al.* (2009).

Henceforth we will impose restrictions on the ranges of the Dörfler marking parameter (5.23) and the threshold parameter ω for GALERKIN and AFEM-SW. We will impose a different restriction later on ω for AFEM-TS.

Assumption 6.14 (marking parameter). Let θ satisfy $\theta \in (0, \theta_0)$ with

$$\theta_0 := \min\big\{(2C_{\mathrm{Lip}}\widetilde{C}_U)^{-1}, 1\big\},$$

where $C_{\mathrm{Lip}}, \widetilde{C}_U$ are the constants in (4.68) and (6.23) respectively.

Assumption 6.15 (restriction on ω). We assume $0 \le \omega \le \omega_0 < 1$, with

$$\omega_0 := \sqrt{\frac{\theta_0^2 - \theta^2}{2 + \theta_0^2 - \theta^2}}.$$

We are now ready to make Stevenson's insight (6.2) precise.

Lemma 6.16 (Dörfler marking). *Let Assumptions 6.14 and 6.15 hold and $0 < \mu \le \frac{1}{2}$. Let $\mathcal{T} \in \mathbb{T}$, and let $\mathcal{T}_* \in \mathbb{T}$ be a refinement of $\mathcal{T}$ with respective Galerkin solutions $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$ and $u_{\mathcal{T}_*} \in \mathbb{V}_{\mathcal{T}_*}$; let $\mathcal{R} = \mathcal{T} \setminus \mathcal{T}_*$ be the refined set. Assume that the oscillation on $\mathcal{T}$ is dominated by the total estimator*

$$\operatorname{osc}_{\mathcal{T}}(f)_{-1} \le \omega\, \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f), \tag{6.24}$$

and that

$$\eta_{\mathcal{T}_*}(u_{\mathcal{T}_*}, f) \le \mu\, \eta_{\mathcal{T}}(u_{\mathcal{T}}, f). \tag{6.25}$$

Then Dörfler marking is valid for any $0 < \theta < \theta_0$:

$$\theta\, \eta_{\mathcal{T}}(u_{\mathcal{T}}, f) \le \eta_{\mathcal{T}}(u_{\mathcal{T}}, f, \mathcal{R}). \tag{6.26}$$

Proof. We invoke Proposition 4.56 (estimator reduction) with $\delta = 1$ along with the localized upper bound (6.23) to write

$$\eta_{\mathcal{T}}(u_{\mathcal{T}}, f)^2 \le 2\, \eta_{\mathcal{T}_*}(u_{\mathcal{T}_*}, f)^2 + 2C_{\mathrm{Lip}}^2 \big(\widetilde{C}_U^2 \mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f, \mathcal{R})^2 + \operatorname{osc}_{\mathcal{T}}(f)_{-1}^2\big).$$

The last term accounts for the presence of $P_{\mathcal{T}} f$ and $P_{\mathcal{T}_*} f$ in the definitions of $\eta_{\mathcal{T}}(u_{\mathcal{T}}, f)$ and $\eta_{\mathcal{T}_*}(u_{\mathcal{T}_*}, f)$. In view of (6.24) and the definition (5.11) of the total estimator, we have

$$\operatorname{osc}_{\mathcal{T}}(f)_{-1} \le \sigma\, \eta_{\mathcal{T}}(u_{\mathcal{T}}), \quad \sigma^2 := \frac{\omega^2}{1 - \omega^2}$$

so that

$$\eta_{\mathcal{T}}(u_{\mathcal{T}}, f)^2 \le 2\eta_{\mathcal{T}_*}(u_{\mathcal{T}_*}, f)^2 + 2C_{\mathrm{Lip}}^2 \widetilde{C}_U^2 \big(\eta_{\mathcal{T}}(u_{\mathcal{T}}, f, \mathcal{R})^2 + 2\sigma^2 \eta_{\mathcal{T}}(u_{\mathcal{T}}, f)^2\big).$$

Using (6.25) and rearranging the above expression, we obtain

$$\big(\theta_0^2 - 2\sigma^2\big)\eta_{\mathcal{T}}(u_{\mathcal{T}}, f)^2 \le \left(\frac{1 - 2\mu^2}{2C_{\mathrm{Lip}}^2 \widetilde{C}_U^2} - 2\sigma^2\right)\eta_{\mathcal{T}}(u_{\mathcal{T}}, f)^2 \le \eta_{\mathcal{T}}(u_{\mathcal{T}}, f, \mathcal{R})^2$$

provided $0 < \mu \le \frac{1}{2}$, because of the definition of θ_0 in Assumption 6.14. Finally, for any $\theta < \theta_0$ we realize that ω_0 from Assumption 6.15 satisfies

$$\sigma_0^2 := \frac{\omega_0^2}{1 - \omega_0^2} = \frac{1}{2}\big(\theta_0^2 - \theta^2\big) \quad \Rightarrow \quad \theta^2 = \theta_0^2 - 2\sigma_0^2 \le \theta_0^2 - 2\sigma^2,$$

and Dörfler marking is valid for $\mathcal{R}$ with parameter θ. □

We remark that Lemma 6.16 requires that the oscillation on $\mathcal{T}$ is dominated by the total estimator to guarantee a Dörfler marking property. This is always the case when f is discrete as in Algorithm 5.4 (GALERKIN), because in that case $\operatorname{osc}_{\mathcal{T}}(f)_{-1} = 0$, or within Algorithm 5.16 (AFEM-SW), which marks elements for refinement only if this property holds.

We also see that θ_0 in Assumption 6.14 corresponds to the choices $\mu = \frac{1}{2}$ and $\delta = 1$. However, the proof reveals that for $\mu \to 0$ we could obtain the largest possible value $\theta_0 = (C_{\mathrm{Lip}}\widetilde{C}_U)^{-1}$, thereby the less restrictive. Since this is just twice the value of θ_0 in Assumption 6.14, the practical choice $\mu = \frac{1}{2}$ is justified.

Lemma 6.16 hinges on two ingredients: the Lipschitz property (4.68) and the localized upper bound (6.23) of the estimator. In particular, it does not rely on the lower *a posteriori* error estimate in (6.22), like the original proofs in Stevenson (2007) and Cascón *et al.* (2008), and easily extends to discontinuous Galerkin methods in Section 9 and inf-sup stable methods in Section 10. The original statement is, however, a bit more insightful: if $\theta_0^2 = C_2^2/2C_1^2$, then for all $0 < \theta < \theta_0$, $\omega^2 \le \theta_0^2 - \theta^2$,

$$|\!|\!| u - u_{\mathcal{T}_*} |\!|\!|_\Omega \le \mu |\!|\!| u - u_{\mathcal{T}} |\!|\!|_\Omega \quad \Rightarrow \quad \eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{R}) \ge \theta\, \eta_{\mathcal{T}}(u_{\mathcal{T}})$$

provided $0 < \mu \le 2^{-1/2}$. We see that the threshold θ_0 is related to the gap between *reliability* constant C_1 and *efficiency* constant C_2 in the *a posteriori* bounds (5.13) in the energy norm; hence the ratio $C_2/C_1 \le 1$ is a quality measure of the estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}})$. It is thus reasonable to be cautious about marking decisions if the constants C_1 and C_2 are very disparate, and thus the ratio C_2/C_1 is far from 1. This justifies the constraint $\theta < \theta_0$.

6.4. *Rate-optimality of one-step* AFEM*s*

Recall that $\mathcal{M}_j$ is the output of the module MARK and that $\mathcal{T}_j, u_j$ are the meshes and associated Galerkin solutions generated within Algorithms 5.4 (GALERKIN) and 5.16 (AFEM-SW). To express the cardinality $N_j(u)$ of $\mathcal{M}_j$ in terms of $|u - u_j|_{H_0^1(\Omega)}$, we must relate the performance of these one-step AFEMs with the approximation classes $\mathbb{A}_s = \mathbb{A}_s(H_0^1(\Omega); \mathcal{T}_0)$ for u and $\mathbb{F}_s = \mathbb{F}_s(H^{-1}(\Omega); \mathcal{T}_0)$ for f, which are never used in the design of these algorithms. Even though this might appear infeasible, the key to unravel this connection is given by Lemma 6.16 (Dörfler marking) and the following assumption.

Assumption 6.17 (cardinality of $\mathcal{M}$). The module MARK selects a set $\mathcal{M}$ in (5.23) with *minimal* cardinality.

According to the equidistribution principle (3.24) and the local lower bound (4.54) in the proof of Theorem 4.45 (modified residual estimator) for discrete coefficients, that is,

$$C_L \eta_{\mathcal{T}_j}(u_j, T) \le C_L \mathcal{E}_{\mathcal{T}_j}(u_j, f, T) \le |u - u_j|_{H^1(\omega_T)},$$

it is natural to mark elements with largest error indicators. This explains the choice of a minimal set $\mathcal{M}$ in Assumption 6.17.

We are now ready to bound the cardinality of $\mathcal{M}_j$ in terms of $|u - u_j|_{H^1_0(\Omega)}$.

Proposition 6.18 (cardinality of $\mathcal{M}_j$). *Let Assumptions 6.14, 6.15 and 6.17 be valid. If $u \in \mathbb{A}_s$ and*

$$\operatorname{osc}_{\mathcal{T}_j}(f)_{-1} \le \omega\, \mathcal{E}_{\mathcal{T}_j}(u_j, f),$$

then the cardinality $N_j(u)$ of $\mathcal{M}_j$ satisfies

$$N_j(u) \lesssim |u|_{\mathbb{A}_s}^{1/s} |u - u_j|_{H^1_0(\Omega)}^{-1/s} \quad \textit{for all } j \ge 0.$$

Proof. Let $C_{\text{Céa}} := \sqrt{C_{\mathcal{B}}/c_{\mathcal{B}}}$ be the quasi-monotonicity constant in (3.8). Let $\delta = \mu(C_L/C_{\text{Céa}})\eta_j(u_j)$ with $\mu \le \frac{1}{2}$. We invoke (6.8) for $u \in \mathbb{A}_s$ to find a mesh $\mathcal{T}_\delta \in \mathbb{T}$ and a Galerkin solution $u_\delta \in \mathbb{V}_{\mathcal{T}_\delta}$, so that

$$|u - u_{\mathcal{T}_\delta}|_{H^1_0(\Omega)} \le \delta, \quad \#\mathcal{T}_\delta \lesssim |u|_{\mathbb{A}_s}^{1/s} \delta^{-1/s}.$$

Since $\mathcal{T}_\delta$ may be totally unrelated to $\mathcal{T}_j$, we introduce the overlay $\mathcal{T}_* = \mathcal{T}_j \oplus \mathcal{T}_\delta$. We exploit that $\mathcal{T}_* \ge \mathcal{T}_\delta$, hence the space nestedness $\mathbb{V}_{\mathcal{T}_\delta} \subset \mathbb{V}_{\mathcal{T}_*}$, along with the property that the Galerkin solution $u_{\mathcal{T}_*} \in \mathbb{V}_{\mathcal{T}_*}$ minimizes the energy error in $\mathbb{V}_{\mathcal{T}_*}$,

$$\eta_{\mathcal{T}_*}(u_{\mathcal{T}_*}) \le \frac{1}{C_L} |u - u_{\mathcal{T}_*}|_{H^1_0(\Omega)} \le \frac{C_{\text{Céa}}}{C_L} |u - u_{\mathcal{T}_\delta}|_{H^1_0(\Omega)} \le \frac{C_{\text{Céa}}}{C_L} \delta = \mu\, \eta_j(u_j),$$

because of the lower bound in (5.12) and (3.8). Therefore Lemma 6.16 (Dörfler marking) implies that the refined set $\mathcal{R} = \mathcal{T} \backslash \mathcal{T}_*$ satisfies Dörfler marking with parameter $\theta < \theta_0$. Since MARK delivers a minimal set $\mathcal{M}_j$ with this property, according to Assumption 6.17, we deduce

$$N_j(u) = \#\mathcal{M}_j \le \#\mathcal{R} \le \#\mathcal{T}_* - \#\mathcal{T} \le \#\mathcal{T}_\varepsilon - \#\mathcal{T}_0 \lesssim |u|_{\mathbb{A}_s}^{1/s} \delta^{-1/s},$$

where we have used Lemma 3.17 (mesh overlay). The assertion follows from $\operatorname{osc}_j(f)_{-1} \le \omega \mathcal{E}_j(u_j, f)$ in Assumption 6.15 and the upper bound in (5.12),

$$|u - u_j|_{H^1_0(\Omega)} \le C_U \mathcal{E}_j(u_j, f) \le \frac{C_U}{\sqrt{1 - \omega^2}} \eta_j(u_j),$$

and completes the proof. □

We next prove rate-optimality of the one-step AFEMs of Algorithm 5.4 and Algorithm 5.16. To this end, we need an additional assumption.

Assumption 6.19 (initial labelling). If the initial mesh $\mathcal{T}_0$ is made of simplices, then let the initial labelling (3.35) for $d = 2$, or that of Stevenson (2008, Section 4) for $d > 2$, be valid.

This assumption ensures the validity of Theorem 3.16 (complexity of REFINE): if $\mathcal{M}_j \subset \mathcal{T}_j$ is a set of marked elements for a sequence $\{\mathcal{T}_j\}_{j=0}^{k-1}$ of consecutive

refinements of $\mathcal{T}_0$, then the cardinality of the kth mesh satisfies

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \le D \sum_{j=0}^{k-1} \#\mathcal{M}_j, \tag{6.27}$$

with a universal constant D depending only on $\mathcal{T}_0$ and d. We always assume that $\#\mathcal{T}_k \ge \frac{3}{2}\#\mathcal{T}_0$, whence $\#\mathcal{T}_k - \#\mathcal{T}_0 \ge \frac{1}{3}\#\mathcal{T}_k$ and, if $\widetilde{D} = 3D$, (6.27) instead reads

$$\#\mathcal{T}_k \le \widetilde{D} \sum_{j=0}^{k-1} \#\mathcal{M}_j, \tag{6.28}$$

Theorem 6.20 (rate-optimality of one-step AFEMs). *For Algorithms 5.4 (with $\mathcal{T} = \mathcal{T}_0$) and 5.16, let Assumptions 6.14, 6.17 and 6.19 be valid, and in addition let the parameter $\omega > 0$ satisfy Assumption 6.15 for Algorithm 5.16. If $u \in \mathbb{A}_s$, then both one-step* AFEM*s give rise to sequences $\{\mathcal{T}_k, \mathbb{V}_k, u_k\}_{k=0}^{\infty}$ such that*

$$|u - u_k|_{H_0^1(\Omega)} \lesssim |u|_{\mathbb{A}_s} (\#\mathcal{T}_k)^{-s}. \tag{6.29}$$

Proof. We first consider Algorithm 5.4, for which the forcing $f \in \mathbb{F}_{\mathcal{T}_0}$ is discrete, whence $\operatorname{osc}_j(f)_{-1} = 0$ for all $j \ge 0$ and $\omega = \sigma = 0$ in Assumption 6.15. In view of (6.28), we apply Proposition 6.18 (cardinality of $\mathcal{M}_j$) to infer that

$$\#\mathcal{T}_k \le \widetilde{D} \sum_{j=0}^{k-1} \#\mathcal{M}_j \lesssim |u|_{\mathbb{A}_s}^{1/s} \sum_{j=0}^{k-1} |u - u_j|_{H_0^1(\Omega)}^{-1/s}.$$

We now recall the inequality $|u-u_k|_{H_0^1(\Omega)} \le C_* \alpha^{k-j} |u-u_j|_{H_0^1(\Omega)}$ from Corollary 5.6 (linear convergence), and replace the sum above with

$$\sum_{j=0}^{k-1} |u - u_j|_{H_0^1(\Omega)}^{-1/s} \le |u - u_k|_{H_0^1(\Omega)}^{-1/s} \sum_{j=0}^{k} \alpha^{(k-j)/s} \le \frac{\alpha^{1/s}}{1 - \alpha^{1/s}} |u - u_k|_{H_0^1(\Omega)}^{-1/s},$$

because $0 < \alpha < 1$ and the geometric series is summable.

We now deal with Algorithm 5.16. If the algorithm calls MARK, then $\operatorname{osc}_j(f)_{-1} \le \omega \mathcal{E}_j(u_j, f)$ and the number of marked elements $N_j(u)$ obeys Proposition 6.18:

$$N_j(u) \lesssim |u|_{\mathbb{A}_s}^{1/s} |u - u_j|_{H_0^1(\Omega)}^{-1/s}.$$

Instead, if the algorithm calls DATA, then $\operatorname{osc}_j(f)_{-1} > \sigma_j = \omega \mathcal{E}_j(u_j, f)$ and DATA returns a mesh $\mathcal{T}_{j+1}$ and reduces the oscillation $\operatorname{osc}_{j+1}(f)_{-1} \le \sigma_j$ with optimal complexity. To quantify the cost, we recall that $u \in \mathbb{A}_s$ yields $f \in \mathbb{F}_s$ according to Lemma 6.9 (relation between approximation classes) and $|f|_{\mathbb{F}_s} \lesssim |u|_{\mathbb{A}_s}$. Therefore the number of marked elements $N_j(f)$ to reduce $\operatorname{osc}_j(f)_{-1}$ to tolerance σ_j satisfies

$$N_j(f) \lesssim |f|_{\mathbb{F}_s}^{1/s} \sigma_j^{-1/s} \lesssim |u|_{\mathbb{A}_s}^{1/s} \mathcal{E}_j(u_j, f)^{-1/s} \lesssim |u|_{\mathbb{A}_s}^{1/s} |u - u_j|_{H_0^1(\Omega)}^{-1/s},$$

because of (5.12). It thus remains to sum over j, again apply (6.28),

$$\#\mathcal{T}_k \le \widetilde{D} \sum_{j=0}^{k-1} (N_j(u) + N_j(f)) \lesssim |u|_{\mathbb{A}_s}^{1/s} \sum_{j=0}^{k-1} |u - u_j|_{H_0^1(\Omega)}^{-1/s},$$

and finally argue as before with the help of Corollary 5.18 (linear convergence of error). □

6.5. *Rate-optimality of two-step* AFEM

The output of $[\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k] = \mathsf{DATA}(\mathcal{T}_k, \mathcal{D}, \omega\varepsilon_k)$, in the k-step of AFEM-TS (Algorithm 5.28), is fed to $[\mathcal{T}_{k+1}, u_{k+1}] = \mathsf{GALERKIN}(\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k, \varepsilon_k)$, which in turn iterates J_k times. We let $(\mathcal{T}_{k,j}, \mathcal{M}_{k,j}, u_{\mathcal{T}_{k,j}})$ denote the triplets of grids, marked sets and discrete solutions computed within $\mathsf{GALERKIN}(\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k, \varepsilon_k)$ for $0 \le j < J_k$. We further assume that

$$\widehat{\varepsilon}_k := \eta_{\mathcal{T}_{k,0}}(u_{\mathcal{T}_{k,0}}, \widehat{\mathcal{D}}_k) > \varepsilon_k,$$

for otherwise the module GALERKIN is skipped. In view of the lower *a posteriori* error estimate in (6.22) for discrete data $\widehat{\mathcal{D}}_k$, we infer that

$$|\widehat{u}_k - u_{\mathcal{T}_{k,0}}|_{H_0^1(\Omega)} \ge C_L \widehat{\varepsilon}_k > C_L \varepsilon_k,$$

where $\widehat{u}_k \in H_0^1(\Omega)$ is the exact solution of (5.5) with approximate data $\widehat{\mathcal{D}}_k$. The module DATA guarantees (5.63) and (5.65), namely

$$\|\mathcal{D} - \widehat{\mathcal{D}}_k\|_{D(\Omega)} \le C_{\mathrm{data}} \omega \varepsilon_k \quad \Rightarrow \quad |u - \widehat{u}_k|_{H_0^1(\Omega)} \le C_D \omega \varepsilon_k, \tag{6.30}$$

where $u = u(\mathcal{D}) \in H_0^1(\Omega)$ is the exact solution of (2.7). We see that the parameter ω controls the discrepancy between u and $\widehat{u}_k = \widehat{u}_k(\mathcal{D}_k)$ relative to ε_k. We now make an assumption on the appropriate size of ω, which replaces Assumption 6.15 for AFEM-SW.

Assumption 6.21 (size of ω). The parameter ω in AFEM-TS satisfies $\omega \in (0, \omega_0]$, where

$$\omega_0 := \frac{\mu C_L}{2 C_D C_{\text{Céa}}}$$

with $C_{\text{Céa}}$ as in (3.8) and the parameter $0 < \mu \le \frac{1}{2}$ appears in Lemma 6.16 (Dörfler marking).

Consequently, if Assumption 6.21 is valid then (6.30) yields for $\omega \le \omega_0$

$$|u - \widehat{u}_k|_{H_0^1(\Omega)} \le \frac{\mu C_L}{2 C_{\text{Céa}}} \varepsilon_k. \tag{6.31}$$

Corollary 6.22 (cardinality of marked sets). *Let Assumptions 6.14, 6.17 and 6.21 hold. If $u \in \mathbb{A}_s(H_0^1(\Omega); \mathcal{T}_0)$ and $\widehat{\varepsilon}_k > \varepsilon_k$, then* GALERKIN *is called and there*

exists a constant C_0 such that, for all $0 \le j < J_k$,

$$\#\mathcal{M}_{k,j} \le C_0 |u|_{\mathbb{A}_s}^{1/s} \, |u - u_{\mathcal{T}_{k,j}}|_{H_0^1(\Omega)}^{-1/s} \tag{6.32}$$

and

$$\#\mathcal{M}_{k,j} \le C_0 |u|_{\mathbb{A}_s}^{1/s} \, \varepsilon_k^{-1/s}. \tag{6.33}$$

Proof. We argue as in Proposition 6.18. Fix $0 \le j < J_k$ and set

$$\delta := \mu \frac{C_L}{C_{\text{Céa}}} \eta_{\mathcal{T}_{k,j}}(u_{\mathcal{T}_{k,j}}) \quad \Rightarrow \quad \delta \ge \mu \frac{C_L}{C_{\text{Céa}}} \varepsilon_k.$$

Since $|u - \widehat{u}_k|_{H_0^1(\Omega)} \le \delta/2$, by virtue of (6.31), we deduce that $\widehat{u}_k$ is an δ-approximation of order s to u according to Lemma 6.13 (ε-approximation of u of order s). Thus there exists an admissible mesh $\mathcal{T}_\delta \in \mathbb{T}$ such that

$$|\widehat{u}_k - u_{\mathcal{T}_\delta}|_{H_0^1(\Omega)} \le \delta, \quad \#\mathcal{T}_\delta \lesssim |u|_{\mathbb{A}_s}^{1/s} \delta^{-1/s},$$

and we proceed exactly as in Proposition 6.18, to show that

$$\#\mathcal{M}_{k,j} \lesssim |u|_{\mathbb{A}_s}^{1/s} \delta^{-1/s} \approx |u|_{\mathbb{A}_s}^{1/s} |u - u_{\mathcal{T}_{k,j}}|_{H_0^1(\Omega)}^{-1/s} \lesssim |u|_{\mathbb{A}_s}^{1/s} \varepsilon_k^{-1/s}.$$

This concludes the proof. □

Corollary 6.23 (quasi-optimality of GALERKIN). *Let Assumptions 6.3, 6.14, 6.17 and 6.21 be valid. Then the number of marked elements $N_k(u)$ within the kth call to* GALERKIN *satisfies*

$$N_k(u) \le J C_0 |u|_{\mathbb{A}_s}^{1/s} \varepsilon_k^{-1/s},$$

where $J \ge J_k$ is a uniform upper bound for the number of iterations of GALERKIN.

Proof. Use $N_k(u) = \sum_{j=0}^{J_k - 1} \#\mathcal{M}_{k,j}$ and combine Corollary 6.22 (cardinality of marked sets) and Proposition 5.27 (computational cost of GALERKIN). □

We finally address the rate-optimality of the two-step algorithm AFEM-TS, by proving the stated bound (6.1).

Theorem 6.24 (rate-optimality of AFEM-TS). *Let Assumptions 6.3 (approximability of u), 6.10 (approximability of data), 6.11 (quasi-optimality of* DATA*), 6.14 (marking parameter), 6.21 (size of ω) and 6.19 (initial labelling) be valid. Then* AFEM-TS *gives rise to a sequence $(\mathcal{T}_k, \mathbb{V}_{\mathcal{T}_k}, u_{\mathcal{T}_k})_{k=0}^K$ such that*

$$|u - u_k|_{H_0^1(\Omega)} \le C(u, \mathcal{D}) (\#\mathcal{T}_k)^{-s}, \quad 1 \le k \le K,$$

where $0 < s = \min\{s_u, s_{\mathcal{D}}\} = \min\{s_u, s_A, s_c, s_f\} \le n/d$ and

$$C(u, \mathcal{D}) = C_* \big(|u|_{\mathbb{A}_{s_u}}^{1/s_u} + |\boldsymbol{A}|_{\mathbb{M}_{s_A}}^{1/s_A} + |c|_{\mathbb{C}_{s_c}}^{1/s_c} + |f|_{\mathbb{F}_{s_f}}^{1/s_f} \big)^s,$$

with constant $C_* > 0$ independent of u and $\mathcal{D}$.

Proof. In view of Assumption 6.3, Corollary 6.23 implies that the number of marked elements $N_k(u)$ within the $(k+1)$th call to GALERKIN satisfies

$$N_k(u) \le C_3 |u|_{\mathbb{A}_{s_u}}^{1/s_u} \varepsilon_k^{-1/s_u},$$

with $s_u \le n/d$ and $C_3 > 0$ a suitable constant. Moreover, by Assumption 6.11 the number of marked elements $N_k(\mathcal{D})$ within the $(k+1)$th call to DATA satisfies

$$N_k(\mathcal{D}) \le C_3 |\mathcal{D}|_{\mathbb{A}_{s_\mathcal{D}}}^{1/s_\mathcal{D}} \varepsilon_k^{-1/s_\mathcal{D}},$$

with $s_\mathcal{D} \le n/d$. The total number of marked elements in the $(k+1)$th loop of AFEM-TS is thus

$$N_k(\mathcal{D}) + N_k(u) \le C_3 \left(|u|_{\mathbb{A}_{s_u}}^{1/s_u} + |\mathcal{D}|_{\mathbb{A}_{s_\mathcal{D}}}^{1/s_\mathcal{D}} \right) \varepsilon_k^{-1/s}.$$

Upon termination, DATA and GALERKIN give

$$|u - \widehat{u}_k|_{H_0^1(\Omega)} \le \frac{\mu C_L}{2C_{\text{Céa}}} \varepsilon_k \le \frac{C_L}{4C_{\text{Céa}}} \varepsilon_k,$$
$$|\widehat{u}_k - u_{k+1}|_{H_0^1(\Omega)} \le C_U \eta_{\mathcal{T}_{k+1}}(u_{k+1}, \widehat{\mathcal{D}}_k) \le C_U \varepsilon_k,$$

because of (6.31), (6.22) and the fact that $\mu < \frac{1}{2}$. This implies by the triangle inequality

$$|u - u_{k+1}|_{H_0^1(\Omega)} \le \left(\frac{C_L}{4C_{\text{Céa}}} + C_U \right) \varepsilon_k = C_4 \varepsilon_k.$$

Therefore, applying Theorem 3.16 (complexity of REFINE), the total amount of elements created by $k+1$ iterations within AFEM-TS, besides those in $\mathcal{T}_0$, obeys the expression

$$\#\mathcal{T}_{k+1} \le \widetilde{D} \sum_{j=0}^{k} (N_j(\mathcal{D}) + N_j(u)) \le \widetilde{D} C_3 \left(|u|_{\mathbb{A}_{s_u}}^{1/s_u} + |\mathcal{D}|_{\mathbb{A}_{s_\mathcal{D}}}^{1/s_\mathcal{D}} \right) \sum_{j=0}^{k} \varepsilon_j^{-1/s}.$$

according to (6.28). Since $\varepsilon_j = 2^{-j}\varepsilon_0$ and

$$\sum_{j=0}^{k-1} (2^{-1/s})^j \le \frac{1}{1 - 2^{-1/s}},$$

we obtain

$$\#\mathcal{T}_{k+1} \le C \left(|u|_{\mathbb{A}_{s_u}}^{1/s_u} + |\mathcal{D}|_{\mathbb{A}_{s_\mathcal{D}}}^{1/s_\mathcal{D}} \right) \varepsilon_k^{-1/s},$$

with

$$C = \frac{\widetilde{D} C_3 \varepsilon_0}{1 - 2^{-1/s}}$$

provided $\#\mathcal{T}_{k+1} \ge \frac{3}{2}\#\mathcal{T}_0$. This together with $|u - u_{k+1}|_{H_0^1(\Omega)} \le C_4 \varepsilon_k$ gives the asserted estimate after $1 \le k+1 \le K$ loops. □

Remark 6.25. The thresholds θ_0, ω_0 play no role in Proposition 5.29 (convergence of AFEM-TS) but are critical in Theorem 6.24 (rate-optimality of AFEM-TS). The former takes care of the discrepancy between error and estimator (Stevenson 2007, Cascón *et al.* 2008, Nochetto *et al.* 2009, Bonito and Nochetto 2010, Nochetto and Veeser 2012). The latter guarantees that the perturbation error (6.30) is much smaller than ε_k and enables GALERKIN to learn the regularity of u from $\widehat{u}_{\widehat{\mathcal{T}}_k}$ (Stevenson 2007, Bonito *et al.* 2013*b*).

Remark 6.26. We claim that the convergence rate $s = \min\{s_u, s_{\mathcal{D}}\}$ cannot be improved to s_u (the optimal rate for approximations of $u \in \mathbb{A}_{s_u}(H_0^1(\Omega); \mathcal{T}_0)$) when $s_{\mathcal{D}} < s_u$ by any algorithm that uses approximations $\widehat{\mathcal{D}} = (\widehat{\boldsymbol{A}}, \widehat{c}, \widehat{f})$ of data $\mathcal{D} = (\boldsymbol{A}, c, f)$. In fact, given any $\delta > 0$, consider the ball

$$B(\mathcal{D}, \delta) := \{\widehat{\mathcal{D}} \in \mathbb{D} \mid \|\mathcal{D} - \widehat{\mathcal{D}}\|_{D(\Omega)} \le \delta\}, \tag{6.34}$$

where $D(\Omega)$ is defined in (5.60). If $u, \widehat{u} \in H_0^1(\Omega)$ are the exact solutions for data $\mathcal{D}, \widehat{\mathcal{D}}$, then there are constants $0 < c_* \le C_*$ such that

$$c_*\delta \le \sup_{\widehat{\mathcal{D}} \in B(\mathcal{D}, \delta)} |u - \widehat{u}|_{H_0^1(\Omega)} \le C_*\delta.$$

The rightmost inequality is a consequence of Lemma 5.20 (continuous dependence on data). For the leftmost inequality, first consider a perturbation $\widehat{f} = (1+\delta)f$ of the source term with coefficients $(\widehat{\boldsymbol{A}}, \widehat{c}) = (\boldsymbol{A}, c)$, whence $\|\mathcal{D} - \widehat{\mathcal{D}}\|_{D(\Omega)} = \delta$. Proceeding as in (2.30), the coercivity and continuity of the bilinear form $\mathcal{B}$ imply

$$c_{\mathcal{B}}\, |u - \widehat{u}|_{H_0^1(\Omega)} \le \|f - \widehat{f}\|_{H^{-1}(\Omega)} = \delta \le C_{\mathcal{B}}\, |u - \widehat{u}|_{H_0^1(\Omega)}.$$

On the other hand, if $\widehat{f} = f$ and $(\widehat{\boldsymbol{A}}, \widehat{c}) = \alpha^{-1}(\boldsymbol{A}, c)$ with $\alpha = 1 + \delta/\|\mathcal{D}\|_{D(\Omega)}$, then

$$\|\mathcal{D} - \widehat{\mathcal{D}}\|_{D(\Omega)} < \delta, \quad |u - \widehat{u}|_{H_0^1(\Omega)} = \frac{|u|_{H_0^1(\Omega)}}{\|\mathcal{D}\|_{D(\Omega)}}\delta \ge \frac{\|f\|_{H^{-1}(\Omega)}}{C_{\mathcal{B}}\, \|\mathcal{D}\|_{D(\Omega)}}\delta.$$

This argument takes care of the multiplicative nature of $(\boldsymbol{A}, c)$ in (2.5), which makes $\widehat{u} = \alpha u$, and proves our claim.

6.6. *Rate-optimality of* AFEM *with other boundary conditions*

The key ingredient for rate-optimality of AFEM, regardless of boundary conditions, is the validity of Lemma 6.16 (Dörfler marking). This lemma provides a bridge between FEM meshes and optimal meshes and, in turn, hinges on three properties of the PDE residual estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}}, f)$: Theorem 4.48 (upper bound for corrections), Lemma 4.54 (Lipschitz property of the estimator) and Lemma 4.55 (estimator dependence on discrete forcing) to account for the possible change in the discrete forcing $P_{\mathcal{T}} f$. Since their proofs are insensitive to boundary conditions, because they do not alter the structure of $\eta_{\mathcal{T}}(u_{\mathcal{T}}, f)$, we conclude their validity as well as for Robin, Neumann and non-homogeneous Dirichlet conditions.

Therefore our three AFEMs based on Dörfler marking deliver the same asymptotic convergence rates associated with the approximations classes $\mathbb{A}_{s_u}$ for the solution $u \in H^1(\Omega)$ and $\mathbb{A}_{s_\mathcal{D}}$ for data $\mathcal{D} = (\boldsymbol{A}, c, p, \ell)$ for Robin and Neumann boundary conditions, and $\mathcal{D} = (\boldsymbol{A}, c, f, g)$ for Dirichlet boundary conditions. We need three new approximation classes for $p \in \mathbb{P}_{s_p}(L^\infty(\partial\Omega); \mathcal{T}_0)$, $\ell \in \mathbb{L}_{s_\ell}(H^1(\Omega)^*; \mathcal{T}_0)$ for Robin or Neumann conditions and $g \in \mathbb{G}_{s_g}(H^{1/2}(\partial\Omega); \mathcal{T}_0)$.

If coefficients $(\boldsymbol{A}, c, p)$ are discrete for the Robin condition, then Lemma 6.9 (relation between approximation classes) extends and yields

$$\langle \ell - \widehat{\ell}, v\rangle = \mathcal{B}[u - \widehat{u}, v] \quad \Rightarrow \quad \|\ell - \widehat{\ell}\|_{\mathbb{V}^*} \le C\|u - \widehat{u}\|_{\mathbb{V}},$$

with $\mathbb{V} = H^1(\Omega)$ and $\mathcal{B} = \widehat{\mathcal{B}}, \widehat{\ell}$ given in (5.85). This in turn implies $|\ell|_{\mathbb{L}_{s_\ell}} \le C|u|_{\mathbb{A}_{s_u}}$, $s_\ell = s_u$ and the validity of Theorem 6.20 (rate-optimality of one-step AFEMs). In AFEM-TS, DATA approximates ℓ along with the other data and Theorem 6.24 (rate-optimality of AFEM-TS) is also valid for Robin and Neumann boundary conditions. We do not explore this matter any further.

For non-homogeneous Dirichlet boundary conditions the analysis is simpler. If g is discrete, then there is no difference to $g = 0$. If not, we note that the solution map $g \mapsto u$ (all other data being fixed) is affine and that the error and augmented total estimator $\mathcal{E}_\mathcal{T}(u_\mathcal{T}, f, g) := \mathcal{E}_\mathcal{T}(u_\mathcal{T}, f) + \operatorname{osc}_\mathcal{T}(g)_{1/2}$ are equivalent (Theorem 4.74). This indicates that the role of g is similar to the role of f. Therefore it suffices to replace $\mathcal{E}_\mathcal{T}(u_\mathcal{T}, f)$ with $\mathcal{E}_\mathcal{T}(u_\mathcal{T}, f, g)$ and $\operatorname{osc}_\mathcal{T}(f)_{-1}$ with $\operatorname{osc}_\mathcal{T}(f)_{-1} + \operatorname{osc}_\mathcal{T}(g)_{1/2}$ in AFEM-SW. For AFEM-TS, the approximation of g is handled by DATA along with the other data. Hence we again conclude that Theorems 6.20 and 6.24 extend to non-vanishing Dirichlet conditions.

6.7. *Rate-optimality of* AFEM *driven by alternative estimators*

We recall the notation $\zeta_\mathcal{T}(u_\mathcal{T})$ of Section 5.6 for any of the three alternative estimators in Section 4.9 and the crucial local properties (5.86) and (5.87).

As already alluded to in Section 6.6, the key instrument for rate-optimality is Lemma 6.16 (Dörfler marking). We now check the validity of its three main pillars: Theorem 4.48 (upper bound for corrections), Lemma 4.54 (Lipschitz property of the estimator) and Lemma 4.55 (estimator dependence on discrete forcing) to account for possible change in the discrete forcing $P_\mathcal{T} f$. It turns out that if they were valid for $\zeta_\mathcal{T}(u_\mathcal{T}) = \zeta_\mathcal{T}(u_\mathcal{T}, f)$, then statements about rates of convergence similar to those for $\eta_\mathcal{T}(u_\mathcal{T})$ would follow for $\zeta_\mathcal{T}(u_\mathcal{T})$.

Lemma 6.27 (localized discrete upper bound). *Let $\mathcal{T}, \mathcal{T}_* \in \mathbb{T}$ and $\mathcal{T}_*$ be a refinement of $\mathcal{T}$. Let the coefficients $(\widehat{\boldsymbol{A}}, \widehat{c})$ be discrete over $\mathcal{T}$ and $f \in H^{-1}(\Omega)$. Then the error between the corresponding Galerkin solutions $u_\mathcal{T} \in \mathbb{V}_\mathcal{T}$ and $u_{\mathcal{T}_*} \in \mathbb{V}_{\mathcal{T}_*}$ is bounded by the indicator in the refined set $\overline{\mathcal{R}}$ plus data oscillation*

$$|u_\mathcal{T} - u_{\mathcal{T}_*}|_{H^1_0(\Omega)} \le \overline{C}_U \left(\left(C_L^{\mathrm{eq}}\right)^{-1} \zeta_\mathcal{T}(u_\mathcal{T}, \overline{\mathcal{R}})^2 + \operatorname{osc}_\mathcal{T}(f)_{-1}^2 \right)^{1/2},$$

where $\overline{\mathcal{R}} := \{z \in \mathcal{V} \mid T \in \mathcal{T}\backslash\mathcal{T}_*, T \subset \omega_z\}$ *collects all vertices whose associated stars change from* $\mathcal{T}$ *to* $\mathcal{T}_*$.

Proof. It suffices to realize that $\mathcal{T}\backslash\mathcal{T}_* \subset \bigcup\{\omega_z \mid z \in \overline{\mathcal{R}}\}$ and appeal to Theorem 4.48 and (5.87) to arrive at

$$\begin{aligned}|u_{\mathcal{T}} - u_{\mathcal{T}_*}|^2_{H^1_0(\Omega)} &\le \overline{C}_U^2\left(\eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{R})^2 + \operatorname{osc}_{\mathcal{T}}(f, \mathcal{R})^2_{-1}\right)\\ &\le \overline{C}_U^2\left(\eta_{\mathcal{T}}(u_{\mathcal{T}}, \overline{\mathcal{R}})^2 + \operatorname{osc}_{\mathcal{T}}(f)^2_{-1}\right)\\ &\le \overline{C}_U^2\left(\left(C_L^{\mathrm{eq}}\right)^{-2}\zeta_{\mathcal{T}}(u_{\mathcal{T}}, \overline{\mathcal{R}})^2 + \operatorname{osc}_{\mathcal{T}}(f)^2_{-1}\right).\end{aligned}$$

This is the desired estimate. □

Lemma 6.28 (Lipschitz property of the estimator). *Let the coefficients* $(\widehat{\boldsymbol{A}}, \widehat{c})$ *be discrete over* $\mathcal{T}$. *There exists* C_{Lip} *such that*

$$|\zeta_{\mathcal{T}}(v_1) - \zeta_{\mathcal{T}}(v_2)| \le C_{\mathrm{Lip}}|v_1 - v_2|_{H^1_0(\Omega)} \quad \textit{for all } v_1, v_2 \in \mathbb{V}_{\mathcal{T}}.$$

Proof. We resort to the star equivalence (5.86) between discrete residual and estimator. It thus suffices to derive the Lipschitz property for $\|P_{\mathcal{T}}R_{\mathcal{T}}(v)\|_{H^{-1}(\omega_z)}$ with respect to $v \in \mathbb{V}_{\mathcal{T}}$ for all $z \in \mathcal{V}$. Since $P_{\mathcal{T}}R_{\mathcal{T}}(v) = P_{\mathcal{T}}f - \widehat{\mathcal{B}}[v, \cdot]$, we get

$$\langle P_{\mathcal{T}}R_{\mathcal{T}}(v_1) - P_{\mathcal{T}}R_{\mathcal{T}}(v_2), w\rangle = \int_{\omega_z} \nabla w \cdot \widehat{\boldsymbol{A}}\nabla(v_1 - v_2) + \widehat{c}(v_1 - v_2)w$$

for all $w \in H^1_0(\omega_z)$. Therefore Lemma 2.2 (first Poincaré inequality) yields

$$\|P_{\mathcal{T}}R_{\mathcal{T}}(v) - P_{\mathcal{T}}R_{\mathcal{T}}(v_2)\|_{H^{-1}(\omega_z)} \le C(\widehat{\boldsymbol{A}}, \widehat{c})\|v_1 - v_2\|_{H^1(\omega_z)},$$

where $C(\widehat{\boldsymbol{A}}, \widehat{c})$ depends on the L^∞-norms of $(\widehat{\boldsymbol{A}}, \widehat{c})$. Finally, using the triangle inequality to accumulate over $z \in \mathcal{V}$ together with (5.86) gives the assertion. □

Lemmas 6.27 and 6.28 lead to Lemma 6.16 (Dörfler marking) for $\zeta_{\mathcal{T}}(u_{\mathcal{T}})$. If we further choose a *minimal* set $\mathcal{M}$ of vertices that satisfies Dörfler property (5.88), then the previous rates of convergence for the three algorithms GALERKIN, AFEM-SW and AFEM-TS but now driven by $\zeta_{\mathcal{T}}(u_{\mathcal{T}})$ are valid provided $u \in \mathbb{A}_s$, the approximation class in Definition 6.1. We do not restate these results.

6.8. *Approximation vs. regularity classes*

The purpose of this section is to reconcile the notion of approximation classes, discussed above, with that of regularity classes. We recall the DeVore diagram of Figure 2.1, which depicts the Sobolev line for the energy space $H^1_0(\Omega)$, namely

$$\operatorname{sob}(H^1_0) = \operatorname{sob}(W^s_p) \quad \Rightarrow \quad s - \frac{d}{p} = 1 - \frac{d}{2}.$$

The differentiability $s \ge 1$ is only limited by the polynomial degree n, so $s \in [1, n+1]$. On the other hand, the integrability p is not restricted to be $p \ge 1$ as is customary

with Sobolev spaces. For example, for $d = 2$ and $s = n+1$, we get $p = 2/(n+1) < 1$ provided $n \geq 2$. Therefore, to take full advantage of nonlinear approximation theory, we need to abandon the framework of Sobolev spaces $W^s_p(\Omega)$ and deal with *Besov spaces* $B^s_{p,q}(\Omega)$ (frequently denoted by $B^s_q(L^p(\Omega))$ or $B^s_q(L_p(\Omega))$ in the literature) with integrability index $p \in (0, \infty]$. The second index $q \in (0, \infty]$ is useful in characterizing special limiting cases; below we will provide a few interesting examples but take $p = q$ most of the time. At this point, we only mention that when s is non-integer and $1 \leq p \leq \infty$, $B^s_{p,p}(\Omega) = W^s_p(\Omega)$, while when r is integer $W^r_p(\Omega)$ for $p \neq 2$ is not a Besov space but it is slightly smaller than $B^r_{p,\infty}(\Omega)$. The case $p = 2$ is special since $B^s_{2,2}(\Omega) = H^s(\Omega)$ even when s is an integer.

This section is devoted to the definition and properties of Besov and Lipschitz spaces, including their close relation to approximation classes. Our presentation closely follows Binev *et al.* (2002) for $n = 1$ and Gaspoz and Morin (2014, 2017) for $n \geq 1$, but it adds a few new ingredients. Since our results involve three different type of spaces to account for the particular cases when the differentiability is integer, it is pertinent to introduce the following abstract space $X^s_p(\Omega)$ with differentiability index $s \in (0, n+1]$ and integrability index $p \in (0, \infty]$:

$$X^s_p(\Omega) := \begin{cases} B^s_{p,p}(\Omega), & s \in (0, n+1), p \in (0, \infty], \\ W^{n+1}_p(\Omega), & s = n+1, p \in [1, \infty], \\ \mathrm{Lip}^{n+1}_p(\Omega), & s = n+1, p \in (0, 1). \end{cases} \tag{6.35}$$

Here $\mathrm{Lip}^s_p(\Omega) = \mathrm{Lip}(s, L^p(\Omega))$, $s \in \mathbb{N}$, are the Lipschitz spaces; see (6.58) below. For $s \in \mathbb{N}$ and $1 < p < \infty$ the Sobolev spaces coincide with the Lipschitz spaces (Leoni 2009, Theorem 10.55), that is,

$$\mathrm{Lip}^s_p(\Omega) = W^s_p(\Omega), \quad s \in \mathbb{N},\ 1 < p < \infty, \tag{6.36}$$

while for $p = 1$ we only have

$$W^s_1(\Omega) \hookrightarrow \mathrm{Lip}^s_1(\Omega), \quad s \in \mathbb{N}. \tag{6.37}$$

We use the following conventions: $X^s_p(\Omega) := L^p(\Omega)$ for $s = 0$; $X^s_p(\Omega; \mathcal{T})$ is the space of functions with piecewise regularity X^s_p over $\mathcal{T} \in \mathbb{T}$; $X^s_p(\Omega; \mathbb{R}^m)$ is the space $X^s_p(\Omega)$ of vector- or matrix-valued functions.

In Section 6.8.4 we will prove the following crucial approximation results for functions in $L^q(\Omega)$ by discontinuous piecewise polynomials $\mathbb{S}^{n,-1}_{\mathcal{T}}$ of degree $n \geq 1$ over conforming refinements $\mathcal{T}$ of $\mathcal{T}_0$. It turns out that this will also allow us to deal with approximations in $H^1_0(\Omega)$ by continuous piecewise polynomials $\mathbb{V}_{\mathcal{T}}$ of degree $n \geq 1$.

Theorem 6.29 (regularity yields approximation). *Let $q \in [1, \infty]$, $p \in (0, \infty]$, $s \in (0, n+1]$ and a function $g \in L^q(\Omega)$ satisfy $g \in X^s_p(\Omega)$ with $s - d/p + d/q > 0$.*

Then there exists a constant $C = C(p, q, s, t, d, \Omega, \mathcal{T}_0)$ *such that*

$$E_n(g, \Omega)_q := \inf_{\mathcal{T} \in \mathbb{T}_N} \inf_{v \in \mathbb{S}_{\mathcal{T}}^{n,-1}} |g - v|_{L^q(\Omega)} \le C|g|_{X_p^s(\Omega)} N^{-s/d}. \tag{6.38}$$

Therefore $g \in \mathbb{A}_{s/d} = \mathbb{A}_{s/d}(L^q(\Omega); \mathcal{T}_0)$ *and*

$$|g|_{\mathbb{A}_{s/d}} \le C|g|_{X_p^s(\Omega)}. \tag{6.39}$$

We see that the decay rate s/d in (6.38) is proportional to the difference of the differentiability indices between the space $X_p^s(\Omega)$ and $L^q(\Omega)$ provided the Sobolev numbers satisfy the relation

$$\mathrm{sob}(X_p^s(\Omega)) > \mathrm{sob}(L^q(\Omega)),$$

which implies that the embedding of $X_p^s(\Omega)$ into $L^q(\Omega)$ is compact. The factor d in the denominator is a manifestation of the so-called *curse of dimensionality.* The limiting case $s = n + 1$ entails dealing with Sobolev spaces $W_p^s(\Omega)$ and Lipschitz spaces $\mathrm{Lip}_p^s(\Omega)$ depending on whether $p \ge 1$ or $p < 1$.

6.8.1. *Modulus of smoothness*

Difference operators. Since we intend to allow $p \in (0, 1)$, the underlying functions in $B_{p,p}^s(\Omega)$ might not be locally integrable, whence they might not be distributions in Ω. Therefore the notion of weak derivative does not apply, which in turn has the disadvantage of being defined for integers and not for fractional numbers. This leads to the most standard definition of Besov spaces $B_{p,q}^s(\Omega)$ using difference operators, which only requires integrability in $L^p(\Omega)$ and is valid for any $s > 0$, $p, q \in (0, \infty]$. Other definitions, which provide equivalent results in the range $1 \le p, q \le \infty$, can be found in Adams and Fournier (2003) and Bergh and Löfström (1976).

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, and a vector $h \in \mathbb{R}^d$, we set

$$\Omega_h := \{x \in \Omega \mid [x, x + h] \subset \Omega\},$$

where $[x, x + h]$ denotes the closed segment connecting x and $x + h$, and define the first-order difference operator to be

$$\Delta_h^1 g(x) = \Delta_h^1(g, x, \Omega) := \begin{cases} g(x + h) - g(x), & x \in \Omega_h, \\ 0, & \text{otherwise}. \end{cases} \tag{6.40}$$

For $k \in \mathbb{N}$, $k \ge 1$, we define the kth difference operator by iteration,

$$\Delta_h^k g(x) := \Delta_h^1 \left(\Delta_h^{k-1}\right) g(x), \quad x \in \Omega_{kh}, \tag{6.41}$$

and observe that it has the explicit form

$$\Delta_h^k g(x) = \begin{cases} \displaystyle\sum_{j=0}^{k} (-1)^{k+j} \binom{k}{j} g(x + jh), & [x, x + kh] \subset \Omega, \\ 0, & \text{otherwise}. \end{cases}$$

Note the property

$$p \in \mathbb{P}_k \quad \Rightarrow \quad \Delta_h^{k+1} p = 0 \quad \text{for all } h. \tag{6.42}$$

Smoothness. Given $p \in (0, \infty]$ and $t > 0$, we define the *modulus of smoothness* of order k in $L^p(\Omega)$ to be

$$\omega_k(g, t)_p = \omega_k(g, t, \Omega)_p := \sup_{|h| \le t} \|\Delta_h^k g\|_{L^p(\Omega)}. \tag{6.43}$$

We note that if $\omega_k(g, t)_p = o(t^{n+1})$ as $t \to 0$, then g is a.e. a polynomial in $\mathbb{P}_n$ and

$$g \notin \mathbb{P}_n \quad \Rightarrow \quad \omega_k(g, t)_p \ge C t^{n+1}, \quad 0 < t \le 1, \tag{6.44}$$

for some $C > 0$ (DeVore and Lorentz 1993, Proposition 7.4). We also observe that the definition (6.43) only requires L^p-integrability of g and leads to the following celebrated Whitney estimate of the *best approximation error*

$$E_n(g, G)_p := \inf_{v \in \mathbb{P}_n} \|g - v\|_{L^p(G)}$$

of g by polynomials of degree $\le n$ in $G \subset \Omega$; see Binev *et al.* (2002), Dekel and Leviatan (2004, Theorem 1.4) and Gaspoz and Morin (2014, 2017, Lemma 4.4).

Lemma 6.30 (Whitney's lemma). *Let $\mathcal{T} \in \mathbb{T}$ be an admissible grid, and let $T \in \mathcal{T}$ be a generic element. If $0 < p \le \infty$ and $n \ge 0$, then*

$$E_n(g, T)_p \le C \omega_{n+1}(g, h_T, T)_p \quad \textit{for all } g \in L^p(T),$$

where $C = C(p, n, d, \mathcal{T}_0)$ but is independent of g and the size of T.

6.8.2. Besov spaces

Given $s > 0$ and $0 < p, q \le \infty$, the Besov space $B^s_{p,q}(\Omega)$ is the set of all functions $v \in L^p(\Omega)$ such that the following quantity is finite:

$$|v|_{B^s_{p,q}(\Omega)} := \begin{cases} \left(\displaystyle\int_0^\infty \left[t^{-s} \omega_k(v, t)_p \right]^q \frac{\mathrm{d}t}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} \left[t^{-s} \omega_k(v, t)_p \right], & q = \infty, \end{cases} \tag{6.45}$$

with $k = [s] + 1 \in \mathbb{N}$ and $[s]$ stands for the integer part of s. If we split the integral in (6.45) for $0 < q < \infty$ in dyadic intervals, we obtain the following equivalent expression for $|v|_{B^s_{p,q}(\Omega)}$:

$$|v|^q_{B^s_{p,q}(\Omega)} = \sum_{m \in \mathbb{Z}} \int_{2^{-m-1}}^{2^{-m}} t^{-sq} \omega_k(v, t)_p^q \frac{\mathrm{d}t}{t} \approx \sum_{m \in \mathbb{Z}} 2^{msq} \omega_k(v, 2^{-m})_p^q. \tag{6.46}$$

Here we have used that both $\omega_k(v, t)_p$ and t^{-s} are monotone functions of t. The hidden constants depend on s and q but are otherwise independent of v, k and p. Note that with obvious changes, (6.46) is also valid for $q = \infty$:

$$|v|_{B^s_{p,\infty}(\Omega)} \approx \sup_{m \in \mathbb{Z}} 2^{ms} \omega_k(v, 2^{-m})_p. \tag{6.47}$$

We point out that $|v|_{B^s_{p,q}(\Omega)}$ is a seminorm for $p, q \geq 1$ and is otherwise a semi-(quasi-)norm in that the triangle inequality is valid up to a constant larger than 1; note that $|1|_{B^s_{p,q}(\Omega)} = 0$. The quasi-norm of $B^s_{p,q}(\Omega)$ is defined to be

$$\|v\|_{B^s_{p,q}(\Omega)} := \|v\|_{L^p(\Omega)} + |v|_{B^s_{p,q}(\Omega)}.$$

If an integer $k' > k$ is chosen in (6.45), then the ensuing quasi-norms $\|v\|_{B^s_{p,q}(\Omega)}$ are equivalent. This hinges on the *Marchaud inequality* (see DeVore and Popov 1988, (2.6), and Ditzian 1988, Theorems 1 and 3)

$$\omega_k(v,t)_p \leq Ct^k \left(\|v\|_{L^p(\Omega)} + \left(\int_t^\infty \left(z^{-k} \omega_{k'}(v,z)_p \right)^p \frac{\mathrm{d}z}{z} \right)^{1/p} \right). \tag{6.48}$$

The following lemma characterizes the precise blow-up of $|v|_{B^s_{p,p}(\Omega)}$ as $s \to n+1$.

Lemma 6.31 (blow-up of $|v|_{B^s_{p,p}(\Omega)}$). *Let $s \in (0, n+1)$, $p \in (0, \infty]$. Then*

$$|v|_{B^s_{p,p}(\Omega)} \leq (p(n+1-s))^{-1/p} \|v\|_{B^{n+1}_{p,p}(\Omega)} \quad \textit{for all } v \in B^{n+1}_{p,p}(\Omega).$$

Proof. We take $p < \infty$ and combine the definition (6.45) with (6.48), after replacing the upper limit of integration with $\operatorname{diam}(\Omega) \approx 1$, to write

$$|v|^p_{B^s_{p,p}(\Omega)} \approx \int_0^1 \left(t^{-s} \omega_{n+1}(v,t)_p \right)^p \frac{\mathrm{d}t}{t} \lesssim I + II,$$

with

$$I = \int_0^1 t^{(n+1-s)p} \int_t^1 \left(z^{-(n+1)} \omega_{n+2}(v,z)_p \right)^p \frac{\mathrm{d}z}{z} \frac{\mathrm{d}t}{t}, \quad II = \int_0^1 t^{(n+1-s)p} \|v\|^p_{L^p(\Omega)} \frac{\mathrm{d}t}{t}.$$

Exchanging the order of integration yields

$$\begin{aligned} I &= \int_0^1 \left(z^{-(n+1)} \omega_{n+2}(v,z)_p \right)^p \left(\int_0^z t^{(n+1-s)p-1} \, \mathrm{d}t \right) \frac{\mathrm{d}z}{z} \\ &= \frac{1}{p(n+1-s)} \int_0^1 \left(z^{-s} \omega_{n+2}(v,z)_p \right)^p \frac{\mathrm{d}z}{z} \leq \frac{1}{p(n+1-s)} |v|^p_{B^{n+1}_{p,p}(\Omega)}. \end{aligned}$$

Since $II = (p(n+1-s))^{-1} \|v\|^p_{L^p(\Omega)}$, the proof is thus complete. □

The following equivalence between Sobolev and Besov spaces is valid for fractional differentiability s (Leoni 2009, Proposition 14.40) (see also Bergh and Löfström 1976, Section 6.4.4, and Adams and Fournier 2003, Sections 7.33, 7.67): for all $s \geq 0$, $s \notin \mathbb{N}$ and $p \in [1, \infty]$,

$$B^s_{p,p}(\Omega) = W^s_p(\Omega). \tag{6.49}$$

However, if $s \in \mathbb{N}$ is an integer, then $B^s_{p,q}(\Omega)$ is defined using $k = s+1$ differences, whereas $W^s_p(\Omega)$ involves s weak derivatives in $L^p(\Omega)$ provided $p \in [1, \infty]$. It turns out that for integer values $s \in \mathbb{N}$ the spaces differ, that is,

$$B^s_{p,p}(\Omega) \neq W^s_p(\Omega), \quad p \neq 2, \tag{6.50}$$

except for the exceptional case $p = 2$ for which $B^s_{2,2}(\Omega) = H^s(\Omega)$ (DeVore 1998).

The Besov seminorm is *sub-additive* in the following sense: if $\{T_i\}_{i=1}^N$ is a disjoint collection of elements $T_i \in \mathcal{T}$ and $\mathcal{T} \in \mathbb{T}$, $p \in (0, \infty]$ and $s > 0$, then there exists a constant C depending on p, s, d and $\mathcal{T}_0$ but independent of N such that

$$\sum_{i=1}^{N} |v|^p_{B^s_{p,p}(T_i)} \leq C|v|^p_{B^s_{p,p}(\Omega)} \quad \text{for all } v \in B^s_{p,p}(\Omega). \tag{6.51}$$

The localization of Besov norms is more general than (6.51). In fact, if $\omega_{\mathcal{T}}(T)$ denotes the patch of elements in $\mathcal{T}$ around $T \in \mathcal{T}$ (first ring), then the following is valid with equivalence constants depending on p, s, d and $\mathcal{T}_0$ but independent of N (Binev *et al.* 2002, Lemmas 4.3, 4.4):

$$\sum_{T \in \mathcal{T}} |v|^p_{B^s_{p,p}(\omega_{\mathcal{T}}(T))} \approx C|v|^p_{B^s_{p,p}(\Omega)} \quad \text{for all } v \in B^s_{p,p}(\Omega). \tag{6.52}$$

The following statements about embeddings between Besov spaces on bounded Lipschitz domains Ω will turn out to be useful below (Triebel 2010, Sections 3.2.4, 3.3.1): if $0 < p \leq \infty$, $0 < q_1, q_2 \leq \infty$ and $s_1, s_2, s > 0$, then

$$\begin{aligned} s_1 > s_2 \quad &\Rightarrow \quad B^{s_1}_{p,q_1}(\Omega) \hookrightarrow B^{s_2}_{p,q_1}(\Omega), \\ q_1 < q_2 \quad &\Rightarrow \quad B^{s}_{p,q_1}(\Omega) \hookrightarrow B^{s}_{p,q_2}(\Omega). \end{aligned} \tag{6.53}$$

Because of the second relation in (6.53), statements valid for all second index q are written for the largest space corresponding to $q = \infty$. In addition, for all $0 < p, q, r \leq \infty$ and $s > 0$, the *discrepancy* between the spaces $B^s_{p,r}(\Omega)$ and $L^q(\Omega)$ is the quantity

$$\delta := s - \frac{d}{p} + \frac{d}{q}. \tag{6.54}$$

The discrepancy δ governs the embedding between these two spaces (Leoni 2009, Theorems 14.29, 14.32, DeVore 1998), namely

$$\delta > 0 \;\Rightarrow\; B^s_{p,\infty}(\Omega) \hookrightarrow L^q(\Omega), \quad \delta = 0 \;\Rightarrow\; B^s_{p,p}(\Omega) \hookrightarrow L^q(\Omega),\; q \neq \infty, \tag{6.55}$$

and the embedding is compact when $\delta > 0$. Notice that $\delta = 0$ determines the Sobolev embedding line of the DeVore diagram in Figure 2.1.

We stress that when $\delta > 0$, the third parameter r in $B^s_{p,r}(\Omega)$ plays no role in (6.55) involving the largest space $B^s_{p,\infty}(\Omega)$; see (6.53). However, it turns out to be useful to quantify regularity in extreme cases. For instance, the characteristic function χ_G of a smooth set $G \subsetneq \Omega$ satisfies

$$\chi_G \in B^{1/p}_{p,\infty}(\Omega) \backslash B^{1/p}_{p,r}(\Omega), \quad 0 < p, r < \infty.$$

Moreover, the Lagrange basis functions $\{\phi_z\}_{z\in\mathcal{N}}$ of $\mathbb{V}_\mathcal{T}$ satisfy for any $0 < p \le \infty$ (Gaspoz and Morin 2014, Proposition 4.7)

$$\omega_{n+1}(\phi_z, t)_p = \begin{cases} |\operatorname{supp}\phi_z|^{(d-1-p)/(dp)} t^{1+1/p}, & 0 < t \le |\operatorname{supp}\phi_z|^{1/d}, \\ |\operatorname{supp}\phi_z|, & t > |\operatorname{supp}\phi_z|^{1/d}. \end{cases}$$

This readily implies that for all $0 < s < 1 + 1/p$ and $0 < q < \infty$,

$$\mathbb{V}_\mathcal{T} \subset B^s_{p,q}(\Omega), \quad \mathbb{V}_\mathcal{T} \subset B^{1+1/p}_{p,\infty}(\Omega). \tag{6.56}$$

6.8.3. Local approximation

We are now in a position to prove a key approximation estimate. In finite element theory it goes by the name of Bramble–Hilbert lemma, whereas in nonlinear approximation theory it is called Jackson's theorem. We distinguish between the case $0 < s < n + 1$ and the limit integral case $s = n + 1$.

Proposition 6.32 (Bramble–Hilbert for Besov spaces). *Let $\mathcal{T} \in \mathbb{T}$ and $T \in \mathcal{T}$. Assume $0 < p, q \le \infty$, $0 < s < n + 1$, and either $s - d/p + d/q \ge 0$, $q < \infty$ or $s > d/p$, $q = \infty$. Set $r = \infty$ when $s - d/p + d/q > 0$ and $r = p$ otherwise. Then we have*

$$\inf_{P\in\mathbb{P}_n} \|v - P\|_{L^q(T)} \le C h_T^{s-d/p+d/q} |v|_{B^s_{p,r}(T)} \quad \textit{for all } v \in B^s_{p,r}(T), \tag{6.57}$$

where the constant $C = C(p, q, s, d, n, \mathcal{T}_0)$ is independent of v and T.

Proof. We first point out that we could use $k = n + 1 \ge [s] + 1$ in the definition of $|v|_{B^s_{p,r}(T)}$ according to (6.48). We next proceed in three steps.

$\boxed{1}$ Suppose first that T is the master element, namely $|T| \approx 1$. If $P \in \mathbb{P}_n$ is an arbitrary polynomial, using that the discrepancy $\delta = s - d/p + d/q \ge 0$ yields

$$E_n(v, T)_q \le \|v - P\|_{L^q(T)} \lesssim \|v - P\|_{B^s_{p,r}(T)} = \|v - P\|_{L^p(T)} + |v - P|_{B^s_{p,r}(T)}$$

due to the embedding (6.55). Since the definition of $\omega_{n+1}(v, t)_p$ involves $n + 1$ differences, we deduce $\Delta_h^{n+1} P = 0$ in view of (6.42), whence $|v - P|_{B^s_{p,r}(T)} = |v|_{B^s_{p,r}(T)}$. We now take P to be the best approximation of v in $L^p(T)$, to derive

$$E_n(v, T)_q \lesssim E_n(v, T)_p + |v|_{B^s_{p,r}(T)}.$$

$\boxed{2}$ We perform a scaling argument from the element $T \in \mathcal{T}$ to the master element $\widehat{T}$. Let $\widehat{x} = |T|^{-1/d} x$ be the change of variables and note that

$$\begin{aligned} \omega_{n+1}(v, t, T)_p &= \sup_{|h|\le t} \|\Delta_h^{n+1} v\|_{L^p(T)} \\ &= |T|^{1/p} \sup_{|h|\le t} \|\Delta^{n+1}_{h|T|^{-1/d}} \widehat{v}\|_{L^p(\widehat{T})} = |T|^{1/p} \omega_{n+1}(\widehat{v}, \widehat{t}, \widehat{T})_p, \end{aligned}$$

with $\widehat{t} = t|T|^{-1/d}$, whence

$$|v|^r_{B^s_{p,r}(T)} = \int_0^\infty (t^{-s}\omega_{n+1}(v,t,T)_p)^r \frac{\mathrm{d}t}{t}$$
$$= |T|^{r/p-sr/d} \int_0^\infty (\widehat{t}^{-s}\omega_{n+1}(\widehat{v},\widehat{t},\widehat{T})_p)^r \frac{\mathrm{d}\widehat{t}}{\widehat{t}} = |T|^{r/p-sr/d}|\widehat{v}|^r_{B^s_{p,r}(\widehat{T})}.$$

Therefore, since $E_n(v,T)_q = |T|^{1/q}E_n(\widehat{v},\widehat{T})_q$, we obtain

$$E_n(v,T)_q \lesssim |T|^{1/q-1/p}E_n(v,T)_p + |T|^{s/d-1/p+1/q}|v|_{B^s_{p,r}(T)}.$$

$\boxed{3}$ It remains to estimate $E_n(v,T)_p$ which, in view of Lemma 6.30 (Whitney's lemma), satisfies $E_n(v,T)_p \le C\omega_{n+1}(v,h_T,T)_p$ with $h_T \approx |T|^{1/d} \approx 2^{-m}$ for some $m \in \mathbb{Z}$. Since $k = n+1 \ge [s]+1$, invoking the equivalent definition (6.46) of $|v|_{B^s_{p,r}(T)}$ yields

$$E_n(v,T)^r_p \lesssim \omega_{n+1}(v,2^{-m},T)^r_p \lesssim h_T^{sr}\sum_{m\in\mathbb{Z}} 2^{msr}\omega_{n+1}(v,2^{-m},T)^r_p = h_T^{sr}|v|^r_{B^s_{p,r}(T)}.$$

Inserting this estimate into that of step $\boxed{2}$ gives (6.57), as asserted. □

We now consider the integer case $s = n+1$. The first thing to notice is that (6.57) cannot possibly be valid: the definition (6.45) requires $k = [s]+1 = n+2$, whence any polynomial $g \in \mathbb{P}_{n+1} \setminus \mathbb{P}_n$ satisfies $E_n(g,T)_p > 0$ as well as $\omega_{n+2}(g,T)_p = 0$ according to (6.42). Lemma 6.31 (blow-up of $|v|_{B^s_{p,p}(\Omega)}$) reveals that replacing the seminorm $|v|_{B^s_{p,p}(\Omega)}$ with the full norm $\|v\|_{B^{n+1}_{p,p}(\Omega)}$ is not a good idea either. To overcome this problem, we now introduce the space $\mathrm{Lip}^s_p(\Omega) := \mathrm{Lip}(s, L^p(\Omega))$ of s-Lipschitz functions with values in $L^p(\Omega)$, $0 < p < \infty$ (DeVore 1998, page 92):

$$|g|_{\mathrm{Lip}^s_p(\Omega)} := \sup_{t>0}(t^{-s}\omega_{n+1}(g,t,\Omega)_p). \tag{6.58}$$

Comparing with (6.45), we realize that $\mathrm{Lip}^s_p(\Omega) = B^s_{p,\infty}(\Omega)$ provided $s \notin \mathbb{N}$ but $\mathrm{Lip}^s_p(\Omega) \ne B^s_{p,\infty}(\Omega)$ when $s \in \mathbb{N}$. Moreover,

$$\delta = s - \frac{d}{p} + \frac{d}{q} > 0,\ s \in \mathbb{N} \quad \Rightarrow \quad \mathrm{Lip}^s_p(\Omega) \hookrightarrow L^q(\Omega), \tag{6.59}$$

with compact embedding. If $\delta = 0$ and $p \ge 1$, $q \ne \infty$, the above embedding is continuous in view of (6.36) and (6.55).

Proposition 6.33 (Bramble–Hilbert for Lipschitz spaces). *Let $\mathcal{T} \in \mathbb{T}$ and $T \in \mathcal{T}$. If $p \in (0,\infty)$, $q \in (0,\infty]$, $k \ge 0$ integer, and $k+1-d/p+d/q \ge 0$, with strict inequality when $p < 1$ or $q = \infty$, then we have*

$$\inf_{P\in\mathbb{P}_k} \|v-P\|_{L^q(T)} \le Ch_T^{k+1-d/p+d/q}|v|_{\mathrm{Lip}^{k+1}_p(T)} \quad \textit{for all } v \in \mathrm{Lip}^{k+1}_p(T), \tag{6.60}$$

where the constant $C = C(p,q,d,k,\mathcal{T}_0)$ is independent of v and T.

Proof. In view of (6.59), we proceed as in the proof of Proposition 6.32, except for the following change in step $\boxed{3}$. For $h_T \approx 2^{-m}$, we instead have

$$\begin{aligned} E_k(v,T)_p &\lesssim \omega_{k+1}(v,2^{-m},T)_p \\ &\lesssim h_T^{k+1} \sup_{m\in\mathbb{Z}} \big(2^{m(k+1)}\omega_{k+1}(v,2^{-m},T)_p\big) = h_T^{k+1}|v|_{\mathrm{Lip}_p^{k+1}(T)}. \end{aligned}$$

This concludes the proof. □

It is instructive to realize that Propositions 6.32 and 6.33 extend to Besov and Lipschitz spaces the usual Bramble–Hilbert lemma for Sobolev spaces (Brenner and Scott 2008, Lemma 4.3.8).

Proposition 6.34 (Bramble–Hilbert for Sobolev spaces). *Let $\mathcal{T} \in \mathbb{T}$ and $T \in \mathcal{T}$. For all $1 \le p, q, \le \infty$ and $0 < s \le n+1$ such that $s - d/p + d/q \ge 0$ with strict inequality when $q = \infty$, then*

$$\inf_{P\in\mathbb{P}_n} \|v - P\|_{L^q(T)} \le C h_T^{s-d/p+d/q} |v|_{W_p^s(T)} \quad \textit{for all } v \in W_p^s(T), \tag{6.61}$$

where the constant $C = C(p,q,s,d,n,\mathcal{T}_0)$ but is independent of v and T.

Proof. When s is fractional, $W_p^s(T) = B_{p,p}^s(T)$ in view of (6.49) and the result follows from Proposition 6.32. Instead, when s is integral, we invoke (6.36) and (6.37) to deduce the result from Proposition 6.33. □

6.8.4. Global approximation: direct estimates

We now collect local contributions from Propositions 6.32, 6.33 and 6.34, depending on the range of parameters p, q, s, to find global error estimates for the solution u as well as the coefficients $(\boldsymbol{A}, c)$ of (2.7). They are trivial consequences of Theorem 6.29, which we prove first. The analysis of the forcing function f is somewhat different, due to the non-locality of the corresponding norm $H^{-1}(\Omega)$, and is postponed to Section 7.3.

Proof of Theorem 6.29. Since the discrepancy $\delta = s - d/p + d/q > 0$, the embedding $X_p^s(\Omega) \hookrightarrow L^q(\Omega)$ is compact according to (6.55) and (6.59). Given $g \in X_p^s(\Omega)$, we consider the surrogate quantity $e_{\mathcal{T}}(g,T) := C h_T^{\delta} |g|_{X_p^s(T)}$, which satisfies

$$E_n(g,T)_q = \inf_{v\in\mathbb{P}_n} \|g - v\|_{L^q(T)} \le e_{\mathcal{T}}(g,T) \quad \text{for all } T \in \mathcal{T}$$

by virtue of Bramble–Hilbert Propositions 6.32, 6.33 and 6.34. We finally combine Proposition 3.19 (abstract greedy error) with the subadditivity property (6.51) to deduce the desired estimate (6.3). The remaining estimate (6.39) follows from the definition of $|g|_{\mathbb{A}_{s/d}}$. This concludes the proof. □

Inspection of this proof reveals that our estimate is stronger than (6.38). In fact, we need the weaker regularity

$$|g|_{X_p^s(\Omega;\mathcal{T})}^p = \sum_{T\in\mathcal{T}} |g|_{X_p^s(T)}^p < \infty,$$

which allows for piecewise Besov smoothness of g very much in the spirit of (3.20). This may accommodate singular behaviour of g aligned with the initial mesh $\mathcal{T}_0$.

Corollary 6.35 (approximation class of u). *Let the solution $u \in H_0^1(\Omega)$ of (2.7) satisfy $u \in X_p^s(\Omega)$ with $s \in (0, n+1]$, $p \in (0, \infty]$ and $s - 1 - d/p + d/2 > 0$, where $X_p^s(\Omega)$ is defined in (6.35). Then $u \in \mathbb{A}_{(s-1)/d}\big(H_0^1(\Omega); \mathcal{T}_0\big)$ and*

$$|u|_{\mathbb{A}_{(s-1)/d}} \lesssim |u|_{X_p^s(\Omega)}. \tag{6.62}$$

Equivalently, $\sigma_N(u)$ defined in (6.3) satisfies

$$\sigma_N(u) \lesssim |u|_{X_p^s(\Omega)} N^{-(s-1)/d}, \quad N \geq \#\mathcal{T}_0. \tag{6.63}$$

Proof. In view of (3.19) of Proposition 3.9 (approximation of gradients), namely

$$\inf_{v \in \mathbb{S}_{\mathcal{T}}^{n,0}} \|\nabla(u - v)\|_{L^2(\Omega)} \lesssim \inf_{v \in \mathbb{S}_{\mathcal{T}}^{n,-1}} \|\nabla(u - v)\|_{L^2(\Omega)},$$

we realize that it suffices to bound the element errors for $g = \nabla u \in L^2(\Omega; \mathbb{R}^d)$ by vector-valued discontinuous piecewise polynomials of degree $\leq n - 1$. Therefore, applying Theorem 6.29 (regularity yields approximation) with n replaced by $n - 1$ gives the desired estimates (6.62) and (6.63). □

We now turn our attention to the coefficients $(\boldsymbol{A}, c)$. Regarding $\boldsymbol{A}$, Lemma 5.20 (continuous dependence on data) shows that the natural function space for $\boldsymbol{A}$ is $L^\infty(\Omega, \mathbb{R}^{d\times d})$ provided $u \in H_0^1(\Omega)$. However, Lemma 5.20 also allows for $\boldsymbol{A} \in L^r(\Omega, \mathbb{R}^{d\times d})$, $2 \leq r < \infty$, provided $u \in W_p^1(\Omega)$ with $2 \leq p = 2r/(r-2) < p_1$, which in turn is guaranteed by Lemma 2.13 (W_p^1-regularity). The latter permits discontinuities of $\boldsymbol{A}$ within elements, which is of practical importance. Therefore we consider the most general situation $2 \leq r \leq \infty$ below.

Corollary 6.36 (approximation class of $\boldsymbol{A}$). *For $0 < \alpha_1 \leq \alpha_2$ and $2 \leq r \leq \infty$ let the diffusion coefficient $\boldsymbol{A} \in M(\alpha_1, \alpha_2)$ of (2.7) satisfy $\boldsymbol{A} \in X_p^s(\Omega; \mathbb{R}^{d\times d})$ with $s \in (0, n]$, $p \in (0, \infty]$ and $s - d/p + d/r > 0$. Then $\boldsymbol{A} \in \mathbb{M}_{s/d} = \mathbb{M}_{s/d}((L^r(\Omega))^{d\times d}; \mathcal{T}_0)$ and*

$$|\boldsymbol{A}|_{\mathbb{M}_{s/d}} \lesssim |\boldsymbol{A}|_{X_p^s(\Omega)}. \tag{6.64}$$

Equivalently, $\widetilde{\delta}_{\mathcal{T}}(\boldsymbol{A})_r$ defined in Section 6.1.2 satisfies

$$\inf_{\mathcal{T} \in \mathbb{T}_N} \widetilde{\delta}_{\mathcal{T}}(\boldsymbol{A})_r \lesssim |\boldsymbol{A}|_{X_p^s(\Omega)} N^{-s/d} \quad \text{for all } \mathcal{T} \in \mathbb{T}_N,\ N \geq \#\mathcal{T}_0, \tag{6.65}$$

and this error decay is achieved by Algorithm 3.18 (greedy algorithm).

Proof. Simply recall the relation (6.12) between the best constrained and unconstrained approximation errors and apply Theorem 6.29 (regularity yields approximation). □

Consider the special case $s = n$ and $r = \infty$ in Corollary 6.36. We readily see that $p > d/n$ which might be less than 1 for $n > d$, hence the need for Besov spaces.

We finally deal with the reaction coefficient $c \in L^\infty(\Omega)$. Given Lemma 5.20 (continuous dependence on data) and the discussion in Section 5.4.2, a natural space for c is $L^q(\Omega)$ with $d/2 < q \le \infty$, $s = 0$; we could take $q = 2$ for $d < 4$. Section 5.4.2 also reveals that the case $n = 1$ is somewhat special in that we can exploit superconvergence in $W_q^{-1}(\Omega)$ with $q > d$. In fact, combining the argument following (5.75) with (5.68) yields

$$\inf_{\widehat{c} \in \mathbb{S}_{\mathcal{T}}^{n-1,-1}} \|c - \widehat{c}\|^2_{W_q^{-1}(\Omega)} \lesssim \sum_{T \in \mathcal{T}} h_T^{2t} \|c - \Pi_T c\|^2_{L^2(T)} \lesssim \sum_{T \in \mathcal{T}} h_T^{2t} \delta_{\mathcal{T}}(c, T)_2^2 = \operatorname{osc}_{\mathcal{T}}(c)_2^2$$

with $0 < t = 1 - d/2 + d/q < 2 - d/2$ provided $d < 4$. This gives the following statement. We note that (5.76) could also be combined with (5.68) for $n = 1$ to obtain a similar result for $\operatorname{osc}_{\mathcal{T}}(c)_\infty$ with $t = 1$ and any $d \ge 2$; however, we do not elaborate further.

Corollary 6.37 (approximation class of c). *Let $0 \le c_1 \le c_2$ and the reaction coefficient $c \in R(c_1, c_2)$ satisfy $c \in X_p^s(\Omega)$ with $s \in (0, n]$, $p \in (0, \infty]$. If $n \ge 1$, $q > d/2$, and $s - d/p + d/q > 0$, then $c \in \mathbb{C}_{s/d} = \mathbb{C}_{s/d}(L^q(\Omega); \mathcal{T}_0)$ and*

$$|c|_{\mathbb{C}_{s/d}} \lesssim |c|_{X_p^s(\Omega)}. \tag{6.66}$$

If instead $n = 1$, $q > d$, $s - d/p + d/2 \ge 0$, $0 < t = 1 - d/2 + d/q < 2 - d/2$ and $d < 4$, then $c \in \mathbb{C}_{(s+t)/d} = \mathbb{C}_{(s+t)/d}(L^2(\Omega); \mathcal{T}_0)$ and

$$|c|_{\mathbb{C}_{(s+t)/d}} \lesssim |c|_{X_p^s(\Omega)}. \tag{6.67}$$

Equivalently, for all $n \ge 1$, $\widetilde{\delta}_{\mathcal{T}}(c)_q$ defined in Section 6.1.2 satisfies

$$\inf_{\mathcal{T} \in \mathbb{T}_N} \widetilde{\delta}_{\mathcal{T}}(c)_q \lesssim |c|_{X_p^s(\Omega)} N^{-(s+t)/d} \quad \text{for all } \mathcal{T} \in \mathbb{T}_N,\ N \ge \#\mathcal{T}_0,$$

with $t = 0$ when $n > 1$. This error decay is achieved by Algorithm 3.18 (greedy algorithm).

Proof. In view of (6.12), inequality (6.66) is a direct application of Theorem 6.29 (regularity yields approximation). The superconvergence rate in (6.67) is a consequence of (6.12) and the proof of Proposition 3.19 (abstract greedy error) with s replaced by $s + t$. □

We finally go back to the abstract space $X_p^s(\Omega)$, defined in (6.35), and introduce the corresponding abstract approximation class $\mathbb{X}_{s/d} = \mathbb{X}_{s/d}(L^q(\Omega); \mathcal{T}_0)$ of functions $v \in L^q(\Omega)$ such that

$$|v|_{\mathbb{X}_{s/d}} = \sup_{N \ge \#\mathcal{T}_0} \left(N^{s/d} \inf_{\mathcal{T} \in \mathbb{T}_N} \operatorname{osc}_{\mathcal{T}}(v)_q \right) < \infty \quad \Rightarrow \quad \inf_{\mathcal{T} \in \mathbb{T}_N} \operatorname{osc}_{\mathcal{T}}(v)_q \le |v|_{\mathbb{X}_{s/d}} N^{-s/d}.$$

Consequently, Theorem 6.29 (regularity yields approximation) implies

$$X_p^s(\Omega) \subset \mathbb{X}_{s/d}, \quad |v|_{\mathbb{X}_{s/d}} \lesssim |v|_{X_p^s(\Omega)}. \tag{6.68}$$

We will utilize this abstract notation and estimates in Section 7 while discussing the approximation of data $\mathcal{D} = (\boldsymbol{A}, c, f)$ by a greedy algorithm.

6.8.5. Global approximation: inverse estimates

Theorem 6.29 gives sufficient regularity properties for a function $g \in L^q(\Omega)$ to belong to an approximation class $\mathbb{A}_{s/d}(L^q(\Omega); \mathcal{T}_0)$; this is called *direct estimate*. Such regularity is written in terms of a Besov space $B^s_{p,p}(\Omega)$, except in the limiting case $s = n+1$. The converse statement is also true and is called an *inverse estimate*: if g belongs to an approximation class $\mathbb{A}_{t/d}(L^q(\Omega); \mathcal{T}_0)$, then it is a member of a Besov space $\widehat{B}^s_{p,p}(\Omega)$ provided $t > s$ and $0 < s < n+1$, $s - d/p + d/q = 0$ (Binev *et al.* 2002, Gaspoz and Morin 2014).

Several comments are in order. The Besov space $\widehat{B}^s_{p,p}(\Omega)$ is defined via a multilevel decomposition of $L^p(\Omega)$ and coincides with $B^s_{p,p}(\Omega)$ only when $s < 1 + 1/p$. This restriction of s is natural because $\mathbb{V}_{\mathcal{T}} \subset \widehat{B}^s_{p,p}(\Omega)$ for all s, but $\mathbb{V}_{\mathcal{T}} \subset B^s_{p,p}(\Omega)$ requires $s < 1+1/p$ according to (6.56). The discrepancy between the spaces $\widehat{B}^s_{p,p}(\Omega)$ and $L^q(\Omega)$ is $\delta = s - d/p + d/q = 0$, but the decay rate t/d of $\mathbb{A}_{t/d}(L^q(\Omega); \mathcal{T}_0)$ is larger than s/d. This accounts for the embedding of $\mathbb{A}_{t/d}(L^q(\Omega); \mathcal{T}_0)$, that is,

$$\sum_{n\in\mathbb{N}} \big(\sigma_{2^n}(g) 2^{(s/d)n}\big)^p \le \sup_{n\in\mathbb{N}} \big(\sigma_{2^n}(g) 2^{(t/d)n}\big)^p \sum_{n\in\mathbb{N}} 2^{((s-t)/d)pn} \lesssim |g|^p_{\mathbb{A}_{t/d}}$$

into a space with decay s/d and summability ℓ^q that in turn embeds into $\widehat{B}^s_{p,p}(\Omega)$ (Binev *et al.* 2002, Gaspoz and Morin 2014). This reveals that there is no complete characterization of the approximation class $\mathbb{A}_{s/d}$ in terms of Besov regularity.

7. Data approximation

This section focuses on the module DATA of Algorithms 5.1 (AFEM-TS) and 5.16 (AFEM-SW). According to Assumption 6.11 (quasi-optimality of DATA), the call

$$[\widehat{\mathcal{T}}, \widehat{\mathcal{D}}] = \mathsf{DATA}(\mathcal{T}, \mathcal{D}, \tau) \tag{7.1}$$

is meant to construct a quasi-optimal conforming refinement $\widehat{\mathcal{T}}$ of $\mathcal{T} \in \mathbb{T}$ and approximate piecewise polynomial data $\widehat{\mathcal{D}} = (\widehat{\boldsymbol{A}}, \widehat{c}, \widehat{f}) \in \mathbb{D}_{\widehat{\mathcal{T}}}$ over $\widehat{\mathcal{T}}$ that satisfies

$$\|\mathcal{D} - \widehat{\mathcal{D}}\|_{D(\Omega)} \le C_{\mathrm{data}} \tau \tag{7.2}$$

as well as the constraints $\widehat{\boldsymbol{A}} \in M(\widehat{\alpha}_1, \widehat{\alpha}_2)$ and $\widehat{c} \in R(\widehat{c}_1, \widehat{c}_2)$ defined in (5.50).

Sections 7.2.2 and 7.2.3 are devoted to the construction of $(\widehat{\boldsymbol{A}}, \widehat{c})$. To approximate the coefficients $(\boldsymbol{A}, c)$, we proceed in two steps. First, we solve an *unconstrained* approximation problem upon computing the L^2-projection $(\widetilde{\boldsymbol{A}}, \widetilde{c})$ of $(\boldsymbol{A}, c)$ onto the space of discontinuous piecewise polynomials of degree $\le n-1$; this step is linear, easily achieves the desired accuracy, but does not guarantee the monotonicity of oscillations with respect to refinement (5.72) and violates the constraints in (5.50) unless $n = 1$. Second, we resort to the nonlinear selection (5.70) of the local L^2 approximation to force the resulting oscillations to be monotone. Third, we solve a *constrained* problem, which modifies $(\widetilde{\boldsymbol{A}}, \widetilde{c})$ locally into $(\widehat{\boldsymbol{A}}, \widehat{c})$ and restores

(5.50) without accuracy degradation; this is a delicate nonlinear procedure executed element by element, introduced and discussed in Section 7.2.

The approximation of the right-hand side $f \in H^{-1}(\Omega)$ is a conceptually different linear process. Without further structural assumptions on f it is not possible to evaluate $\mathrm{osc}_{\mathcal{T}}(f)_{-1}$ and reduce it. Hence we introduce surrogate estimators $\widetilde{\mathrm{osc}}_{\mathcal{T}}(f)_{-1}$, which are larger than $\mathrm{osc}_{\mathcal{T}}(f)_{-1}$, but computable, for several classes of forcing functions f relevant in practice. We discuss this in Section 7.3.

We start in Section 7.1 with a presentation and assessment of quasi-optimal GREEDY algorithms to reduce the data error. An important consideration is that the local error estimators $\{\mathrm{osc}_{\mathcal{T}}(v,T)\}_{T\in\mathcal{T}}$ may accumulate in ℓ^∞ as well as in ℓ^q for $q<\infty$. Both are handled via a GREEDY algorithm similar to Algorithm 3.18 but with different stopping criteria when the local errors accumulate in ℓ^q with $q<\infty$. The module DATA combines both: its structure is displayed in Algorithm 7.23 and its performance is elucidated in Corollary 7.24 below.

7.1. *Quasi-optimal* GREEDY *algorithms for data reduction*

Algorithm 3.18 (greedy algorithm) is well suited to dealing with local error estimators $\mathrm{osc}_{\mathcal{T}}(v,T)_q$ that accumulate with respect to $T\in\mathcal{T}$ in the space ℓ^∞. This is the framework for approximating coefficients $v = \boldsymbol{A}, c$ in $L^\infty(\Omega)$, in which case the local error estimators $\mathrm{osc}_{\mathcal{T}}(v,T)_\infty$ are defined in (5.73) for $r=q=\infty$. This requires $v=\boldsymbol{A},c$ to be piecewise uniformly continuous on $\mathcal{T}$ for $\mathrm{osc}_{\mathcal{T}}(v;T)_\infty \to 0$ as $h_T \to 0$. However, for discontinuous $(\boldsymbol{A},c)$ and the forcing function f, the accumulation of $\mathrm{osc}_{\mathcal{T}}(v,T)_q$ for $v=\boldsymbol{A},c,f$ is in ℓ^q for $q<\infty$. In this case, Algorithm 3.18 does not provide a direct relation between a desired output tolerance τ for the total error

$$E_{\mathcal{T}}(v)_q := \|\{\mathrm{osc}_{\mathcal{T}}(v,T)_q\}_{T\in\mathcal{T}}\|_{\ell^q} = \left(\sum_{T\in\mathcal{T}} \mathrm{osc}_{\mathcal{T}}(v,T)_q^q\right)^{1/q}$$

and the threshold δ; recall that $\mathrm{osc}_{\mathcal{T}}(v,T)_q := \|v-\widehat{v}\|_{L^q(T)}$ for $T\in\mathcal{T}$.

Another subtle difference from Algorithm 3.18 is that the algorithm GREEDY below starts not from $\mathcal{T}_0$ but from any $\mathcal{T}\in\mathbb{T}$. Since DATA and thus GREEDY are called repeatedly within AFEM, it seems advantageous to exploit the mesh refinement already performed in the adaptive process rather than restart from scratch; this then improves the computational efficiency.

Algorithm 7.1 (GREEDY). Given a tolerance $\tau>0$, $0<q\le\infty$, a number of bisections $b\ge 1$ performed per element to be refined, and an arbitrary conforming grid $\mathcal{T}\in\mathbb{T}$, not necessarily $\mathcal{T}_0$, GREEDY finds a conforming refinement $\widehat{\mathcal{T}}\ge\mathcal{T}$ of $\mathcal{T}$ by bisection and $\widehat{v}\in\mathbb{S}^{n-1,-1}_{\mathcal{T}}$ such that $E_{\mathcal{T}}(v)_q\le\tau$:

$[\widehat{\mathcal{T}},\widehat{v}] = \mathsf{GREEDY}(\mathcal{T},\tau,q,b,v)$
  $[\widehat{v}] = \mathsf{PROJECT}(\mathcal{T},v)$
  while $E_{\mathcal{T}}(v)_q > \tau$

$[\mathcal{M}] = \arg\max\{\mathrm{osc}_{\mathcal{T}}(v,T)_q : T \in \mathcal{T}\}$
$[\mathcal{T}] = \mathsf{REFINE}(\mathcal{T}, \mathcal{M}, b)$
$[\widehat{v}] = \mathsf{PROJECT}(\mathcal{T}, v)$
return $\mathcal{T}, \widehat{v}$

In GREEDY above, the element T with largest error is refined as long as the total error $E_{\mathcal{T}}(v)_q$ exceeds the target tolerance τ. When the largest error is achieved by several elements, an *ad hoc* criterion such as lexical order is used to break ties. We also recall that the routine REFINE bisects all the elements in $\mathcal{M}$ (in this case only one) b times and performs additional refinements necessary to produce a conforming subdivision. PROJECT computes the local approximations $\widehat{v}$ of v needed to evaluate $\mathrm{osc}_{\mathcal{T}}(v,T)_q$; refer to Section 5.4.2 and (5.70) for the definition of $\widehat{v}$. The dependence on $\widehat{v}$ in $\mathrm{osc}_{\mathcal{T}}(v,T)_q$ and $E_{\mathcal{T}}(v)$ is not indicated.

To discuss the performances of the GREEDY algorithm, we recall that $X^s_p(\Omega;\mathcal{T}_0)$ is the abstract space defined in (6.35), which satisfies

$$\sum_{T\in\mathcal{T}} |v|^p_{X^s_p(T)} \lesssim |v|^p_{X^s_p(\Omega;\mathcal{T}_0)} \tag{7.3}$$

for all $\mathcal{T} \in \mathbb{T}$ and $v \in X^s_p(\Omega;\mathcal{T}_0)$.

The GREEDY algorithm analysed in Proposition 3.19 (abstract greedy error) relies on the abstract assumptions (3.40), (3.41) and (3.42). With the aim of reducing the data oscillations, we make these assumptions more concrete.

Assumption 7.2 (admissible set of parameters for GREEDY). We say that the set of parameters (v,s,t,p,q) is admissible for GREEDY with local oscillations $\{\mathrm{osc}_{\mathcal{T}}(v,T)_q\}_{T\in\mathcal{T}}$ if $0 < p,q \le \infty$, $s,t \ge 0$ satisfy

(i) $v \in X^s_p(\Omega;\mathcal{T}_0)$,

(ii) $t+s>0$, $s-d/p+d/q \ge 0$ with strict inequality when $q=\infty$ or $s=n+1$, $p<1$,

(iii) for $r := t+s-d/p+d/q > 0$,

$$\mathrm{osc}_{\mathcal{T}}(v,T)_q \lesssim h_T^r |v|_{X^s_p(T)} \quad \text{for all } T\in\mathcal{T},\ \mathcal{T}\in\mathbb{T}. \tag{7.4}$$

When the local oscillations considered are clear from the context, we say that (v,s,t,p,q) is admissible for GREEDY.

Relation (7.4) replaces (3.41) and is a regularity assumption guaranteeing a convergence rate when approximating v by $\widehat{v} = \mathsf{PROJECT}(\mathcal{T}, v)$ (appearing in the definition of $\mathrm{osc}_{\mathcal{T}}(v,T)_q$). We refer to Propositions 6.32, 6.33 and 6.34 for examples where Assumption 7.2 holds. Note that in view of (6.55) and (6.59), condition (ii) in Assumption 7.2 guarantees $v \in X^s_p(\Omega) \subset L^q(\Omega)$. The parameter $t \ge 0$ reflects a possible additional power of h in the oscillation term; see e.g. (5.75), (5.76) and (5.78). Furthermore, in view of (7.3), assumption (6.51) is always satisfied by the $X^s_p(\Omega;\mathcal{T}_0)$ seminorms, and (3.28) is no longer needed.

As alluded to above, the case $q < \infty$ is more complex to analyse and cannot rely solely on the decay property (7.4) as in the proof of Proposition 3.19. It requires the local oscillations to be monotone with respect to refinements.

Assumption 7.3 (monotonicity of local oscillations). We say that for $0 < q \leq \infty$, the local errors satisfy the monotonicity property in ℓ^q if, for any $v \in L^q(\Omega)$, any $\mathcal{T}, \mathcal{T}_* \in \mathbb{T}$ with $\mathcal{T}_* \geq \mathcal{T}$ and any $T_* \in \mathcal{T}_*, T \in \mathcal{T}$ with $T_* \subset T$, we have

$$\operatorname{osc}_{\mathcal{T}_*}(v, T_*)_q \leq \operatorname{osc}_{\mathcal{T}}(v, T)_q. \tag{7.5}$$

In view of Lemma 5.26 (monotonicity of oscillation), Assumption 7.3 holds for the oscillations on $\boldsymbol{A}$ and c given in (5.73) and (5.75), but not for the oscillations (5.77) of f. However, in Section 7.3 below we derive computable surrogates for the local error $\operatorname{osc}_{\mathcal{T}}(f, T)_{-1}$. These surrogates satisfy the monotonicity property and are used in turn to drive the GREEDY algorithm. In passing, we note that we refrain from using the right-hand side of inequality (7.4) as a surrogate for the local oscillation. In fact it is monotone with respect to refinements but at the expense of being difficult to evaluate because it involves the seminorm $|v|_{X^s_p(T)}$.

The following result is the counterpart of Proposition 3.19 (abstract greedy error) for GREEDY with errors accumulating in ℓ^q, $0 < q \leq \infty$, and still starting from $\mathcal{T}_0$. We address the case where $\mathcal{T} \neq \mathcal{T}_0$ in Lemma 7.5 below.

Proposition 7.4 (performance of GREEDY). *Let the initial subdivision $\mathcal{T}_0$ of $\Omega \subset \mathbb{R}^d$ satisfy Assumption 6.19 (initial labelling). Let $\tau > 0$ be the target tolerance and let $b \geq 1$ be the number of bisections performed on each marked element. Let (v, s, t, p, q) satisfy Assumption 7.2 (admissible set of parameters for* GREEDY*) with local errors $\{\operatorname{osc}_{\mathcal{T}}(v, T)_q\}_{T \in \mathcal{T}}$ which in turn verify Assumption 7.3 (monotonicity of local oscillations) in ℓ^q. Then* GREEDY$(\mathcal{T}_0, \tau, q, b, v)$ *terminates in a finite number of iterations and*

$$E_{\mathcal{T}}(v)_q \leq \tau \leq C|v|_{X^s_p(\Omega;\mathcal{T}_0)}(\#\mathcal{T})^{-(s+t)/d}, \tag{7.6}$$

with a constant $C = C(p, q, s, b, d, \Omega, \mathcal{T}_0)$. Furthermore, $v \in \mathbb{X}_{(s+t)/d}$ and $|v|_{\mathbb{X}_{(s+t)/d}} \lesssim |v|_{X^s_p(\Omega;\mathcal{T}_0)}$. Moreover, the estimate (7.6) is valid for tensor-valued functions v.

Proof. Since the proof is similar to that of Proposition 3.19 (abstract greedy error) with $e_{\mathcal{T}}(v, T)_q = \operatorname{osc}_{\mathcal{T}}(v, T)_q$, we only report the new ingredients. We recall that we use the convention $1/\infty = 0$. Let $\mathcal{T}_1, \ldots, \mathcal{T}_k$ be the sequence of refinements produced by GREEDY, and let $T_1, \ldots T_k$ be the sequence of marked elements. We need to estimate $\#\mathcal{M} = k$ with $\mathcal{M} = \{T_1, \ldots, T_k\}$. Set

$$\delta_i := \operatorname{osc}_{\mathcal{T}_i}(v, T_i)_q \quad (1 \leq i \leq k) \quad \text{and} \quad \delta := \delta_{k-1}.$$

Then

$$E_{\mathcal{T}_k}(v)_q \leq \tau < E_{\mathcal{T}_{k-1}}(v)_q \leq \delta\,(\#\mathcal{T}_{k-1})^{1/q} \leq \delta\,(\#\mathcal{T}_k)^{1/q}. \tag{7.7}$$

On the other hand, as REFINE does not increase the element estimators $\mathrm{osc}_{\mathcal{T}_i}(v, T_i)$ thanks to (7.5), we have $\delta_i \geq \delta$ for any i, whence

$$\mathrm{osc}_{\mathcal{T}_i}(v, T_i)_q = \delta_i \geq \delta \quad \text{for all } 1 \leq i < k.$$

Let us now partition $\mathcal{M}$ into disjoint subsets $\mathcal{P}_j$ as in the proof of Proposition 3.19. If $T_i \in \mathcal{P}_j$, (7.4) implies

$$\delta \leq \mathrm{osc}_{\mathcal{T}_i}(v, T_i)_q \lesssim h_T^r |v|_{X_p^s(T_i)} \leq 2^{-jr/2} |v|_{X_p^s(T_i)},$$

whence, exploiting the ℓ^p summability (7.3) gives

$$\#\mathcal{P}_j \lesssim \delta^{-p} 2^{-jrp/2} |v|^p_{X_p^s(\Omega;\mathcal{T}_0)}.$$

which is similar to (3.45). Recalling (3.44), and proceeding as in the proof of Proposition 3.19, yields

$$\delta \lesssim |v|_{X_p^s(\Omega;\mathcal{T}_0)} \left(\#\mathcal{T}_k - \#\mathcal{T}_0\right)^{-(s+t)/d-1/q}.$$

We conclude the proof using (7.7) and the bound $\#\mathcal{T}_k \geq c_0 \#\mathcal{T}_0$ for $c_0 > 1$. □

In contrast to Section 3, and most of the existing literature, Algorithm 7.1 starts from a refinement $\mathcal{T}$ of $\mathcal{T}_0$ rather than $\mathcal{T}_0$ and thus exploits the mesh refinement already performed in the adaptive process. We now give a simple argument, updated from Bonito *et al.* (2013*b*), that shows that the number of elements $N(\mathcal{T}, \tau, b, v)$ marked by GREEDY starting from $\mathcal{T}$ with target tolerance τ and refined $b \geq 1$ times is dominated by $N(\mathcal{T}_0, \tau, 1, v)$, namely

$$N(\mathcal{T}, \tau, b, v) \leq N(\mathcal{T}_0, \tau, 1, v). \tag{7.8}$$

Estimate (7.8) is crucial because it avoids studying the cardinality of GREEDY starting from $\mathcal{T} \neq \mathcal{T}_0$ directly, and simplifies the analysis. Even though (7.8) is plausible, the fact that the output of $\mathsf{GREEDY}(\mathcal{T}_0, \tau, q, 1, v)$ is unrelated to $\mathcal{T}$ makes it non-obvious. In fact, note that we do not claim that $N(\mathcal{T}, \tau, b, v) \leq N(\mathcal{T}_0, \tau, b, v)$, which is unclear. The proof presented below hinges on the fact that all the elements refined within $\mathsf{GREEDY}(\mathcal{T}_0, \tau, q, 1, v)$ are either refined because they are marked by GREEDY (and thus of largest oscillation) or because their refinement is necessary to guarantee conformity of the resulting subdivision. For our purposes, (7.8) suffices.

Lemma 7.5 (GREEDY starting from $\mathcal{T}$). *Let $\tau > 0$ be a target tolerance and let $b \geq 1$ be the number of bisections per marked element. Assume that the local errors employed by* GREEDY *satisfy Assumption 7.3 (monotonicity of local oscillations) in ℓ^q. Then the number of elements $N(\mathcal{T}, \tau, b, v)$ marked by* $\mathsf{GREEDY}(\mathcal{T}, \tau, q, b, v)$ *satisfies* (7.8) *for any admissible refinement $\mathcal{T} \in \mathbb{T}$ of $\mathcal{T}_0$.*

Proof. We simply write $\mathsf{GREEDY}(\mathcal{T}_0, \tau, 1)$ and $\mathsf{GREEDY}(\mathcal{T}, \tau, b)$ because v and q are fixed. Let $N := N(\mathcal{T}_0, \tau, 1, v)$, and recall that the bisection rules define a unique forest $\mathbb{T}$ emanating from $\mathcal{T}_0$ and a unique sequence of elements $\{T_i\}_{i=1}^N$ marked by $\mathsf{GREEDY}(\mathcal{T}_0, \tau, 1)$. We let $\{\mathcal{T}^i\}_{i=1}^N$ denote the sequence of intermediate subdivisions built within $\mathsf{GREEDY}(\mathcal{T}_0, \tau, 1)$ starting with $\mathcal{T}^0 = \mathcal{T}_0$: $T_i \in \mathcal{T}^{i-1}$ is bisected

once by REFINE, which also produces the smallest conforming refinement $\mathcal{T}^i$ of $\mathcal{T}^{i-1}$ containing the two children of T_i. We thus say that GREEDY$(\mathcal{T}_0, \tau, 1)$ satisfies the *minimality property* that all the elements refined are either marked elements because their error is largest or necessary to guarantee conforming subdivisions. Notice that this is not true for GREEDY$(\mathcal{T}_0, \tau, b)$ when $b > 1$.

For any $\mathcal{T} \in \mathbb{T}$, we let $\Lambda_{\mathcal{T}}$ be the set of indices $j \in \{1, \dots, N\}$ such that T_j is never refined in the process to create $\mathcal{T}$, that is, T_j is either an element of $\mathcal{T}$ or a successor of an element of $\mathcal{T}$. We show that

$$N(\mathcal{T}, \tau, b, v) \leq \#\Lambda_{\mathcal{T}} \tag{7.9}$$

by induction on $\#\Lambda_{\mathcal{T}}$. If $\#\Lambda_{\mathcal{T}} = 0$ then $\mathcal{T}$ is a refinement of $\mathcal{T}^N$, whence the monotonicity of the total error

$$E_{\mathcal{T}}(v)_q \leq E_{\mathcal{T}^N}(v)_q \leq \tau,$$

guaranteed by (7.5), implies that $N(\mathcal{T}, \tau, b, v) = 0$; this satisfies (7.9) as desired.

We now assume that (7.9) is valid for any $\mathcal{T} \in \mathbb{T}$ such that $\#\Lambda_{\mathcal{T}} \leq k$, a non-negative integer, and deduce that it must also hold for any $\mathcal{T} \in \mathbb{T}$ such that $\#\Lambda_{\mathcal{T}} \leq k + 1$. Let $\mathcal{T} \in \mathbb{T}$ be one such mesh, namely $\#\Lambda_{\mathcal{T}} = k + 1$. If $E_{\mathcal{T}}(v)_q \leq \tau$, then $N(\mathcal{T}, \tau, b, v) = 0$ and $N(\mathcal{T}, \tau, b, v) \leq \#\Lambda_{\mathcal{T}}$ holds trivially.

When instead $E_{\mathcal{T}}(v)_q > \tau$, we let j be the smallest index in $\Lambda_{\mathcal{T}}$ and show that $T_j \in \mathcal{T}$ using the minimality property of GREEDY$(\mathcal{T}_0, \tau, 1)$. Assume by contradiction that $T_j \notin \mathcal{T}$ but T_j belongs to a refinement $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ and is thus a successor of an element $T \in \mathcal{T}$. Note that T is refined by GREEDY$(\mathcal{T}_0, \tau, 1)$ to produce T_j but was not marked, because otherwise $T = T_i$ for some $i < j$ and $i \in \Lambda_{\mathcal{T}}$, which would contradict the minimality of j. Hence T must have been refined by the REFINE routine to guarantee conformity when bisecting a marked element T_ℓ, $\ell < j$. Invoking the minimality of j again yields that $\ell \notin \Lambda_{\mathcal{T}}$ and T_ℓ cannot be in $\mathcal{T}$ because T_ℓ has been refined to get to $\mathcal{T}$ by definition of $\Lambda_{\mathcal{T}}$. Since REFINE refines the minimal number of non-marked elements to guarantee conformity, and $\mathcal{T}$ is conforming, T must have been refined as well when refining T_ℓ in the process of constructing $\mathcal{T}$ and therefore cannot be in $\mathcal{T}$. This is a contradiction and $T_j \in \mathcal{T}$.

Therefore $\mathcal{T}$ is a refinement of $\mathcal{T}^{j-1}$ because all the elements marked or refined to ensure conformity by GREEDY$(\mathcal{T}_0, \tau, 1)$ have been refined in the process of creating $\mathcal{T}$. Moreover, $T_j \in \mathcal{T}$ is the element with largest error $\operatorname{osc}_{\mathcal{T}}(v, T_j)$ within $\mathcal{T}$ (with *ad hoc* criteria to break ties), because $\operatorname{osc}_{\mathcal{T}^{j-1}}(v, T_j)$ is largest in $\mathcal{T}^{j-1}$ by definition of T_j and monotonicity of the local error (7.5); hence T_j must be the first element marked by GREEDY$(\mathcal{T}, \tau, b)$. Let $\mathcal{T}^*$ be the subdivision obtained from $\mathcal{T}$ upon bisecting b times T_j. Notice that $\Lambda_{\mathcal{T}^*}$ is a strict subset of $\Lambda_{\mathcal{T}}$, because $j \notin \Lambda_{\mathcal{T}^*}$, so that the induction assumption yields

$$N(\mathcal{T}, \tau, b, v) = 1 + N(\mathcal{T}^*, \tau, b, v) \leq 1 + \#\Lambda_{\mathcal{T}^*} \leq \#\Lambda_{\mathcal{T}}.$$

This proves (7.9), and (7.8) follows immediately since $\#\Lambda_{\mathcal{T}} \leq N(\mathcal{T}_0, \tau, 1, v)$. $\square$

Estimate (7.8) is critical to analysing the performances of GREEDY starting from any admissible subdivision $\mathcal{T} \in \mathbb{T}$. We emphasize that the complexity estimate provided by Corollary 7.6 is expressed in terms of the number of marked elements $N(\mathcal{T}, \tau, q, b, v)$ and tolerance τ instead of error and cardinality of $\mathcal{T}$. This is why GREEDY can start from any mesh $\mathcal{T} \in \mathbb{T}$.

Corollary 7.6 (performance of GREEDY). *Let the initial subdivision $\mathcal{T}_0$ of $\Omega \subset \mathbb{R}^d$ satisfy Assumption 6.19 (initial labelling) and let $\mathcal{T} \in \mathbb{T}$ be any admissible refinement of $\mathcal{T}_0$. Let $\tau > 0$ be the target tolerance and let $b \geq 1$ be the number of bisections performed on each marked element. Let (v, s, t, p, q) satisfy Assumption 7.2 (admissible set of parameters for* GREEDY*) with local errors $\{\mathrm{osc}_{\mathcal{T}}(v, T)_q\}_{T \in \mathcal{T}}$ which in turn verify Assumption 7.3 (monotonicity of local oscillations) in ℓ^q. The number of marked elements $N(\mathcal{T}, \tau, q, b, v)$ by* GREEDY$(\mathcal{T}, \tau, q, b, v)$ *satisfies*

$$N(\mathcal{T}, \tau, q, b, v) \leq C|v|_{X^s_p(\Omega)}^{d/(s+t)} \tau^{-d/(s+t)}, \tag{7.10}$$

with a constant $C = C(p, q, s, b, d, \Omega, \mathcal{T}_0)$. Moreover, the estimate (7.10) is valid for tensor-valued functions v.

Proof. Invoking Proposition 7.4 (performance of GREEDY), which gives rise to a mesh $\widehat{\mathcal{T}}$, and Lemma 7.5 (GREEDY starting from $\mathcal{T}$), we readily deduce

$$N(\mathcal{T}, \tau, q, b, v) \leq N(\mathcal{T}_0, \tau, q, 1, v) \leq \#\widehat{\mathcal{T}} \leq C|v|_{X^s_p(\Omega)}^{d/(s+t)} \tau^{-d/(s+t)},$$

which is the desired inequality (7.10). □

7.2. Constrained approximations

We discuss how the approximations produced by GREEDY (see Corollary 7.6) can be modified to satisfy the structural assumption (5.51) without sacrificing their accuracy.

7.2.1. Constrained approximations of scalar functions

The approximate data $\widetilde{\mathcal{D}} = (\widetilde{\boldsymbol{A}}, \widetilde{c}, \widetilde{f})$ constructed in the previous sections using the GREEDY algorithm are not guaranteed to satisfy the necessary conditions for perturbed problem (5.5) with $\widehat{\mathcal{D}} = \widetilde{\mathcal{D}}$ to have a solution $\widehat{u} = \widehat{u}(\widehat{\mathcal{D}}) \in H^1_0(\Omega)$. Recall that the data $\mathcal{D} = (\boldsymbol{A}, c, f) \in D(\Omega)$ is assumed to satisfy the structural assumption (5.50), i.e. $\boldsymbol{A} \in M(\alpha_1, \alpha_2)$ and $c \in R(c_1, c_2)$ with $0 < \alpha_1 \leq \alpha_2$ and $0 \leq c_1 \leq c_2$. It turns out that constructing approximate data $\widehat{\mathcal{D}}$ with the same constraints is a difficult task. We follow Bonito *et al.* (2013*b*) and modify the data $\widetilde{\mathcal{D}}$ to obtain $\widehat{\mathcal{D}} = (\widehat{\boldsymbol{A}}, \widehat{c}, \widehat{f})$ in such a way that the approximation property of $\widetilde{\mathcal{D}}$ is preserved, that is,

$$\|\mathcal{D} - \widehat{\mathcal{D}}\|_{D(\Omega)} \leq C_{\mathrm{data}} \|\mathcal{D} - \widetilde{\mathcal{D}}\|_{D(\Omega)},$$

while ensuring that

$$\widehat{A} \in M\left(\frac{\alpha_1}{2}, C\widehat{\alpha}_2\right), \quad \widehat{c} \in R\left(-\frac{\alpha_1}{4C_P^2}, C\widehat{c}_2\right). \tag{7.11}$$

Here C is a constant independent of relevant quantities (we make this more precise below). In particular, the data $\widehat{\mathcal{D}}$ satisfies the structural assumption (7.11) which guarantees that the perturbed problem (5.5) has a unique solution. Note that the general case is more subtle than when the data are approximated by piecewise constant approximations (5.74), which are directly satisfying the structural assumption and used as motivation in Section 5.4.2.

We start by discussing a process modifying the approximation of a strictly positive scalar function $v \in L^\infty(\Omega)$, i.e. $v \in R(c_1, c_2)$ for some $0 < c_1 \le c_2$; see (5.49). Because the polynomial degree used to approximate the data might differ depending on the application, we use $m \in \mathbb{N}$ to denote a generic polynomial degree. We think of $\widetilde{v} \in \mathbb{S}_{\mathcal{T}}^{m,-1}$ as an approximation to v not necessarily strictly positive. The following process modifies $\widetilde{v}$ locally to construct $\widehat{v} \in \mathbb{S}_{\mathcal{T}}^{m,-1}$. It involves a parameter $L > 2$ responsible for the truncation of $\widetilde{v}$ whenever it is too large, i.e. $\widetilde{v} \ge Lc_2$. For $T \in \mathcal{T}$, we set $\widehat{v}|_T := \widehat{v}_T$, where

$$\widehat{v}_T := \begin{cases} c_2, & \text{when } \|\widetilde{v}\|_{L^\infty(T)} \ge Lc_2, \\ \widetilde{v}|_T - \min_{x\in T} \widetilde{v}(x) + \dfrac{c_1}{2}, & \text{when otherwise } \min_{x\in T} \widetilde{v}(x) < \dfrac{c_1}{2}, \\ \widetilde{v}|_T, & \text{otherwise.} \end{cases} \tag{7.12}$$

Corollary 7.9 below is in essence Proposition 3 of Bonito *et al.* (2013*b*), and states that the constructed $\widehat{v}$ satisfies

$$0 < \frac{c_1}{2} \le \widehat{v} \le \left(\frac{1}{2} + 2L\right)c_2 \quad \text{a.e. in } \Omega.$$

This is at the expense of inflating the approximation error in L^q, $1 \le q \le \infty$, by a multiplicative constant C depending only on d, m, c_2/c_1, q, L and the shape regularity of $\mathbb{T}$,

$$\|v - \widehat{v}\|_{L^q(T)} \le C\|v - \widetilde{v}\|_{L^q(T)}.$$

In preparation for this result, we introduce the following notation. We let C_I denote the smallest constant such that for any $T \in \mathcal{T}$ and any polynomial $P \in \mathbb{P}_m(T)$, we have the inverse inequality

$$\|\nabla P\|_{L^\infty(T)} \le C_I \|P\|_{L^\infty(T)} |T|^{-1/d}. \tag{7.13}$$

The inverse inequality constant C_I depends only on the shape regularity of $\mathbb{T}$, m and d. Note that for such a polynomial $P \in \mathbb{P}_m(T)$, we have

$$|P(x) - P(y)| \le C_I \|P\|_{L^\infty(T)} |T|^{-1/d} |x - y|, \quad \text{for all } x, y \in T.$$

Consequently, for any $\rho > 0$ and $x \in T$, we define

$$T(x,\rho) := T \cap \overline{B(x, \rho|T|^{1/d}/C_I)},$$

which is motivated by the fact that for $x \in T$ and $y \in T(x,\rho)$ we have

$$|P(x) - P(y)| \le C_I \|P\|_{L^\infty(T)} |T|^{-1/d} |x-y| \le \rho \|P\|_{L^\infty(T)}. \tag{7.14}$$

Critical for the analysis below is the existence of a constant $0 < C_S(\rho) \le 1$ depending on ρ but also on d, m and the shape regularity of $\mathbb{T}$, such that

$$|T(x,\rho)| \ge C_S(\rho)|T| \quad \text{for all } x \in T,\ T \in \mathcal{T}. \tag{7.15}$$

This constant $C_S(\rho)$ assesses the area of a subset of T where the polynomial P varies no more than $\rho\|P\|_{L^\infty(T)}$ away from $P(x)$.

We are now in a position to analyse the effect of the nonlinear correction (7.12). We proceed locally over each $T \in \mathcal{T}$ and start with the case where $\|\widetilde{v}\|_{L^\infty(T)}$ is large (Lemma 7.7). We then discuss the case where $\widetilde{v}(x)$ is small on T (Lemma 7.8), while for the remaining case the function $\widetilde{v}$ does not need to be modified on T. These three cases are collected in Corollary 7.9 for scalar-valued functions and in Corollary 7.11 for matrix-valued functions. In all the arguments below we used the convention $a^{1/\infty} = 1$ for any $a > 0$.

Lemma 7.7 (locally enforcing constraints for large approximations). *Let $\mathcal{T} \in \mathbb{T}$ be any conforming refinement of $\mathcal{T}_0$ satisfying Assumption 6.19 (initial labelling). Let $c_2 > 0$, $T \in \mathcal{T}$ and $v_T \in L^\infty(T)$ satisfying $0 < v_T \le c_2$ a.e. in T. Furthermore, for $m \ge 0$ and $L > 2$, assume that $\widetilde{v}_T \in \mathbb{P}_m(T)$ satisfies*

$$\|\widetilde{v}_T\|_{L^\infty(T)} \ge Lc_2. \tag{7.16}$$

Then, for the constant function $\widehat{v}_T := c_2 \in \mathbb{P}_m(T)$,

$$\frac{c_1}{2} < \widehat{v}_T < Lc_2 < \left(\frac{1}{2} + 2L\right)c_2.$$

Moreover, for $1 \le q \le \infty$, we have

$$\|v_T - \widehat{v}_T\|_{L^q(T)} \le C_2^+ \|v_T - \widetilde{v}_T\|_{L^q(T)},$$

where

$$C_2^+ := \frac{4C_S^{-1/q}}{L-2}$$

and $C_S = C_S(1/2)$ is the constant appearing in (7.15) with $\rho = 1/2$.

Proof. Let $x_0 \in T$ and $\widetilde{c}_{2,T}$ defined by the relation

$$\widetilde{c}_{2,T} := |\widetilde{v}_T(x_0)| := \|\widetilde{v}_T\|_{L^\infty(T)}.$$

In view of the Lipschitz property (7.14) applied to $P = \widetilde{v}_T$ and with $\rho = \frac{1}{2}$, we have

$$|\widetilde{v}_T(x) - \widetilde{v}_T(x_0)| \le \frac{\widetilde{c}_{2,T}}{2}$$

for $x \in T_0 := T(x_0, \frac{1}{2}) \subset T$. Recall (7.15), which implies that $|T_0| \geq \widetilde{C}_S|T|$ for some constant $\widetilde{C}_S := C_S(1/2)$ depending only on d, n and the shape regularity of $\mathbb{T}$. On the one hand, this implies that $\widetilde{v}_T|_{T_0}$ is bounded below with $|\widetilde{v}_T(x)| \geq \widetilde{c}_{2,T}/2$ for $x \in T_0$ and, on the other hand, v_T is bounded from above by

$$0 \leq v_T(x) \leq c_2 \leq L^{-1}\widetilde{c}_{2,T}, \quad x \in T.$$

Consequently, for $x \in T_0$ and since $L > 2$, we have

$$0 \leq v_T(x) \leq L^{-1}\widetilde{c}_{2,T} \leq \frac{\widetilde{c}_{2,T}}{2} \leq |\widetilde{v}_T(x)|$$

and thus

$$|v_T(x) - \widetilde{v}_T(x)| \geq |\widetilde{v}_T(x)| - v_T(x) \geq \left(\frac{1}{2} - \frac{1}{L}\right)\widetilde{c}_{2,T} = \frac{L-2}{2L}\widetilde{c}_{2,T},$$

which indicates that v_T and $\widetilde{v}_T$ are sufficiently far apart on a substantial portion T_0 of T. This is responsible for the L^q-bound below. In fact, we have

$$\|v_T - \widetilde{v}_T\|_{L^q(T)} \geq \|v_T - \widetilde{v}_T\|_{L^q(T_0)} \geq \frac{L-2}{2L}\widetilde{c}_{2,T}|T_0|^{1/q}, \tag{7.17}$$

whence, from the definition $\widehat{v}_T := c_2$ and using (7.15), we deduce

$$\|v_T - \widehat{v}_T\|_{L^q(T)} \leq 2c_2|T|^{1/q} \leq 2L^{-1}\widetilde{c}_{2,T}|T|^{1/q} \leq \frac{4C_S^{-1/q}}{L-2}\|v_T - \widetilde{v}_T\|_{L^q(T)}$$

as desired. □

Lemma 7.8 (locally enforcing constraints for small approximations). *Let $\mathcal{T} \in \mathbb{T}$ be any conforming refinement of $\mathcal{T}_0$ satisfying Assumption 6.19 (initial labelling). Let $0 < c_1 \leq c_2$, $T \in \mathcal{T}$ and $v_T \in L^\infty(T)$ satisfying $c_1 < v_T \leq c_2$ a.e. in T. Furthermore, for $m \geq 0$ and $L > 2$ assume that $\widetilde{v}_T \in \mathbb{P}_m(T)$ satisfies*

$$\|\widetilde{v}_T\|_{L^\infty(T)} \leq Lc_2 \tag{7.18}$$

and

$$\min_{x \in T} \widetilde{v}_T(x) < \frac{c_1}{2}. \tag{7.19}$$

Then the function $\widehat{v}_T := c_1/2 + \widetilde{v}_T - \min_{x\in T}\widetilde{v}_T(x) \in \mathbb{P}_n(T)$ is such that

$$\frac{c_1}{2} \leq \widehat{v}_T \leq 2Lc_2 + \frac{c_1}{2} \leq \left(2L + \frac{1}{2}\right)c_2$$

and

$$\|v_T - \widehat{v}_T\|_{L^q(T)} \leq C_1^+\|v_T - \widetilde{v}_T\|_{L^q(T)},$$

where

$$C_1^+ := \left(1 + C_S^{-1/q}(\rho)\right)$$

and $C_S(\rho)$ is the constant appearing in (7.15) with $\rho = c_1/(2Lc_2)$.

Proof. We define $x_0 \in T$, $\widetilde{c}_{1,T} \in \mathbb{R}$ by the relations

$$\widetilde{c}_{1,T} := \widetilde{v}_T(x_0) := \min_{x \in T} \widetilde{v}_T(x).$$

From the Lipschitz property (7.14) and the assumption (7.18), we find that

$$|\widetilde{v}_T(x) - \widetilde{v}_T(x_0)| \le \frac{c_1}{2}$$

for $x \in T_0 := T(x_0, \rho)$ with $\rho := c_1/(2Lc_2)$. Recall (7.15), which implies that $|T_0| \ge \widetilde{C}_S|T|$ for some constant $\widetilde{C}_S := C_S(\rho)$ depending only on d, m, c_2/c_1, L and the shape regularity of $\mathbb{T}$.

For $x \in T_0$, we proceed by estimating the difference

$$v_T(x) - \widetilde{v}_T(x) = v_T(x) - (\widetilde{v}_T(x) - \widetilde{v}_T(x_0)) - \widetilde{v}_T(x_0) \ge c_1 - \frac{c_1}{2} - \widetilde{c}_{1,T} = \frac{c_1}{2} - \widetilde{c}_{1,T} > 0,$$

because $\widetilde{c}_{1,T} < c_1/2$ by assumption (7.19). This implies that

$$|T_0|^{1/q}\left(\frac{c_1}{2} - \widetilde{c}_{1,T}\right) \le \|v_T - \widetilde{v}_T\|_{L^q(T)},$$

and v_T and $\widetilde{v}_T$ are uniformly far apart in the substantial part T_0 of T. Therefore $\widehat{v}_T := \widetilde{v}_T + (c_1/2 - \widetilde{c}_{1,T})$ satisfies

$$\frac{c_1}{2} \le \widehat{v}_T \le 2Lc_2 + \frac{c_1}{2}$$

because $\widetilde{c}_{1,T} \ge -\|\widetilde{v}\|_{L^\infty(T)} \ge -Lc_2$ by assumption (7.18), and

$$\begin{aligned}
\|v_T - \widehat{v}_T\|_{L^q(T)} &\le \|v_T - \widetilde{v}_T\|_{L^q(T)} + \left(\frac{c_1}{2} - \widetilde{c}_{1,T}\right)|T|^{1/q} \\
&\le \|v_T - \widetilde{v}_T\|_{L^q(T)} + \left(\frac{c_1}{2} - \widetilde{c}_{1,T}\right)\widetilde{C}_S^{-1/q}|T_0|^{1/q} \\
&\le \left(1 + \widetilde{C}_S^{-1/q}\right)\|v_T - \widetilde{v}_T\|_{L^q(T)}.
\end{aligned}$$

This proves the assertions. □

Corollary 7.9 (locally enforcing constraints). *Let $\mathcal{T} \in \mathbb{T}$ be any conforming refinement of $\mathcal{T}_0$ satisfying Assumption 6.19 (initial labelling). Let $0 < c_1 \le c_2$, $T \in \mathcal{T}$ and $v_T \in L^\infty(T)$ satisfying $c_1 \le v_T \le c_2$ a.e. in T. Then, for $m \ge 0$, $L > 2$, and $\widetilde{v}_T \in \mathbb{P}_m(T)$, the function $\widehat{v}_T \in \mathbb{P}_m(T)$ defined in (7.12) satisfies*

$$\frac{c_1}{2} \le \widehat{v}_T \le \left(\frac{1}{2} + 2L\right)c_2 \quad \text{a.e. in } T.$$

Moreover, for $1 \le q \le \infty$, we have

$$\|v_T - \widehat{v}_T\|_{L^q(T)} \le \max(C_1^+, C_2^+)\|v_T - \widetilde{v}_T\|_{L^q(T)} \quad \text{for all } T \in \mathcal{T},$$

where C_1^+ and C_2^+ are the constants appearing in Lemmas 7.8 and 7.7, which depend only on d, m, c_2/c_1, L and the shape regularity of $\mathbb{T}$.

Proof. The desired results follow from Lemma 7.7 when

$$\|\widetilde{v}_T\|_{L^\infty(T)} \geq Lc_2,$$

and from Lemma 7.8 when

$$\|\widetilde{v}_T\|_{L^\infty(T)} < Lc_2 \quad \text{and} \quad \min_{x\in T} \widetilde{v}_T < \frac{c_1}{2}.$$

In the remaining case,

$$\|\widetilde{v}_T\|_{L^\infty(T)} < Lc_2 \quad \text{and} \quad \min_{x\in T} \widetilde{v}_T(x) \geq \frac{c_1}{2},$$

since $\widehat{v}_T = \widetilde{v}_T$ satisfies the desired constraints, there is nothing to prove. □

7.2.2. Constrained approximation of the diffusion coefficients

For matrix-valued functions, the constraints are on the eigenvalues of the matrix rather than on the coefficients themselves. Although this requires a few adjustments, the process is similar to the scalar case. We recall that for $0 < \alpha_1 \leq \alpha_2$, $M(\alpha_1, \alpha_2) \subset L^\infty\left(\Omega; \mathbb{R}^{d\times d}_{\text{sym}}\right)$ denotes the class of symmetric matrix-valued functions whose eigenvalues lie between α_1 and α_2; see (5.48).

Algorithm CONSTRAINT-A is based on (7.12) and modifies approximations $\widetilde{\boldsymbol{A}} \in \left(\mathbb{S}^{n_A,-1}_{\mathcal{T}}\right)^{d\times d}$ of $\boldsymbol{A} \in M(\alpha_1, \alpha_2)$ to produce uniformly positive definite approximations $\widehat{\boldsymbol{A}} \in \left(\mathbb{S}^{n_A,-1}_{\mathcal{T}}\right)^{d\times d}$ of $\boldsymbol{A}$.

Algorithm 7.10 (CONSTRAINT-A). Given a threshold parameter $L > 2$, $0 < \alpha_1 \leq \alpha_2$, a conforming refinement $\mathcal{T} \in \mathbb{T}$ of $\mathcal{T}_0$, and $\widetilde{\boldsymbol{A}} \in \left(\mathbb{S}^{n_A,-1}_{\mathcal{T}}\right)^{d\times d}$, this routine constructs a positive definite $\widehat{\boldsymbol{A}} \in \left(\mathbb{S}^{n_A,-1}_{\mathcal{T}}\right)^{d\times d}$.

$[\widehat{\boldsymbol{A}}]$ = CONSTRAINT-A($\mathcal{T}, \alpha_1, \alpha_2, L, \widetilde{\boldsymbol{A}}$)
  For $T \in \mathcal{T}$
    $\widetilde{\alpha}_{1,T} = \inf\{y^t \widetilde{\boldsymbol{A}}(x) y,\ x \in T,\ |y| = 1\}$
    $\widetilde{\alpha}_{2,T} = \sup\{|y^t \widetilde{\boldsymbol{A}}(x) y|,\ x \in T,\ |y| = 1\}$
    if $\widetilde{\alpha}_{2,T} \geq L\alpha_2$
      $\widehat{\boldsymbol{A}}|_T = \alpha_2 \boldsymbol{I}_d$
    else if $\widetilde{\alpha}_{1,T} < \alpha_1/2$
      $\widehat{\boldsymbol{A}}|_T = \widetilde{\boldsymbol{A}}|_T - (\alpha_1/2 - \widetilde{\alpha}_{1,T})\boldsymbol{I}_d$
    else
      $\widehat{\boldsymbol{A}}|_T = \widetilde{\boldsymbol{A}}|_T$
  return $\widehat{\boldsymbol{A}}$

Notice that CONSTRAINT-A preserves symmetry, that is, if $\widetilde{\boldsymbol{A}}$ is symmetric then so is the output $\widehat{\boldsymbol{A}}$. In addition, when $n = 1$ and $\widetilde{\boldsymbol{A}} \in \left(\mathbb{S}^{0,-1}_{\mathcal{T}}\right)^{d\times d}$ is the piecewise constant local average of $\boldsymbol{A}$, the output $\widehat{\boldsymbol{A}}$ of CONSTRAINT-A is $\widehat{\boldsymbol{A}} = \widetilde{\boldsymbol{A}}$ since in

that case the parameters $\widetilde{\alpha}_{1,T}$ and $\widetilde{\alpha}_{2,T}$ satisfy

$$\widetilde{\alpha}_{1,T} \geq \alpha_1 > \frac{\alpha_1}{2} \quad \text{and} \quad \widetilde{\alpha}_{2,T} \leq \alpha_2 < L\alpha_2 \quad \text{for all } T \in \mathcal{T}.$$

This is consistent with the observation made in Section 5.4.2.

The next corollary hinges on Corollary 7.9 (locally enforcing constraints) to derive properties of CONSTRAINT-A. In passing, we recall that for $\boldsymbol{A} \in L^p(\Omega; \mathbb{R}^{d\times d})$ we write

$$\|\boldsymbol{A}\|_{L^p(\Omega)} := \||\boldsymbol{A}|\|_{L^p(\Omega)},$$

where for $x \in \Omega$, $|\boldsymbol{A}(x)|$ is the spectral norm of $\boldsymbol{A}(x)$.

Corollary 7.11 (locally enforcing constraints for matrices). *Let the threshold be $L > 2$, $0 < \alpha_1 \leq \alpha_2$ and $\boldsymbol{A} \in M(\alpha_1, \alpha_2)$. Let $\mathcal{T} \in \mathbb{T}$ be any conforming refinement of $\mathcal{T}_0$ and let $\widetilde{\boldsymbol{A}} \in \left(\mathbb{S}_{\mathcal{T}}^{n_A,-1}\right)^{d\times d}$ be a symmetric approximation of $\boldsymbol{A}$. Then the output $[\widehat{\boldsymbol{A}}] = \text{CONSTRAINT-A}(\mathcal{T}, \alpha_1, \alpha_2, L, \widetilde{\boldsymbol{A}})$ is symmetric and satisfies*

$$\frac{\alpha_1}{2} \leq \lambda_j(\widehat{\boldsymbol{A}}) \leq \left(\frac{1}{2} + 2L\right)\alpha_2 \quad \textit{a.e. in } \Omega,\ 1 \leq j \leq d.$$

Moreover, for $1 \leq q \leq \infty$, we have

$$\|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|_{L^q(T)} \leq C_{\text{data}}\|\boldsymbol{A} - \widetilde{\boldsymbol{A}}\|_{L^q(T)} \quad \textit{for all } T \in \mathcal{T},$$

where $C_{\text{data}} := \max(C_1^+, C_2^+)$ and C_1^+ and C_2^+ are the constants appearing in Lemmas 7.8 and 7.7, which depend only on d, n_A, α_2/α_1, L and the shape regularity of $\mathbb{T}$.

Proof. We observe that $\widetilde{\boldsymbol{A}}$ is not assumed to be positive semidefinite. We argue locally and fix $T \in \mathcal{T}$. Let $\widetilde{\alpha}_{2,T} > 0$ and $y_0 \in \mathbb{R}^d$ be such that $|y_0| = 1$ and

$$\widetilde{\alpha}_{2,T} := \sup_{x\in T} |y_0^t \widetilde{\boldsymbol{A}}(x) y_0| := \sup_{x\in T} \sup_{y\in\mathbb{R}^d, |y|=1} |y^t \widetilde{\boldsymbol{A}}(x) y|.$$

We first consider the case $\widetilde{\alpha}_{2,T} \geq L\alpha_2$ for which $\widehat{\boldsymbol{A}}|_T := \alpha_2 \boldsymbol{I}_d$. For $x \in T$, we set

$$a(x) := y_0^t \boldsymbol{A}(x) y_0 \quad \text{and} \quad \widetilde{a}_T(x) = y_0^t \widetilde{\boldsymbol{A}}(x) y_0 \in \mathbb{P}_{n_A}(T).$$

This notation allows us to reduce to the scalar case upon noting that

$$\|a - \widetilde{a}_T\|_{L^q(T)} \leq \|\boldsymbol{A} - \widetilde{\boldsymbol{A}}\|_{L^q(T)}$$

and $\alpha_1 \leq a \leq \alpha_2$ a.e. in Ω. Because $\widetilde{\alpha}_{2,T} \geq L\alpha_2$, Lemma 7.7 with $m = n_A$ guarantees that $\widehat{a}_T := \alpha_2$ satisfies

$$\|a - \widehat{a}_T\|_{L^q(T)} \leq C_2^+ \|a - \widetilde{a}_T\|_{L^q(T)} \leq C_2^+ \|\boldsymbol{A} - \widetilde{\boldsymbol{A}}\|_{L^q(T)}.$$

Consequently, the matrix-valued approximation $\widehat{\boldsymbol{A}}|_T := \alpha_2 \boldsymbol{I}_d$ satisfies

$$\|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|_{L^q(T)} = \|a - \widehat{a}_T\|_{L^q(T)} \leq C_2^+ \|\boldsymbol{A} - \widetilde{\boldsymbol{A}}\|_{L^q(T)}.$$

This proves the desired result when $\widetilde{\alpha}_{2,T} \geq L\alpha_2$.

We now consider the case where $\widetilde{\alpha}_{2,T} < L\alpha_2$ and define $\widetilde{\alpha}_{1,T} \in \mathbb{R}$, $y_1 \in \mathbb{R}^d$ with $|y_1| = 1$ by the relations

$$\widetilde{\alpha}_{1,T} = \inf_{x\in T} y_1^t \widetilde{\boldsymbol{A}}(x) y_1 = \inf_{x\in T} \inf_{|y|=1} y^t \widetilde{\boldsymbol{A}}(x) y.$$

We also redefine the associated scalar functions for $x \in T$ using y_1 instead of y_0:

$$a(x) := y_1^t \boldsymbol{A}(x) y_1 \quad \text{and} \quad \widetilde{a}_T(x) = y_1^t \widetilde{\boldsymbol{A}}(x) y_1 \in \mathbb{P}_{n_A}(T).$$

If $\widetilde{\alpha}_{1,T} < \alpha_1/2$ then $\widehat{\boldsymbol{A}}|_T = \widetilde{\boldsymbol{A}}|_T + (\alpha_1/2 - \widetilde{\alpha}_{1,T})\boldsymbol{I}_d$. Lemma 7.8 with $m = n_A$ ensures that $\widehat{a}_T = \widetilde{a}_T + \alpha_1/2 - \widetilde{\alpha}_{1,T}$ satisfies

$$\frac{\alpha_1}{2} \leq \widehat{a}_T \leq \left(\frac{1}{2} + 2M\right)\alpha_2$$

and

$$\|a - \widehat{a}_T\|_{L^q(T)} \leq C_1^+ \|a - \widetilde{a}_T\|_{L^q(T)} \leq C_1^+ \|\boldsymbol{A} - \widetilde{\boldsymbol{A}}\|_{L^q(T)}.$$

Thus $\widehat{\boldsymbol{A}}|_T$ satisfies the desired properties provided $\widetilde{\alpha}_{2,T} \geq L\alpha_2$ as well.

It the remaining case $\widetilde{\alpha}_{2,T} < L\alpha_2$ and $\widetilde{\alpha}_{1,T} \geq \alpha_1/2$, the function $\widehat{\boldsymbol{A}}|_T = \widetilde{\boldsymbol{A}}|_T$ satisfies the desired properties and there is nothing to prove. □

As a corollary, we report the complexity of an algorithm that concatenates the linear approximation of GREEDY with the nonlinear correction into the constraint of CONSTRAINT-A. We recall from Corollary 6.36 (approximation class of $\boldsymbol{A}$) that the admissible set of parameters of $\boldsymbol{A}$ for GREEDY are $n_A \leq n - 1$:

$$s_A \in (0, n_A], \quad p_A \in (0, \infty], \quad q_A \in [2, \infty], \quad s_A - \frac{d}{p_A} + \frac{d}{q_A} > 0, \quad t_A = 0.$$

Corollary 7.12 (complexity of constrained GREEDY for A). *Let the initial mesh $\mathcal{T}_0$ of $\Omega \subset \mathbb{R}^d$ satisfy Assumption 6.19 (initial labelling) and let $\mathcal{T} \in \mathbb{T}$ be any admissible refinement of $\mathcal{T}_0$. Let $\tau > 0$ be the target tolerance, let $b \geq 1$ be the number of bisections performed on each marked element, and let $L > 2$ be a threshold parameter. Furthermore, assume that $(\boldsymbol{A}, s_A, t_A, p_A, q_A)$ satisfies Assumption 7.2 (admissible set of parameters for* GREEDY*) with local oscillations $\{\|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|_{L^{q_A}(T)}\}_{T\in\mathcal{T}}$ and, in addition, $\boldsymbol{A} \in M(\alpha_1, \alpha_2)$ for some $0 < \alpha_1 \leq \alpha_2$. The algorithm*

$[\widehat{\mathcal{T}}, \widetilde{\boldsymbol{A}}] = \mathsf{GREEDY}(\mathcal{T}, \tau, q_A, b, \boldsymbol{A})$
$[\widehat{\boldsymbol{A}}] = \mathsf{CONSTRAINT\text{-}A}(\widehat{\mathcal{T}}, \alpha_1, \alpha_2, L, \widetilde{\boldsymbol{A}})$

where GREEDY *is applied to the $d(d+1)/2$ distinct components of $\boldsymbol{A}$, marks N elements of $\mathcal{T}$ for refinement with*

$$N \leq C|\boldsymbol{A}|_{X_{p_A}^{s_A}(\Omega;\mathcal{T}_0)}^{d/s_A} \tau^{-d/s_A} \tag{7.20}$$

and $C = C(p_A, q_A, s_A, b, d, n_A, \alpha_2/\alpha_1, L, \Omega, \mathcal{T}_0)$. *Moreover,* $\widehat{\boldsymbol{A}} \in \left(\mathbb{S}_{\mathcal{T}}^{n_A,-1}\right)^{d\times d}$ *satisfies*

$$\widehat{\boldsymbol{A}} \in M(\widehat{\alpha}_1, \widehat{\alpha}_2): \quad \widehat{\alpha}_1 = \frac{\alpha_1}{2}, \quad \widehat{\alpha}_2 = (1+4L)\frac{\alpha_2}{2}, \tag{7.21}$$

and there is a constant $C_{\text{data}} > 0$ *such that*

$$\|\boldsymbol{A} - \widehat{\boldsymbol{A}}\|_{L^q(\Omega)} \le C_{\text{data}}\tau.$$

Proof. This result follows upon invoking Corollary 7.6 (performance of GREEDY) and Corollary 7.11 (locally enforcing constraints for matrices). □

Remark 7.13 (constrained approximation class of matrices). As a consequence of Corollary 7.12, we realize that for $A \in M(\alpha_1, \alpha_2)$,

$$\delta_{\mathcal{T}}(\boldsymbol{A})_r \le \widetilde{\delta}_{\mathcal{T}}(\boldsymbol{A})_r \le C_{\text{data}}\delta_{\mathcal{T}}(\boldsymbol{A})_r,$$

where the best approximation error $\delta_{\mathcal{T}}(\boldsymbol{A})_r$ and best constrained approximation error $\widetilde{\delta}_{\mathcal{T}}(\boldsymbol{A})_r$ are defined in (6.9) and (6.10).

7.2.3. Constrained approximation of the reaction coefficients

If the reaction coefficient $c \in R(c_1, c_2)$ is strictly positive ($c_1 > 0$), then Corollary 7.9 (locally enforcing constraints) with $m = n_c$ directly applies to $v_T = c|_T$, $T \in \mathcal{T}$, and guarantees that the approximate coefficient $\widehat{c} \in \mathbb{S}_{\mathcal{T}}^{n_c,-1}$ defined on $T \in \mathcal{T}$ by $\widehat{c}|_T := \widehat{v}_T$ satisfies

$$\widehat{c} \in R(\widehat{c}_1, \widehat{c}_2): \quad \widehat{c}_1 = \frac{c_1}{2}, \quad \widehat{c}_2 = (1+4L)\frac{c_2}{2}.$$

However, reaction coefficients are not necessarily strictly positive on $\overline{\Omega}$, and Corollary 7.9 cannot be invoked directly. Instead, we take advantage of the fact that the perturbed problem (5.5) is still well-posed provided $\widehat{c} \ge -\widehat{\alpha}_1/(2C_P^2)$ and the approximate diffusion coefficient $\widehat{\boldsymbol{A}} \in M(\widehat{\alpha}_1, \widehat{\alpha}_2)$ of $\boldsymbol{A} \in M(\alpha_1, \alpha_2)$ satisfies $\widehat{\alpha}_1 \ge \alpha_1/2$ according to (5.52); hence $\widehat{c} \ge -\alpha_1/(4C_P^2)$. Therefore we apply Corollary 7.9 to the shifted reaction coefficient $v = c + \widehat{\alpha}_1/C_P^2$, which satisfies

$$v_1 := c_1 + \frac{\widehat{\alpha}_1}{C_P^2} \le v \le c_2 + \frac{\widehat{\alpha}_1}{C_P^2} =: v_2. \tag{7.22}$$

Below is the proposed algorithm for the construction of $\widehat{c}$ in the general case $c_1 \ge 0$.

Algorithm 7.14 (CONSTRAINT-c). Given $L > 2$, $\widehat{\alpha}_1 > 0$, a conforming refinement $\mathcal{T} \in \mathbb{T}$ of $\mathcal{T}_0$, and $\widetilde{c} \in \mathbb{S}_{\mathcal{T}}^{n_c,-1}$, this routine constructs $\widehat{c} \in \mathbb{S}_{\mathcal{T}}^{n_c,-1}$ as follows:

$[\widehat{c}]$ = CONSTRAINT-c($\mathcal{T}, \widehat{\alpha}_1, L, \widetilde{c}$)
  $\widetilde{v} = \widetilde{c} + \widehat{\alpha}_1/C_P^2$
  For $T \in \mathcal{T}$
    if $\|\widetilde{v}\|_{L^\infty(T)} \ge Lv_2$
      $\widehat{v}|_T = v_2$

```
    else if min_{x∈T} ṽ(x) < v_1/2
        v̂|_T = ṽ|_T − min_{x∈T} ṽ(x) + v_1/2
    else
        v̂|_T = ṽ|_T
ĉ = v̂ − α̂_1/C_P^2
return ĉ
```

We note that if $n_c = 0$, then $\widetilde{c}$ is the piecewise average of c and CONSTRAINT-c does not modify $\widetilde{c}$, which already satisfies the structural assumption (7.11).

The next result shows that the output $\widehat{c}$ of CONSTRAINT-c is a modification of $\widetilde{c}$ which satisfies $\widehat{c} \in R(\widehat{c}_1, \widehat{c}_2)$, with

$$\widehat{c}_1 := \frac{c_1}{2} - \frac{\widehat{\alpha}_1}{2C_P^2} \quad \text{and} \quad \widehat{c}_2 := (1+4L)\frac{c_2}{2} + (4L-1)\frac{\widehat{\alpha}_1}{2C_P^2} \tag{7.23}$$

without affecting the approximation of c in L^q, $1 \le q \le \infty$ (up to a multiplicative constant). In particular, $\widehat{c}_1 \ge -\widehat{\alpha}_1/(2C_P^2)$, which is necessary for the well-posedness of the perturbed problem (5.5) when $\widehat{A} \in M(\widehat{\alpha}_1, \widehat{\alpha}_2)$.

Corollary 7.15 (locally enforcing constraints for non-negative scalar functions). *Let $\boldsymbol{A} \in M(\widehat{\alpha}_1, \widehat{\alpha}_2)$ with $0 < \widehat{\alpha}_1 \le \widehat{\alpha}_2$, and $c \in R(c_1, c_2)$ with $0 \le c_1 \le c_2$. Let $L > 2$ and $v_1 \le v_2$ be defined in (7.22). Let $\mathcal{T} \in \mathbb{T}$ be any conforming refinement of $\mathcal{T}_0$ and $\widetilde{c} \in \mathbb{S}_{\mathcal{T}}^{n_c,-1}$. Then the output $[\widehat{c}] = \text{CONSTRAINT-c}(\mathcal{T}, \widehat{\alpha}_1, L, \widetilde{c})$ satisfies*

$$\widehat{c}_1 \le \widehat{c} \le \widehat{c}_2 \quad \textit{a.e. in } \Omega,$$

where $\widehat{c}_1$ and $\widehat{c}_2$ are given by (7.23). Moreover, for $0 < q \le \infty$, we have

$$\|c - \widehat{c}\|_{L^q(T)} \le C_{\text{data}} \|c - \widetilde{c}\|_{L^q(T)} \quad \textit{for all } T \in \mathcal{T},$$

where C_{data} is a constant depending only on d, n, v_2/v_1, Ω, L and the shape regularity of $\mathbb{T}$.

Proof. Set $\kappa := \widehat{\alpha}_1/C_P^2$ and $v := c + \kappa \in R(c_1 + \kappa, c_2 + \kappa)$ so that $c_1 + \kappa > 0$. On each $T \in \mathcal{T}$, we invoke Corollary 7.9 (locally enforcing constraints) with $m = n_c$, $\widetilde{v}_T = \widetilde{c}|_T + \kappa$ and where c_1, c_2 are replaced by $c_1 + \kappa$, $c_2 + \kappa$ respectively. Hence we deduce that the function $\widehat{v}$ constructed within CONSTRAINT-c satisfies

$$\frac{c_1 + \kappa}{2} \le \widehat{v} \le (1+4L)\frac{c_2 + \kappa}{2}.$$

and

$$\|v - \widehat{v}\|_{L^q(T)} \le C_{\text{data}} \|v - \widetilde{v}\|_{L^q(T)} \quad \text{for all } T \in \mathcal{T}, \tag{7.24}$$

with a constant C_{data} depending on $d, n, v_2/v_1, L$, and the shape regularity of $\mathbb{T}$. Shifting back, $c = v - \kappa$ and $\widehat{c} := \widehat{v} - \kappa$, we find that the approximation $\widehat{c}$ constructed by CONSTRAINT-c satisfies

$$\frac{c_1 + \kappa}{2} - \kappa \le \widehat{c} \le (1+4L)\frac{c_2 + \kappa}{2} - \kappa$$

or equivalently

$$\frac{c_1}{2} - \frac{k}{2} \le \widehat{c} \le (1+4L)\frac{c_2}{2} + (4L-1)\frac{\kappa}{2}.$$

In view of (7.23) and $\kappa = \widehat{\alpha}_1 / C_P^2$, this is the first desired inequality in disguise.

Furthermore, the second desired inequality follows from (7.24) because for $T \in \mathcal{T}$ we have $c - \widehat{c} = v - \widehat{v}$ and $c - \widetilde{c} = v - \widetilde{v}$. □

The next corollary combines the linear approximation of GREEDY together with the nonlinear correction into the constraint of CONSTRAINT-c. We recall from Corollary 6.37 (approximation class of c) that the admissible set of parameters of c for GREEDY are $n_c \le n-1$, $s_c \in (0, n_c]$, $p_c \in (0, \infty]$, where

$$\begin{aligned} n_c > 0 \quad &\Rightarrow \quad q_c > \frac{d}{2}, \quad s_c - \frac{d}{p_c} + \frac{d}{q_c} > 0, \quad t_c = 0, \\ n_c = 0 \quad &\Rightarrow \quad q_c = 2, \quad s_c - \frac{d}{p_c} + \frac{d}{2} > 0, \quad 0 < t_c < 2 - \frac{d}{2}. \end{aligned}$$

Corollary 7.16 (complexity of constrained GREEDY for c). *Let the initial subdivision $\mathcal{T}_0$ of $\Omega \subset \mathbb{R}^d$ satisfy Assumption 6.19 (initial labelling) and $\mathcal{T} \in \mathbb{T}$ be any admissible refinement of $\mathcal{T}_0$. Let $\tau > 0$ be the target tolerance, $b \ge 1$ be the number of bisections performed on each marked element, let $L > 2$ be the threshold parameter, and $\widehat{\alpha}_1 > 0$. Furthermore, assume that (c, s_c, t_c, p_c, q_c) satisfies Assumption 7.2 (admissible set of parameters for* GREEDY*) with local oscillations $\{\|c - \widehat{c}\|_{L^{q_c}(T)}\}_{T \in \mathcal{T}}$ and that $c \in R(c_1, c_2)$ for some $0 \le c_1 \le c_2$. The algorithm*

$[\widehat{\mathcal{T}}, \widetilde{c}] = \mathsf{GREEDY}(\mathcal{T}, \tau, q_c, b, c)$
$[\widehat{c}] = \mathsf{CONSTRAINT\text{-}c}(\widehat{\mathcal{T}}, \widehat{\alpha}_1, L, \widetilde{c})$

marks N elements of $\mathcal{T}$ for refinement with

$$N \le C |c|_{X^{s_c}_{p_c}(\Omega;\mathcal{T}_0)}^{d/(s_c+t_c)} \tau^{-d/(s_c+t_c)} \tag{7.25}$$

and a constant $C = C(p_c, q_c, s_c, b, d, n_c, v_2/v_1, L, \Omega, \mathcal{T}_0)$ with $v_1 \le v_2$ defined in (7.22) to construct $\widehat{\mathcal{T}}$. The function $\widehat{c} \in \mathbb{S}^{n_c,-1}_{\widehat{\mathcal{T}}}$ is a piecewise polynomial of degree $\le n_c$ over $\widehat{\mathcal{T}}$ and satisfies

$$\widehat{c} \in R(\widehat{c}_1, \widehat{c}_2),$$

where $\widehat{c}_1 \le \widehat{c}_2$ are given by (7.23). Moreover, for $1 < q_c \le \infty$, there is a constant C_{data} depending only on d, n_c, v_2/v_1, Ω, L and the shape regularity of $\mathbb{T}$ such that

$$\|c - \widehat{c}\|_{L^{q_c}(\Omega)} \le C_{\text{data}} \tau.$$

Proof. Simply apply Corollary 7.6 (performance of GREEDY) and Corollary 7.15 (locally enforcing constraints for non-negative scalar functions). □

Remark 7.17 (constrained approximation class of scalars). Corollary 7.16 implies that for $c \in R(c_1, c_2)$

$$\delta_{\mathcal{T}}(c)_q \le \widetilde{\delta}_{\mathcal{T}}(c)_q \le C_{\text{data}} \delta_{\mathcal{T}}(c)_q,$$

where the best approximation error $\delta_{\mathcal{T}}(c)_q$ and best constrained approximation error $\widetilde{\delta}_{\mathcal{T}}(c)_q$ are defined in (6.9) and (6.10).

7.3. *Approximation of the load term f*

We now turn our attention to the question of designing a practical algorithm for reducing the global oscillation

$$E_{\mathcal{T}}(f)_{-1}^2 := \sum_{T \in \mathcal{T}} \|f - P_{\mathcal{T}} f\|_{H^{-1}(\omega_T)}^2 \approx \sum_{z \in \mathcal{V}} \|f - P_{\mathcal{T}} f\|_{H^{-1}(\omega_z)}^2, \tag{7.26}$$

where the projection $P_{\mathcal{T}}$ is defined in (4.34). The approximation of functionals in $H^{-1}(\Omega)$ is rather intricate and out of reach without assuming additional structure enabling practical evaluation of their actions on polynomial functions.

We examine three cases of independent interest. In Section 7.3.1 we consider $f \in L^q(\Omega)$ for q satisfying $2d/(d+2) < q \le \infty$, which includes the most common setting $f \in L^2(\Omega)$. Sections 7.3.2 and 7.3.3 present examples of right-hand sides not in L^1. In Section 7.3.2 we treat the case $f = g\delta_\Gamma$, where Γ is a hypersurface not necessarily captured by the faces of the subdivisions and $g \in L^q(\Gamma)$, $q \ge 2$, while in Section 7.3.3 we consider $f = \operatorname{div} \boldsymbol{g}$ for some $\boldsymbol{g} \in L^2(\Omega; \mathbb{R}^d)$. In all cases, the total error $E_{\mathcal{T}}(f)_{-1}$ is estimated by a surrogate $\widetilde{E}_{\mathcal{T}}(f)_{-1}$, namely $E_{\mathcal{T}}(f)_{-1} \le C_{\text{data}} \widetilde{E}_{\mathcal{T}}(f)_{-1}$,

$$\widetilde{E}_{\mathcal{T}}(f)_{-1}^2 := \sum_{T \in \mathcal{T}} \widetilde{\operatorname{osc}}_{\mathcal{T}}(f, T)_q^2,$$

with a definition of $\widetilde{\operatorname{osc}}_{\mathcal{T}}(f, T)_q$ depending on the situation but local to $T \in \mathcal{T}$ (and not on stars). This allows Algorithm 7.1 (GREEDY) to reduce $\widetilde{E}_{\mathcal{T}}(f)_{-1}$.

Before starting, we recall relevant definitions and results from Section 4 (*a posteriori* error analysis). For $z \in \mathcal{V}$, we let $\mathcal{T}_z \subset \mathcal{T}$ denote all the elements in ω_z and $\mathcal{F}_z \subset \mathcal{F}$ all the faces in ω_z. For $\ell \in H^{-1}(\Omega)$, the restriction $P_{\mathcal{T}} \ell|_{\omega_z}$ belongs to the space $\mathbb{F}(\mathcal{T}_z) = \mathbb{F}_{m_1, m_2}(\mathcal{T}_z)$ of functionals whose action against $w \in H_0^1(\omega_z)$ reads

$$\langle \ell, w \rangle = \sum_{T \in \mathcal{T}_z} \int_T q_T \, w + \sum_{F \in \mathcal{F}_z} \int_F q_F \, w \tag{7.27}$$

for some $q_F \in P_{m_1}(F)$, $F \in \mathcal{F}_z$ and $q_T \in P_{m_2}(T)$, $T \in \mathcal{T}_z$. The polynomial degrees are chosen to be $m_1 = n - 1$ and $m_2 = n - 2$ but can be general in this discussion.

Corollary 4.31 (local near-best approximation) guarantees that $P_{\mathcal{T}} \ell|_{\omega_z}$ is the quasi-best discrete functional in $\mathbb{F}(\mathcal{T}_z)$, namely

$$\|\ell - P_{\mathcal{T}} \ell\|_{H^{-1}(\omega_z)} \le C_P \inf_{\chi \in \mathbb{F}(\mathcal{T}_z)} \|\ell - \chi\|_{H^{-1}(\omega_z)}. \tag{7.28}$$

This will be used repeatedly to replace $P_{\mathcal{T}}\ell$ with more tractable quantities and justify the use of GREEDY algorithms to reduce (7.26).

7.3.1. The case $f \in L^q(\Omega)$

In this section we show how to reduce the oscillation error (7.26) when $f \in L^q(\Omega)$, with $q > 2d/(d+2)$ to guarantee that $L^q(\Omega)$ compactly embeds in $H^{-1}(\Omega)$. Note that this not only includes the most treated case in the literature $f \in L^2(\Omega)$ but also the more intricate cases $q < 2$ originally analysed by Cohen *et al.* (2012).

If $\Pi_{\mathcal{T}} f$ is the L^2-projection of f into the space $\mathbb{S}_{\mathcal{T}}^{n_f,-1}$ of discontinuous piecewise polynomials of degree n_f, let $\widehat{f} \in \mathbb{S}_{\mathcal{T}}^{n_f,-1}$ be defined by (5.70); $n_f = n-1$ in some applications but not always. Since $\widehat{f}|_{\omega_z} \in \mathbb{F}(\mathcal{T}_z)$ by taking $q_F = 0$ and $q_T = \widehat{f}|_T$ in (7.27), the local near-best approximation property (7.28) of $P_{\mathcal{T}}$ implies

$$\|f - P_{\mathcal{T}} f\|_{H^{-1}(\omega_z)} \le C_P \|f - \widehat{f}\|_{H^{-1}(\omega_z)}.$$

Furthermore, for $v \in H_0^1(\omega_z)$ we have

$$\langle f - \widehat{f}, v\rangle \le \|f - \widehat{f}\|_{L^q(\omega_z)} \|v\|_{L^{\widetilde{q}}(\omega_z)}$$

where $1/q + 1/\widetilde{q} = 1$. Note that the restriction $q > 2d/(d+2)$ guarantees that $1 \le \widetilde{q} < 2d/(d-2)$ and thus $\mathrm{sob}(H^1) > \mathrm{sob}(L^{\widetilde{q}})$. Therefore Lemma 2.2 (first Poincaré inequality) yields

$$\|f - \widehat{f}\|_{H^{-1}(\omega_z)} \lesssim \mathrm{diam}(\omega_z)^{1+d(1/2-1/q)} \|f - \widehat{f}\|_{L^q(\omega_z)}.$$

Returning to (7.26), after rearranging the terms element-wise and invoking the shape regularity of $\mathbb{T}$, we obtain $E_{\mathcal{T}}(f)_{-1} \le C_{\mathrm{data}} \widetilde{E}_{\mathcal{T}}(f)_{-1}$, where

$$\widetilde{E}_{\mathcal{T}}(f)_{-1}^2 := \sum_{T \in \mathcal{T}} \widetilde{\mathrm{osc}}_{\mathcal{T}}(f,T)_q^2, \tag{7.29}$$

and $\widetilde{\mathrm{osc}}_{\mathcal{T}}(f,T)_q := h_T^t \|f - \widehat{f}\|_{L^q(T)}$ with $t := 1 + d(1/2 - 1/q) > 0$.

In view of the definition (5.70), the local oscillations $\widetilde{\mathrm{osc}}(f,T)_q$ satisfy Assumption 7.3 (monotonicity of local oscillations) in ℓ^2 and we can now employ Algorithm 7.1 (GREEDY) with local errors $\widetilde{\mathrm{osc}}_{\mathcal{T}}(f,T)_q$ accumulating in ℓ^2. Recall that we use the convention $X_q^0(\Omega;\mathcal{T}_0) = L^q(\Omega)$.

Corollary 7.18 (approximation class of $f \in L^q(\Omega)$). *Let the initial subdivision $\mathcal{T}_0$ of $\Omega \subset \mathbb{R}^d$ satisfy Assumption 6.19 (initial labelling) and let $\mathcal{T} \in \mathbb{T}$ be any admissible refinement of $\mathcal{T}_0$. Let $\tau > 0$ be the target tolerance and let $b \ge 1$ be the number of bisections performed on each marked element. Let $2d/(d+2) < q \le \infty$ and set $t = 1 + d(1/2 - 1/q)$. Let $(f,s,t,p,2)$ satisfy Assumption 7.2 (admissible set of parameters for* GREEDY*) with local oscillations $\{\widetilde{\mathrm{osc}}(f,T)_q\}_{T\in\mathcal{T}}$. Then $[\widehat{\mathcal{T}}, \widehat{f}] =$ GREEDY$(\mathcal{T},\tau,2,b,f)$ terminates in a finite number of steps with $\widetilde{E}_{\widehat{\mathcal{T}}}(f)_{-1} \le \tau$, whence*

$$E_{\widehat{\mathcal{T}}}(f)_{-1} \le C_{\mathrm{data}}\tau.$$

Moreover, the number N of marked elements by GREEDY *satisfies*

$$N \lesssim |f|_{X_p^s(\Omega;\mathcal{T}_0)}^{d/(s+t)} \tau^{-d/(s+t)}. \tag{7.30}$$

In particular, $f \in \mathbb{F}_{d/(s+t)}$ with $|f|_{\mathbb{F}_{d/(s+t)}} \lesssim \|f\|_{X_p^s(\Omega;\mathcal{T}_0)}$.

Proof. Directly apply Corollary 7.6 (performance of GREEDY). □

7.3.2. The case $f = g\delta_{\mathcal{C}}$

We now consider the case where the right-hand side data f is a density supported on a Lipschitz hyper-surface $\mathcal{C} \subset \Omega$ in $\mathbb{R}^d$ with $(d-1)$-measure $|\mathcal{C}| < \infty$.

The intricate interactions between bulk and interface contributions on $P_{\mathcal{T}}$ makes it difficult to analyse when $f = g\delta_{\mathcal{C}}$ with density $g \in L^q(\mathcal{C})$. We take a simpler approach, likely suboptimal when $n > 1$ and $d > 2$, which discards $P_{\mathcal{T}}$ in view of the near-best approximation property (7.28):

$$\|f - P_{\mathcal{T}} f\|_{H^{-1}(\omega_z)} \lesssim \|f\|_{H^{-1}(\omega_z)}. \tag{7.31}$$

The right-hand side of the above estimate is the starting point of the analysis by Cohen *et al.* (2012) assuming $n = 1$ and $d = 2$.

We start with the derivation of a first upper bound for the local error $\|f\|_{H^{-1}(\omega_z)}$.

Lemma 7.19 (local oscillation). *Let $\mathcal{T} \in \mathbb{T}$, $z \in \mathcal{N}$, and $q > 2(d-1)/d$. If $g \in L^q(\mathcal{C})$ and $t := d/2 - (d-1)/q > 0$, then*

$$\|f\|_{H^{-1}(\omega_z)} \lesssim |\omega_z \cap \mathcal{C}|^{t/(d-1)} \|g\|_{L^q(\omega_z \cap \mathcal{C})} \lesssim \sum_{T \subset \omega_z} h_T^t \|g\|_{L^q(T \cap \mathcal{C})}. \tag{7.32}$$

Proof. For $v \in H_0^1(\omega_z)$ and $1/q + 1/\widetilde{q} = 1$, we have

$$\langle f, v \rangle = \int_{\omega_z \cap \mathcal{C}} gv \le \|g\|_{L^q(\omega_z \cap \mathcal{C})} \|v\|_{L^{\widetilde{q}}(\omega_z \cap \mathcal{C})}. \tag{7.33}$$

We realize that $H^{1/2}(\omega_z \cap \mathcal{C})$ compactly embeds in $L^{\widetilde{q}}(\omega_z \cap \mathcal{C})$ because

$$\begin{aligned} t &:= \mathrm{sob}(H^{1/2}(\omega_z \cap \mathcal{C})) - \mathrm{sob}(L^{\widetilde{q}}(\omega_z \cap \mathcal{C})) \\ &= \frac{1}{2} - (d-1)\left(\frac{1}{2} - \frac{1}{\widetilde{q}}\right) \\ &= \frac{d}{2} - \frac{1}{q}(d-1) > 0, \end{aligned}$$

provided $q > 2(d-1)/d$. Consequently, we find that

$$\|v\|_{L^{\widetilde{q}}(\omega_z \cap \mathcal{C})} \lesssim |\omega_z \cap \mathcal{C}|^{t/(d-1)} \|v\|_{H^{1/2}(\omega_z \cap \mathcal{C})}.$$

It remains to invoke the continuity (2.4) of the trace operator to write

$$\|v\|_{L^{\widetilde{q}}(\omega_z \cap \mathcal{C})} \lesssim |\omega_z \cap \mathcal{C}|^{t/(d-1)} \|v\|_{H^1(\omega_z)},$$

which, together with (7.33), yields the first estimate in (7.32). To deduce the second

estimate, it suffices to note that $|\omega_z \cap \mathcal{C}| \lesssim \operatorname{diam}(\omega_z)^{d-1} \lesssim h_T^{d-1}$ for $T \subset \omega_z$ and that $\|g\|_{L^q(\omega_z \cap \mathcal{C})} \leq \sum_{T \subset \omega_z} \|g\|_{L^q(T \cap \mathcal{C})}$. □

Estimate (7.31) and Lemma 7.19 provide a surrogate for data oscillation

$$\widetilde{E}_{\mathcal{T}}(f)_{-1}^2 := \sum_{T \in \mathcal{T}} \widetilde{\operatorname{osc}}_{\mathcal{T}}(g,T)_q^2, \quad \widetilde{\operatorname{osc}}_{\mathcal{T}}(g,T)_q := h_T^t \|g\|_{L^q(T \cap \mathcal{C})}, \tag{7.34}$$

where $t = d/2 - (d-1)/q$. The quantity $\widetilde{\operatorname{osc}}_{\mathcal{T}}(g,T)_q$ verifies Assumption 7.3 (monotonicity of local oscillations) with Ω replaced by $\mathcal{C}$ because of its element-wise structure. Therefore Proposition 7.4 (performance of GREEDY) states that Algorithm 7.1 (GREEDY) can reduce $\widetilde{E}_{\mathcal{T}}(f)_{-1}$. This is in contrast to the star-wise GREEDY algorithm analysed in Cohen *et al.* (2012), which requires that all marked stars are refined d times to ensure all the faces in the marked stars are refined.

We now discuss the performance of GREEDY with local indicators $\widetilde{\operatorname{osc}}_{\mathcal{T}}(g,T)_q$.

Lemma 7.20 (approximation class of $f = g\delta_{\mathcal{C}}$). *Let $\mathcal{C} \subset \Omega$ be a Lipschitz hypersurface. Let the initial subdivision $\mathcal{T}_0$ of $\Omega \subset \mathbb{R}^d$ satisfy Assumption 6.19 (initial labelling) and let $\mathcal{T} \in \mathbb{T}$ be any admissible refinement of $\mathcal{T}_0$. Let $\tau > 0$ be the target tolerance and let $b \geq 1$ be the number of bisections performed on each marked element, and $2(d-1)/d < q \leq \infty$. Then $[\widehat{\mathcal{T}}, \widehat{f}] = \mathsf{GREEDY}(\mathcal{T}, \tau, 2, b, f)$ terminates in a finite number of steps with surrogate estimator $\widetilde{E}_{\widehat{\mathcal{T}}}(f)_{-1} \leq \tau$ defined in (7.34), whence*

$$E_{\widehat{\mathcal{T}}}(f)_{-1} \leq C_{\text{data}}\tau.$$

Moreover, the number N of marked elements by GREEDY *satisfies*

$$N \lesssim \|g\|_{L^q(\mathcal{C})}^{2(d-1)} \tau^{-2(d-1)}. \tag{7.35}$$

In particular, $f = g\delta_{\mathcal{C}} \in \mathbb{F}_{1/(2(d-1))}$ with $|f|_{\mathbb{F}_{1/(2(d-1))}} \lesssim \|g\|_{L^q(\mathcal{C})}$.

Proof. This proof mainly follows the proof of Proposition 7.4 (performance of GREEDY) but requires a few modifications to account for the geometry of the problem. Since in turn the proof of Proposition 7.4 describes modifications to the proof of Proposition 3.19 (abstract greedy error), we now provide a complete proof. We proceed in several steps. We first consider the call $\mathsf{GREEDY}(\mathcal{T}_0, \tau, 2, 1, f)$ from $\mathcal{T}_0$ with one bisection $b = 1$ and accumulation in ℓ^2, and discuss the general call from $\mathcal{T}$ with $b \geq 1$ in the last step of this proof.

$\boxed{1}$ *Termination.* Since h_T decreases monotonically to 0 with bisection, so does $\widetilde{\operatorname{osc}}_{\mathcal{T}}(g,T)_q$. Consequently, GREEDY terminates in a finite number $k \geq 1$ of iterations. Let $T_1, \dots T_k$ be the sequence of marked elements, with $\mathcal{M} = \{T_1, \dots, T_k\}$ and let $\mathcal{T}_1, \dots, \mathcal{T}_k$ be the sequence of refinements produced by GREEDY starting from $\mathcal{T}_0$. Upon termination, the surrogate error satisfies $\widetilde{E}_{\mathcal{T}_k}(f)_{-1} \leq \tau$, whence $E_{\mathcal{T}_k}(f)_{-1} \leq C_{\text{data}}\tau$.

$\boxed{2}$ *Counting.* To estimate the cardinality of $\mathcal{T}_k$, we need to count $\#\mathcal{M}$. Set

$$\delta_i := \widetilde{\operatorname{osc}}_{\mathcal{T}_i}(g,T_i)_q, \quad 1 \leq i \leq k \quad \text{and} \quad \delta := \delta_{k-1}.$$

Then we obtain

$$\widetilde{E}_{\mathcal{T}_k}(f)_{-1} \le \tau < \widetilde{E}_{\mathcal{T}_{k-1}}(f)_{-1} \le \delta\,(\#\mathcal{T}_{k-1})^{1/2} \le \delta\,(\#\mathcal{T}_k)^{1/2}. \tag{7.36}$$

We organize the elements in $\mathcal{M}$ by size in such a way that allows for a counting argument. Let $\mathcal{P}_j$ be the set of elements T of $\mathcal{M}$ with size

$$2^{-(j+1)} \le |T| < 2^{-j} \quad \Rightarrow \quad 2^{-(j+1)/d} \le h_T < 2^{-j/d}.$$

We first observe that all the T in $\mathcal{P}_j$ are *disjoint*. This is because if $T_1, T_2 \in \mathcal{P}_j$ and $\mathring{T}_1 \cap \mathring{T}_2 \neq \emptyset$, then one of them is contained in the other, say $T_1 \subset T_2$, due to the bisection procedure which works in any dimension $d \ge 1$; see Section 3.5. Hence

$$|T_1| \le \frac{1}{2}\,|T_2|,$$

contradicting the definition of $\mathcal{P}_j$. On the one hand, this implies the first bound

$$2^{-(j+1)(d-1)/d}\,\#\mathcal{P}_j \lesssim |\mathcal{C}| \quad \Rightarrow \quad \#\mathcal{P}_j \lesssim |\mathcal{C}|\,2^{(j+1)(d-1)/d}, \tag{7.37}$$

where we used that $h_T^{d-1} \approx |\omega_T \cap \mathcal{C}|$ since $T \cap \mathcal{C} \neq \emptyset$ for all marked elements. Recall that ω_T stands for the patch of elements around T.

On the other hand, the monotonicity of the local error indicators $\widetilde{\mathrm{osc}}_{\mathcal{T}_i}(g,T)_q = h_T^t\,\|g\|_{L^q(T\cap\mathcal{C})}$ implies that REFINE does not increase $\widetilde{\mathrm{osc}}_{\mathcal{T}_i}(g,T)_q$ and thus

$$\delta \le \delta_i = \widetilde{\mathrm{osc}}_{\mathcal{T}_i}(g,T_i)_q, \quad 1 \le i \le k-1,$$

where $t = d/2 - (d-1)/q$. In view of (7.34), if $T_i \in \mathcal{P}_j$, then we obtain

$$\delta \le \widetilde{\mathrm{osc}}_{\mathcal{T}_i}(g,T_i)_q \lesssim 2^{-jt/d}\|g\|_{L^q(T_i\cap\mathcal{C})}.$$

Therefore, accumulating these quantities in ℓ^q yields

$$\delta^q\,\#\mathcal{P}_j \lesssim 2^{-jtq/d}\|g\|^q_{L^q(\mathcal{C})}$$

and gives rise to the second bound

$$\#\mathcal{P}_j \lesssim \delta^{-q}\,2^{-jtq/d}\,\|g\|^q_{L^q(\mathcal{C})}. \tag{7.38}$$

$\boxed{3}$ *Cardinality.* The two bounds for $\#\mathcal{P}$ in (7.37) and (7.38) are complementary. The first one is good for j small whereas the second is suitable for j large (think of $\delta \ll 1$). The crossover takes place for j_0 such that

$$2^{(j_0+1)(d-1)/d}|\mathcal{C}| \approx \delta^{-q}\,2^{-j_0tq/d}\|g\|^q_{L^q(\mathcal{C})} \quad \Rightarrow \quad 2^{j_0} \approx |\mathcal{C}|^{-2/q}\delta^{-2}\|g\|^2_{L^q(\mathcal{C})},$$

upon using the expression for t. We now compute

$$k = \#\mathcal{M} = \sum_j \#\mathcal{P}_j \lesssim \sum_{j\le j_0} 2^{j(d-1)/d}|\mathcal{C}| + \delta^{-q}\,\|g\|^q_{L^q(\mathcal{C})} \sum_{j>j_0} 2^{-(tq/d)j}.$$

Since

$$\sum_{j\le j_0} 2^{j(d-1)/d} \approx 2^{j_0(d-1)/d}, \quad \sum_{j>j_0}(2^{-tq/d})^j \lesssim 2^{-tqj_0/d},$$

we can write

$$\#\mathcal{M} \lesssim |\mathcal{C}|^{1-2(d-1)/(qd)} \big(\delta^{-1}\|g\|_{L^q(\mathcal{C})}\big)^{2(d-1)/d}.$$

We finally apply Theorem 3.16 (complexity of REFINE), to arrive at

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \lesssim \#\mathcal{M} \lesssim |\mathcal{C}|^{1-2(d-1)/(qd)} \big(\delta^{-1}\|g\|_{L^q(\mathcal{C})}\big)^{2(d-1)/d},$$

or equivalently

$$\delta \lesssim |\mathcal{C}|^{d/(d-1)-2/q}\|g\|_{L^q(\mathcal{C})}(\#\mathcal{T} - \#\mathcal{T}_0)^{-d/(2(d-1))}.$$

We deduce from (7.36) that

$$\tau \lesssim \delta(\#\mathcal{T})^{1/2} \lesssim |\mathcal{C}|^{d/(d-1)-2/q}\|g\|_{L^q(\mathcal{C})}(\#\mathcal{T} - \#\mathcal{T}_0)^{-d/(2(d-1))+1/2}$$

or equivalently

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \lesssim \|g\|_{L^q(\mathcal{C})}^{2(d-1)}\tau^{-2(d-1)}, \tag{7.39}$$

From this we conclude that $f = g\delta_{\mathcal{C}} \in \mathbb{F}_{1/(2(d-1))}$ with $|f|_{\mathbb{F}_{1/(2(d-1))}} \lesssim \|g\|_{L^q(\mathcal{C})}$, as desired.

$\boxed{4}$ *Starting from* $\mathcal{T}$. To derive similar properties for GREEDY starting from $\mathcal{T} \in \mathbb{T}$, we proceed as in the proof of Corollary 7.6 (performance of GREEDY). We distinguish the output $[\widetilde{\mathcal{T}}, \widetilde{f}] = \mathsf{GREEDY}(\mathcal{T}_0, \tau, 2, 1, f)$ starting from $\mathcal{T}_0$ and performing $b = 1$ bisection per marked element with $[\widehat{\mathcal{T}}, \widehat{f}] = \mathsf{GREEDY}(\mathcal{T}, \tau, 2, b, f)$ starting from $\mathcal{T} \in \mathbb{T}$ and performing $b \geq 1$ bisections per marked element. Lemma 7.5 (GREEDY starting from $\mathcal{T}$) guarantees that $\mathsf{GREEDY}(\mathcal{T}, \tau, 2, b, f)$ terminates with $\widetilde{E}_{\widehat{\mathcal{T}}}(f)_{-1} \leq \tau$. Moreover, Lemma 7.5 also ensures that the number of marked elements satisfies

$$N \leq \#\widetilde{\mathcal{T}} - \#\mathcal{T}_0 \lesssim \|g\|_{L^q(\mathcal{C})}^{2(d-1)}\tau^{-2(d-1)},$$

where we used (7.39) to derive the last inequality. This ends the proof. □

7.3.3. The case $f = \operatorname{div} \boldsymbol{g}$ *with* $\boldsymbol{g} \in L^2(\Omega; \mathbb{R}^d)$

A characterization of distributions in $H^{-1}(\Omega)$ is given in Evans (2010, Section 5.9.1): they are of the form

$$f = f_0 + \operatorname{div} \boldsymbol{g}$$

with $f_0 \in L^2(\Omega), \boldsymbol{g} \in L^2(\Omega; \mathbb{R}^d)$. Since we have already treated separately the ubiquitous case $\boldsymbol{g} = \boldsymbol{0}$ in Section 7.3.1, we now consider the case $f_0 = 0$. Therefore

$$\langle f, v\rangle = -\int_\Omega \boldsymbol{g} \cdot \nabla v \quad \text{for all } v \in H_0^1(\Omega) \tag{7.40}$$

gives the action of f on v, and its norm is (Evans 2010, Section 5.9.1)

$$\|f\|_{H^{-1}(\Omega)} = \inf\{\|\boldsymbol{g}\|_{L^2(\Omega)} \mid \boldsymbol{g} \in L^2(\Omega; \mathbb{R}^d) \text{ satisfies (7.40)}\}. \tag{7.41}$$

Since adding the curl of a smooth vector field to $\boldsymbol{g}$ does not change (7.40), we realize that the actual computation of (7.41) is problematic. We assume here that $\boldsymbol{g}$ is given and simply deal directly with $\boldsymbol{g}$, thereby exploiting the relation

$$\|f\|_{H^{-1}(\Omega)} \leq \|\boldsymbol{g}\|_{L^2(\Omega)}; \tag{7.42}$$

this leads to a surrogate estimator. We first approximate $\boldsymbol{g}$ by discontinuous piecewise polynomials of degree $n_f \leq n-1$, namely, we compute the L^2-projection $\boldsymbol{g}_{\mathcal{T}} = \Pi_{\mathcal{T}}\boldsymbol{g}$ onto $\left[\mathbb{S}_{\mathcal{T}}^{n_f,-1}\right]^d$, then we let $f_{\mathcal{T}} := \operatorname{div}\boldsymbol{g}_{\mathcal{T}} \in \mathbb{F}_{\mathcal{T}} \subset H^{-1}(\Omega)$ be the approximation of f:

$$\langle f_{\mathcal{T}}, v\rangle = -\sum_{T\in\mathcal{T}}\int_T \operatorname{div}\boldsymbol{g}_{\mathcal{T}} v - \sum_{F\in\mathcal{F}}\int_F [[\boldsymbol{g}_{\mathcal{T}}]]\cdot\boldsymbol{n}_F v \quad \text{for all } v \in H_0^1(\Omega).$$

We see that for $z \in \mathcal{V}$, $f_{\mathcal{T}}|_{\omega_z}$ has the form of a functional in $\mathbb{F}(\mathcal{T}_z)$ (see (7.27)) with $q_T = \operatorname{div}\boldsymbol{g}_{\mathcal{T}}|_T \in \mathbb{P}_{n_f-1}, q_F = [[\boldsymbol{g}_{\mathcal{T}}]]\cdot\boldsymbol{n}_F \in \mathbb{P}_{n_f}$ for all $T \in \mathcal{T}, F \in \mathcal{F}$, but with smaller polynomial degree than functions in $\mathbb{F}(\mathcal{T}_z)$. We next exploit the local near-best approximation (7.28) to replace $P_{\mathcal{T}}f$ by $f_{\mathcal{T}}$,

$$\|f - P_{\mathcal{T}}f\|_{H^{-1}(\omega_z)} \leq C_P\|f - f_{\mathcal{T}}\|_{H^{-1}(\omega_z)} \leq C_P\|\boldsymbol{g} - \boldsymbol{g}_{\mathcal{T}}\|_{L^2(\omega_z)}, \tag{7.43}$$

by virtue of (7.42) with Ω replaced by ω_z. This leads to the surrogate element-wise oscillation $\widetilde{\operatorname{osc}}_{\mathcal{T}}(\boldsymbol{g},T)_2 := \|\boldsymbol{g} - \boldsymbol{g}_{\mathcal{T}}\|_{L^2(T)}$, which satisfies Assumption 7.3 (monotonicity of local oscillations). We thus have the global surrogate

$$\widetilde{E}_{\mathcal{T}}(f)_{-1}^2 := \sum_{T\in\mathcal{T}}\widetilde{\operatorname{osc}}_{\mathcal{T}}(\boldsymbol{g},T)_2^2.$$

Corollary 7.21 (approximation class of $\operatorname{div}\boldsymbol{g}$**).** *Let the initial subdivision $\mathcal{T}_0$ of $\Omega \subset \mathbb{R}^d$ satisfy Assumption 6.19 (initial labelling) and let $\mathcal{T} \in \mathbb{T}$ be any admissible refinement of $\mathcal{T}_0$. Let $\tau > 0$ be the target tolerance and let $b \geq 1$ be the number of bisections performed on each marked element. Let $(\boldsymbol{g}, s, 0, p, 2)$ satisfy Assumption 7.2 (admissible set of parameters for* GREEDY*) with local oscillations $\{\widetilde{\operatorname{osc}}(\boldsymbol{g},T)_2\}_{T\in\mathcal{T}}$. Then $[\widehat{\mathcal{T}}, \widehat{f}] = \mathsf{GREEDY}(\mathcal{T}, \tau, 2, b, f)$ terminates in a finite number of steps with $\widetilde{E}_{\widehat{\mathcal{T}}}(f)_{-1} \leq \tau$, whence $E_{\widehat{\mathcal{T}}}(f)_{-1} \leq C_{\text{data}}\tau$. Moreover, the number N of marked elements by* GREEDY *satisfies*

$$N \lesssim \|\boldsymbol{g}\|_{X_p^s(\Omega)}^{d/s}\tau^{-d/s}.$$

In particular, $f = \operatorname{div}\boldsymbol{g} \in \mathbb{F}_{s/d}$ with

$$|f|_{\mathbb{F}_{s/d}} \lesssim \|\boldsymbol{g}\|_{X_p^s(\Omega)}.$$

Proof. Apply Corollary 7.6 (performance of GREEDY) with $q = 2$ to $\boldsymbol{g}$. □

7.4. DATA *module*

We now summarize in one single algorithm, called DATA, all the developments in Sections 7.2.2, 7.2.3 and 7.3. We first recall that Corollaries 7.12 (complexity of

constrained GREEDY for $\boldsymbol{A}$) and 7.16 (complexity of constrained GREEDY for c) deliver piecewise polynomial approximations $(\widehat{\boldsymbol{A}}, \widehat{c})$ of the coefficients $(\boldsymbol{A}, c)$ over an admissible mesh $\widehat{\mathcal{T}}$ that satisfies both the global errors estimates

$$E_{\widehat{\mathcal{T}}}(\boldsymbol{A})_{q_A} \leq C_{\text{data}}\tau, \quad E_{\widehat{\mathcal{T}}}(c)_{q_c} \leq C_{\text{data}}\tau,$$

where $2 \leq q_A, q_c \leq \infty$ are the corresponding integrability indices, as well as the structural constraint (5.51).

The situation for the load f is more intricate due to the evaluation of the non-local norm $H^{-1}(\Omega)$, which requires further structure of f besides regularity. Section 7.3 provides three examples of practical significance that allow for computable surrogate errors $\widetilde{E}_{\mathcal{T}}(f)_{-1}$ larger than the desired oscillations $E_{\mathcal{T}}(f)_{-1}$. Since these examples have different requirements for the approximation procedure to work, we gather the salient structural points in the following assumption.

Assumption 7.22 (structure of f). Let (s_f, p_f) denote the additional regularity–integrability indices of f beyond the basic H^{-1}-regularity, which are required by Assumption 7.2 (admissible set of parameters for GREEDY). Let $|f|_{\widetilde{X}^{s_f}_{p_f}(\Omega;\mathcal{T}_0)}$ be a measure of piecewise regularity of f in $\mathcal{T}_0$ expressed below in terms of surrogates. Assume that exactly one of the following cases holds, and note that all accumulate local oscillations in ℓ^2.

- $f \in L^q(\Omega)$, with $2d/(d+2) < q \leq \infty$. Let $\widetilde{\text{osc}}_{\mathcal{T}}(f,T)_q = h_T^{t_f}\|f - \widehat{f}\|_{L^q(T)}$ be the local oscillation with $t_f = 1 + d(1/2 - 1/q) \geq 0$ and $(f, s_f, t_f, p_f, 2)$ satisfy Assumption 7.2, and set $|f|_{\widetilde{X}^{s_f}_{p_f}(\Omega;\mathcal{T}_0)} := |f|_{X^{s_f}_{p_f}(\Omega;\mathcal{T}_0)}$.
- $f = g\delta_{\mathcal{C}}$ where $\mathcal{C} \subset \Omega$ is a Lipschitz hyper-surface and $g \in L^q(\mathcal{C})$ with $2(d-1)/d < q \leq \infty$. Let $\widetilde{\text{osc}}_{\mathcal{T}}(g,T)_q = h_T^r\|g\|_{L^q(T\cap\mathcal{C})}$ be the local oscillation with $r = d/2 - (d-1)/q > 0$. Set $s_f = 0$, $t_f = d/(2(d-1))$, $p_f = q$, and $|f|_{\widetilde{X}^{s_f}_{p_f}(\Omega;\mathcal{T}_0)} := \|g\|_{L^q(\mathcal{C})}$.
- $f = \operatorname{div}\boldsymbol{g}$ with $\boldsymbol{g} \in L^2(\Omega;\mathbb{R}^d)$. Let $\widetilde{\text{osc}}_{\mathcal{T}}(f,T)_2 = \|\boldsymbol{g} - \Pi_{\mathcal{T}}\boldsymbol{g}\|_{L^2(T)}$ be the local oscillation, $t_f = 0$, and $(\boldsymbol{g}, s_f, t_f, p_f, 2)$ satisfy Assumption 7.3, and set $|f|_{\widetilde{X}^{s_f}_{p_f}(\Omega;\mathcal{T}_0)} := \|\boldsymbol{g}\|_{X^{s_f}_{p_f}(\Omega;\mathcal{T}_0)}$.

In all these cases, GREEDY algorithms with tolerance $\tau > 0$ reduce the surrogate error $\widetilde{E}_{\mathcal{T}}(f)^2_{-1}$ and eventually guarantee that

$$E_{\mathcal{T}}(f)_{-1} \leq C_{\text{data}}\tau,$$

where $C_{\text{data}} \geq 1$ is the constant appearing in Corollary 7.18, Lemma 7.20 or Corollary 7.21 depending on Assumption 7.22 (structure of f).

Algorithm 7.23 (DATA). Given a tolerance $\tau > 0$ and an arbitrary conforming grid $\mathcal{T} \in \mathbb{T}$, not necessarily $\mathcal{T}_0$, DATA finds a conforming refinement $\widehat{\mathcal{T}} \geq \mathcal{T}$ of $\mathcal{T}$

and approximate data $\widehat{\mathcal{D}} = (\widehat{\boldsymbol{A}}, \widehat{c}, \widehat{f}) \in \mathbb{D}_{\widehat{\mathcal{T}}}$ over $\widehat{\mathcal{T}}$ such that

$$\|\mathcal{D} - \widehat{\mathcal{D}}\|_{D(\Omega)} = E_{\widehat{\mathcal{T}}}(A)_{q_A} + E_{\widehat{\mathcal{T}}}(c)_{q_c} + E_{\widehat{\mathcal{T}}}(f)_{-1} \le C_{\text{data}}\tau.$$

$[\widehat{\mathcal{T}}, \widehat{\mathcal{D}}] = \mathsf{DATA}(\mathcal{T}, \tau, \mathcal{D})$
  $[\mathcal{T}_{\boldsymbol{A}}, \widehat{\boldsymbol{A}}] = \mathsf{GREEDY}(\mathcal{T}, \tau/3, q_A, b, \boldsymbol{A})$
  $\widehat{\boldsymbol{A}} = \mathsf{CONSTRAINT\text{-}A}(\mathcal{T}_A, \alpha_1, \alpha_2, L, \widetilde{\boldsymbol{A}})$
  Set $\widehat{\alpha}_1 = \frac{1}{2}\alpha_1$ and $\widehat{\alpha}_2 = (1 + 4L)\alpha_2/2$
  $[\mathcal{T}_c, \widetilde{c}] = \mathsf{GREEDY}(\mathcal{T}_A, \tau/3, q_c, b, c)$
  $\widehat{c} = \mathsf{CONSTRAINT\text{-}c}(\mathcal{T}_c, \widehat{\alpha}_1, L, \widetilde{c})$
  $[\widehat{\mathcal{T}}, \widehat{f}] = \mathsf{GREEDY}(\mathcal{T}_c, \tau/3, 2, b, f)$
  return $\widehat{\mathcal{T}}, \widehat{\mathcal{D}}$

Note that DATA depends on the threshold parameter $L > 2$ used in CONSTRAINT-A and CONSTRAINT-c, although for simplicity it is not listed among the input parameters.

The next result summarizes the properties of DATA.

Corollary 7.24 (performance of DATA). *Let the initial subdivision $\mathcal{T}_0$ of $\Omega \subset \mathbb{R}^d$ satisfy Assumption 6.19 (initial labelling) and let $\mathcal{T} \in \mathbb{T}$ be any admissible refinement of $\mathcal{T}_0$. Let $b \ge 1$ be the number of bisections performed on each marked element. Let the assumptions of Corollaries 7.12 and 7.16 for the coefficients $(\boldsymbol{A}, c)$ be valid, and let f satisfy Assumption 7.22.*

For any target tolerance $\tau > 0$ and any threshold parameter $L > 2$, $[\widehat{\mathcal{T}}, \widehat{\mathcal{D}}] = \mathsf{DATA}(\mathcal{T}, \tau, \mathcal{D})$ terminates in a finite number of iterations and outputs $\widehat{\mathcal{D}}$, $\widehat{\mathcal{T}} \in \mathbb{T}$ such that $\widehat{\boldsymbol{A}}$ is symmetric and

$$\widehat{\boldsymbol{A}} \in M(\widehat{\alpha}_1, \widehat{\alpha}_2), \quad \widehat{c} \in R(\widehat{c}_1, \widehat{c}_2),$$

where $\widehat{\alpha}_1, \widehat{\alpha}_2$ are given by (7.21) while $\widehat{c}_1, \widehat{c}_2$ are given by (7.23). Moreover, there is a constant $C_{\text{data}} \ge 1$ such that DATA *terminates with*

$$\|\mathcal{D} - \widehat{\mathcal{D}}\|_{D(\Omega)} \le C_{\text{data}}\tau,$$

and the number N of elements marked to construct $\widehat{\mathcal{T}}$ satisfies

$$N \lesssim |\mathcal{D}|_{\mathbb{X}_{s_{\mathcal{D}}/d}}^{d/s_{\mathcal{D}}} \tau^{-d/s_{\mathcal{D}}}, \tag{7.44}$$

with $s_{\mathcal{D}} := \min\{s_A, s_c + t_c, s_f + t_f\}$, and

$$|\mathcal{D}|_{\mathbb{X}_{s_{\mathcal{D}}/d}} = \left(|\boldsymbol{A}|_{X_{p_A}^{s_A}(\Omega,\mathcal{T}_0)}^{d/s_A} + |c|_{X_{p_c}^{s_c}(\Omega,\mathcal{T}_0)}^{d/s_c} + |f|_{\widetilde{X}_{p_f}^{s_f}(\Omega,\mathcal{T}_0)}^{d/s_f} \right)^{s_{\mathcal{D}}/d}.$$

Proof. Since the local oscillations for $\boldsymbol{A}$ and c satisfy Assumption 7.3 (monotonicity of local oscillations), we deduce that global oscillations do not increase upon refinement, namely, for $\widehat{\mathcal{T}} \ge \mathcal{T}_c \ge \mathcal{T}_{\boldsymbol{A}}$,

$$E_{\widehat{\mathcal{T}}}(\boldsymbol{A})_{q_A} + E_{\widehat{\mathcal{T}}}(c)_{q_c} \le E_{\mathcal{T}_A}(\boldsymbol{A})_{q_A} + E_{\mathcal{T}_c}(c)_{q_c}.$$

In view of Corollaries 7.12 and 7.16, this in turn implies

$$E_{\widehat{\mathcal{T}}}(\boldsymbol{A})_{q_A} + E_{\widehat{\mathcal{T}}}(c)_{q_c} \leq C_{\text{data}} \frac{2}{3}\tau.$$

For the load term f, we invoke Corollary 7.18, Lemma 7.20 or Corollary 7.21, depending on Assumption 7.22 (structure of f), to infer that

$$E_{\widehat{\mathcal{T}}}(f)_{-1} \leq C_{\text{data}} \frac{1}{3}\tau.$$

Hence

$$\|\mathcal{D} - \widehat{\mathcal{D}}\|_{D(\Omega)} = E_{\widehat{\mathcal{T}}}(\boldsymbol{A})_{q_A} + E_{\widehat{\mathcal{T}}}(c)_{q_c} + E_{\widehat{\mathcal{T}}}(f)_{-1} \leq C_{\text{data}}\tau$$

as desired. The complexity estimate (7.44) directly follows from the complexity estimates given in Corollaries 7.12 and 7.16 for $(\boldsymbol{A}, c)$, and Corollary 7.18, Lemma 7.20 or Corollary 7.21 for f depending on its structure. □

Similar ideas apply to approximate non-vanishing Dirichlet data or boundary flux conditions for Robin or Neumann problems, but we do not elaborate on this.

8. Mesh refinement: the bisection method

This section is devoted to the complexity analysis of REFINE for Λ-admissible triangulations. More precisely, we prove the existence of a constant $D > 0$ such that

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \leq D \sum_{j=0}^{k-1} \#\mathcal{M}_j, \quad k \geq 0.$$

This kind of result holds for conforming meshes ($\Lambda = 0$) and was stated in Theorem 3.16, and for non-conforming meshes ($\Lambda > 1$) as anticipated in Theorem 3.29. The results of Sections 8.1 and 8.2 are valid for $d = 2$ but the proofs of the cited theorems extend to $d > 2$. We refer to the survey by Nochetto *et al.* (2009) for a full discussion for $d \geq 2$.

8.1. Conforming meshes

8.1.1. Chains and labelling for $d = 2$

In order to study non-local effects of bisection for $d = 2$, we now introduce the concept of chain (Binev *et al.* 2004); this concept is inadequate for $d > 2$ (Nochetto *et al.* 2009, Stevenson 2008). Recall that $E(T)$ denotes the edge of T assigned for refinement. To each $T \in \mathcal{T}$ we associate the element $F(T) \in \mathcal{T}$ sharing the edge $E(T)$ if $E(T)$ is interior and $F(T) = \emptyset$ if $E(T)$ is on $\partial\Omega$. A *chain* $\mathcal{C}(T, \mathcal{T})$, with starting element $T \in \mathcal{T}$, is a sequence $\{T, F(T), \ldots, F^m(T)\}$ with no repetitions of elements and with

$$F^{m+1}(T) = F^k(T) \text{ for some } k \in \{0, \ldots, m-1\} \text{ or } F^{m+1}(T) = \emptyset;$$

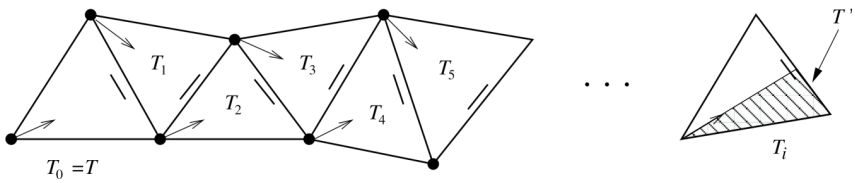


Figure 8.1. Typical chain $\mathcal{C}(T, \mathcal{T}) = \{T_j\}_{j=0}^{i}$ emanating from $T = T_0 \in \mathcal{T}$ with $T_j = F(T_{j-1})$, $j \geq 1$.

see Figure 8.1. We observe that if an element T belongs to two different grids, then the corresponding chains may be different as well. Two adjacent elements $T, T' = F(T)$ are *compatibly divisible* (or equivalently T, T' form a *compatible bisection patch*) if $F(T') = T$. Hence $\mathcal{C}(T, \mathcal{T}) = \{T, T'\}$, and a bisection of either T or T' does not propagate outside the patch.

Example (chains). Let $\mathcal{F} = \{T_i\}_{i=1}^{12}$ be the forest of Figure 3.5. Then $\mathcal{C}(T_6, \mathcal{T}) = \{T_6, T_7\}$, $\mathcal{C}(T_9, \mathcal{T}) = \{T_9\}$ and $\mathcal{C}(T_{10}, \mathcal{T}) = \{T_{10}, T_8, T_2\}$ are chains, but only $\mathcal{C}(T_6, \mathcal{T})$ is a compatible bisection patch.

To study the structure of chains we rely on the initial labelling (3.35) and the bisection rule of Section 3.5 (see Figure 3.7):

Every triangle $T \in \mathcal{T}$ with generation $g(T) = i$ receives the label $(i+1, i+1, i)$ with i corresponding to the refinement edge $E(T)$, its side i is bisected and both new sides as well as the bisector are labelled $i+2$ whereas the remaining labels do not change. (8.1)

We first show that once the initial labelling and bisection rule are set, the resulting master forest $\mathbb{F}$ is uniquely determined: the label of an edge is independent of the elements sharing this edge and no ambiguity arises in the recursion process.

Lemma 8.1 (labelling). *Let the initial labelling (3.35) for $\mathcal{T}_0$ and the above bisection rule be enforced. If $\mathcal{T}_0 \leq \mathcal{T}_1 \leq \cdots \leq \mathcal{T}_n$ are generated according to (8.1), then each side in $\mathcal{T}_k$ has a unique label independent of the two triangles sharing this edge.*

Proof. We argue by induction over $\mathcal{T}_k$. For $k = 0$ the assertion is valid due to the initial labelling. Suppose the statement is true for $\mathcal{T}_k$. An edge S in $\mathcal{T}_{k+1}$ can be obtained in two ways. The first is that S is a bisector and so a new edge, in which case there is nothing to prove about its label being unique. The second possibility is that S was obtained by bisecting an edge $S' \in \mathcal{S}_k$. Let T, $T' \in \mathcal{T}_k$ be the elements sharing S', and let us assume that $E(T') = S'$. Let $(i+1, i+1, i)$ be the label of T', which means that S is assigned the label $i+2$. By induction assumption over $\mathcal{T}_k$, the label of S' as an edge of T is also i. There are two possible cases for the label of T.

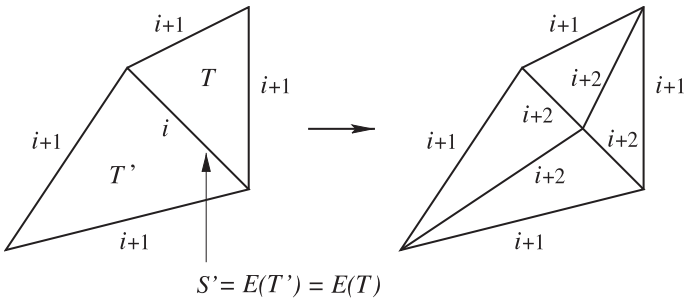


Figure 8.2. T and T' form a compatible patch, as they share the generation.

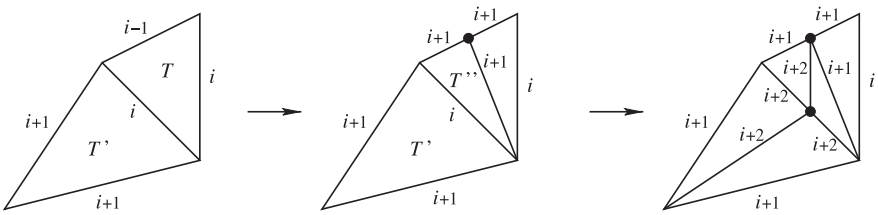


Figure 8.3. T' form a compatible patch with the child T'' of T; indeed T has a lower generation than T'.

- Label $(i+1, i+1, i)$: this situation is symmetric, $E(T) = S'$, and S' is bisected with both halves getting the label $i+2$. This is depicted in Figure 8.2.
- Label $(i, i, i-1)$: a bisection of side $E(T)$ with label $i-1$ creates a child T'' with label $(i+1, i+1, i)$ that is compatibly divisible with T'. Joining the new node of T with the midpoint of S' creates a conforming partition with level $i+2$ assigned to S. This is depicted in Figure 8.3.

Therefore, in both cases the label $i+2$ assigned to S is the same from both sides, as asserted. □

The two possible configurations displayed in the two figures above lead readily to the following statement about generations.

Corollary 8.2 (generation of consecutive elements). *For any $\mathcal{T} \in \mathbb{T}$ and $T, T' \in \mathcal{T}$ with $T = F(T')$, we have either*

(a) *$g(T) = g(T')$ and T, T' are compatibly divisible, or*

(b) *$g(T) = g(T') - 1$ and T' is compatibly divisible with a child of T.*

Corollary 8.3 (generations within a chain). *For all $\mathcal{T} \in \mathbb{T}$ and $T \in \mathcal{T}$, its chain $\mathcal{C}(T, \mathcal{T}) = \{T_k\}_{k=0}^{m}$ with $T_k = F^k(T)$ has the property*

$$g(T_k) = g(T) - k, \quad 0 \le k \le m-1,$$

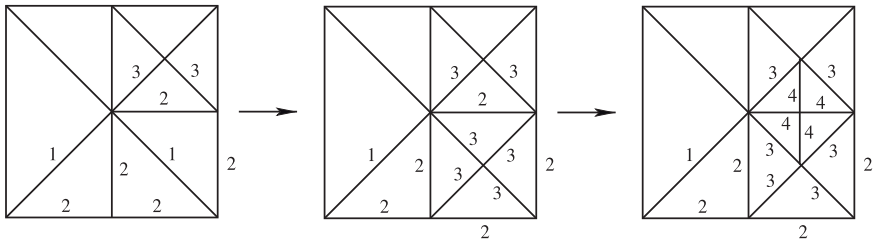


Figure 8.4. The recursive refinement of $T_{10} \in \mathcal{T}$ in Figure 3.4 using REFINE_RECURSIVE. This entails refining the chain $\mathcal{C}(T_{10}, \mathcal{T}) = \{T_{10}, T_8, T_2\}$, starting from the last element $T_2 \in \mathcal{T}$, which forms a compatible bisection patch on its own because its refinement edge is on the boundary, and continuing with $T_8 \in \mathcal{T}$ and finally $T_{10} \in \mathcal{T}$. Note that the successive meshes are always conforming and that REFINE_RECURSIVE bisects elements in $\mathcal{C}(T_{10}, \mathcal{T})$ twice before getting back to T_{10}.

and $T_m = F^m(T)$ has generation $g(T_m) = g(T_{m-1})$ or it is a boundary element with lowest labelled edge on $\partial\Omega$. In the first case, T_{m-1} and T_m are compatibly divisible.

Proof. Apply Corollary 8.2 repeatedly to consecutive elements of $\mathcal{C}(T, \mathcal{T})$. □

8.1.2. Recursive bisection

Given an element $T \in \mathcal{M}$ to be refined, the routine REFINE_RECURSIVE$(\mathcal{T}, T)$ recursively refines the chain $\mathcal{C}(T, \mathcal{T})$ of T, from the end back to T, and creates a minimal conforming partition $\mathcal{T}_* \geq \mathcal{T}$ such that T is bisected once. This procedure reads as follows:

```
[T*] = REFINE_RECURSIVE(T, T)
    if g(F(T)) < g(T)
        [T] = REFINE_RECURSIVE(T, F(T))
    else
        bisect the compatible bisection patch C(T, T)
        update T
    return T
```

We let $\mathcal{C}_*(T, \mathcal{T}) \subset \mathcal{T}_*$ denote the recursive refinement of $\mathcal{C}(T, \mathcal{T})$ (or completion of $\mathcal{C}(T, \mathcal{T})$) caused by bisection of T. Since REFINE_RECURSIVE refines solely compatible bisection patches, intermediate meshes are always conforming.

We refer to Figure 8.4 for an example of recursive bisection $\mathcal{C}_*(T_{10}, \mathcal{T})$ of $\mathcal{C}(T_{10}, \mathcal{T}) = \{T_{10}, T_8, T_2\}$ in Figure 3.4: REFINE_RECURSIVE starts bisecting from the end of $\mathcal{C}(T_{10}, \mathcal{T})$, namely T_2, which is a boundary element, and goes back the chain bisecting elements twice until it gets to T_{10}.

We now establish a fundamental property of REFINE_RECURSIVE$(\mathcal{T}, T)$ relating the generation of elements within $\mathcal{C}_*(T, \mathcal{T})$ (Binev *et al.* 2004).

Lemma 8.4 (recursive refinement). *Let $\mathcal{T}_0$ satisfy the labelling (3.35), and let $\mathcal{T} \in \mathbb{T}$ be a conforming refinement of $\mathcal{T}_0$. A call to* REFINE_RECURSIVE$(\mathcal{T}, T)$ *terminates, for all T in the set $\mathcal{M}$ of marked elements, and outputs the smallest conforming refinement $\mathcal{T}_*$ of $\mathcal{T}$ such that T is bisected. In addition, all newly created $T' \in \mathcal{C}_*(T, \mathcal{T})$ satisfy*

$$g(T') \leq g(T) + 1. \tag{8.2}$$

Proof. We first observe that T has maximal generation within $\mathcal{C}(T, \mathcal{T})$. So recursion is applied to elements with generation $\leq g(T)$, whence the recursion terminates. We also note that this procedure creates children of T and either children or grandchildren of triangles $T_k \in \mathcal{C}(T, \mathcal{T}) = \{T_i\}_{i=0}^m$ with $k \geq 1$. If T' is a child of T there is nothing to prove. If not, we first consider $m = 1$, in which case T' is a child of T_1 because T_0 and T_1 are compatibly divisible and so have the same generation; thus $g(T') = g(T_1) + 1 = g(T_0) + 1$. Finally, if $m > 1$, then $g(T_k) < g(T)$ and we apply Corollary 8.3 to deduce

$$g(T') \leq g(T_k) + 2 \leq g(T) + 1,$$

as asserted. □

The following crucial lemma links generation and distance between T and $T' \in \mathcal{C}_*(T, \mathcal{T})$, the latter being defined as (Binev *et al.* 2004)

$$\operatorname{dist}(T', T) := \inf_{x' \in T', x \in T} |x' - x|.$$

Lemma 8.5 (distance and generation). *Let $T \in \mathcal{M}$. Any newly created $T' \in \mathcal{C}_*(T, \mathcal{T})$ by* REFINE_RECURSIVE$(\mathcal{T}, T)$ *satisfies*

$$\operatorname{dist}(T', T) \leq D_2 \frac{2}{\sqrt{2} - 1} 2^{-g(T')/2}, \tag{8.3}$$

where $D_2 > 0$ is the constant in (3.34).

Proof. Suppose $T' \subset T_i \in \mathcal{C}(T, \mathcal{T})$ has been created by subdividing T_i (see Figure 8.1). If $i \leq 1$ then $\operatorname{dist}(T', T) = 0$ and there is nothing to prove. If $i > 1$, then we observe that $\operatorname{dist}(T', T_{i-1}) = 0$, whence

$$\begin{aligned} \operatorname{dist}(T', T) &\leq \operatorname{dist}(T_{i-1}, T) + \operatorname{diam}(T_{i-1}) \leq \sum_{k=1}^{i-1} \operatorname{diam}(T_k) \\ &\leq D_2 \sum_{k=1}^{i-1} 2^{-g(T_k)/2} < D_2 \frac{1}{1 - 2^{-1/2}} 2^{-g(T_{i-1})/2}, \end{aligned}$$

because the generations decrease exactly by 1 along the chain $\mathcal{C}(T)$ according to Corollary 8.2(b). Since T' is a child or grandchild of T_i, we deduce

$$g(T') \leq g(T_i) + 2 = g(T_{i-1}) + 1,$$

whence

$$\operatorname{dist}(T', T) < D_2 \frac{2^{1/2}}{1 - 2^{-1/2}} 2^{-g(T')/2}.$$

This is the desired estimate. □

The recursive procedure REFINE_RECURSIVE is the core of the routine REFINE of Section 3.5: given a conforming mesh $\mathcal{T} \in \mathbb{T}$ and a subset $\mathcal{M} \subset \mathcal{T}$ of marked elements, REFINE creates a conforming refinement $\mathcal{T}_* \geq \mathcal{T}$ of $\mathcal{T}$ such that all elements of $\mathcal{M}$ are bisected at least once:

$[\mathcal{T}_*] = \mathsf{REFINE}(\mathcal{T}, \mathcal{M})$
  for all $T \in \mathcal{M} \cap \mathcal{T}$ do
    $[\mathcal{T}] = \mathsf{REFINE\_RECURSIVE}(\mathcal{T}, T)$
  return $\mathcal{T}$

It may happen that an element $T' \in \mathcal{M}$ is scheduled prior to T for refinement and $T \in \mathcal{C}(T', \mathcal{T})$. Since the call $\mathsf{REFINE\_RECURSIVE}(\mathcal{T}, T')$ bisects T, its two children replace T in $\mathcal{T}$. This implies that $T \notin \mathcal{M} \cap \mathcal{T}$, which prevents further refinement of T.

In practice, we often like to bisect selected elements several times: for instance, each marked element is scheduled for $b \geq 1$ bisections. This can be done by assigning the number $b(T) = b$ of bisections that have to be executed for each marked element T. If T is bisected then we assign $b(T) - 1$ as the number of pending bisections to its children and the set of marked elements is $\mathcal{M} := \{T \in \mathcal{T} \mid b(T) > 0\}$.

8.1.3. Complexity of bisection for conforming meshes

Figure 8.4 reveals that the issue of propagation of mesh refinement to keep conformity is rather delicate. In particular, an estimate of the form

$$\#\mathcal{T}_k - \#\mathcal{T}_{k-1} \leq C \, \#\mathcal{M}_{k-1}$$

is not valid with a constant C independent of k; in fact the constant can be proportional to k according to Figure 8.4.

Binev, Dahmen and DeVore (2004) for $d = 2$ and Stevenson (2008) for $d > 2$ show that control of the propagation of refinement by bisection is possible when considering the collective effect:

$$\#\mathcal{T}_k - \#\mathcal{T}_0 \leq D \sum_{j=0}^{k-1} \#\mathcal{M}_j. \tag{8.4}$$

This can be heuristically motivated as follows. Consider the set $\mathcal{M} := \bigcup_{j=0}^{k-1} \mathcal{M}_j$ used to generate the sequence $\mathcal{T}_0 \leq \mathcal{T}_1 \leq \cdots \leq \mathcal{T}_k =: \mathcal{T}$. Suppose that each element $T_* \in \mathcal{M}$ is assigned a fixed amount C_1 of money to spend on refined

elements in $\mathcal{T}$, i.e. on $T \in \mathcal{T} \setminus \mathcal{T}_0$. Assume further that $\lambda(T, T_*)$ is the portion of money spent by T_* on T. Then it must hold that

$$\sum_{T \in \mathcal{T} \setminus \mathcal{T}_0} \lambda(T, T_*) \le C_1 \quad \text{for all } T_* \in \mathcal{M}. \tag{8.5a}$$

In addition, we suppose that the investment of all elements in $\mathcal{M}$ is fair in the sense that each $T \in \mathcal{T} \setminus \mathcal{T}_0$ gets at least a fixed amount C_2, whence

$$\sum_{T_* \in \mathcal{M}} \lambda(T, T_*) \ge C_2 \quad \text{for all } T \in \mathcal{T} \setminus \mathcal{T}_0. \tag{8.5b}$$

Therefore, summing up (8.5b) and using the upper bound (8.5a), we readily obtain

$$C_2(\#\mathcal{T} - \#\mathcal{T}_0) \le \sum_{T \in \mathcal{T} \setminus \mathcal{T}_0} \sum_{T_* \in \mathcal{M}} \lambda(T, T_*) = \sum_{T_* \in \mathcal{M}} \sum_{T \in \mathcal{T} \setminus \mathcal{T}_0} \lambda(T, T_*) \le C_1 \, \#\mathcal{M},$$

which proves (8.4) for $\mathcal{T}$ and $\mathcal{M}$. In the remainder of this section we design such an allocation function $\lambda \colon \mathcal{T} \times \mathcal{M} \to \mathbb{R}^+$ in several steps and prove that recurrent refinement by bisection yields (8.5) provided $\mathcal{T}_0$ satisfies (3.35), thereby establishing Theorem 3.16 (complexity of REFINE).

Construction of the allocation function. The function $\lambda(T, T_*)$ is defined with the help of two sequences $(a(\ell))_{\ell=-1}^{\infty}, (b(\ell))_{\ell=0}^{\infty} \subset \mathbb{R}^+$ of positive numbers satisfying

$$\sum_{\ell \ge -1} a(\ell) = A < \infty, \quad \sum_{\ell \ge 0} 2^{-\ell/2} \, b(\ell) = B < \infty, \quad \inf_{\ell \ge 1} b(\ell) \, a(\ell) = c_* > 0,$$

and $b(0) \ge 1$. Valid instances are $a(\ell) = (\ell + 2)^{-2}$ and $b(\ell) = 2^{\ell/3}$.

With these settings we are prepared to define $\lambda \colon \mathcal{T} \times \mathcal{M} \to \mathbb{R}^+$ by

$$\lambda(T, T_*) := \begin{cases} a(g(T_*) - g(T)), & \operatorname{dist}(T, T_*) < D_3 \, B \, 2^{-g(T)/d} \text{ and } g(T) \le g(T_*) + 1, \\ 0, & \text{else}, \end{cases}$$

where $D_3 := D_2(1 + 2(\sqrt{2} - 1)^{-1})$. Therefore the investment of money by $T_* \in \mathcal{M}$ is restricted to cells T that are sufficiently close and are of generation $g(T) \le g(T_*) + 1$. Only elements of these generations can be created during refinement of T_* according to Lemma 8.4. We stress that except for the definition of B, this construction is multidimensional, and we refer to Nochetto *et al.* (2009) and Stevenson (2008) for details.

The following lemma shows that the total amount of money spent by the allocation function $\lambda(T, T_*)$ per marked element T_* is bounded.

Lemma 8.6 (upper bound). *There exists a constant $C_1 > 0$ depending only on $\mathcal{T}_0$ such that λ satisfies (8.5a), that is,*

$$\sum_{T \in \mathcal{T} \setminus \mathcal{T}_0} \lambda(T, T_*) \le C_1 \quad \textit{for all } T_* \in \mathcal{M}.$$

Proof. We proceed in two steps.

1 Given $T_* \in \mathcal{M}$, we set $g_* = g(T_*)$ and we let $0 \le g \le g_* + 1$ be a generation of interest in the definition of λ. We claim that for such g the cardinality of the set

$$\mathcal{T}(T_*, g) = \{T \in \mathcal{T} \mid \operatorname{dist}(T, T_*) < D_3\, B\, 2^{-g/2} \text{ and } g(T) = g\}$$

is uniformly bounded, i.e. $\#\mathcal{T}(T_*, g) \le C$ with C depending solely on D_1, D_2, D_3, B.

From (3.34) we learn that $\operatorname{diam}(T_*) \le D_2 2^{-g_*/2} \le 2D_2 2^{-(g_*+1)/2} \le 2D_2 2^{-g/2}$ as well as $\operatorname{diam}(T) \le D_2 2^{-g/2}$ for any $T \in \mathcal{T}(T_*, g)$. Hence all elements of the set $\mathcal{T}(T_*, g)$ lie inside a ball centred at the barycentre of T_* with radius $(D_3 B + 3D_2)2^{-g/2}$. Again relying on (3.34), we thus conclude that

$$\#\mathcal{T}(T_*, g) D_1 2^{-g} \le \sum_{T \in \mathcal{T}(T_*, g)} |T| \le c(D_3 B + 3D_2)^2 2^{-g},$$

whence $\#\mathcal{T}(T_*, g) \le c\, D_1^{-1}\, (D_3 B + 3D_2)^2 =: C$.

2 Accounting only for non-zero contributions $\lambda(T, T_*)$, we deduce

$$\sum_{T \in \mathcal{T} \setminus \mathcal{T}_0} \lambda(T, T_*) = \sum_{g=0}^{g_*+1} \sum_{T \in \mathcal{T}(T_*, g)} a(g_* - g) \le C \sum_{\ell=-1}^{\infty} a(\ell) = CA =: C_1,$$

which is the desired upper bound. □

The definition of λ also implies that each refined element receives a fixed amount of money. We show this next.

Lemma 8.7 (lower bound). *There exists a constant $C_2 > 0$ depending only on $\mathcal{T}_0$ such that λ satisfies (8.5b), that is,*

$$\sum_{T_* \in \mathcal{M}} \lambda(T, T_*) \ge C_2 \quad \textit{for all } T \in \mathcal{T} \setminus \mathcal{T}_0.$$

Proof. We proceed in several steps.

1 Fix an arbitrary $T_0 \in \mathcal{T} \setminus \mathcal{T}_0$. Then there is an iteration count $1 \le k_0 \le k$ such that $T_0 \in \mathcal{T}_{k_0}$ and $T_0 \notin \mathcal{T}_{k_0-1}$. Therefore there exists an $T_1 \in \mathcal{M}_{k_0-1} \subset \mathcal{M}$ such that T_0 is generated during REFINE_RECURSIVE($\mathcal{T}_{k_0-1}, T_1$). Iterating this process, we construct a sequence $\{T_j\}_{j=1}^J \subset \mathcal{M}$ with corresponding iteration counts $\{k_j\}_{j=1}^J$ such that T_j is created by REFINE_RECURSIVE($\mathcal{T}_{k_j-1}, T_{j+1}$). The sequence is finite since the iteration counts are strictly decreasing and thus $k_J = 0$ for some $J > 0$, or equivalently $T_J \in \mathcal{T}_0$.

Since T_j is created during refinement of T_{j+1}, we infer from (8.2) that

$$g(T_{j+1}) \ge g(T_j) - 1.$$

Accordingly, $g(T_{j+1})$ can decrease the previous value of $g(T_j)$ by at most 1. Since $g(T_J) = 0$, there exists a smallest value s such that $g(T_s) = g(T_0) - 1$. Note that for $j = 1, \ldots, s$ we have $\lambda(T_0, T_j) > 0$ if $\operatorname{dist}(T_0, T_j) \le D_3 B g^{-g(T_0)/d}$.

2 We next estimate the distance $\operatorname{dist}(T_0, T_j)$. For $1 \le j \le s$ and $\ell \ge 0$ we define the set

$$\mathcal{T}(T_0, \ell, j) := \{T \in \{T_0, \dots, T_{j-1}\} \mid g(T) = g(T_0) + \ell\}$$

and denote its cardinality by $m(\ell, j)$. The triangle inequality combined with an induction argument yields

$$\begin{aligned}\operatorname{dist}(T_0, T_j) &\le \operatorname{dist}(T_0, T_1) + \operatorname{diam}(T_1) + \operatorname{dist}(T_1, T_j)\\ &\le \sum_{i=1}^{j} \operatorname{dist}(T_{i-1}, T_i) + \sum_{i=1}^{j-1} \operatorname{diam}(T_i).\end{aligned}$$

We apply (8.3) for the terms of the first sum and (3.34) for the terms of the second sum, to obtain

$$\begin{aligned}\operatorname{dist}(T_0, T_j) &< D_2 \frac{2}{\sqrt{2}-1} \sum_{i=1}^{j} 2^{-g(T_{i-1})/2} + D_2 \sum_{i=1}^{j-1} 2^{-g(T_i)/2}\\ &\le D_2 \left(1 + \frac{2}{\sqrt{2}-1}\right) \sum_{i=0}^{j-1} 2^{-g(T_i)/2}\\ &= D_3 \sum_{\ell=0}^{\infty} m(\ell, j)\, 2^{-(g(T_0)+\ell)/2}\\ &= D_3 2^{-g(T_0)/2} \sum_{\ell=0}^{\infty} m(\ell, j)\, 2^{-\ell/2}.\end{aligned}$$

To establish the lower bound we distinguish two cases depending on the size of $m(\ell, s)$. This is done next.

3 *Case 1: $m(\ell, s) \le b(\ell)$ for all $\ell \ge 0$.* From this we conclude

$$\operatorname{dist}(T_0, T_s) < D_3 2^{-g(T_0)/2} \sum_{\ell=0}^{\infty} b(\ell)\, 2^{-\ell/2} = D_3 B\, 2^{-g(T_0)/2},$$

and the definition of λ then readily implies

$$\sum_{T_* \in \mathcal{M}} \lambda(T_0, T_*) \ge \lambda(T_0, T_s) = a(g(T_s) - g(T_0)) = a(-1) > 0.$$

4 *Case 2: there exists $\ell \ge 0$ such that $m(\ell, s) > b(\ell)$.* For each of these ℓ there exists a smallest $j = j(\ell)$ such that $m(\ell, j(\ell)) > b(\ell)$. We let ℓ^* be the index ℓ that gives rise to the smallest $j(\ell)$, and set $j^* = j(\ell^*)$. Consequently

$$m(\ell, j^* - 1) \le b(\ell) \quad \text{for all } \ell \ge 0,\ m(\ell^*, j^*) > b(\ell^*).$$

As in Case 1, we see $\operatorname{dist}(T_0, T_i) < D_3 B\, 2^{-g(T_0)/2}$ for all $i \le j^* - 1$, or equivalently

$$\operatorname{dist}(T_0, T_i) < D_3 B\, 2^{-g(T_0)/2} \quad \text{for all } T_i \in \mathcal{T}(T_0, \ell, j^*).$$

We next show that the elements in $\mathcal{T}(T_0, \ell^*, j^*)$ spend enough money on T_0. We first consider $\ell^* = 0$ and note that $T_0 \in \mathcal{T}(T_0, 0, j^*)$. Since $m(0, j^*) > b(0) \ge 1$ we discover $j^* \ge 2$. Hence there is an $T_i \in \mathcal{T}(T_0, 0, j^*) \cap \mathcal{M}$, which yields the estimate

$$\sum_{T_* \in \mathcal{M}} \lambda(T_0, T_*) \ge \lambda(T_0, T_i) = a(g(T_i) - g(T_0)) = a(0) > 0.$$

For $\ell^* > 0$ we see that $T_0 \notin \mathcal{T}(T_0, \ell^*, j^*)$, whence $\mathcal{T}(T_0, \ell^*, j^*) \subset \mathcal{M}$. In addition, $\lambda(T_0, T_i) = a(\ell^*)$ for all $T_i \in \mathcal{T}(T_0, \ell^*, j^*)$. From this we conclude

$$\begin{aligned}\sum_{T_* \in \mathcal{M}} \lambda(T_0, T_*) &\ge \sum_{T_* \in \mathcal{T}(T_0, \ell^*, j^*)} \lambda(T_0, T_*) = m(\ell^*, j^*)\, a(\ell^*) \\ &> b(\ell^*)\, a(\ell^*) \ge \inf_{\ell \ge 1} b(\ell)\, a(\ell) = c_* > 0.\end{aligned}$$

$\boxed{5}$ In summary, we have proved the assertion, since for any $T_0 \in \mathcal{T} \setminus \mathcal{T}_0$

$$\sum_{T_* \in \mathcal{M}} \lambda(T_0, T_*) \ge \min\{a(-1), a(0), c_*\} =: C_2 > 0. \tag{8.6}$$

This completes the proof. □

Remark 8.8 (complexity with $b > 1$ bisections). To show the complexity estimate when REFINE performs $b > 1$ bisections, the set $\mathcal{M}_k$ is to be understood as a sequence of *single* bisections recorded in sets $\{\mathcal{M}_k(j)\}_{j=1}^b$, which belong to intermediate triangulations between $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$ with $\#\mathcal{M}_k(j) \le 2^{j-1}\#\mathcal{M}_k$, $j = 1, \dots, b$. Then we also obtain Theorem 3.16 because

$$\sum_{j=1}^{b} \#\mathcal{M}_k(j) \le \sum_{j=1}^{b} 2^{j-1}\#\mathcal{M}_k = (2^b - 1)\#\mathcal{M}_k.$$

In practice, it is customary to take $b = d$ (Siebert 2012).

8.2. *Non-conforming meshes*

In this subsection we consider two kinds of non-conforming meshes undergoing a refinement process: (a) quadrilateral meshes with at most one hanging node per edge ($\Lambda = 1$ in the definition of Λ-admissible meshes), and (b) triangular meshes having global index bounded by a fixed, but arbitrary $\Lambda > 1$.

8.2.1. *Complexity of bisection for non-conforming quadrilateral meshes*

We briefly examine the refinement process for quadrilaterals with one hanging node per edge, which gives rise to the so-called 1-*meshes*. The refinement of $T \in \mathcal{T}$

might affect four elements of $\mathcal{T}$ for $d = 2$ (or 2^d elements for any dimension $d \geq 2$), all contained in the *refinement patch* $R(T, \mathcal{T})$ of T in $\mathcal{T}$. The latter is defined as

$$R(T, \mathcal{T}) := \{T' \in \mathcal{T} \mid T' \text{ and } T \text{ share an edge and } g(T') \leq g(T)\},$$

and is called *compatible* provided $g(T') = g(T)$ for all $T' \in R(T, \mathcal{T})$. The generation gap between elements sharing an edge, in particular those in $R(T, \mathcal{T})$, is always ≤ 1 for 1-meshes, and is 0 if $R(T, \mathcal{T})$ is compatible. The element size satisfies

$$h_T = 2^{-g(T)} h_{T_0} \quad \text{for all } T \in \mathcal{T},$$

where $T_0 \in \mathcal{T}_0$ is the ancestor of T in the initial mesh $\mathcal{T}_0$. Lemma 3.15 is thus valid:

$$h_T < \bar{h}_T \leq D_2 2^{-g(T)} \quad \text{for all } T \in \mathcal{T}. \tag{8.7}$$

Given an element $T \in \mathcal{M}$ to be refined, the routine REFINE_RECURSIVE($\mathcal{T}, T$) refines $R(T, \mathcal{T})$ recursively in such a way that the intermediate meshes are always 1-meshes, and reads as follows:

```
[𝒯*] = REFINE_RECURSIVE(𝒯, T)
    if g = min{g(T''): T'' ∈ R(T, 𝒯)} < g(T)
        let T' ∈ R(T, 𝒯) satisfy g(T') = g
        [𝒯] = REFINE_RECURSIVE(𝒯, T')
    else
        subdivide T
        update 𝒯 upon replacing T with its children
    return 𝒯
```

The conditional prevents the generation gap within $R(T, \mathcal{T})$ from getting larger than 1. If it fails, then the refinement patch $R(T, \mathcal{T})$ is compatible, and refining T increases the generation gap from 0 to 1 without violating the 1-mesh structure. This implies a variant of Lemma 8.4: REFINE_RECURSIVE($\mathcal{T}, T$) creates a minimal 1-mesh $\mathcal{T}_* \geq \mathcal{T}$ refinement of $\mathcal{T}$ so that for all newly created elements $T' \in \mathcal{T}_*$,

$$g(T') \leq g(T) + 1 \tag{8.8}$$

and T is subdivided only *once*. This yields Lemma 8.5: there exist a geometric constant $D_g > 0$ such that for all newly created elements $T' \in \mathcal{T}_*$

$$\operatorname{dist}(T, T') \leq D_g 2^{g(T')}. \tag{8.9}$$

The procedure REFINE_RECURSIVE is the core of REFINE, which is conceptually identical to that in Section 8.1.2. Suppose that each marked element $T \in \mathcal{M}$ is to be subdivided $b \geq 1$ times. We assign a flag $q(T)$ to each element T which is initialized $q(T) = b$ if $T \in \mathcal{M}$ and $q(T) = 0$ otherwise. The marked set $\mathcal{M}$ is then the set of elements T with $q(T) > 0$, and every time T is subdivided it is removed from $\mathcal{T}$ and replaced by its children, which inherit the flag $q(T) - 1$. This

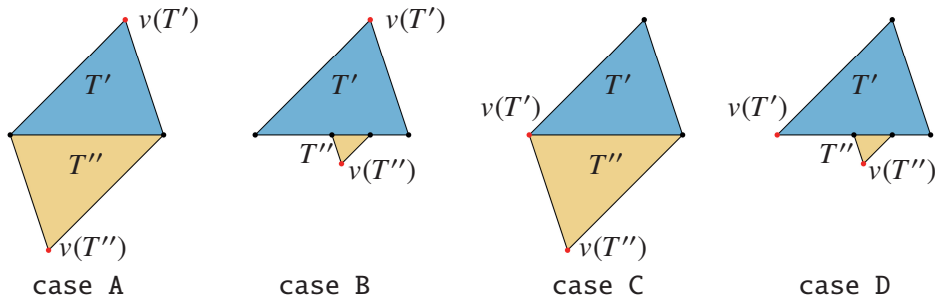


Figure 8.5. The elements T' and T'' are adjacent in cases A to D. They are compatible in cases A and B, and non-compatible in cases C and D.

avoids the conflict of again subdividing an element that has been previously refined by REFINE_RECURSIVE. The procedure REFINE($\mathcal{T}, \mathcal{M}$) reads

$[\mathcal{T}_*]$ = REFINE($\mathcal{T}, \mathcal{M}$)
  for all $T \in \mathcal{M} \cap \mathcal{T}$ do
    $[\mathcal{T}]$ = REFINE_RECURSIVE($\mathcal{T}, T$);
  end
  return $\mathcal{T}$

and its output is a minimal 1-mesh $\mathcal{T}_* \geq \mathcal{T}$ refinement of $\mathcal{T}$, so that all marked elements of $\mathcal{M}$ are refined at least b times. Since $\mathcal{T}_*$ has one hanging node per side it is thus admissible in the sense of (3.47). However, the refinement may spread outside $\mathcal{M}$ and the issue of complexity of REFINE again becomes non-trivial.

With the above ingredients in place, a statement similar to Theorem 3.16 (complexity of REFINE) for non-conforming quadrilateral meshes follows along the lines of Section 8.1.3.

8.2.2. Complexity of bisection for Λ-admissible triangular meshes

Let $\mathcal{T} \in \mathbb{T}^\Lambda$ be a Λ-admissible simplicial mesh. Given any $T \in \mathcal{T}$, let us again denote by $E(T)$ the edge of T assigned for refinement, i.e. the edge opposite to the newest vertex $v(T)$. Let $x(T)$ denote the midpoint of the edge $E(T)$.

Two elements $T', T'' \in \mathcal{T}$ are said to be *adjacent* if $E = T' \cap T''$ is an edge for at least one element, and are said to be *compatible* if they are adjacent and both $E(T')$ and $E(T'')$ belong to the same line (see Figure 8.5, cases A and B).

The following technical results will be helpful in the design of the refinement procedure.

Lemma 8.9 (global index of a hanging node). *Consider an edge $E = [x', x'']$ of the partition $\mathcal{T}$. If $x \in \mathcal{H} \cap \operatorname{int} E$ is generated by $m \geq 1$ bisections of E, then its global index $\lambda(x)$ satisfies*

$$\lambda(x) = \max(\lambda(x'), \lambda(x'')) + m.$$

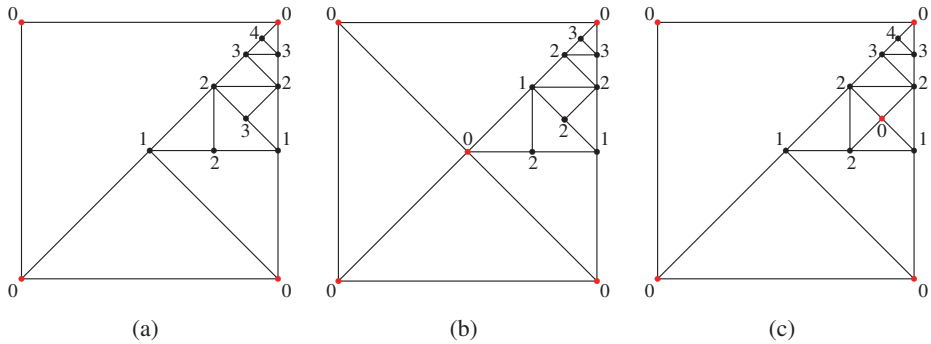


Figure 8.6. Three examples of distributions of proper nodes (red) and hanging nodes (black), with associated global indices λ. The bisection added in (b) converts the centre node into a proper node, and induces non-local changes of global indices on chains associated with it; if $\Lambda = 3$, then mesh (a) is not admissible and this procedure is instrumental in restoring admissibility. Mesh (c) illustrates the creation of a proper node without non-local effects on global indices.

Proof. If $m = 1$, then $x = x_M$ is the midpoint of E, and the formula is just Definition 3.24 (global index of a node). If $m > 1$, then x is generated by bisecting some interval $[z', z''] \subset E$, and $\lambda(x) = \max(\lambda(z'), \lambda(z'')) + 1$. Exactly one between z', z'' has been generated by $m - 1$ bisections, whereas the other one has been generated by less than $m - 1$ bisections. Hence we conclude by induction. □

Lemma 8.10 (reducing the global index of hanging nodes). *Let $\mathcal{H} \cap \operatorname{int} E$ contain at least the midpoint x_M of E. Assume that a bisection of some element in $\mathcal{T}$ transforms x_M into a proper node, and let λ_{new} denote the new global-index mapping of the nodes in $\mathcal{H} \cap \operatorname{int} E$ after the bisection. Then we have*

$$\lambda_{\text{new}}(x) \le \lambda(x) - 1 \quad \textit{for all } x \in \mathcal{H} \cap \operatorname{int} E.$$

Proof. If $x = x_M$, then trivially $\lambda_{\text{new}}(x) = 0 \le \lambda(x) - 1$. If $x \in \mathcal{H} \cap \operatorname{int} E$ is contained, say, in (x', x_M) and has been generated by $m > 1$ successive bisections of E, then it is generated by $m - 1$ successive bisections of $[x', x_M]$. Thus, by applying Lemma 8.9, we get

$$\begin{aligned}\lambda_{\text{new}}(x) &\le \max(\lambda_{\text{new}}(x'), \lambda_{\text{new}}(x_M)) + m - 1\\ &= \max(\lambda(x'), 0) + m - 1 = \lambda(x') + m - 1\\ &\le \max((\lambda(x'), \lambda(x'')) + m - 1 = \lambda(x) - 1.\end{aligned}$$

This gives the desired estimate. □

The result just established is the motivation for the proposed refinement strategy, introduced by Beirão da Veiga *et al.* (2024). Indeed, it ensures that in order to reduce the global index of a hanging node sitting on an edge, it is enough to transform the midpoint of the edge into a proper node. The situation is well represented in Figure 8.6.

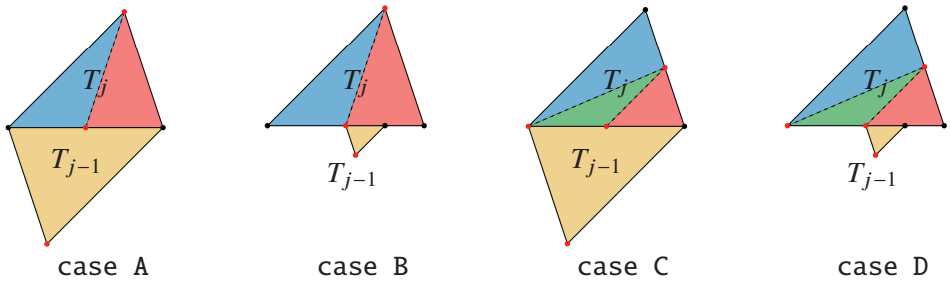


Figure 8.7. Two elements T_{j-1} and T_j in the chain $\mathcal{C}(\mathcal{T}, T)$: T_{j-1} can be bisected in a Λ-admissible way, only after T_j is refined once (cases A and B) or twice (cases C and D).

The following remark will be useful below.

Remark 8.11 (facing element). Given a Λ-admissible mesh $\mathcal{T}$ and $T \in \mathcal{T}$, let $x(T)$ be the midpoint of $E(T)$, and suppose that $\lambda(x(T)) > \Lambda$. Then $x(T)$ is not a node of $\mathcal{T}$, whence the edge $E(T)$ cannot contain any hanging node in its interior. We conclude that there exists a unique adjacent element $\widetilde{T} \in \mathcal{T}$, $\widetilde{T} \neq T$, such that $T \cap \widetilde{T} = E(T)$. This element will be called the element *facing* T, and denoted by $F(T)$.

Given an element $T \in \mathcal{T}$ which has been marked for refinement, we are ready to identify those elements in $\mathcal{T}$ that need be bisected with T in order to create a Λ-admissible refinement of $\mathcal{T}$. Figure 8.7 illustrates the possible situations.

Definition 8.12 (chain of elements to be refined). Define by recurrence the chain of elements starting at T,

$$\mathcal{C}(T, \mathcal{T}) = \{T_0, T_1, \ldots, T_k\},$$

for some $k \geq 0$, as follows. First set $T_0 = T$. Assuming we have defined T_j for $j \geq 0$, then

(i) if $\lambda(x(T_j)) \leq \Lambda$, set $k = j$ and stop;

(ii) if $\lambda(x(T_j)) = \Lambda + 1$ and the facing element $F(T_j)$ is compatible with T_j, set $T_{j+1} = F(T_j)$, $k = j + 1$ and stop;

(iii) if $\lambda(x(T_j)) = \Lambda + 1$ and the facing element $F(T_j)$ is not compatible with T_j, set $T_{j+1} = F(T_j)$ and continue.

Lemma 8.13 (properties of the chain of refinement). *The chain $\mathcal{C}(T, \mathcal{T})$ has finite length; precisely, we have $k \leq g(T) + 1$, where $g(T)$ is the generation of T, defined in Section 3.5. Furthermore, the sequence of element generations $\{g(T_j)\}_{j=0}^{k}$ is not increasing.*

Proof. We claim that step (iii) in Definition 8.12 reduces the generation by at least one. In fact T_j coincides with or is a refinement of a triangle $\hat{T} \in \mathbb{T}$ sharing a full edge with T_{j+1}; thus $g(T_j) \geq g(\hat{T})$. Such a triangle $\hat{T}$ satisfies $g(\hat{T}) = g(T_{j+1}) + 1$, whence

$$g(T_{j+1}) = g(\hat{T}) - 1 \leq g(T_j) - 1. \tag{8.10}$$

Therefore, for as long as case (iii) is active, i.e. for all $j < k$, we have $g(T_j) \leq g(T_0) - j$ and

$$0 \leq g(T_{k-1}) \leq g(T_0) - (k-1),$$

which gives the first statement of the lemma. The monotonicity of $\{g(T_j)\}_{j=0}^{k}$ follows from (8.10) and the fact that $g(T_{k-1}) = g(T_k)$ in case (ii). □

Once the chain $\mathcal{C}(\mathcal{T}, T)$ is defined, all its elements are refined, starting from the last one and proceeding backwards. This is accomplished in the following procedure.

```
[𝒯*] = REFINE_RECURSIVE(𝒯, T, Λ)
    if λ(x(T)) ≤ Λ
        bisect T
        update 𝒯
    else if F(T) is compatible with T
        bisect F(T) and T
        update 𝒯
    else
        [𝒯] = REFINE_RECURSIVE(𝒯, F(T), Λ)
    return 𝒯
```

Proposition 8.14 (properties of REFINE_RECURSIVE). *If $\mathcal{T}$ is Λ-admissible, the call* $[\mathcal{T}_*]$ = REFINE_RECURSIVE$(\mathcal{T}, T, \Lambda)$ *outputs the smallest Λ-admissible refinement $\mathcal{T}_*$ of $\mathcal{T}$ such that T is bisected. In addition, every element $T' \in \mathcal{T}_*$ generated by this call satisfies*

$$g(T') \leq g(T) + 1. \tag{8.11}$$

Proof. Let $\mathcal{C}(T, \mathcal{T}) = \{T_j\}_{j=0}^{k}$ and observe that, for $j \geq 1$, one or two bisections of T_j convert the midpoint of the edge E of T_j shared with T_{j-1} into a proper node. Therefore Lemma 8.10 (reducing the global index of hanging nodes) implies that the global indices of all interior nodes to E decrease by at least 1, and makes the bisection of T_{j-1} Λ-admissible as desired.

To prove (8.11) we take $j \geq 1$ and consider the following two mutually exclusive cases. If T_j and T_{j-1} are compatible, then T_j is replaced by two elements $T' \in \mathcal{T}_*$ of generation

$$g(T') = g(T_j) + 1 \leq g(T) + 1,$$

according to Lemma 8.13 (properties of the chain of refinement). On the other

hand, if T_j and T_{j-1} are not compatible, then T_j is replaced by one element of generation $g(T_j)+1$ and two elements $T' \in \mathcal{T}_*$ of generation

$$g(T') = g(T_j) + 2 \le g(T_{j-1}) + 1 \le g(T) + 1$$

because of (8.10). Finally, the element $T_0 = T$ is replaced by two elements of generation $g(T)+1$. □

If we consider the chains starting at any element $T \in \mathcal{M}$, we obtain the procedure $\mathsf{REFINE}(\mathcal{T}, \mathcal{M}, \Lambda)$, which reads

$[\mathcal{T}_*] = \mathsf{REFINE}(\mathcal{T}, \mathcal{M}, \Lambda)$
  for all $T \in \mathcal{M} \cap \mathcal{T}$ do
    $[\mathcal{T}] = \mathsf{REFINE\_RECURSIVE}(\mathcal{T}, T, \Lambda)$
  return $\mathcal{T}$

and outputs a minimal Λ-admissible mesh $\mathcal{T}_* \ge \mathcal{T}$, refinement of $\mathcal{T}$, so that all marked elements of $\mathcal{M}$ are refined.

Proof of Theorem 3.29 (complexity of REFINE *for Λ-admissible meshes).* The arguments given in Section 8.1.3 for the conforming case can be easily adapted to the current situation. The two crucial properties needed are the relation (8.3) between the distance of two elements in a chain and their generation, which is valid for bisection grids regardless of Λ-admissibility, and the relation (8.11) between generations of elements. □

8.2.3. Mesh overlay and Λ-admissibility

Given two partitions $\mathcal{T}_A$ and $\mathcal{T}_B$, let $\mathcal{T}_A \oplus \mathcal{T}_B$ denote the *overlay* of $\mathcal{T}_A$ and $\mathcal{T}_B$, i.e. the partition whose associated tree is the union of the trees of $\mathcal{T}_A$ and $\mathcal{T}_B$. The following property holds.

Proposition 8.15 (mesh overlay is Λ-admissible). *If $\mathcal{T}_A$ and $\mathcal{T}_B$ are Λ-admissible, then $\mathcal{T}_A \oplus \mathcal{T}_B$ remains Λ-admissible.*

Proof. Let $\mathcal{N}$ denote the set of all nodes obtained by newest-vertex bisection from the root partition $\mathcal{T}_0$. Let $\mathcal{N}_0, \mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{A+B}$, respectively, be the set of nodes of the partitions $\mathcal{T}_0, \mathcal{T}_A, \mathcal{T}_B, \mathcal{T}_A \oplus \mathcal{T}_B$. It is easily seen that for each $x \in \mathcal{N} \setminus \mathcal{N}_0$ there exists a unique set $\mathcal{B}(x) = \{x', x''\} \subset \mathcal{N}$ such that x is generated by the bisection of the segment $[x', x'']$. Furthermore, if $x \in \mathcal{N}_{A+B}$ is a proper node of $\mathcal{T}_A$ (resp. $\mathcal{T}_B$), then it is also a proper node of $\mathcal{T}_A \oplus \mathcal{T}_B$.

Let λ_A, λ_B, λ_{A+B}, respectively, denote the global-index mappings defined on $\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{A+B}$. It is convenient to extend the definition of λ_A and λ_B to the whole $\mathcal{N}_{A+B}$ by setting

$$\lambda_A(x) = +\infty \quad \text{if } x \in \mathcal{N}_{A+B} \setminus \mathcal{N}_A, \qquad \lambda_B(x) = +\infty \quad \text{if } x \in \mathcal{N}_{A+B} \setminus \mathcal{N}_B.$$

With this notation at hand, we are going to prove the inequality

$$\lambda_{A+B}(x) \le \min(\lambda_A(x), \lambda_B(x)) \quad \text{for all } x \in \mathcal{N}_{A+B}, \tag{8.12}$$

from which the thesis immediately follows.

We proceed by induction on $k = \lambda_{A+B}(x)$, $x \in \mathcal{N}_{A+B}$. If $k = 0$, the inequality is trivial since $\lambda_A(x), \lambda_B(x) \geq 0$. So suppose (8.12) holds up to some $k \geq 0$. If $x \in \mathcal{N}_{A+B}$ satisfies $\lambda_{A+B}(x) = k + 1 > 0$, then it is a hanging node of $\mathcal{T}_A \oplus \mathcal{T}_B$ by the definition of global index, so it is a hanging node of $\mathcal{T}_A$ or $\mathcal{T}_B$; without loss of generality, suppose it is a hanging node of $\mathcal{T}_A$. If x is generated by the bisection of the segment $[x', x'']$, then again by the definition of global index it holds that

$$k + 1 = \lambda_{A+B}(x) = \max(\lambda_{A+B}(x'), \lambda_{A+B}(x'')) + 1,$$

which implies

$$\lambda_{A+B}(x') \leq k, \quad \lambda_{A+B}(x'') \leq k.$$

By induction,

$$\lambda_{A+B}(x') \leq \min(\lambda_A(x'), \lambda_B(x')), \quad \lambda_{A+B}(x'') \leq \min(\lambda_A(x''), \lambda_B(x'')),$$

from which we obtain

$$\lambda_{A+B}(x) \leq \max(\lambda_A(x'), \lambda_A(x'')) + 1 = \lambda_A(x)$$

since x is a hanging node of $\mathcal{T}_A$. On the other hand, either $x \in \mathcal{N}_B$ or $x \notin \mathcal{N}_B$. In the latter case $\lambda_B(x) = +\infty$, and (8.12) is proved. In the former case, necessarily x is a hanging node of $\mathcal{T}_B$, hence as above

$$\lambda_{A+B}(x) \leq \max(\lambda_B(x'), \lambda_B(x'')) + 1 = \lambda_B(x),$$

and the thesis is proved. □

9. Discontinuous Galerkin methods

So far we have studied conforming finite element approximations. In this section we present and analyse a two-step AFEM for discontinuous Galerkin methods (dG). The core PDE routine GALERKIN is thereby replaced by GALERKIN-DG, which hinges on the interior penalty discontinuous FEM. We regard dG as a prototype non-conforming method of practical importance and thus the natural first step to investigate the effects of non-conformity within adaptivity.

Finite element functions, being discontinuous, allow for non-conforming meshes to support them. We consider Λ-admissible subdivisions, according to Definition 3.25, where $\Lambda \geq 0$ restricts the level of non-conformity, and let $\mathbb{T}^\Lambda$ denote the collection of all Λ-admissible refinements of an initial subdivision $\mathcal{T}_0$; we refer to Section 8 for details. However, we further assume that $\mathcal{T}_0$ is conforming, to limit the level of technicalities.

There are several novel but characteristic aspects of dG. The most notable one is the appearance of jumps in its formulation, to compensate for the lack of H^1-conformity, as well as in the *a posteriori* upper bounds and the comparison of Galerkin solutions on different meshes. The lack of monotonicity of these jumps

presents a formidable obstruction to the available proof techniques in adaptivity. However, we show in Lemma 9.11 that they are controlled by the residual estimator, thereby enabling us to loosely follow the roadmap of the conforming method, namely Sections 4, 5 and 6. Our approach is based on Bonito and Nochetto (2010) for the one-step AFEM.

The extra flexibility provided by non-conforming meshes, and corresponding discontinuous functions, does not yield a better asymptotic rate in H^1. An early manifestation of this fact, although written for conforming subdivisions, is Proposition 6.2 (equivalence of classes). We extend this result below for general Λ-admissible partitions.

One advantage of the two-step AFEM is that its design and analysis allows for $f \in H^{-1}(\Omega)$ without added difficulties: the function f is replaced by the discrete functional $\widehat{f} = P_{\mathcal{T}} f \in \mathbb{F}_{\mathcal{T}}$, which applies to functions in $\mathbb{S}_{\mathcal{T}}^{n,-1}$. This is in contrast to f, which cannot be applied to functions in $\mathbb{S}_{\mathcal{T}}^{n,-1}$. We exploit this property and thereby extend the applicability of dG to load functions in $H^{-1}(\Omega)$.

Our intention is to analyse the following algorithm for the approximation of the solution $u \in H_0^1(\Omega)$ to the coercive problem (2.7).

Algorithm 9.1 (AFEM-DG-TS). Given an initial tolerance $\varepsilon_0 > 0$, a target tolerance `tol` and initial mesh $\mathcal{T}_0$, as well as a safety parameter $\omega \in (0, 1]$, AFEM-DG-TS is a two-step algorithm alternating between the resolution of data $\mathcal{D}$ and the Galerkin solution $u_{\mathcal{T}}$:

$[\mathcal{T}, u_{\mathcal{T}}] = \mathsf{AFEM\text{-}DG\text{-}TS}(\mathcal{T}_0, \varepsilon_0, \omega, \texttt{tol})$
  set $k = 0$
  do
    $[\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k] = \mathsf{DATA}(\mathcal{T}_k, \mathcal{D}, \omega\, \varepsilon_k)$
    $[\mathcal{T}_{k+1}, u_{k+1}] = \mathsf{GALERKIN\text{-}DG}(\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k, \varepsilon_k)$
    $\varepsilon_{k+1} = \frac{1}{2}\varepsilon_k$
    $k \leftarrow k + 1$
  while $\varepsilon_{k-1} > \texttt{tol}$
  return $\mathcal{T}_k, u_k$

In AFEM-DG-TS, the module $\mathsf{DATA}(\mathcal{T}, \mathcal{D}, \tau)$ is the same as described in Section 5.4.2, except that it produces approximate data $\widehat{\mathcal{D}} \in \mathbb{D}_{\widehat{\mathcal{T}}}$, defined in (5.2), subordinate to a Λ-admissible refinement $\widehat{\mathcal{T}}$ of $\mathcal{T}_0$ for $\Lambda \geq 0$, rather than $\Lambda = 0$ (conforming). The discrete data $\widehat{\mathcal{D}}$ also satisfies the structural assumption (5.51) as discussed in Section 7. It is worth pointing out that the projection $P_{\mathcal{T}}$ used to approximate the right-hand side $f \in H^{-1}(\Omega)$ as well as all the results and algorithms presented in Section 7.4 are restricted to conforming subdivisions $\mathbb{T}$. We briefly discuss in Section 9.7 the extension of $P_{\mathcal{T}}$ and DATA to Λ-admissible subdivisions. Algorithm 9.17 describes the module GALERKIN-DG, the counterpart of GALERKIN for dG formulations.

In Section 9.1 we introduce notation and tools relevant for the characterization of discontinuous finite elements. Among them is the operator $\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}}$, which projects piecewise polynomial functions onto globally continuous piecewise polynomial functions. It is instrumental in deriving a Poincaré inequality on the discontinuous spaces and guarantees that the approximation classes $\mathbb{A}_s^{-1}$ for the solution u using discontinuous approximation on Λ-admissible subdivisions are equivalent to their conforming counterparts $\mathbb{A}_s^0$ introduced in Section 6. We present the discontinuous Galerkin method in Section 9.2. We start with the standard symmetric interior penalty, discuss its drawbacks regarding the unnecessary regularity beyond $H_0^1(\Omega)$ imposed on the exact solution u, and describe a reformulation valid in $H_0^1(\Omega)$. The latter suffers from lack of consistency that needs to be accounted for. The *a posteriori* estimates for the perturbed problem (5.5) are derived in Section 9.3. Because the data is polynomial within GALERKIN-DG, the *a posteriori* estimators are oscillation-free. The GALERKIN-DG module is analysed in Section 9.4 while the discussion of rate-optimality of AFEM-DG-TS is reserved for Section 9.5.

9.1. Discontinuous Galerkin setting

We start with an initial conforming subdivision $\mathcal{T}_0$ made of simplices or hexahedra satisfying Assumption 6.19 (initial labelling). Given $\Lambda > 0$, the refinement procedure REFINE is designed to produce a Λ-admissible sequence of meshes $\mathbb{T}^\Lambda$ obeying Theorem 3.29 (complexity of REFINE for Λ-admissible meshes). From now on, we do not specify the dependence on Λ in the constants.

9.1.1. Basic setting

For $\mathcal{T} \in \mathbb{T}^\Lambda$, we let

$$\mathbb{V}_{\mathcal{T}}^{-1} := \mathbb{S}_{\mathcal{T}}^{n,-1} := \prod_{T \in \mathcal{T}} \mathbb{P}_n(T)$$

denote the space of piecewise polynomials of degree at most $n \geq 1$ subordinate to a partition $\mathcal{T}$. In contrast to the conforming spaces

$$\mathbb{V}_{\mathcal{T}}^{0} := \mathbb{S}_{\mathcal{T}}^{n,0} \cap H_0^1(\Omega)$$

considered earlier, the space $\mathbb{V}_{\mathcal{T}}^{-1}$ consists of (possibly) discontinuous functions across the elements $T \in \mathcal{T}$ and do not necessarily satisfy the vanishing boundary condition. Continuity across elements and vanishing boundary condition will be weakly imposed in the discontinuous Galerkin formulations.

We recall from Section 3.7 that for a proper (interior) node $P \in \mathcal{P}$, the *domain of influence* $\omega_{\mathcal{T}}(P) = \operatorname{supp}(\psi_P)$ is the support of the Lagrange basis function $\psi_P \in \mathbb{V}_{\mathcal{T}}^0$ associated with the node P; we refer to Figure 3.11. Since the sequence of meshes is Λ-admissible, Proposition 3.27 (size of the domain of influence) shows that the number of elements $T \in \mathcal{T}$ such that $T \subset \omega_{\mathcal{T}}(P)$ is uniformly bounded for $\mathcal{T} \in \mathbb{T}^\Lambda$.

The set of faces associated with a subdivision $\mathcal{T} \in \mathbb{T}^\Lambda$ is denoted $\mathcal{F}^+ := \mathcal{F}^+(\mathcal{T})$, and it contains boundary faces as well as interior faces. The set of interior faces is

denoted $\mathcal{F}$. For a face $F \in \mathcal{F}^+$, we let $\{\{v\}\}|_F$ and $[[.]]|_F$ denote the average and jump operators across a face F. To define them precisely, we associate for each face $F \in \mathcal{F}^+$ one of the two unit normals $\boldsymbol{n}_F$. The choice of $\boldsymbol{n}_F$ is fixed but irrelevant as long as the outward pointing normal to Ω is chosen for boundary faces. Let $T_\pm \in \mathcal{T}$ be the elements that share the interior face F, namely $F = T_- \cap T_+$, and let $\mp \boldsymbol{n}_F$ be their outward pointing normals. Now, given $v \in \mathbb{V}_{\mathcal{T}}^{-1}$, let $v_\pm := v|_{T_\pm}$ and define for an interior face F

$$\{\{v\}\}|_F := \frac{1}{2}(v_- + v_+)|_F, \quad [[v]]|_F := (v_- - v_+)|_F, \tag{9.1}$$

By convention, we set $\{\{v\}\}|_F := v_-$ and $[[v]]|_F := v_-$ whenever F is a boundary face. These definitions extend readily to vector-valued functions.

We use the subscript $\mathcal{T}$ to denote the piecewise version of differential operators. For instance, the *broken gradient* $\nabla_{\mathcal{T}}$ is the piecewise gradient $\nabla_{\mathcal{T}} v|_{\mathring{T}} = \nabla v|_{\mathring{T}}$ for $T \in \mathcal{T}$ and $v \in \mathbb{V}_{\mathcal{T}}^{-1}$. For simplicity, we write

$$\|v\|^2_{L^2(\tau)} := \sum_{T \in \tau} \|v\|^2_{L^2(T)}$$

for any subset $\tau \subset \mathcal{T}$ of elements, and

$$\|v\|^2_{L^2(\sigma)} := \sum_{F \in \sigma} \|v\|^2_{L^2(F)}$$

for any subset $\sigma \subset \mathcal{F}^+$ of faces. We also define a mesh size function $h := h_{\mathcal{T}} : \overline{\Omega} \to (0, \infty)$ such that $h|_{\mathring{T}} \approx \operatorname{diam}(T)$ for $T \in \mathcal{T}$ and $h|_F \approx \operatorname{diam}(F)$ for $F \in \mathcal{F}^+$. With this notation at hand, the broken H^1 space

$$\mathbb{E}_{\mathcal{T}} := H^1(\Omega; \mathcal{T}) = \prod_{T \in \mathcal{T}} H^1(T)$$

is endowed with the mesh-dependent seminorm

$$\|v\|^2_{a,\mathcal{T}} := \|\nabla_{\mathcal{T}} v\|^2_{L^2(\mathcal{T})} + a\|h^{-1/2}[[v]]\|^2_{L^2(\mathcal{F}^+)}, \tag{9.2}$$

where a is some positive parameter. We will prove below that this is indeed a norm.

With this notation we can extend functionals $\widehat{f} \in \mathbb{F}_{\widehat{\mathcal{T}}}$ in Definition 4.17 to $\mathbb{V}_{\mathcal{T}}^{-1}$ for $\mathcal{T} \geq \widehat{\mathcal{T}}$. Before doing so, recall that for $\widehat{f} \in \mathbb{F}_{\widehat{\mathcal{T}}}$ and $v \in H^1_0(\Omega)$ we have

$$\langle \widehat{f}, v \rangle = \sum_{\widehat{T} \in \widehat{\mathcal{T}}} \int_{\widehat{T}} \widehat{f} v + \sum_{\widehat{F} \in \mathcal{F}(\widehat{\mathcal{T}})} \int_{\widehat{F}} \widehat{f} v,$$

where, compared to Definition 4.17, we slightly abused the notation

$$\widehat{f}|_{\widehat{T}} = \widehat{f}_T \in \mathbb{P}_{2n-2}(\widehat{T}) \quad \widehat{f}|_{\widehat{F}} = \widehat{f}_{\widehat{F}} \in \mathbb{P}_{2n-1}(\widehat{F}).$$

In view of this, we can extend the duality pairing to $\mathbb{V}_{\mathcal{T}}^{-1}$ by setting

$$\langle \widehat{f}, v \rangle_{\mathcal{T}} := \sum_{T \in \mathcal{T}} \int_T \widehat{f} v + \sum_{F \in \mathcal{F}} \int_F \widehat{f} \{\{v\}\} \tag{9.3}$$

so that consistency in $H^1_0(\Omega)$ is preserved, that is,

$$\langle \widehat{f}, v\rangle_{\mathcal{T}} = \langle \widehat{f}, v\rangle \quad \text{for all } v \in H^1_0(\Omega). \tag{9.4}$$

9.1.2. Interpolation operator $\mathcal{I}^{\mathrm{dG}}_{\mathcal{T}}$

We shall need the interpolation operator $\mathcal{I}^{\mathrm{dG}}_{\mathcal{T}} \colon \mathbb{E}_{\mathcal{T}} \to \mathbb{V}^0_{\mathcal{T}}$ from Bonito and Nochetto (2010). Its construction is based on an original idea of Clément (1975); see also Bernardi and Girault (1998) and other alternatives (Brenner 2003, Bonito, Nochetto and Ntogkas 2021).

Before embarking on the construction of $\mathcal{I}^{\mathrm{dG}}_{\mathcal{T}}$, we introduce some notation. For an interior or boundary proper node $P \in \mathcal{P}$ of the subdivision $\mathcal{T}$, we let

$$\mathbb{V}^0_{\omega_{\mathcal{T}}(P)} := H^1_0(\Omega) \cap \prod_{T \subset \omega_{\mathcal{T}}(P)} \mathbb{P}_n(T) \tag{9.5}$$

denote the space of continuous piecewise polynomial with support on the domain of influence $\omega_{\mathcal{T}}(P)$ of P and vanishing on $\partial\Omega$. When the underlying grid $\mathcal{T}$ is clear from the context, we will simplify the notation and write $\mathbb{V}^0_P := \mathbb{V}^0_{\omega_{\mathcal{T}}(P)}$ and $\omega_P := \omega_{\mathcal{T}}(P)$; we refer to Figures 3.9 and 8.6.

We now construct $\mathcal{I}^{\mathrm{dG}}_{\mathcal{T}}$ in two steps. First, we define $V_P \in \mathbb{V}^0_P$ locally as satisfying

$$\int_{\omega_P} (v - V_P) w = 0 \quad \text{for all } w \in \mathbb{V}^0_P. \tag{9.6}$$

The value $V_P(P)$ is then used as the nodal value of $\mathcal{I}^{\mathrm{dG}}_{\mathcal{T}} v$, namely

$$\mathcal{I}^{\mathrm{dG}}_{\mathcal{T}} v := \sum_{P \in \mathcal{P}} V_P(P) \psi_P, \tag{9.7}$$

and we recall that $\{\psi_P\}_{P\in\mathcal{P}}$ is a basis of $\mathbb{V}^0_{\mathcal{T}}$ (see Section 3); note that $\mathcal{I}^{\mathrm{dG}}_{\mathcal{T}} v = 0$ on $\partial\Omega$ for all $v \in \mathbb{E}_{\mathcal{T}}$. Moreover, including boundary proper nodes in the definition (9.6)–(9.7) and replacing $H^1_0(\Omega)$ with $H^1(\Omega)$ in the definition (9.5), $\mathcal{I}^{\mathrm{dG}}_{\mathcal{T}}$ easily extends to $\mathbb{S}^{n,0}_{\mathcal{T}}$ without zero trace; we denote this operator by $\mathcal{I}^{\mathrm{dG}}_{\mathcal{T},+} \colon \mathbb{E}_{\mathcal{T}} \to \mathbb{S}^{n,0}_{\mathcal{T}}$. An immediate property of $\mathcal{I}^{\mathrm{dG}}_{\mathcal{T}}$ is *local invariance*,

$$v \in \mathbb{V}^0_{\omega_T} \quad \Rightarrow \quad v = \mathcal{I}^{\mathrm{dG}}_{\mathcal{T}} v \quad \text{in } T, \tag{9.8}$$

where $\omega_T := \bigcup\{\omega_P \mid P \in \mathcal{P}, T \subset \omega_P\}$; a similar property is valid for $\mathcal{I}^{\mathrm{dG}}_{\mathcal{T},+}$. We next gather a few more properties satisfied by $\mathcal{I}^{\mathrm{dG}}_{\mathcal{T}}$.

Lemma 9.2 (interpolation operator). *Let Assumption 6.19 (initial labelling) hold and let $\mathcal{T} \in \mathbb{T}^{\Lambda}$. For $v \in H^1_0(\Omega)$,*

$$\|v - \mathcal{I}^{\mathrm{dG}}_{\mathcal{T}} v\|_{L^2(T)} \lesssim \|h\nabla v\|_{L^2(\omega_T)}, \quad \|\nabla \mathcal{I}^{\mathrm{dG}}_{\mathcal{T}} v\|_{L^2(T)} \lesssim \|\nabla v\|_{L^2(\omega_T)}, \tag{9.9}$$

where ω_T is defined above. Instead, for $v \in \mathbb{E}_{\mathcal{T}}$,

$$\begin{aligned}\|v - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v\|_{L^2(T)} &+ \|h\nabla_{\mathcal{T}}\big(v - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v\big)\|_{L^2(T)} \\ &\lesssim \|h^{1/2}[[v]]\|_{L^2(\mathcal{F}^+\cap\omega_T)} + \|h\nabla_{\mathcal{T}}(v - \Pi_{\mathcal{T}} v)\|_{L^2(\omega_T)},\end{aligned} \tag{9.10}$$

where $\Pi_{\mathcal{T}}$ is the L^2 projection operator onto $\mathbb{V}_{\mathcal{T}}^{-1} = \mathbb{S}_{\mathcal{T}}^{n,-1}$.

Proof. We start with (9.9) and let $v \in H_0^1(\Omega)$. The definition (9.6) of the local projection $V_P \in \mathbb{V}_P^0$ yields for all $P \in \mathcal{P}$

$$\|V_P\|_{L^2(\omega_P)} \le \|v\|_{L^2(\omega_P)} \quad \Rightarrow \quad \|V_P\|_{L^\infty(\omega_P)} \lesssim \operatorname{diam}(\omega_P)^{-d/2}\|v\|_{L^2(\omega_P)}.$$

Proposition 3.27 (size of the domain of influence) gives $\operatorname{diam}\omega_T \le Ch_T$, whence the number of ω_P containing T is uniformly bounded. Combining this with the definition (9.7) of $\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}}$ implies

$$\|\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v\|_{L^2(T)} \lesssim \sum_{P\in\mathcal{P}:\, T\subset\omega_P} |V_P(P)|\,\|\psi_P\|_{L^2(T)} \lesssim \|v\|_{L^2(\omega_T)} \quad \text{for all } T \in \mathcal{T}. \tag{9.11}$$

Since $\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}}$ reproduces constants exactly locally, according to (9.8), the first relation in (9.9) follows from invoking the local L^2-stability property (9.11) together with Proposition 6.34 (Bramble–Hilbert for Sobolev spaces). The second relation is proved using the same arguments and an inverse inequality

$$\|\nabla\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v\|_{L^2(T)} \lesssim h_T^{-1} \inf_{v_0\in\mathbb{R}} \|\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}}(v - v_0)\|_{L^2(T)} \lesssim \|\nabla v\|_{L^2(\omega_T)}. \tag{9.12}$$

We now consider $v \in \mathbb{E}_{\mathcal{T}}$ and let $\widehat{v} = \Pi_{\mathcal{T}} v \in \mathbb{V}_{\mathcal{T}}^{-1}$. We intend to prove (9.10) by dealing with $v - \widehat{v}$ and $\widehat{v}$ separately and applying the triangle inequality. Since $v - \widehat{v}$ has zero mean in T according to (5.66), we apply Lemma 2.3 (second Poincaré inequality) to deduce

$$\|v - \widehat{v}\|_{L^2(T)} \lesssim h_T\,\|\nabla(v - \widehat{v})\|_{L^2(T)},$$

whence, combining an inverse estimate with (9.11), we further infer that

$$h_T\,\|\nabla\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}}(v-\widehat{v})\|_{L^2(T)} \lesssim \|\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}}(v-\widehat{v})\|_{L^2(T)} \lesssim \|v-\widehat{v}\|_{L^2(\omega_T)} \lesssim h_T\,\|\nabla(v-\widehat{v})\|_{L^2(\omega_T)}.$$

This argument yields the inequality (9.10) for $v - \widehat{v}$. It remains to deal with $\widehat{v}$.

We scale ω_T to a reference domain with unit diameter. Estimate (3.49) on the size of the domains of influence guarantees that the number of such reference patches is uniformly finite over $\mathbb{T}^\Lambda$. We relabel $\widehat{v}$ as v and examine the seminorm $\|[[v]]\|_{L^2(\mathcal{F}^+\cap\omega_P)}$ on the space of discontinuous piecewise polynomials

$$\{v \in \Pi_{T\subset\omega_P}\mathbb{P}_n(T) \mid V_P = 0\},$$

where V_P is defined by (9.6). If this seminorm vanishes then v is continuous in ω_P and thus $v \in \mathbb{V}_P^0$, whence the seminorm dominates any norm in this finite-dimensional space. Consequently, scaling back gives

$$\|v - V_P\|_{L^2(\omega_P)} + \|h\nabla_{\mathcal{T}}(v - V_P)\|_{L^2(\omega_P)} \lesssim \|h^{1/2}[[v]]\|_{L^2(\mathcal{F}^+\cap\omega_P)}. \tag{9.13}$$

We now deduce corresponding estimates for $\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}}$. For $T \in \mathcal{T}$ and $P, Q \in \omega_T \cap \mathcal{P}$, (9.13) implies that

$$\|V_P - V_Q\|_{L^2(T)} + \|h\nabla_{\mathcal{T}}(V_P - V_Q)\|_{L^2(T)} \lesssim \|h^{1/2}[[v]]\|_{L^2(\mathcal{F}^+\cap\omega_T)}. \tag{9.14}$$

Consequently, the definition (9.7) of $\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}}$ yields

$$v - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v = v - \sum_{P\in\omega_T\cap\mathcal{P}} V_P\psi_P = (v - V_Q) - \sum_{P\in\omega_T\cap\mathcal{P}} (V_P - V_Q)\psi_P,$$

which, combined with (9.13) and (9.14), implies

$$\|v - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v\|_{L^2(T)} + \|h\nabla_{\mathcal{T}}\big(v - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v\big)\|_{L^2(T)} \lesssim \|h^{1/2}[[v]]\|_{L^2(\mathcal{F}^+\cap\omega_T)}.$$

This is the desired estimate (9.10) for $v = \widehat{v} \in \mathbb{V}_{\mathcal{T}}^{-1}$. To finish the proof we still need to express the right-hand side of the last inequality in terms of $v \in \mathbb{E}_{\mathcal{T}}$. Applying the triangle inequality, we are left with estimating $\|[[v - \widehat{v}]]\|_{L^2(F)}$ for any $F \in \mathcal{F}^+ \cap \omega_T$. If $T_F \in \mathcal{T}$ is an element within ω_T that contains F in its boundary, we employ the scaled trace inequality to arrive at

$$h_T^{1/2}\|v - \widehat{v}\|_{L^2(F)} \lesssim h_T\|\nabla(v - \widehat{v})\|_{L^2(T_F)} + \|v - \widehat{v}\|_{L^2(T_F)} \lesssim h_T\|\nabla(v - \widehat{v})\|_{L^2(T_F)}.$$

Finally, collecting all the estimates completes the proof. □

We now discuss consequences of Lemma 9.2. The first one is that jumps are solely responsible for controlling the discrepancy between $v \in \mathbb{V}_{\mathcal{T}}^{-1}$ and $\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v \in \mathbb{V}_{\mathcal{T}}^{0}$:

$$\|v - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v\|_{L^2(T)} + \|h\nabla_{\mathcal{T}}\big(v - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v\big)\|_{L^2(T)} \lesssim \|h^{1/2}[[v]]\|_{L^2(\mathcal{F}^+\cap\omega_T)}, \tag{9.15}$$

because $\Pi_{\mathcal{T}} v = v$ in ω_T. We next observe that (9.10) is also valid for $\mathcal{I}_{\mathcal{T},+}^{\mathrm{dG}}$ with the same proof. We can thus apply (9.10) for $\mathcal{I}_{\mathcal{T},+}^{\mathrm{dG}}$ to $w = v - \mathcal{I}_{\mathcal{T},+}^{\mathrm{dG}} v$, use the invariance of $\mathcal{I}_{\mathcal{T},+}^{\mathrm{dG}}$ in $\mathbb{S}_{\mathcal{T}}^{n,0}$, and its continuity across internal faces in $\mathcal{F}$, to deduce

$$\begin{aligned}\|v - \mathcal{I}_{\mathcal{T},+}^{\mathrm{dG}} v\|_{L^2(T)} + \|h\nabla_{\mathcal{T}}\big(v - \mathcal{I}_{\mathcal{T},+}^{\mathrm{dG}} v\big)\|_{L^2(T)} &\lesssim \|h\nabla_{\mathcal{T}}(v - \Pi_{\mathcal{T}} v)\|_{L^2(\omega_T)} \\ &\quad + \|h^{1/2}[[v]]\|_{L^2(\mathcal{F}\cap\omega_T)} + \|h^{1/2}\big(v - \mathcal{I}_{\mathcal{T},+}^{\mathrm{dG}} v\big)\|_{L^2(\partial\Omega\cap\omega_T)}.\end{aligned} \tag{9.16}$$

A third consequence of (9.10) is the following Poincaré-type inequality on $\mathbb{E}_{\mathcal{T}}$.

Lemma 9.3 (Poincaré-type inequality on $\mathbb{E}_{\mathcal{T}}$). *Let $\mathcal{T} \in \mathbb{T}^{\Lambda}$ be a Λ-admissible refinement of $\mathcal{T}_0$ satisfying Assumption 6.19 (initial labelling). There exists $C_P = C_P(\Omega, \mathcal{T}_0)$, such that, for all $v \in \mathbb{E}_{\mathcal{T}}$,*

$$\|v\|_{L^2(\Omega)} \le C_P\big(\|\nabla_{\mathcal{T}} v\|_{L^2\mathcal{T})} + \|h^{-1/2}[[v]]\|_{L^2(\mathcal{F})} + \|h^{-1/2} v\|_{L^2(\partial\Omega)}\big). \tag{9.17}$$

In particular, if $v = 0$ on $\partial\Omega$ then (9.17) is a dG version of (2.2).

Proof. We argue locally with (9.10). First we realize that an argument similar to (9.12) yields $\|\nabla_{\mathcal{T}}(v - \Pi_{\mathcal{T}} v)\|_{L^2(\omega_T)} \lesssim \|\nabla_{\mathcal{T}} v\|_{L^2(\omega_T)}$, whence adding over $T \in \mathcal{T}$ we obtain

$$\|v - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v\|_{L^2(\Omega)} + \|\nabla\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v\|_{L^2(\Omega)} \lesssim \|\nabla_{\mathcal{T}} v\|_{L^2(\Omega)} + \|h^{-1/2}[[v]]\|_{L^2(\mathcal{F}^+)}.$$

It thus suffices to write

$$\|v\|_{L^2(\Omega)} \le \|\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v\|_{L^2(\Omega)} + \|v - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v\|_{L^2(\Omega)},$$

and invoke (2.2) for $\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v \in H_0^1(\Omega)$ together with the preceding inequality. □

Another important property obtained using the interpolation operator $\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}}$ is that the approximation classes $\mathbb{A}_s^0 := \mathbb{A}_s(H_0^1(\Omega); \mathcal{T}_0)$ defined using globally continuous piecewise polynomial approximations of degree $\le n$ on conforming subdivisions are equivalent to those without global continuity on Λ-admissible subdivisions $\mathcal{T} \in \mathbb{T}^\Lambda$, provided $\|\cdot\|_{1,\mathcal{T}}$ (defined in (9.2)) is used as norm on $\mathbb{E}_{\mathcal{T}}$. We define

$$\sigma_N^{n,-1}(v) := \inf_{\mathcal{T} \in \mathbb{T}_N^\Lambda} \inf_{v_{\mathcal{T}} \in \mathbb{S}_{\mathcal{T}}^{n,-1}} \|v - v_{\mathcal{T}}\|_{1,\mathcal{T}} \tag{9.18}$$

and $\mathbb{A}_s^{-1} := \mathbb{A}_s^{-1}(H_0^1(\Omega); \mathcal{T}_0)$ to be the class of functions $v \in H_0^1(\Omega)$ such that

$$|v|_{\mathbb{A}_s^{-1}} := \sup_{N \ge \#\mathcal{T}_0} \left(N^s \sigma_N^{n,-1}(v)\right) < \infty \quad \Rightarrow \quad \sigma_N^{n,-1}(v) \le |v|_{\mathbb{A}_s^{-1}} N^{-s}.$$

Note that the scaling parameter a for jumps in the definition of $\sigma_N^{n,-1}$ is just $a = 1$.

The following result can be traced back to Bonito and Nochetto (2010).

Proposition 9.4 (equivalence of classes for u). *Let $\mathcal{T}_0$ be an initial conforming subdivision satisfying Assumption 6.19 (initial labelling). There are two constants $m \in \mathbb{N}$ and $C \ge 1$ such that, for all $N \ge \#\mathcal{T}_0$ and all $v \in H_0^1(\Omega)$,*

$$\sigma_N^{n,-1}(v) \le \sigma_N^{n,0}(v) \quad \textit{and} \quad \sigma_{mN}^{n,0}(v) \le C\sigma_N^{n,-1}(v).$$

In particular, the approximation classes coincide: $\mathbb{A}_s^0 \equiv \mathbb{A}_s^{-1}$, $s \ge 0$.

Proof. We start with the first inequality. For $v \in H_0^1(\Omega)$ and $N \ge \#\mathcal{T}_0$, we let $\mathcal{T} \in \mathbb{T}_N$ be a conforming subdivision of $\mathcal{T}_0$ and let $v_{\mathcal{T}}^0 \in \mathbb{V}_{\mathcal{T}}^0 \subset \mathbb{V}_{\mathcal{T}}^{-1}$ be such that

$$\sigma_N^{n,0}(v) = |v - v_{\mathcal{T}}^0|_{H_0^1(\Omega)}.$$

Because $v - v_{\mathcal{T}}^0 \in H_0^1(\Omega)$, we have $\|v - v_{\mathcal{T}}^0\|_{1,\mathcal{T}} = |v - v_{\mathcal{T}}^0|_{H_0^1(\Omega)}$ and thus

$$\sigma_N^{n,-1}(v) \le |v - v_{\mathcal{T}}^0|_{H_0^1(\Omega)} = \sigma_N^{n,0}(v).$$

We now prove the second inequality. For $v \in H_0^1(\Omega)$ and $N \ge \#\mathcal{T}_0$, let $\mathcal{T} \in \mathbb{T}_N^\Lambda$ be a Λ-admissible mesh with N elements and let $v_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}^{-1}$ be such that

$$\|v - v_{\mathcal{T}}\|_{1,\mathcal{T}} = \sigma_N^{n,-1}(v).$$

We first show that $\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}^0$ satisfies

$$|v - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v_{\mathcal{T}}|_{H_0^1(\Omega)} \lesssim \sigma_N^{n,-1}(v).$$

Indeed, using the triangle inequality we obtain

$$\|v - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v_{\mathcal{T}}\|_{1,\mathcal{T}} \le \|v - v_{\mathcal{T}}\|_{1,\mathcal{T}} + \|v_{\mathcal{T}} - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v_{\mathcal{T}}\|_{1,\mathcal{T}}.$$

Interpolation estimate (9.15) yields

$$\|v_{\mathcal{T}} - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v_{\mathcal{T}}\|_{1,\mathcal{T}} \lesssim \|h^{-1/2}[[v_{\mathcal{T}}]]\|_{L^2(\mathcal{F}^+)}, \tag{9.19}$$

because $v_{\mathcal{T}} - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}^{-1}$, whence

$$|v - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v_{\mathcal{T}}|_{H_0^1(\Omega)} = \|v - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v_{\mathcal{T}}\|_{1,\mathcal{T}} \le C\sigma_N^{n,-1}(v)$$

as claimed for a constant $C \ge 1$ independent of v and N. To assert an estimate on $\sigma_N^{n,0}(v)$, we now exhibit a conforming refinement $\overline{\mathcal{T}}$ of $\mathcal{T}$ with a comparable number of elements. To do this, we note that because $\mathcal{T} \in \mathbb{T}^\Lambda$ is Λ-admissible, it is the product of successive calls $[\mathcal{T}_j] = \mathsf{REFINE}(\mathcal{T}_{j-1}, T_{j-1})$, $j = 1, \dots, J$, where $\mathcal{T}_j$ is the smallest Λ-*admissible* refinement of $\mathcal{T}_{j-1}$ such that the element $T_{j-1} \in \mathcal{T}_{j-1}$ is bisected once. We now let $\overline{\mathcal{T}} \in \mathbb{T}$ be the conforming subdivision obtained from the successive calls $[\overline{\mathcal{T}}_j] = \mathsf{REFINE}(\overline{\mathcal{T}}_{j-1}, \{T_{j-1}\} \cap \overline{\mathcal{T}}_{j-1})$ with $\overline{\mathcal{T}}_0 = \mathcal{T}_0$, but where this time REFINE produces the smallest *conforming* refinement of $\overline{\mathcal{T}}_{j-1}$, where the element of T_{j-1} is bisected once if $T_{j-1} \in \overline{\mathcal{T}}_{j-1}$ or otherwise $\overline{\mathcal{T}}_j = \overline{\mathcal{T}}_{j-1}$. A simple induction argument, exploiting the minimality of the meshes generated by REFINE, reveals that $\overline{\mathcal{T}}_j \ge \mathcal{T}_j$ for $0 \le j \le J$. Consequently, Theorem 3.16 (complexity of REFINE) guarantees that

$$\#\overline{\mathcal{T}} - \#\mathcal{T}_0 \le D \sum_{j=0}^{J-1} \#(\{T_{j-1}\} \cap \overline{\mathcal{T}}_{j-1}) \le DJ \le D(\#\mathcal{T} - \#\mathcal{T}_0),$$

whence $\#\overline{\mathcal{T}} \le D\#\mathcal{T} \le mN$ with $m := \lceil D \rceil$ because $D \ge 1$.

Therefore $\mathbb{V}_{\mathcal{T}}^0 \subset \mathbb{V}_{\overline{\mathcal{T}}}^0$ because $\overline{\mathcal{T}}$ is a conforming refinement of $\mathcal{T}$. Since $\#\overline{\mathcal{T}} \le mN$ and $\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}^0$, we deduce

$$\sigma_{mN}^{n,0}(v) \le |v - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} v_{\mathcal{T}}|_{H_0^1(\Omega)} \le C\sigma_N^{n,-1}(v),$$

which is the desired inequality. Finally, the equivalence of moduli of approximation yields $\mathbb{A}_s^0 \equiv \mathbb{A}_s^{-1}$ and completes the proof. □

Remark 9.5 (equivalence of classes for $\mathcal{D}$). The approximation classes for data $\mathcal{D} = (\boldsymbol{A}, c, f)$, namely $\mathbb{M}_s((L^r(\Omega))^{d\times d}); \mathcal{T}_0$, $\mathbb{C}_s(L^q(\Omega); \mathcal{T}_0)$ and $\mathbb{F}_s(H^{-1}(\Omega); \mathcal{T}_0)$, are defined for conforming subdivisions in Section 6. However, repeating the construction of the smallest conforming refinement $\overline{\mathcal{T}}$ of any Λ-admissible subdivision $\mathcal{T}$, and using the fact that $\#\overline{\mathcal{T}} \approx \#\mathcal{T}$ proved above, we deduce that these classes are equivalent to their counterparts on non-conforming meshes. Therefore we do not repeat the proof here, and from now on we use the same notation to denote the approximation classes on Λ-admissible subdivisions.

9.2. *Discontinuous Galerkin formulation*

This section discusses the SOLVE routine at the core of the module $\mathsf{GALERKIN\text{-}DG}$. Recall that within the two-step method $\mathsf{AFEM\text{-}DG\text{-}TS}$, data $\mathcal{D} = (\boldsymbol{A}, c, f)$ is approximated by $\widehat{\mathcal{D}} = (\widehat{\boldsymbol{A}}, \widehat{c}, \widehat{f}) \in \mathbb{D}_{\widehat{\mathcal{T}}}$ subordinate to a partition $\widehat{\mathcal{T}} \in \mathbb{T}^\Lambda$. For a subdivision

$\mathcal{T} \in \mathbb{T}^\Lambda$, $\mathcal{T} \geq \widehat{\mathcal{T}}$, the Galerkin solution $[u_\mathcal{T}] = \mathsf{SOLVE}(\mathcal{T})$ is constructed to approximate $\widehat{u} = u(\widehat{\mathcal{D}}) \in H^1_0(\Omega)$, the exact weak solution of the perturbed problem (5.5) with approximate data $\widehat{\mathcal{D}} = (\widehat{\boldsymbol{A}}, \widehat{c}, \widehat{f})$ constructed using Algorithm 7.23 (DATA). Corollary 7.24 (performance of DATA) guarantees that the output $[\widehat{\mathcal{D}}, \widehat{\mathcal{T}}]$ of DATA satisfies the structural assumption

$$\widehat{\boldsymbol{A}} \in M(\widehat{\alpha}_1, \widehat{\alpha}_2), \quad \widehat{c} \in R(\widehat{c}_1, \widehat{c}_2) \tag{9.20}$$

with $0 < \widehat{\alpha}_1 \leq \widehat{\alpha}_2$ and $-\widehat{\alpha}_1/(2C_P^2) \leq \widehat{c}_1 \leq \widehat{c}_2$ upon replacing the Poincaré constant C_P with the larger constant C_P appearing in Lemma 9.3 in Algorithm 7.14 (CONSTRAINT-c). We do not specify the dependence on $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{c}_1$ and $\widehat{c}_2$ of the constants appearing in the analysis below. We also emphasize that the constants involved in (9.20) do not depend on $\widehat{\mathcal{T}}$ and are thus uniform among all the discrete data constructed within AFEM-DG-TS.

Relation (9.20) not only ensures the existence and uniqueness of a solution $\widehat{u} \in H^1_0(\Omega)$ satisfying the perturbed problem (5.5) but also, as we shall see in Corollary 9.8, the existence and uniqueness of its discontinuous Galerkin approximation. We first present the standard symmetric interior penalty method and point out that its consistency requires the exact solution $u \in H^s(\Omega)$, $s > 3/2$. To circumvent this rather restrictive assumption, we introduce lifting operators allowing a reformulation valid in $H^1(\Omega)$. However, this reformulation is only consistent on the conforming subspace $\mathbb{V}^0_\mathcal{T} = \mathbb{V}^{-1}_\mathcal{T} \cap H^1_0(\Omega)$, and requires our analysis to decompose the discrete space $\mathbb{V}^{-1}_\mathcal{T}$ into $\mathbb{V}^0_\mathcal{T}$ and its complement $\mathbb{V}^\perp_\mathcal{T}$ with respect to an appropriate scalar product.

9.2.1. The symmetric interior penalty method

The symmetric interior penalty (SIP) formulation is the most standard discontinuous Galerkin method. For $\mathcal{T} \in \mathbb{T}^\Lambda$, it consists in finding $u_\mathcal{T} \in \mathbb{V}^{-1}_\mathcal{T}$ satisfying

$$\mathcal{B}_\mathcal{T}[u_\mathcal{T}, v] = \langle \widehat{f}, v \rangle_\mathcal{T} \quad \text{for all } v \in \mathbb{V}^{-1}_\mathcal{T}, \tag{9.21}$$

where $\mathcal{B}_\mathcal{T} \colon \mathbb{V}^{-1}_\mathcal{T} \times \mathbb{V}^{-1}_\mathcal{T} \to \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned}\mathcal{B}_\mathcal{T}[w, v] \coloneqq \int_\Omega (\nabla_\mathcal{T} v \cdot \widehat{\boldsymbol{A}} \nabla_\mathcal{T} w + \widehat{c} w v) - \sum_{F \in \mathcal{F}^+} \int_F [\![v]\!] \boldsymbol{n}_F \cdot \{\!\{\widehat{\boldsymbol{A}} \nabla_\mathcal{T} w\}\!\} \\ - \sum_{F \in \mathcal{F}^+} \int_F [\![w]\!] \boldsymbol{n}_F \cdot \{\!\{\widehat{\boldsymbol{A}} \nabla_\mathcal{T} v\}\!\} + \kappa \sum_{F \in \mathcal{F}^+} \int_F h_F^{-1} [\![w]\!] [\![v]\!].\end{aligned} \tag{9.22}$$

The parameter $\kappa > 0$ is responsible for keeping the discontinuity of the Galerkin solution under control and its value is discussed below. Unless specified otherwise, all the constants appearing in the discussion below are independent of κ, and the notation $A \lesssim B$ signifies $A \leq CB$ with a constant C independent of the discretization parameters and κ.

A few comments regarding the weak formulation (9.21) are in order. An integration by parts reveals that the method is consistent whenever the exact solution

satisfies the additional regularity $u \in H^s(\Omega)$, $s > 3/2$. However, we do not make this assumption in the analysis below but rather extend the formulation to the energy space $\mathbb{E}_{\mathcal{T}} \supset \mathbb{V}_{\mathcal{T}}^{-1}$ using lifting operators. The same integration by parts also indicates that the term $\sum_{F\in\mathcal{F}^+}\int_F [\![w]\!]\boldsymbol{n}_F \cdot \{\{\widehat{\boldsymbol{A}}\nabla v\}\}$ is not necessary but included to achieve a symmetric formulation. Recall that $\widehat{\boldsymbol{A}}$ constructed by DATA is symmetric. In addition, the presence of $\langle \widehat{f}, v\rangle_{\mathcal{T}}$ is not standard but allows for right-hand sides $\widehat{f} \in \mathbb{F}_{\widehat{\mathcal{T}}}$ and in turn for $f \in H^{-1}(\Omega)$ within the AFEM-DG-TS algorithm.

9.2.2. Lifting operators

The interior penalty bilinear form (9.22) includes inter-element terms

$$\sum_{F\in\mathcal{F}^+}\int_F [\![v]\!]\boldsymbol{n}_F \cdot \{\{\widehat{\boldsymbol{A}}\nabla_{\mathcal{T}} w\}\} + \sum_{F\in\mathcal{F}^+}\int_F [\![w]\!]\boldsymbol{n}_F \cdot \{\{\widehat{\boldsymbol{A}}\nabla_{\mathcal{T}} v\}\}, \tag{9.23}$$

which are not defined on $H^1(\Omega)$ but on $H^s(\Omega)$, $s > 3/2$. In turn, the method is consistent when $u \in H^s(\Omega)$, $s > 3/2$. The key ingredient to extending $\mathcal{B}_{\mathcal{T}}[w, v]$ to $\mathbb{E}_{\mathcal{T}} \times \mathbb{E}_{\mathcal{T}}$ without additional regularity is a lifting operator (Brezzi *et al.* 2000, Arnold, Brezzi, Cockburn and Marini 2002, Perugia and Schötzau 2003, Houston, Schötzau and Wihler 2004, 2007, Bonito and Nochetto 2010) introduced in this section.

For $n' > 0$, we define $\mathcal{L}_{\mathcal{T}}^{n'} \colon \mathbb{E}_{\mathcal{T}} \to [\mathbb{S}_{\mathcal{T}}^{n',-1}]^d$ by the relations

$$\int_\Omega \mathcal{L}_{\mathcal{T}}^{n'}[v] \cdot \widehat{\boldsymbol{A}}\boldsymbol{w} = \sum_{F\in\mathcal{F}^+}\int_F [\![v]\!]\boldsymbol{n}_F \cdot \{\{\widehat{\boldsymbol{A}}\boldsymbol{w}\}\} \quad \text{for all } \boldsymbol{w} \in \left[\mathbb{S}_{\mathcal{T}}^{n',-1}\right]^d. \tag{9.24}$$

From this definition, we easily deduce an L^2-stability estimate.

Lemma 9.6 (stability of lift). *Let $\mathcal{T} \in \mathbb{T}^\Lambda$ be a Λ-admissible subdivision of $\mathcal{T}_0$ satisfying Assumption 6.19 (initial labelling). Assume $\widehat{\boldsymbol{A}} \in M(\widehat{\alpha}_1, \widehat{\alpha}_2)$ with $0 < \widehat{\alpha}_1 \le \widehat{\alpha}_2$. For $n' \ge 0$ and all $v \in \mathbb{S}_{\mathcal{T}}^{n',-1}$,*

$$\|\mathcal{L}_{\mathcal{T}}^{n'}[v]\|_{L^2(\Omega)} \le C\|h^{-1/2}[\![v]\!]\|_{L^2(\mathcal{F}^+)}, \tag{9.25}$$

where $C = C(\widehat{\alpha}_2/\widehat{\alpha}_1, \mathcal{T}_0, n')$.

Proof. Let $v \in \mathbb{S}_{\mathcal{T}}^{n',-1}$ and set $\boldsymbol{w} = \mathcal{L}_{\mathcal{T}}^{n'}[v]$ in (9.24) to write

$$\begin{aligned}
\|\widehat{\boldsymbol{A}}^{1/2}\mathcal{L}_{\mathcal{T}}^{n'}[v]\|_{L^2(\Omega)}^2 &= \int_\Omega \mathcal{L}_{\mathcal{T}}^{n'}[v] \cdot \widehat{\boldsymbol{A}}\mathcal{L}_{\mathcal{T}}^{n'}[v] \\
&= \sum_{F\in\mathcal{F}^+}\int_F h^{-1/2}[\![v]\!]\boldsymbol{n}_F \cdot h^{1/2}\{\{\widehat{\boldsymbol{A}}\mathcal{L}_{\mathcal{T}}^{n'}[v]\}\} \\
&\le \|h^{-1/2}[\![v]\!]\|_{L^2(\mathcal{F}^+)}\|h^{1/2}\{\{\widehat{\boldsymbol{A}}\mathcal{L}_{\mathcal{T}}^{n'}[v]\}\}\|_{L^2(\mathcal{F}^+)}.
\end{aligned}$$

A local inverse estimate along with the eigenvalue bounds for $\widehat{\boldsymbol{A}} \in M(\widehat{\alpha}_1, \widehat{\alpha}_2)$ yields

$$\|h^{1/2}\{\{\widehat{\boldsymbol{A}}\mathcal{L}_{\mathcal{T}}^{n'}[v]\}\}\|_{L^2(\mathcal{F}^+)} \le C\widehat{\alpha}_2\|\mathcal{L}_{\mathcal{T}}^{n'}[v]\|_{L^2(\Omega)},$$

where C depends only on the shape regularity constant of $\mathcal{T}_0$ and n'. Combining the above two inequalities and again taking advantage of the assumption $\widehat{\boldsymbol{A}} \in M(\widehat{\alpha}_1, \widehat{\alpha}_2)$ implies (9.25). □

We record two estimates based on (9.25) and used multiple times in the analysis below. Combining the estimate (9.25) on the lifting operator with assumption (9.20) and a Cauchy–Schwarz inequality, we find that

$$\int_\Omega \mathcal{L}_{\mathcal{T}}^{n'}[v] \cdot \widehat{\boldsymbol{A}} \nabla_{\mathcal{T}} w \le C_{\text{lift}} \|h^{-1/2}[[v]]\|_{L^2(\mathcal{F}^+)} \|\nabla_{\mathcal{T}} w\|_{L^2(\mathcal{T})} \quad \text{for all } v, w \in \mathbb{E}_{\mathcal{T}}, \tag{9.26}$$

for a constant $C_{\text{lift}} = C_{\text{lift}}(\widehat{\alpha}_1, \widehat{\alpha}_2, \mathcal{T}_0, n')$ and in particular independent of the discretization parameters and κ. This, together with a Young inequality, yields for any $\epsilon > 0$ the second estimate for all $v, w \in \mathbb{E}_{\mathcal{T}}$:

$$\int_\Omega \mathcal{L}_{\mathcal{T}}^{n'}[v] \cdot \widehat{\boldsymbol{A}} \nabla_{\mathcal{T}} w \le \frac{C_{\text{lift}}^2}{2\epsilon} \|h^{-1/2}[[v]]\|_{L^2(\mathcal{F}^+)}^2 + \frac{\epsilon}{2} \|\nabla_{\mathcal{T}} w\|_{L^2(\mathcal{T})}^2. \tag{9.27}$$

We now return to the SIP weak formulation (9.21) and take advantage of the lifting operators to deduce an equivalent expression of the bilinear form $\mathcal{B}_{\mathcal{T}}$ on $\mathbb{V}_{\mathcal{T}}^{-1}$, which is well-defined on $\mathbb{E}_{\mathcal{T}}$. The problematic inter-element terms (9.23) are equivalently rewritten as

$$\int_\Omega \mathcal{L}_{\mathcal{T}}^{n'}[v] \cdot \widehat{\boldsymbol{A}} \nabla_{\mathcal{T}} w + \int_\Omega \mathcal{L}_{\mathcal{T}}^{n'}[w] \cdot \widehat{\boldsymbol{A}} \nabla_{\mathcal{T}} v, \tag{9.28}$$

provided

$$\nabla_{\mathcal{T}} \mathbb{V}_{\mathcal{T}}^{-1} \subset \left[\mathbb{S}_{\mathcal{T}}^{n',-1}\right]^d.$$

The above condition is satisfied when $n' \ge n - 1$ for subdivisions $\mathcal{T}$ made of simplices and $n' \ge n$ for hexahedra. To continue with an analysis incorporating both cases, we set $n' = n$ and write $\mathcal{L}_{\mathcal{T}} := \mathcal{L}_{\mathcal{T}}^{n}$. With this choice, the bilinear form $\mathcal{B}_{\mathcal{T}}$ in the symmetric interior penalty method (9.21) reads

$$\begin{aligned} \mathcal{B}_{\mathcal{T}}[w, v] = a_{\mathcal{T}}[w, v] &- \int_\Omega \mathcal{L}_{\mathcal{T}}[v] \cdot \widehat{\boldsymbol{A}} \nabla_{\mathcal{T}} w \\ &- \int_\Omega \mathcal{L}_{\mathcal{T}}[w] \cdot \widehat{\boldsymbol{A}} \nabla_{\mathcal{T}} v + \kappa \sum_{F \in \mathcal{F}^+} \int_F h_F^{-1} [[w]] [[v]], \end{aligned} \tag{9.29}$$

for all $w, v \in \mathbb{V}_{\mathcal{T}}^{-1}$ and where we used

$$a_{\mathcal{T}}[w, v] := \int_\Omega \nabla_{\mathcal{T}} v \cdot \widehat{\boldsymbol{A}} \nabla_{\mathcal{T}} w + \widehat{c} w v \tag{9.30}$$

to denote the bilinear form related to the conforming method.

Expression (9.29) is well-defined for $w, v \in \mathbb{E}_{\mathcal{T}}$ and the weak formulation (9.21) is well-posed. These two claims follow from Corollary 9.8 below, which in turn is a consequence of the next result focusing on the bilinear form $a_{\mathcal{T}}$; we recall (5.52).

Lemma 9.7 (properties of $a_{\mathcal{T}}$). *Let $\mathcal{T} \in \mathbb{T}^{\Lambda}$ be a Λ-admissible refinement of $\mathcal{T}_0$ satisfying Assumption 6.19 (initial labelling). Furthermore, assume that $\widehat{\boldsymbol{A}}$ and $\widehat{c}$ satisfy the structural assumption (9.20). Then we have*

$$a_{\mathcal{T}}[w, v] \leq \left(\widehat{\alpha}_2 + |\widehat{c}_2| C_P^2\right) \|v\|_{1,\mathcal{T}} \|w\|_{1,\mathcal{T}} \quad \textit{for all } v, w \in \mathbb{E}_{\mathcal{T}} \tag{9.31}$$

and

$$a_{\mathcal{T}}[v, v] \geq \frac{\widehat{\alpha}_1}{2} \|\nabla_{\mathcal{T}} v\|^2_{L^2(\Omega)} + \min(0, \widehat{c}_1) C_P^2 \|h^{-1/2} [\![v]\!]\|^2_{L^2(\mathcal{F}^+)} \quad \textit{for all } v \in \mathbb{E}_{\mathcal{T}}, \tag{9.32}$$

where C_P is the constant in Lemma 9.3 (Poincaré-type inequality in $\mathbb{E}_{\mathcal{T}}$).

Proof. We start with the continuity estimate (9.31). The assumption on the discretized coefficients implies that for $v, w \in \mathbb{E}_{\mathcal{T}}$ we have

$$a_{\mathcal{T}}[w, v] \leq \alpha_2 \|\nabla_{\mathcal{T}} w\|_{L^2(\mathcal{T})} \|\nabla_{\mathcal{T}} v\|_{L^2(\mathcal{T})} + |c_2| \|w\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}.$$

It remains to invoke Lemma 9.3 (Poincaré-type inequality on $\mathbb{E}_{\mathcal{T}}$) to deduce (9.31).

Similarly, for the partial coercivity estimate (9.32) we have

$$a_{\mathcal{T}}[v, v] \geq \widehat{\alpha}_1 \|\nabla_{\mathcal{T}} v\|^2_{L^2(\Omega)} + \widehat{c}_1 \|v\|^2_{L^2(\Omega)} \geq \widehat{\alpha}_1 \|\nabla_{\mathcal{T}} v\|^2_{L^2(\Omega)} + \min(0, \widehat{c}_1) C_P^2 \|v\|^2_{1,\mathcal{T}},$$

and the desired estimate follows from the assumption $-\widehat{\alpha}_1/(2C_P^2) \leq \widehat{c}_1$. □

For the next result, we recall that the discrete norm $\|\cdot\|_{\kappa,\mathcal{T}}$ is defined in (9.2).

Corollary 9.8 (properties of $\mathcal{B}_{\mathcal{T}}$). *Let $\mathcal{T} \in \mathbb{T}^{\Lambda}$ be a Λ-admissible refinement of $\mathcal{T}_0$ satisfying Assumption 6.19 (initial labelling). Furthermore, assume that $\widehat{\boldsymbol{A}}$ and $\widehat{c}$ satisfy the structural assumption (9.20). There exists a constant C_{cont} such that*

$$\mathcal{B}_{\mathcal{T}}[w, v] \leq C_{\mathrm{cont}} \|v\|_{\kappa,\mathcal{T}} \|w\|_{\kappa,\mathcal{T}} \quad \textit{for all } v, w \in \mathbb{E}_{\mathcal{T}}. \tag{9.33}$$

Moreover, there are constants $\overline{\kappa}_{\mathrm{stab}}, C_{\mathrm{coer}} > 0$ such that for all $\kappa > \overline{\kappa}_{\mathrm{stab}}$ we have

$$C_{\mathrm{coer}} \|v\|^2_{\kappa,\mathcal{T}} \leq \mathcal{B}_{\mathcal{T}}[v, v] \quad \textit{for all } v \in \mathbb{E}_{\mathcal{T}}. \tag{9.34}$$

In particular, the Galerkin formulation (9.21) has a unique solution $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}^{-1}$.

Proof. The continuity estimate is a direct consequence of the continuity estimate (9.31), estimate (9.26) for the lifting terms, and Cauchy–Schwarz inequality

$$\kappa \sum_{F \in \mathcal{F}^+} \int_F h_F^{-1} [\![w]\!] [\![v]\!] \leq \kappa \|h^{-1/2} [\![w]\!]\|_{L^2(\mathcal{F}^+)} \|h^{-1/2} [\![v]\!]\|_{L^2(\mathcal{F}^+)},$$

which holds for all $v, w \in \mathbb{E}_{\mathcal{T}}$.

We now focus on the coercivity estimate (9.34) and start from (9.32), which we write for $v \in \mathbb{E}_{\mathcal{T}}$ as

$$\frac{\widehat{\alpha}_1}{2} \|\nabla_{\mathcal{T}} v\|^2_{L^2(\mathcal{T})} - \max\left(0, -c_1 \widehat{C}_P^2\right) \|h^{-1/2} [\![v]\!]\|^2_{L^2(\mathcal{F}^+)} \leq a_{\mathcal{T}}[v, v]. \tag{9.35}$$

Furthermore, the terms involving the lifting operators in the definition (9.29) of the bilinear form $\mathcal{B}_{\mathcal{T}}$ reduce to $-2\int_\Omega \mathcal{L}_{\mathcal{T}}[v]\cdot\widehat{\boldsymbol{A}}\nabla_{\mathcal{T}}v$ when $w=v$. Hence the estimate (9.27) with $\epsilon=\widehat{\alpha}_1/4$ implies that

$$2\left|\int_\Omega \mathcal{L}_{\mathcal{T}}[v]\cdot\widehat{\boldsymbol{A}}\nabla_{\mathcal{T}}v\right| \le \frac{\widehat{\alpha}_1}{4}\|\nabla_{\mathcal{T}}v\|^2_{L^2(\mathcal{T})} + \frac{4C_{\mathcal{L}}^2}{\widehat{\alpha}_1}\|h^{-1/2}[[v]]\|^2_{L^2(\mathcal{F}^+)}.$$

Gathering the above inequalities and recalling definition (9.29) of $\mathcal{B}_{\mathcal{T}}$, we find that

$$\frac{\alpha_1}{4}\|\nabla_{\mathcal{T}}v\|^2_{L^2(\mathcal{T})} + (\kappa-\overline{\kappa}_{\text{stab}})\|h^{-1/2}[[v]]\|^2_{L^2(\mathcal{F}^+)} \le \mathcal{B}_{\mathcal{T}}[v,v],$$

with

$$\overline{\kappa}_{\text{stab}} := \frac{4C_{\mathcal{L}}^2}{\alpha_1} + \max\left(0, -\widehat{c}_1 C_P^2\right).$$

The desired coercivity estimate directly follows provided $\kappa > \overline{\kappa}_{\text{stab}}$. □

9.2.3. Partial consistency and role of the conforming Galerkin solution

From now on we shall use the expression of $\mathcal{B}_{\mathcal{T}}$ in (9.29) extending $\mathcal{B}_{\mathcal{T}}$ to $\mathbb{E}_{\mathcal{T}}\times\mathbb{E}_{\mathcal{T}}$. This reformulation comes at the price of partial consistency. Since $\mathcal{L}_{\mathcal{T}}[v]=0$ whenever $v\in H^1_0(\Omega)$ and the duality product $\langle\cdot,\cdot\rangle_{\mathcal{T}}$ satisfies the consistency (9.4), we have

$$\mathcal{B}_{\mathcal{T}}[\widehat{u},v] = a_{\mathcal{T}}[\widehat{u},v] = \langle\widehat{f},v\rangle_{\mathcal{T}} \quad \text{for all } v\in H^1_0(\Omega), \tag{9.36}$$

which indicates that the reformulation (9.29) using lifts is consistent on $H^1_0(\Omega)$. However, (9.36) does not hold for all $v\in\mathbb{V}^{-1}_{\mathcal{T}}$.

This suggests splitting $\mathbb{V}^{-1}_{\mathcal{T}}$ into a conforming space where the consistency holds and its orthogonal complement. We decompose the discontinuous space as

$$\mathbb{V}^{-1}_{\mathcal{T}} = \mathbb{V}^0_{\mathcal{T}} \oplus \mathbb{V}^{\perp}_{\mathcal{T}}, \tag{9.37}$$

where $\mathbb{V}^0_{\mathcal{T}} = \mathbb{V}^{-1}_{\mathcal{T}}\cap H^1_0(\Omega)$ is the finest conforming subspace of $\mathbb{V}^{-1}_{\mathcal{T}}$ and $\mathbb{V}^{\perp}_{\mathcal{T}}$ is the orthogonal complement with respect to the $\mathcal{B}_{\mathcal{T}}[\cdot,\cdot]$ scalar product. Note that the latter is well-defined provided the assumption on the penalty parameter $\kappa>\overline{\kappa}_{\text{stab}}$, required by Corollary 9.8, is satisfied. From now on we assume this is the case, and point out that although the constants appearing in the analysis below do not depend on κ, they may depend on $\overline{\kappa}_{\text{stab}}$.

We also emphasize that there might not be a conforming subdivision associated with $\mathbb{V}^0_{\mathcal{T}}$. The latter is the span of the basis functions associated with proper nodes; see Figure 3.11 for an illustration and refer to Section 8 for more details. Consequently, the analysis provided below relies on the decomposition (9.37) of the space $\mathbb{V}_{\mathcal{T}}$ rather than on a subdivision $\mathcal{T}$. It is also worth pointing out that the conforming part $u^0_{\mathcal{T}}\in\mathbb{V}^0_{\mathcal{T}}$ of the Galerkin solution $u_{\mathcal{T}}\in\mathbb{V}^{-1}_{\mathcal{T}}$ satisfies

$$\mathcal{B}_{\mathcal{T}}\left[u^0_{\mathcal{T}},v\right] = \langle\widehat{f},v\rangle \quad \text{for all } v\in\mathbb{V}^0_{\mathcal{T}}. \tag{9.38}$$

Hence $u^0_{\mathcal{T}}$ is the conforming Galerkin approximation on $\mathbb{V}^0_{\mathcal{T}}$. As we shall see, this finest coarser conforming Galerkin solution plays a critical role in the convergence of AFEM-DG-TS. This justifies the orthogonal decomposition (9.37) associated with the $\mathcal{B}_{\mathcal{T}}$ scalar product.

Another advantage of using the $\mathcal{B}_{\mathcal{T}}$-orthogonal decomposition (9.37) is that it offers a control on the non-conforming component of $v \in \mathbb{V}^{-1}_{\mathcal{T}}$ by its scaled jumps. To achieve this, the operator $\mathcal{I}^{\mathrm{dG}}_{\mathcal{T}}$ defined by (9.7) is instrumental.

Lemma 9.9 (control of non-conformity). *Let $\mathcal{T} \in \mathbb{T}^{\Lambda}$ be a Λ-admissible refinement of $\mathcal{T}_0$ satisfying Assumption 6.19 (initial labelling). Assume that $\widehat{\boldsymbol{A}}$ and $\widehat{c}$ satisfy the structural assumption (9.20). For $\kappa > \overline{\kappa}_{\mathrm{stab}}$, if $v = v^0 + v^{\perp} \in \mathbb{V}^{-1}_{\mathcal{T}}$ according to (9.37), then*

$$\|v^{\perp}\|_{\kappa,\mathcal{T}} \lesssim \kappa^{1/2}\|h^{-1/2}[[v^{\perp}]]\|_{L^2(\mathcal{F}^+)} = \kappa^{1/2}\|h^{-1/2}[[v]]\|_{L^2(\mathcal{F}^+)}.$$

Proof. Because $\mathcal{I}^{\mathrm{dG}}_{\mathcal{T}} v \in \mathbb{V}^0_{\mathcal{T}}$, the orthogonal decomposition (9.37) implies that

$$\mathcal{B}_{\mathcal{T}}[v^{\perp}, v^{\perp}] \le \mathcal{B}_{\mathcal{T}}\big[v - \mathcal{I}^{\mathrm{dG}}_{\mathcal{T}} v, v - \mathcal{I}^{\mathrm{dG}}_{\mathcal{T}} v\big].$$

The desired result follows from the coercivity (9.34) and continuity (9.33) of $\mathcal{B}_{\mathcal{T}}$ along with the interpolation estimate (9.10). □

9.3. A posteriori *error estimates*

We derive a residual error estimate for the discontinuous Galerkin method. Because the data $\widehat{\mathcal{D}} \in \mathbb{D}_{\mathcal{T}}$ is discrete, the analysis is free from data oscillation. In the notation introduced in Section 4, this means $\mathcal{E}_{\mathcal{T}} = \eta_{\mathcal{T}}$, where for $v \in \mathbb{E}_{\mathcal{T}}$

$$\eta^2_{\mathcal{T}}(v) := \sum_{T \in \mathcal{T}} \eta_{\mathcal{T}}(v, T)^2,$$

and

$$\eta_{\mathcal{T}}(v, T)^2 := h_T \sum_{F \subset \partial T \setminus \partial\Omega} \|j_{\mathcal{T}}(v) - \widehat{f}\|^2_{L^2(F)} + h^2_T \|r_{\mathcal{T}}(v)\|^2_{L^2(T)},$$

with $j_{\mathcal{T}}(v)|_F := \boldsymbol{n}_F \cdot [[\widehat{\boldsymbol{A}}\nabla_{\mathcal{T}} v]]$ and $r_{\mathcal{T}}(v)|_T := \widehat{f} - \widehat{c}v + \mathrm{div}_{\mathcal{T}}(\widehat{\boldsymbol{A}}\nabla_{\mathcal{T}} v)$.

We start with a result mimicking the conforming argument, and then discuss its drawbacks and remedies.

Lemma 9.10 (*a posteriori* error estimates). *Let $\mathcal{T} \in \mathbb{T}^{\Lambda}$ be a Λ-admissible refinement of $\mathcal{T}_0$ satisfying Assumption 6.19 (initial labelling). Assume that $\widehat{\boldsymbol{A}}$ and $\widehat{c}$ satisfy the structural assumption (9.20). If $\kappa > \overline{\kappa}_{\mathrm{stab}}$, then*

$$\|\widehat{u} - u_{\mathcal{T}}\|^2_{\kappa,\mathcal{T}} \lesssim \eta_{\mathcal{T}}(u_{\mathcal{T}})^2 + \kappa\|h^{-1/2}[[u_{\mathcal{T}}]]\|^2_{L^2(\mathcal{F}^+)} \tag{9.39}$$

and

$$C_L\, \eta_{\mathcal{T}}(u_{\mathcal{T}}) \le \|\widehat{u} - u_{\mathcal{T}}\|_{\kappa,\mathcal{T}}, \tag{9.40}$$

for some constant C_L.

Proof. We start with the upper bound (9.39). To exploit the consistency (9.36) in $\mathbb{V}^0_{\mathcal{T}}$, we decompose the error $e := \widehat{u} - u_{\mathcal{T}} \in \mathbb{E}_{\mathcal{T}}$ into a conforming part $e^0 := \widehat{u} - u^0_{\mathcal{T}} \in H^1_0(\Omega)$ and a non-conforming part $e^\perp := -u^\perp_{\mathcal{T}} \in \mathbb{V}^{-1}_{\mathcal{T}}$ according to (9.37). The proof thus relies on techniques used in the conforming theory coupled with Lemma 9.9 (control of non-conformity). We let C denote a generic constant independent of the discretization and κ but possibly depending on $\overline{\kappa}_{\text{stab}}$.

Using the coercivity (9.34) and partial consistency (9.36), we get

$$C_{\text{coer}}\|e\|^2_{\kappa,\tau} \le \mathcal{B}_{\mathcal{T}}[e,e] = \mathcal{B}_{\mathcal{T}}[e, e^0 - \mathcal{I}_{\mathcal{T}} e^0] - \mathcal{B}_{\mathcal{T}}[e, u^\perp_{\mathcal{T}}], \tag{9.41}$$

where $I_{\mathcal{T}}$ is the Scott–Zhang interpolant provided in Proposition 3.5. For the first term, note that since $e^0 - I_{\mathcal{T}} e^0 \in H^1_0(\Omega)$ we have

$$\mathcal{B}_{\mathcal{T}}\big[e, e^0 - I_{\mathcal{T}} e^0\big] = a_{\mathcal{T}}[e, e^0 - I_{\mathcal{T}} e^0] - \int_\Omega \mathcal{L}_{\mathcal{T}}[e] \cdot \boldsymbol{A}\nabla(e^0 - I_{\mathcal{T}} e^0).$$

For the term involving the bilinear form $a_{\mathcal{T}}$, we proceed as in the conforming case, to arrive at

$$a_{\mathcal{T}}[e, e^0 - I_{\mathcal{T}} e^0] \lesssim \eta_{\mathcal{T}}(u_{\mathcal{T}})\|\nabla e^0\|_{L^2(\Omega)}.$$

This, combined with estimate (9.26) on the lifting operators and the H^1-stability of the Scott–Zhang interpolant, yields

$$\mathcal{B}_{\mathcal{T}}\big[e, e^0 - I_{\mathcal{T}} e^0\big] \lesssim \big(\eta_{\mathcal{T}}(u_{\mathcal{T}})^2 + \|h^{-1/2}[\![u_{\mathcal{T}}]\!]\|^2_{L^2(\mathcal{F}^+)}\big)^{1/2}\|\nabla e^0\|_{L^2(\Omega)}.$$

We rewrite $e^0 = e + u^\perp_{\mathcal{T}}$ and use the estimate on the non-conforming component provided by Lemma 9.9 along with a Young inequality, to write

$$\mathcal{B}_{\mathcal{T}}\big[e, e^0 - I_{\mathcal{T}} e^0\big] \le C\big(\eta_{\mathcal{T}}(u_{\mathcal{T}})^2 + \kappa\|h^{-1/2}[\![u_{\mathcal{T}}]\!]\|^2_{L^2(\mathcal{F}^+)}\big) + \frac{C_{\text{coer}}}{4}\|\nabla e\|^2_{L^2(\Omega)}.$$

For the second term in (9.41), the continuity (9.33) of the bilinear form $\mathcal{B}_{\mathcal{T}}$, Lemma 9.9 again, and a Young inequality yield

$$\mathcal{B}_{\mathcal{T}}[e, u^\perp_{\mathcal{T}}] \le \frac{C_{\text{coer}}}{4}\|e\|^2_{\kappa,\mathcal{T}} + C\kappa\|h^{-1/2}[\![u_{\mathcal{T}}]\!]\|^2_{L^2(\mathcal{F}^+)}.$$

Returning to (9.41), we find that

$$\|e\|^2_{\mathcal{T}} \lesssim \eta_{\mathcal{T}}(u_{\mathcal{T}})^2 + \kappa\|h^{-1/2}[\![u_{\mathcal{T}}]\!]\|^2_{L^2(\mathcal{F}^+)},$$

which is the desired upper bound.

We finally deal with the lower bound (9.40). For $T \in \mathcal{T}$ and $v \in H^1_0(T)$, we get

$$\int_T (-\operatorname{div}_{\mathcal{T}}(\widehat{\boldsymbol{A}}\nabla_{\mathcal{T}} u_{\mathcal{T}}) + \widehat{c}u_{\mathcal{T}} - \widehat{f})v = \int_T \nabla v \cdot \widehat{\boldsymbol{A}}\nabla_{\mathcal{T}}(\widehat{u} - u_{\mathcal{T}})v + \widehat{c}(\widehat{u} - u_{\mathcal{T}})v.$$

For an interior face $F \in \mathcal{F}$, $v \in H^1_0(\omega_F)$ with $\omega_F := \{T \in \mathcal{T} \mid T \cap F \neq \emptyset\}$, we have

$$\int_F ([\![\widehat{\boldsymbol{A}}\nabla_{\mathcal{T}} u_{\mathcal{T}}]\!] - \widehat{f})v = \int_{\omega_F} (-\operatorname{div}_{\mathcal{T}}(\widehat{\boldsymbol{A}}\nabla_{\mathcal{T}} u_{\mathcal{T}}) + \widehat{c}u_{\mathcal{T}} - \widehat{f})v - \int_{\omega_F} \nabla v \cdot \widehat{\boldsymbol{A}}\nabla_{\mathcal{T}}(\widehat{u} - u_{\mathcal{T}}) - \widehat{c}(\widehat{u} - u_{\mathcal{T}})v.$$

The desired lower bound follows from the same arguments as in the conforming case; we refer to Proposition 4.12 (partial lower bound). □

Upper bound (9.39) may suggest adding the jump term $\kappa^{1/2}\|h^{-1/2}[\![u_{\mathcal{T}}]\!]\|_{L^2(\mathcal{F}^+)}$ to the residual estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}})$. This would result in a clean upper bound but, because of the presence of negative powers of the mesh size, it would be at the expense of destroying the monotonicity property of the estimator; see e.g. Proposition 4.56 (estimator reduction). The latter is instrumental to the analysis provided below.

The next result mitigates the effect of the additional jump term by showing that $\|h^{-1/2}[\![u_{\mathcal{T}}]\!]\|_{L^2(\mathcal{F}^+)}$ can be bounded by $\eta_{\mathcal{T}}(u_{\mathcal{T}})/\kappa$ and can thus be absorbed by the estimator in the upper bound provided κ is sufficiently large. We follow the proof provided in Bonito and Nochetto (2010) and refer to Karakashian and Pascal (2007, (3.20)) for an alternative (original) proof.

Lemma 9.11 (discontinuity control). *Let $\mathcal{T} \in \mathbb{T}^\Lambda$ be a Λ-admissible refinement of $\mathcal{T}_0$ satisfying Assumption 6.19 (initial labelling). Assume that $\widehat{\boldsymbol{A}}$ and $\widehat{c}$ satisfy the structural assumption (9.20). There exists a constant $\overline{\kappa}_{\text{jump}} \geq \overline{\kappa}_{\text{stab}} > 0$ such that if $\kappa \geq \overline{\kappa}_{\text{jump}}$, then*

$$\|h^{-1/2}[\![u_{\mathcal{T}}]\!]\|_{L^2(\mathcal{F}^+)} \lesssim \kappa^{-1}\eta_{\mathcal{T}}(u_{\mathcal{T}}).$$

Proof. For $v^0_{\mathcal{T}} \in \mathbb{V}^0_{\mathcal{T}}$ we realize that because $[\![v^0_{\mathcal{T}}]\!] = 0$, the coercivity estimate (9.34) implies that

$$C_{\text{coerc}}\kappa\|h^{-1/2}[\![u_{\mathcal{T}}]\!]\|^2_{L^2(\mathcal{F}^+)} \leq \mathcal{B}_{\mathcal{T}}\big[u_{\mathcal{T}} - v^0_{\mathcal{T}}, u_{\mathcal{T}} - v^0_{\mathcal{T}}\big]. \tag{9.42}$$

We now rewrite the right-hand side of (9.42) to produce residual terms. Since $u_{\mathcal{T}}$ solves (9.21), we have

$$\mathcal{B}_{\mathcal{T}}\big[u_{\mathcal{T}} - v^0_{\mathcal{T}}, u_{\mathcal{T}} - v^0_{\mathcal{T}}\big] = \langle \widehat{f}, u_{\mathcal{T}} - v^0_{\mathcal{T}}\rangle_{\mathcal{T}} - \mathcal{B}_{\mathcal{T}}\big[v^0_{\mathcal{T}}, u_{\mathcal{T}} - v^0_{\mathcal{T}}\big]. \tag{9.43}$$

We concentrate for the moment on the second term. Since $[\![v^0_{\mathcal{T}}]\!] = 0$, the stabilization term vanishes as well:

$$\kappa \sum_{F \in \mathcal{F}^+} \int_F [\![v^0_{\mathcal{T}}]\!][\![u_{\mathcal{T}} - v^0_{\mathcal{T}}]\!] = 0.$$

Hence, writing $v_{\mathcal{T}}^0 = u_{\mathcal{T}} + (v_{\mathcal{T}}^0 - u_{\mathcal{T}})$, we deduce that

$$\begin{aligned}\mathcal{B}_{\mathcal{T}}\big[v_{\mathcal{T}}^0, u_{\mathcal{T}} - v_{\mathcal{T}}^0\big] &= a_{\mathcal{T}}[u_{\mathcal{T}}, u_{\mathcal{T}} - v_{\mathcal{T}}^0] \\ &\quad - a_{\mathcal{T}}[u_{\mathcal{T}} - v_{\mathcal{T}}^0, u_{\mathcal{T}} - v_{\mathcal{T}}^0] - \int_\Omega \nabla v_{\mathcal{T}}^0 \cdot \widehat{\boldsymbol{A}}\mathcal{L}_{\mathcal{T}}[u_{\mathcal{T}}],\end{aligned}$$

where we again invoked the property $[[v_{\mathcal{T}}^0]] = 0$ to infer that $\mathcal{L}_{\mathcal{T}}[v_{\mathcal{T}}^0] = 0$. Integrating the first term on the right-hand side by parts, adding it to the first term on the right-hand side of (9.43), and using the extended definition (9.3) of the duality pairing leads to the following expression involving the residuals $r_{\mathcal{T}}(u_{\mathcal{T}})$, $j_{\mathcal{T}}(u_{\mathcal{T}})$:

$$\begin{aligned}&\mathcal{B}_{\mathcal{T}}\big[u_{\mathcal{T}} - v_{\mathcal{T}}^0, u_{\mathcal{T}} - v_{\mathcal{T}}^0\big] \\ &\quad = \int_\Omega r_{\mathcal{T}}(u_{\mathcal{T}})(u_{\mathcal{T}} - v_{\mathcal{T}}^0) + \int_{\mathcal{F}^+} (j_{\mathcal{T}}(u_{\mathcal{T}}) - \widehat{f})\{\{u_{\mathcal{T}} - v_{\mathcal{T}}^0\}\} \\ &\qquad + a_{\mathcal{T}}[u_{\mathcal{T}} - v_{\mathcal{T}}^0, u_{\mathcal{T}} - v_{\mathcal{T}}^0] - \int_\Omega \nabla(u_{\mathcal{T}} - v_{\mathcal{T}}^0) \cdot \widehat{\boldsymbol{A}}\mathcal{L}_{\mathcal{T}}[u_{\mathcal{T}}].\end{aligned}$$

We point out that we have also employed the definition (9.24) of lift to rewrite the resulting face terms. Inserting this estimate back in (9.42), together with the bound (9.26) for lifts and the continuity estimate (9.31) of $a_{\mathcal{T}}$, gives

$$\begin{aligned}\kappa\|h^{-1/2}[[u_{\mathcal{T}}]]\|_{L^2(\mathcal{F}^+)}^2 &\lesssim \|u_{\mathcal{T}} - v_{\mathcal{T}}^0\|_{1,\mathcal{T}}^2 \\ &\quad + \eta_{\mathcal{T}}(u_{\mathcal{T}})\big(\|h^{-1}(u_{\mathcal{T}} - v_{\mathcal{T}}^0)\|_{L^2(\Omega)} + \|h^{-1/2}\{\{u_{\mathcal{T}} - v_{\mathcal{T}}^0\}\}\|_{L^2(\mathcal{F}^+)}\big).\end{aligned}$$

Note that the presence of $\|\cdot\|_{1,\mathcal{T}}$ rather than $\|\cdot\|_{\kappa,\mathcal{T}}$ on the right-hand side of the above estimate is critical for the argument below. The former is independent of κ and can thus be absorbed on the left-hand side for sufficiently large κ provided $v_{\mathcal{T}}^0 = \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} u_{\mathcal{T}}$. In fact the interpolation estimates (9.15) in turn imply

$$\begin{aligned}&\|h^{-1}(u_{\mathcal{T}} - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} u_{\mathcal{T}})\|_{L^2(\Omega)} + \|h^{-1/2}\{\{u_{\mathcal{T}} - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} u_{\mathcal{T}}\}\}\|_{L^2(\mathcal{F}^+)} \\ &\quad + \|u_{\mathcal{T}} - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} u_{\mathcal{T}}\|_{1,\mathcal{T}} \lesssim \|h^{-1/2}[[u_{\mathcal{T}}]]\|_{L^2(\mathcal{F}^+)}.\end{aligned}$$

Hence a Young's inequality yields

$$(\kappa - \overline{\kappa}_{\mathrm{stab}})\|h^{-1/2}[[u_{\mathcal{T}}]]\|_{L^2(\mathcal{F}^+)}^2 \lesssim \kappa^{-1}\eta_{\mathcal{T}}(u_{\mathcal{T}})^2 + \|h^{-1/2}[[u_{\mathcal{T}}]]\|_{L^2(\mathcal{F}^+)}^2,$$

and the desired estimate follows provided κ is sufficiently large. □

As a direct consequence of the previous lemma, we obtain a simpler practical upper bound.

Corollary 9.12 (stabilization-free *a posteriori* upper bound). *If we make the same assumptions as Lemma 9.11, there exists a constant C_U such that for all $\kappa \geq \overline{\kappa}_{\mathrm{jump}}$ we have*

$$\|\widehat{u} - u_{\mathcal{T}}\|_{\kappa,\mathcal{T}} \leq C_U\, \eta_{\mathcal{T}}(u_{\mathcal{T}}). \tag{9.44}$$

Proof. Combine the upper bound (9.39) and Lemma 9.11. □

The partial consistency (9.36) leads to *partial Galerkin orthogonality*

$$\mathcal{B}_{\mathcal{T}}[\widehat{u} - u_{\mathcal{T}}, v] = 0 \quad \text{for all } v \in \mathbb{V}_{\mathcal{T}}^{0}. \tag{9.45}$$

This would suggest that a quasi-best approximation (Céa's lemma) result in the full space $\mathbb{V}_{\mathcal{T}}^{-1}$ is questionable. However, the lack of consistency is built into the jump terms, which are in turn controlled by the estimator weighted by a negative power of the penalty parameter κ. It thus remains to resort to the lower bound to return to the error and derive a quasi-best approximation estimate for sufficiently large κ. We prove this result next, which expresses the important fact that dG is quasi-optimal with respect to the norm $\|\cdot\|_{\kappa,\mathcal{T}}$ defined in (9.2). This has two significant consequences: first it leads to quasi-monotonicity of the error upon refinement (see Corollary 9.14 below), and second it dictates the approximation class for dG already alluded to in Proposition 9.4 (equivalence of classes for u).

Corollary 9.13 (Céa's lemma). *Under the assumptions of Lemma 9.11, there is* $\overline{\kappa}_{\text{Céa}} \geq \overline{\kappa}_{\text{jump}}$ *such that, for* $\kappa \geq \overline{\kappa}_{\text{Céa}}$,

$$\|\widehat{u} - u_{\mathcal{T}}\|_{\kappa,\mathcal{T}} \leq C_{\text{Céa}} \inf_{v_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}^{-1}} \|\widehat{u} - v_{\mathcal{T}}\|_{\kappa,\mathcal{T}}. \tag{9.46}$$

Proof. We combine the orthogonal decomposition (9.37) and the partial Galerkin orthogonality (9.45) to write

$$\mathcal{B}_{\mathcal{T}}[\widehat{u} - u_{\mathcal{T}}, \widehat{u} - u_{\mathcal{T}}] = \mathcal{B}_{\mathcal{T}}[\widehat{u} - u_{\mathcal{T}}, \widehat{u} - v_{\mathcal{T}}] - \mathcal{B}_{\mathcal{T}}[\widehat{u} - u_{\mathcal{T}}, u_{\mathcal{T}}^{\perp}] + \mathcal{B}_{\mathcal{T}}[\widehat{u} - u_{\mathcal{T}}, v_{\mathcal{T}}^{\perp}]$$

for all $v_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$. Invoking the coercivity and continuity of $\mathcal{B}_{\mathcal{T}}$ in Lemma 9.8 (properties of $\mathcal{B}_{\mathcal{T}}$) in conjunction with Lemma 9.9 (control of non-conformity) yields

$$\|\widehat{u} - u_{\mathcal{T}}\|_{\kappa,\mathcal{T}} \lesssim \|\widehat{u} - v_{\mathcal{T}}\|_{\kappa,\mathcal{T}} + \kappa^{1/2}\|h^{-1/2}[[u_{\mathcal{T}}]]\|_{L^2(\mathcal{F}^+)} + \kappa^{1/2}\|h^{-1/2}[[v_{\mathcal{T}}]]\|_{L^2(\mathcal{F}^+)}.$$

Now applying Lemma 9.11 (discontinuity control) results in

$$\|h^{-1/2}[[u_{\mathcal{T}}]]\|_{L^2(\mathcal{F}^+)} \lesssim \kappa^{-1}\eta_{\mathcal{T}}(u_{\mathcal{T}}) \lesssim \kappa^{-1}\|\widehat{u} - u_{\mathcal{T}}\|_{\kappa,\mathcal{T}}$$

because of the lower bound (9.40). We thus end up with

$$\|\widehat{u} - u_{\mathcal{T}}\|_{\kappa,\mathcal{T}} \lesssim \|\widehat{u} - v_{\mathcal{T}}\|_{\kappa,\mathcal{T}} + \kappa^{-1/2}\|\widehat{u} - u_{\mathcal{T}}\|_{\kappa,\mathcal{T}},$$

which for κ sufficiently large gives the desired bound. □

With this best approximation result, we deduce the following crucial property.

Corollary 9.14 (quasi-monotonicity). *Under the assumptions of Lemma 9.11, there is a constant* C_{Mo} *independent of the discretization parameters and* κ *such that for all* $\kappa \geq \overline{\kappa}_{\text{jump}}$ *and* $\mathcal{T}_* \geq \mathcal{T}$ *we have*

$$\|\widehat{u} - u_{\mathcal{T}_*}\|_{\kappa,\mathcal{T}_*} \leq C_{\text{Mo}}\|\widehat{u} - u_{\mathcal{T}}\|_{\kappa,\mathcal{T}}. \tag{9.47}$$

Proof. We rely on the orthogonal decomposition (9.37) to write $u_{\mathcal{T}} = u^0_{\mathcal{T}} + u^\perp_{\mathcal{T}}$ and on Corollary 9.13 (Céa's lemma). Since $u^0_{\mathcal{T}} \in \mathbb{V}^0_{\mathcal{T}} \subset \mathbb{V}^{-1}_{\mathcal{T}_*}$, we see that

$$C^{-1}_{\text{Céa}} \|\widehat{u} - u_{\mathcal{T}_*}\|_{\kappa,\mathcal{T}_*} \le \|\widehat{u} - u^0_{\mathcal{T}}\|_{\kappa,\mathcal{T}_*} = |\widehat{u} - u^0_{\mathcal{T}}|_{H^1_0(\Omega)} = \|\widehat{u} - u^0_{\mathcal{T}}\|_{\kappa,\mathcal{T}}.$$

Therefore, adding and subtracting $u^\perp_{\mathcal{T}}$ and making use of Lemma 9.9 (control of non-conformity) together with Lemma 9.11 (discontinuity control) implies

$$\|\widehat{u} - u_{\mathcal{T}_*}\|_{\kappa,\mathcal{T}_*} \lesssim \|\widehat{u} - u_{\mathcal{T}}\|_{\kappa,\mathcal{T}} + \kappa^{-1/2}\eta_{\mathcal{T}}(u_{\mathcal{T}}).$$

It remains to invoke the lower bound (9.40) to deduce the desired result. □

Corollary 9.14 assumes the same data. In estimating the cost of GALERKIN-DG we need a variant of this result that allows for different data. We establish this next.

Corollary 9.15 (quasi-monotonicity with different data). *Let $\mathcal{T}_* \ge \mathcal{T}$ and $\widehat{\mathcal{D}}_*, \widehat{\mathcal{D}}$ be discrete data on these meshes. Let $\widehat{u}_* = u(\widehat{\mathcal{D}}_*)$, $\widehat{u} = u(\widehat{\mathcal{D}}) \in H^1_0(\Omega)$ and let $u_{\mathcal{T}_*} \in \mathbb{V}^{-1}_{\mathcal{T}_*}$, $u_{\mathcal{T}} \in \mathbb{V}^{-1}_{\mathcal{T}}$ be the corresponding exact and Galerkin solutions. Under the assumptions of Lemma 9.11, for $\kappa \ge \overline{\kappa}_{\text{jump}}$ we have*

$$\|\widehat{u}_* - u_{\mathcal{T}_*}\|_{\kappa,\mathcal{T}_*} \le C_{\text{Mo}}\big(\|\widehat{u} - u_{\mathcal{T}}\|_{\kappa,\mathcal{T}} + |\widehat{u}_* - \widehat{u}|_{H^1_0(\Omega)}\big). \tag{9.48}$$

Proof. We proceed as in the proof of Corollary 9.14 with $\widehat{u}_*$, but in the last step use the fact that $u_{\mathcal{T}}$ and $\widehat{u}$ are the functions associated with the same data $\widehat{\mathcal{D}}$ and thereby satisfy $C_L\eta_{\mathcal{T}}(u_{\mathcal{T}}) \le \|\widehat{u} - u_{\mathcal{T}}\|_{\kappa,\mathcal{T}}$ according to (9.40). Applying the triangle inequality and the property $\|\widehat{u}_* - \widehat{u}\|_{\kappa,\mathcal{T}} = |\widehat{u}_* - \widehat{u}|_{H^1_0(\Omega)}$ concludes the proof. □

We end this section with the dG counterpart of Theorem 4.48 (upper bound for corrections). One striking difference is that the lack of consistency prevents the discrete lower bound in the dG context from localizing to the refined set $\mathcal{T} \setminus \mathcal{T}_*$ for $\mathcal{T}_* \ge \mathcal{T}$. Rather, it contains a global jump term that expresses the lack of conformity and vanishes as $\kappa \to \infty$ in view of Lemma 9.11 (discontinuity control). This is consistent with the upper bound (9.39). We use the notation $\omega_{\mathcal{T}}(\tau)$ for a set of elements $\tau \in \mathcal{T}$ to denote τ augmented by one layer of elements

$$\omega(\tau) := \omega_{\mathcal{T}}(\tau) = \bigcup_{T \in \tau} \omega_{\mathcal{T}}(T).$$

Lemma 9.16 (quasi-localized discrete upper bound). *Assume $\mathcal{T}, \mathcal{T}_* \in \mathbb{T}^\Lambda$, with $\mathcal{T}_* \ge \mathcal{T}$, are two Λ-admissible refinements of $\mathcal{T}_0$ satisfying Assumption 6.19 (initial labelling). Assume that $\widehat{\boldsymbol{A}}$ and $\widehat{c}$ satisfy the structural assumption (9.20), $\widehat{f} \in \mathbb{F}_{\mathcal{T}}$, and let $u_{\mathcal{T}} \in \mathbb{V}^{-1}_{\mathcal{T}}$, $u_{\mathcal{T}_*} \in \mathbb{V}^{-1}_{\mathcal{T}_*}$ denote the two Galerkin solutions associated with $\mathcal{T}$, $\mathcal{T}_*$ respectively. There is a constant C_{LU} such that for all $\kappa > \overline{\kappa}_{\text{stab}}$ we have*

$$\|u^0_{\mathcal{T}_*} - u_{\mathcal{T}}\|^2_{\kappa,\mathcal{T}} \le C^2_{LU}\big(\eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \omega(\mathcal{T} \setminus \mathcal{T}_*)) + \kappa\|h^{-1/2}[[u_{\mathcal{T}}]]\|^2_{L^2(\mathcal{F}^+)}\big),$$

where $u_{\mathcal{T}_*} = u^0_{\mathcal{T}_*} + u^\perp_{\mathcal{T}_*}$ is the orthogonal decomposition according to (9.37).

Moreover, if $\kappa \geq \overline{\kappa}_{\text{jump}}$, *then*

$$\|u^0_{\mathcal{T}_*} - u_{\mathcal{T}}\|^2_{\kappa,\mathcal{T}} \leq C^2_{LU}\big(\eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \omega(\mathcal{T} \setminus \mathcal{T}_*)) + \kappa^{-1}\eta^2_{\mathcal{T}}(u_{\mathcal{T}})\big). \tag{9.49}$$

Proof. We decompose $u_{\mathcal{T}_*} = u^0_{\mathcal{T}_*} + u^\perp_{\mathcal{T}_*}$ according to (9.37), exploit the partial consistency (9.36) for $u^0_{\mathcal{T}_*}$ with $v^0 \in \mathbb{V}^0_{\mathcal{T}}$, and $\widehat{f} \in \mathbb{F}_{\mathcal{T}} \subset \mathbb{F}_{\mathcal{T}_*}$ to obtain

$$\mathcal{B}_{\mathcal{T}_*}[u_{\mathcal{T}_*}, v^0] = a_{\mathcal{T}_*}[u_{\mathcal{T}_*}, v^0] = a_{\mathcal{T}}[u_{\mathcal{T}_*}, v^0] = \mathcal{B}_{\mathcal{T}}[u_{\mathcal{T}_*}, v^0] = \langle \widehat{f}, v^0\rangle_{\mathcal{T}}.$$

Since $\mathcal{B}_{\mathcal{T}}[u_{\mathcal{T}}, v^0] = \langle \widehat{f}, v^0\rangle_{\mathcal{T}}$, we readily see that

$$\mathcal{B}_{\mathcal{T}}[u^0_{\mathcal{T}_*} - u_{\mathcal{T}}, v^0] = 0 \quad \text{for all } v^0 \in \mathbb{V}^0_{\mathcal{T}}.$$

We rely on this reduced form of Galerkin orthogonality to prove the assertions. To this end, we write $u^0_{\mathcal{T}_*} - u_{\mathcal{T}} = e^0_* - u^\perp_{\mathcal{T}}$, with $e^0_* := u^0_{\mathcal{T}_*} - u^0_{\mathcal{T}}$. Using the coercivity estimate (9.34) for $v = u^0_{\mathcal{T}_*} - u_{\mathcal{T}} \in \mathbb{E}_{\mathcal{T}}$ yields for $\kappa \geq \overline{\kappa}_{\text{stab}}$

$$\begin{aligned}\|u^0_{\mathcal{T}_*} - u_{\mathcal{T}}\|^2_{\kappa,\mathcal{T}} &\lesssim \mathcal{B}_{\mathcal{T}}[u^0_{\mathcal{T}_*} - u_{\mathcal{T}}, u^0_{\mathcal{T}_*} - u_{\mathcal{T}}] \\ &= \mathcal{B}_{\mathcal{T}}[u^0_{\mathcal{T}_*} - u_{\mathcal{T}}, e^0_*] - \mathcal{B}_{\mathcal{T}}[u^0_{\mathcal{T}_*} - u_{\mathcal{T}}, u^\perp_{\mathcal{T}}].\end{aligned} \tag{9.50}$$

Note that the last term cannot be localized, and accounts for the lack of consistency of the dG method. However, it can be made arbitrarily small by increasing the penalty parameter κ. In fact, combining the continuity (9.33) with Lemma 9.9 (control of non-conformity) gives

$$\mathcal{B}_{\mathcal{T}}[u^0_{\mathcal{T}_*} - u_{\mathcal{T}}, u^\perp_{\mathcal{T}}] \lesssim \kappa^{1/2}\|h^{-1/2}[[u_{\mathcal{T}}]]\|_{L^2(\mathcal{F}^+)}\|u^0_{\mathcal{T}_*} - u_{\mathcal{T}}\|_{\kappa,\mathcal{T}}.$$

To localize $\mathcal{B}_{\mathcal{T}}[u^0_{\mathcal{T}_*} - u_{\mathcal{T}}, e^0_*]$, we choose $v^0 = \mathcal{I}^{\text{dG}}_{\mathcal{T}} e^0_*$, where the interpolation operator $\mathcal{I}^{\text{dG}}_{\mathcal{T}}$ is given by (9.7), and exploit the reduced Galerkin orthogonality. Since $e^0_* - \mathcal{I}^{\text{dG}}_{\mathcal{T}} e^0_* \in H^1_0(\Omega)$, the decomposition (9.29) of the bilinear form $\mathcal{B}_{\mathcal{T}}$ reads

$$\begin{aligned}&\mathcal{B}_{\mathcal{T}}\big[u^0_{\mathcal{T}_*} - u_{\mathcal{T}}, e^0_*\big] \\ &\quad = \mathcal{B}_{\mathcal{T}}\big[u^0_{\mathcal{T}_*} - u_{\mathcal{T}}, e^0_* - \mathcal{I}^{\text{dG}}_{\mathcal{T}} e^0_*\big] \\ &\quad = a_{\mathcal{T}}\big[u^0_{\mathcal{T}_*} - u_{\mathcal{T}}, e^0_* - \mathcal{I}^{\text{dG}}_{\mathcal{T}} e^0_*\big] - \int_\Omega \mathcal{L}_{\mathcal{T}}\big[u^0_{\mathcal{T}_*} - u_{\mathcal{T}}\big] \cdot \widehat{\boldsymbol{A}}\nabla_{\mathcal{T}}(e^0_* - \mathcal{I}^{\text{dG}}_{\mathcal{T}} e^0_*).\end{aligned}$$

We handle the first term as in the conforming case (Theorem 4.48), namely

$$a_{\mathcal{T}}(u^0_{\mathcal{T}_*} - u_{\mathcal{T}}, e^0_*) \lesssim \eta_{\mathcal{T}}(u_{\mathcal{T}}, \omega(\mathcal{T} \setminus \mathcal{T}_*))\, |u^0_{\mathcal{T}_*} - u^0_{\mathcal{T}}|_{H^1_0(\Omega)}.$$

Note that the interpolation estimate (9.9) for $\mathcal{I}^{\text{dG}}_{\mathcal{T}}$ is responsible for the appearance of $\omega(\mathcal{T} \setminus \mathcal{T}_*)$ rather than the smaller set $\mathcal{T} \setminus \mathcal{T}_*$.

For the second term, we use the lift estimate (9.26) along with the H^1-stability of $\mathcal{I}^{\text{dG}}_{\mathcal{T}}$ and $[[u^0_{\mathcal{T}_*}]] = 0$ to write

$$\int_\Omega \mathcal{L}_{\mathcal{T}}[u^0_{\mathcal{T}_*} - u_{\mathcal{T}}] \cdot \widehat{\boldsymbol{A}}\nabla_{\mathcal{T}} e^0_* \lesssim \|h^{-1/2}[[u_{\mathcal{T}}]]\|_{L^2(\mathcal{F}^+)}\, |u^0_{\mathcal{T}_*} - u^0_{\mathcal{T}}|_{H^1_0(\Omega)}.$$

Inserting the estimates into (9.50), and recalling that $1 \lesssim \overline{\kappa}_{\text{stab}} \leq \kappa$, we find that

$$\|u^0_{\mathcal{T}_*} - u_{\mathcal{T}}\|^2_{\kappa,\mathcal{T}} \lesssim \kappa^{1/2}\|h^{-1/2}[[u_{\mathcal{T}}]]\|_{L^2(\mathcal{F}^+)}\|u^0_{\mathcal{T}_*} - u_{\mathcal{T}}\|_{\kappa,\mathcal{T}} + \big(\eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{T} \setminus \mathcal{T}_*) + \kappa^{1/2}\|h^{1/2}[[u_{\mathcal{T}}]]\|_{L^2(\mathcal{F}^+)}\big)|u^0_{\mathcal{T}_*} - u^0_{\mathcal{T}}|_{H^1_0(\Omega)}.$$

Notice that $u^0_{\mathcal{T}_*} - u^0_{\mathcal{T}} = u^0_{\mathcal{T}_*} - u_{\mathcal{T}} + u^{\perp}_{\mathcal{T}}$, so that in view of Lemma 9.9 (control of non-conformity), we have

$$|u^0_{\mathcal{T}_*} - u^0_{\mathcal{T}}|_{H^1_0(\Omega)} = \|u^0_{\mathcal{T}_*} - u^0_{\mathcal{T}}\|_{\kappa,\mathcal{T}} \lesssim \|u^0_{\mathcal{T}_*} - u_{\mathcal{T}}\|_{\kappa,\mathcal{T}} + \kappa^{1/2}\|h^{-1/2}[[u_{\mathcal{T}}]]\|_{L^2(\mathcal{F}^+)}.$$

The first desired inequality follows from the last two estimates. For the second inequality, it suffices to further invoke Lemma 9.11 (discontinuity control). □

9.4. *Module* GALERKIN-DG

The main ingredients for the *a posteriori* estimation have been derived in the previous section and we can now turn our attention to the adaptive method. In essence, it is the same as in the conforming case (Algorithm 5.4) but accounting for the perturbation arising from the non-conforming setting. Compared to Algorithm 5.4, SOLVE($\mathcal{T}$) determines the discontinuous Galerkin solution to (9.21) and REFINE($\mathcal{T}, \mathcal{M}$) produces the smallest Λ-admissible refinement of $\mathcal{T}$ where all the marked elements $\mathcal{M}$ are refined at least $b \geq 1$ times.

Algorithm 9.17 (GALERKIN-DG). Let $\widehat{\mathcal{T}} \geq \mathcal{T}_0$ be a Λ-admissible refinement, $\Lambda \geq 0$, of a suitable initial mesh $\mathcal{T}_0$. Let data $\widehat{\mathcal{D}} = (\widehat{\boldsymbol{A}}, \widehat{c}, \widehat{f}) \in \mathbb{D}_{\widehat{\mathcal{T}}}$ be discrete on $\widehat{\mathcal{T}}$ and let $\varepsilon > 0$ be a stopping tolerance. The following routine creates a Λ-admissible refinement $\mathcal{T} \geq \widehat{\mathcal{T}}$ and discontinuous Galerkin solution $u_{\mathcal{T}} \in \mathbb{V}^{-1}_{\mathcal{T}}$ for data $\widehat{\mathcal{D}}$ such that $\eta_{\mathcal{T}}(u_{\mathcal{T}}) \leq \varepsilon$.

$[\mathcal{T}, u_{\mathcal{T}}]$ = GALERKIN-DG($\widehat{\mathcal{T}}, \widehat{\mathcal{D}}, \varepsilon$)
  set $j = 0, \mathcal{T}_0 = \widehat{\mathcal{T}}$ and do
    $[u_j]$ = SOLVE($\mathcal{T}_j$);
    $[\{\eta_j(u_j, T)\}_{T \in \mathcal{T}_j}]$ = ESTIMATE($u_j, \mathcal{T}_j, \widehat{\mathcal{D}}$);
    if $\eta_j(u_j) \leq \varepsilon$;
      return $(\mathcal{T}_j, u_j)$
    $[\mathcal{M}_j]$ = MARK($\{\eta_j(u_j, T)\}_{T \in \mathcal{T}_j}, \mathcal{T}_j, \theta$);
    $[\mathcal{T}_{j+1}]$ = REFINE($\mathcal{T}_j, \mathcal{M}_j$);
    $j \leftarrow j + 1$;
  while true

We start the analysis of GALERKIN-DG by investigating how the energy norm $\mathcal{B}_{\mathcal{T}}[v, v]^{1/2}$ changes upon refining $\mathcal{T}$. Note that in the conforming case, Lemma 5.2 (Pythagoras) directly provides the relation $|||\widehat{u} - u_{\mathcal{T}_*}|||_{\Omega} \leq |||\widehat{u} - u_{\mathcal{T}}|||_{\Omega}$. In the non-conforming setting, the constant on the right-hand side is no longer 1 and jump terms are present in the estimate. Regardless, it is possible to assess the effect of refinement in the energy norm and, in turn, compare two consecutive Galerkin

solutions $u_{\mathcal{T}}$ and $u_{\mathcal{T}_*}$, where $\mathcal{T} \in \mathbb{T}^\Lambda$ and $\mathcal{T}_* = \mathsf{REFINE}(\mathcal{T}, \mathcal{M})$ for some $\mathcal{M} \subset \mathcal{T}$. This is the subject of the next three results, but before embarking on this path, we mention a key ingredient for this comparison to hold: *the routine* REFINE *does not refine elements in* $\mathcal{T}$ *more than* d *times for* $b = 1$; see Corollary 3.31. This implies that for any $F \in \mathcal{F}^+$ and $F_* \in \mathcal{F}_*^+$ with $F_* \subset F$, we have

$$h_F \lesssim h_{F_*}. \tag{9.51}$$

Lemma 9.18 (mesh perturbation). *Let* $\mathcal{T} \in \mathbb{T}^\Lambda$ *be a* Λ*-admissible refinement of* $\mathcal{T}_0$ *satisfying Assumption 6.19 (initial labelling),* $\mathcal{M} \subset \mathcal{T}$ *and* $\mathcal{T}_* = \mathsf{REFINE}(\mathcal{T}, \mathcal{M})$. *Assume that* $\widehat{\boldsymbol{A}}$ *and* $\widehat{c}$ *satisfy the structural assumption* (9.20). *There is a constant* C *such that, for* $0 < \varepsilon < 1$ *and all* $v \in \mathbb{E}_{\mathcal{T}}$,

$$\mathcal{B}_{\mathcal{T}_*}[v, v] \le (1 + \varepsilon)\mathcal{B}_{\mathcal{T}}[v, v] + C\varepsilon^{-1}\kappa\|h^{-1/2}[\![v]\!]\|^2_{L^2(\mathcal{F}^+)}. \tag{9.52}$$

Proof. Because $\nabla_{\mathcal{T}_*} v = \nabla_{\mathcal{T}} v$ when $v \in \mathbb{E}_{\mathcal{T}}$, we directly deduce that

$$\begin{aligned}\mathcal{B}_{\mathcal{T}_*}[v, v] = \mathcal{B}_{\mathcal{T}}[v, v] + 2\int_\Omega (\mathcal{L}_{\mathcal{T}}[v] - \mathcal{L}_{\mathcal{T}_*}[v]) \cdot \widehat{\boldsymbol{A}}\nabla_{\mathcal{T}} v \\ + \kappa\|h_*^{-1/2}[\![v]\!]\|^2_{L^2(\mathcal{F}_*^+)} - \kappa\|h^{-1/2}[\![v]\!]\|^2_{L^2(\mathcal{F}^+)},\end{aligned} \tag{9.53}$$

where $h_* := h_{\mathcal{T}_*}$ denotes the mesh size function of $\mathcal{T}_*$.

Unlike the broken gradients, the lifting operators are affected by refinements. However, this effect is controlled by the scaled jumps, as we now show. Using estimate (9.27) twice with $\epsilon = \varepsilon C_{\mathrm{coer}}/2$ yields

$$\begin{aligned}&2\int_\Omega (\mathcal{L}_{\mathcal{T}}[v] - \mathcal{L}_{\mathcal{T}_*}[v]) \cdot \widehat{\boldsymbol{A}}\nabla_{\mathcal{T}} v \\ &\quad \le \varepsilon C_{\mathrm{coer}}\|\nabla_{\mathcal{T}} v\|^2_{L^2(\mathcal{T})} + \frac{2C_{\mathrm{lift}}^2}{\varepsilon C_{\mathrm{coer}}}\big(\|h_*^{-1/2}[\![v]\!]\|^2_{L^2(\mathcal{F}_*^+)} + \|h^{-1/2}[\![v]\!]\|^2_{L^2(\mathcal{F}^+)}\big).\end{aligned}$$

Hence the coercivity estimate (9.34) gives

$$\begin{aligned}&2\int_\Omega (\mathcal{L}_{\mathcal{T}}[v] - \mathcal{L}_{\mathcal{T}_*}[v]) \cdot \widehat{\boldsymbol{A}}\nabla_{\mathcal{T}} v \\ &\quad \le \epsilon\mathcal{B}_{\mathcal{T}}[v, v] + \frac{2C_{\mathrm{lift}}^2}{\varepsilon C_{\mathrm{coer}}}\big(\|h^{-1/2}[\![v]\!]\|^2_{L^2(\mathcal{F}^+)} + \|h_*^{-1/2}[\![v]\!]\|^2_{L^2(\mathcal{F}_*^+)}\big).\end{aligned}$$

Inserting this back into (9.53), and using the fact that the jumps of v occur only on $\mathcal{F}^+ \subset \mathcal{F}_*^+$, the mesh size relation (9.51) proves the desired estimate. □

Lemma 9.19 (comparison of solutions). *Let* $\mathcal{T} \in \mathbb{T}^\Lambda$ *be a* Λ*-admissible refinement of* $\mathcal{T}_0$ *satisfying Assumption 6.19 (initial labelling),* $\mathcal{M} \subset \mathcal{T}$, *and* $\mathcal{T}_* = \mathsf{REFINE}(\mathcal{T}, \mathcal{M})$. *Assume that* $\widehat{\boldsymbol{A}}$ *and* $\widehat{c}$ *satisfy the structural assumption* (9.20). *Let* $\widehat{u} = u(\widehat{\mathcal{D}}) \in H_0^1(\Omega)$ *be the solution of the perturbed problem* (5.5) *with discrete data* $\widehat{\mathcal{D}}$ *and let* $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}^{-1}$, $u_{\mathcal{T}_*} \in \mathbb{V}_{\mathcal{T}_*}^{-1}$ *denote the Galerkin solutions associated to* $\mathcal{T}$, $\mathcal{T}_*$ *respectively with data* $\widehat{\mathcal{D}}$. *Let* $\overline{\kappa}_{\mathrm{jump}}$ *be as in Lemma 9.11. There exists a*

constant C_{comp} *such that for all* $\kappa \geq \overline{\kappa}_{\mathrm{jump}}$ *and all* $0 < \epsilon < 1$ *we have*

$$\begin{aligned}
&\mathcal{B}_{\mathcal{T}_*}[\widehat{u} - u_{\mathcal{T}_*}, \widehat{u} - u_{\mathcal{T}_*}] \\
&\quad \leq (1+\epsilon)\mathcal{B}_{\mathcal{T}}[\widehat{u} - u_{\mathcal{T}}, \widehat{u} - u_{\mathcal{T}}] \\
&\qquad - \frac{C_{\mathrm{coer}}}{2}\|\nabla_{\mathcal{T}_*}(u_{\mathcal{T}_*} - u_{\mathcal{T}})\|^2_{L^2(\mathcal{T}_*)} + \frac{C_{\mathrm{comp}}}{\epsilon\kappa}\big(\eta_{\mathcal{T}}(u_{\mathcal{T}})^2 + \eta_{\mathcal{T}_*}(u_{\mathcal{T}_*})^2\big).
\end{aligned}$$

Proof. We invoke the partial Galerkin orthogonality (9.45) of $\widehat{u} - u_{\mathcal{T}_*}$ upon testing with $v^0 := u^0_{\mathcal{T}_*} - u^0_{\mathcal{T}} \in \mathbb{V}^0_{\mathcal{T}_*}$:

$$\mathcal{B}_{\mathcal{T}_*}[\widehat{u} - u_{\mathcal{T}_*}, \widehat{u} - u_{\mathcal{T}_*}] = \mathcal{B}_{\mathcal{T}_*}[\widehat{u} - u_{\mathcal{T}_*} + v^0, \widehat{u} - u_{\mathcal{T}_*} + v^0] - \mathcal{B}_{\mathcal{T}_*}[v^0, v^0].$$

Note that

$$\widehat{u} - u_{\mathcal{T}_*} + v^0 = \widehat{u} - u_{\mathcal{T}} + u^{\perp}_{\mathcal{T}} - u^{\perp}_{\mathcal{T}_*}$$

and $\|v^0\|_{\kappa,\mathcal{T}_*} = \|\nabla_{\mathcal{T}_*} v^0\|_{L^2(\mathcal{T}_*)}$, which is critical to the argument below. Hence, from the coercivity and continuity of $\mathcal{B}_{\mathcal{T}_*}$ (Corollary 9.8), we deduce that

$$\begin{aligned}
&\mathcal{B}_{\mathcal{T}_*}[\widehat{u} - u_{\mathcal{T}_*}, \widehat{u} - u_{\mathcal{T}_*}] \\
&\quad \leq \mathcal{B}_{\mathcal{T}_*}[\widehat{u} - u_{\mathcal{T}}, \widehat{u} - u_{\mathcal{T}}] + 2C^{1/2}_{\mathrm{cont}}\mathcal{B}_{\mathcal{T}_*}[\widehat{u} - u_{\mathcal{T}}, \widehat{u} - u_{\mathcal{T}}]^{1/2}\|u^{\perp}_{\mathcal{T}} - u^{\perp}_{\mathcal{T}_*}\|_{\kappa,\mathcal{T}_*} \\
&\qquad + C_{\mathrm{cont}}\|u^{\perp}_{\mathcal{T}} - u^{\perp}_{\mathcal{T}_*}\|^2_{\kappa,\mathcal{T}_*} - C_{\mathrm{coer}}\|\nabla_{\mathcal{T}_*}(u^0_{\mathcal{T}_*} - u^0_{\mathcal{T}})\|^2_{L^2(\mathcal{T}_*)}.
\end{aligned}$$

We now apply the reverse triangle inequality and Young's inequality

$$\|\nabla_{\mathcal{T}_*}(u^0_{\mathcal{T}_*} - u^0_{\mathcal{T}})\|^2_{L^2(\mathcal{T}_*)} \geq \frac{1}{2}\|\nabla_{\mathcal{T}_*}(u_{\mathcal{T}_*} - u_{\mathcal{T}})\|^2_{L^2(\mathcal{T}_*)} - \|\nabla_{\mathcal{T}_*}(u^{\perp}_{\mathcal{T}_*} - u^{\perp}_{\mathcal{T}})\|^2_{L^2(\mathcal{T}_*)}$$

to deduce that for any $0 < \varepsilon < 1$

$$\begin{aligned}
\mathcal{B}_{\mathcal{T}_*}[\widehat{u} - u_{\mathcal{T}_*}, \widehat{u} - u_{\mathcal{T}_*}] \leq &(1+\varepsilon)\mathcal{B}_{\mathcal{T}_*}[\widehat{u} - u_{\mathcal{T}}, \widehat{u} - u_{\mathcal{T}}] \\
&- \frac{C_{\mathrm{coer}}}{2}\|\nabla_{\mathcal{T}_*}(u_{\mathcal{T}_*} - u_{\mathcal{T}})\|^2_{L^2(\mathcal{T}_*)} + \frac{C}{\varepsilon}\|u^{\perp}_{\mathcal{T}_*} - u^{\perp}_{\mathcal{T}}\|^2_{\kappa,\mathcal{T}_*},
\end{aligned}$$

where C is for the remainder of this proof a constant independent of the discretization parameters and κ.

To bound the last term we recall Lemma 9.9 (control of non-conformity),

$$\|u^{\perp}_{\mathcal{T}_*} - u^{\perp}_{\mathcal{T}}\|_{\kappa,\mathcal{T}_*} \lesssim \kappa^{1/2}\|h_*^{-1/2}[[u_{\mathcal{T}_*}]]\|_{L^2(\mathcal{F}^+_*)} + \kappa^{1/2}\|h_*^{-1/2}[[u_{\mathcal{T}}]]\|_{L^2(\mathcal{F}^+_*)},$$

and notice that the last integral over $\mathcal{F}^+_*$ has weights relative to the local mesh size of $\mathcal{T}_* \geq \mathcal{T}$. Since for consecutive meshes the local mesh sizes are comparable, according to (9.51), we can write $\|h^{-1/2}[[u_{\mathcal{T}}]]\|_{L^2(\mathcal{F}^+)}$ instead. Inserting these expressions in the preceding estimate, and using Lemma 9.18 (mesh perturbation) to replace $\mathcal{B}_{\mathcal{T}_*}$ with $\mathcal{B}_{\mathcal{T}}$ on the right-hand side, yields

$$\begin{aligned}
\mathcal{B}_{\mathcal{T}_*}[\widehat{u} - u_{\mathcal{T}_*}, \widehat{u} - u_{\mathcal{T}_*}] \leq (1+\varepsilon)\mathcal{B}_{\mathcal{T}}[\widehat{u} - u_{\mathcal{T}}, \widehat{u} - u_{\mathcal{T}}] - \frac{C_{\mathrm{coer}}}{2}\|\nabla_{\mathcal{T}_*}(u_{\mathcal{T}_*} - u_{\mathcal{T}})\|^2_{L^2(\mathcal{T}_*)} \\
+ C\frac{\kappa}{\varepsilon}\big(\|h_*^{-1/2}[[u_{\mathcal{T}_*}]]\|^2_{L^2(\mathcal{F}^+_*)} + \|h^{-1/2}[[u_{\mathcal{T}}]]\|^2_{L^2(\mathcal{F}^+)}\big),
\end{aligned}$$

where 2ε has been relabelled ε. Finally, to derive the desired estimate, it remains to invoke Lemma 9.11 (discontinuity control). □

Combining Lemma 9.19 (comparison of solutions) with Lemma 9.10 (*a posteriori* error estimates), we derive the following dG version of Lemma 5.2 (Pythagoras).

Corollary 9.20 (quasi-orthogonality of dG errors). *If we make the same assumptions as Lemma 9.19, then for all*

$$\kappa \geq \kappa_{QO} := \frac{C_{\mathrm{comp}}}{\varepsilon^2 C_L} \quad \textit{and} \quad 0 < \varepsilon \leq \frac{1}{4}$$

we obtain

$$\|\widehat{u} - u_{\mathcal{T}_*}\|^2_{\kappa,\mathcal{T}_*} \leq (1+4\varepsilon)\,\|\widehat{u} - u_{\mathcal{T}}\|^2_{\kappa,\mathcal{T}} - \frac{C_{\mathrm{coer}}}{2}\,\|\nabla_{\mathcal{T}_*}(u_{\mathcal{T}_*} - u_{\mathcal{T}})\|^2_{L^2(\mathcal{T}_*)}.$$

Proof. We make use of the lower bound (9.40), and set

$$D := \frac{C_{\mathrm{comp}}}{\varepsilon C_L},$$

to rewrite the estimate of Lemma 9.19 as follows:

$$\left(1 - \frac{D}{\kappa}\right)\|\widehat{u} - u_{\mathcal{T}_*}\|^2_{\kappa,\mathcal{T}_*} \leq \left(1 + \varepsilon + \frac{D}{\kappa}\right)\|\widehat{u} - u_{\mathcal{T}}\|^2_{\kappa,\mathcal{T}} - \frac{C_{\mathrm{coer}}}{2}\|\nabla_{\mathcal{T}_*}(u_{\mathcal{T}_*} - u_{\mathcal{T}})\|^2_{L^2(\mathcal{T}_*)}.$$

For $\kappa \geq \kappa_{QO} = D/\varepsilon$ this inequality implies

$$\|\widehat{u} - u_{\mathcal{T}_*}\|^2_{\kappa,\mathcal{T}_*} \leq \frac{1+2\varepsilon}{1-\varepsilon}\|\widehat{u} - u_{\mathcal{T}}\|^2_{\kappa,\mathcal{T}} - \frac{C_{\mathrm{coer}}}{2(1-\varepsilon)}\|\nabla_{\mathcal{T}_*}(u_{\mathcal{T}_*} - u_{\mathcal{T}})\|^2_{L^2(\mathcal{T}_*)}.$$

It remains to realize that $(1+2\varepsilon)/(1-\varepsilon) \leq 1 + 4\varepsilon$ provided $\varepsilon \leq \frac{1}{4}$. □

The last ingredient to prove convergence of GALERKIN-DG is a dG version of Proposition 4.56 (estimator reduction) with $f = f_* \in \mathbb{F}_{\mathcal{T}}$. It turns out that the same estimate and proof are valid for dG except that the H^1_0-seminorm is to be replaced by the broken H^1_0-seminorm. We thus state the result without proof.

Proposition 9.21 (estimator reduction). *Given $\mathcal{T} \in \mathbb{T}^\Lambda$ and a subset $\mathcal{M} \subset \mathcal{T}$ of elements marked for refinement, let $\mathcal{T}_* = \mathsf{REFINE}(\mathcal{T}, \mathcal{M})$. If $f = P_{\mathcal{T}} f \in \mathbb{F}_{\mathcal{T}}$, then there is a constant C_{Lip} such that, for all $v \in \mathbb{V}_{\mathcal{T}}$, $v_* \in \mathbb{V}_{\mathcal{T}_*}$ and any $\delta > 0$,*

$$\eta_{\mathcal{T}_*}(v_*, \mathcal{T}_*)^2 \leq (1+\delta)\big(\eta_{\mathcal{T}}(v, \mathcal{T})^2 - \lambda\,\eta_{\mathcal{T}}(v, \mathcal{M})^2\big) + (1+\delta^{-1})\,C^2_{\mathrm{Lip}}\,\|\nabla_{\mathcal{T}_*}(v_* - v)\|^2_{L^2(\mathcal{T}_*)}.$$

We are now in a position to prove a contraction property between two consecutive iterations of the adaptive loop GALERKIN-DG.

Theorem 9.22 (contraction property). *Let $\widehat{\mathcal{T}}$ be a Λ-admissible refinement of $\mathcal{T}_0$ satisfying Assumption 6.19 (initial labelling). Let $\widehat{\mathcal{D}} \in \mathbb{D}_{\widehat{\mathcal{T}}}$ be such that $\widehat{\boldsymbol{A}}$ and $\widehat{c}$ satisfy the structural assumption (9.20). Let $\theta \in (0, 1]$ be the Dörfler marking parameter used in the* MARK *module and let $\{\mathcal{T}_j, \mathbb{V}_j, u_j\}_{j=0}^J$ be a sequence of*

conforming meshes, finite element spaces and discrete solutions $u_j = u_j(\widehat{\mathcal{D}}) \in \mathbb{V}_j$ *created within* GALERKIN-DG. *If* $\widehat{u} \in H^1_0(\Omega)$ *is the exact solution of* (5.5) *with discrete data* $\widehat{\mathcal{D}}$, *then there exist constants* $\overline{\kappa}_{\mathrm{conv}} \geq 0$, $\gamma > 0$, *and* $0 < \alpha < 1$ *independent of the discretization parameters and* κ, *such that, for all* $\kappa \geq \overline{\kappa}_{\mathrm{conv}}$ *and* $0 \leq j < J$,

$$B_{j+1}^2 + \gamma \eta_{\mathcal{T}_{j+1}}^2(u_{j+1}) \leq \alpha^2 \big(B_j^2 + \gamma \eta_{\mathcal{T}_j}^2(u_j)\big), \tag{9.54}$$

where $B_j := (\mathcal{B}_{\mathcal{T}_j}[\widehat{u} - u_j, \widehat{u} - u_j])^{1/2}$ *is the dG norm of* $\widehat{u} - u_j$.

Proof. In essence, we proceed as in Theorem 5.8 (general contraction property) for the conforming case but with minor changes that account for non-conformity. We only explain the differences below. For $j \geq 0$, we shorten the notation and write $\eta_j := \eta_{\mathcal{T}_j}(u_j)$ and $E_j := \|\nabla_{\mathcal{T}_{j+1}}(u_{j+1} - u_j)\|_{L^2(\mathcal{T}_{j+1})}$.

Corollary 9.20 (quasi-orthogonality of dG errors) gives for any $0 < \varepsilon \leq \frac{1}{4}$

$$B_{j+1}^2 \leq (1 + 4\varepsilon) B_j^2 - \frac{C_{\mathrm{coer}}}{2} E_j^2.$$

Combining Proposition 9.21 (estimator reduction), written in terms of $\mathcal{T} = \mathcal{T}_j$, $\mathcal{T}_* = \mathcal{T}_{j+1}$, $v = u_j$ and $v_* = u_{j+1}$, with Dörfler marking $\eta_j(u_j, \mathcal{M}_j) \geq \theta \eta_j$, yields

$$\eta_{j+1}^2 \leq (1 + \delta)(1 - \lambda\theta^2)\eta_j^2 + (1 + \delta^{-1}) C_{\mathrm{Lip}}^2 E_j^2$$

for any $\delta > 0$. We now multiply this inequality by $\gamma > 0$ and add it to the previous one with the following choice of parameters:

$$\delta = \frac{1 - \lambda\theta^2/2}{1 - \lambda\theta^2} - 1, \quad \gamma = \frac{C_{\mathrm{coer}}}{2C_{\mathrm{Lip}}^2(1 + \delta^{-1})}.$$

Consequently, the terms involving E_j^2 cancel out and we end up with

$$\begin{aligned} B_{j+1}^2 + \gamma \eta_{j+1}^2 &\leq (1 + 4\varepsilon) B_j^2 + \gamma(1 + \delta)(1 - \lambda\theta^2)\eta_j^2 \\ &= \left(1 + 4\varepsilon - \frac{\gamma\lambda\theta^2}{4}\right) B_j^2 + \gamma\left(1 - \frac{\lambda\theta^2}{4}\right)\eta_j^2. \end{aligned}$$

We finally choose

$$\varepsilon := \frac{\gamma\lambda\theta^2}{32}$$

to obtain (9.54) with

$$\alpha^2 = \max\left\{1 - \frac{\gamma\lambda\theta^2}{8}, 1 - \frac{\lambda\theta^2}{4}\right\},$$

and conclude the proof. □

Corollary 9.23 (linear convergence). *Under the assumptions of Theorem 9.22, and if* $0 < \alpha < 1$, $\gamma > 0$, $\overline{\kappa}_{\mathrm{conv}} > 0$ *are the constants in* (9.54), *then for all* $\kappa \geq \overline{\kappa}_{\mathrm{conv}}$

we obtain

$$\|\widehat{u} - u_k\|_{\kappa,\mathcal{T}_k} \le C_* \alpha^{k-j} \|\widehat{u} - u_j\|_{\kappa,\mathcal{T}_j},$$

for some constant C_* independent of the discretization parameters and κ.

Proof. Let $e_k^2 = \|\widehat{u} - u_k\|_{\kappa,\mathcal{T}_k}^2$ and $B_k^2 = \mathcal{B}_{\mathcal{T}_k}[\widehat{u} - u_k, \widehat{u} - u_k]^2$, and use the coercivity estimate (9.34), the contraction property (9.54), the continuity estimate (9.33) and the lower bound (9.40), to arrive at

$$C_{\mathrm{coer}} e_k^2 \le B_k^2 \le \alpha^{2(k-j)} \left(B_j^2 + \gamma \eta_j^2\right) \le \alpha^{2(k-j)} \left(C_{\mathrm{cont}} + \frac{\gamma}{C_L^2}\right) e_j^2.$$

This is the desired estimate in disguise with

$$C_* := \frac{1}{C_{\mathrm{coer}}} \left(C_{\mathrm{cont}} + \frac{\gamma}{C_L^2}\right)^{1/2}. \qquad \square$$

We end the discussion of GALERKIN-DG by deriving the optimality property of the Dörfler marking strategy. We mimic the proof of Lemma 6.16 (Dörfler marking) but directly use the optimal parameter $\mu = \frac{1}{2}$ to simplify the argument. We refer to the discussion after Lemma 6.16 for the role of μ and its influence on θ_0. Notice that θ_0 depends on κ^{-1} because of its appearance in the perturbed localized upper bound (9.49). It plays a similar role to σ in Assumption 6.15 (restriction on ω) in the presence of oscillations (one-step method with switch).

Lemma 9.24 (Dörfler marking). *Let $\mathcal{T}_* \ge \mathcal{T}$ be two Λ-admissible refinements of $\mathcal{T}_0$ satisfying Assumption 6.19 (initial labelling). Let $\widehat{\mathcal{D}} \in \mathbb{D}_{\widehat{\mathcal{T}}}$ be such that $\widehat{\boldsymbol{A}}$ and $\widehat{c}$ satisfy the structural assumption (9.20). Let $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}^{-1}$, $u_{\mathcal{T}_*} \in \mathbb{V}_{\mathcal{T}_*}^{-1}$ denote the Galerkin solutions associated with $\mathcal{T}$, $\mathcal{T}_*$, respectively, and let $\widehat{u} \in H_0^1(\Omega)$ denotes the solution to (5.5) with discrete data $\widehat{\mathcal{D}}$. Assume*

$$\kappa > \overline{\kappa}_D := \max\left(\overline{\kappa}_{\mathrm{stab}}, 4C_{\mathrm{Lip}}^2 C_{LU}^2\right).$$

If

$$\eta_{\mathcal{T}_*}(u_{\mathcal{T}_*}) \le \frac{1}{2} \eta_{\mathcal{T}}(u_{\mathcal{T}}), \tag{9.55}$$

then the refined set $\mathcal{T} \setminus \mathcal{T}_*$ satisfies the Dörfler property

$$\eta_{\mathcal{T}}(u_{\mathcal{T}}, \omega(\mathcal{T} \setminus \mathcal{T}_*)) \ge \theta_0 \, \eta_{\mathcal{T}}(u_{\mathcal{T}}), \tag{9.56}$$

with

$$0 < \theta_0^2 := \theta_0^2(\kappa) := \frac{1 - 4C_{\mathrm{Lip}}^2 C_{LU}^2 \kappa^{-1}}{4C_{\mathrm{Lip}}^2 C_{LU}^2} < \frac{1}{4C_{\mathrm{Lip}}^2 C_{LU}^2}.$$

Proof. To relate $\eta_{\mathcal{T}}$ to $\eta_{\mathcal{T}_*}$, we invoke Proposition 9.21 (estimator reduction) with $\delta = 1$, along with the localized upper bound (9.49), to write

$$\eta_{\mathcal{T}}(u_{\mathcal{T}})^2 \le 2\eta_{\mathcal{T}_*}(u_{\mathcal{T}_*})^2 + 2C_{\mathrm{Lip}}^2 C_{LU}^2 \left(\eta_{\mathcal{T}}(u_{\mathcal{T}}, \omega(\mathcal{T} \setminus \mathcal{T}_*))^2 + \kappa^{-1} \eta_{\mathcal{T}}(u_{\mathcal{T}})^2\right).$$

This, combined with (9.55), yields

$$\left(\frac{1}{2} - 2C_{\mathrm{Lip}}^2 C_{LU}^2 \kappa^{-1}\right)\eta_{\mathcal{T}}(u_{\mathcal{T}})^2 \le 2C_{\mathrm{Lip}}^2 C_{LU}^2 \eta_{\mathcal{T}}(u_{\mathcal{T}}, \omega(\mathcal{T} \setminus \mathcal{T}_*))^2$$

for $\kappa \ge \overline{\kappa}_{\mathrm{D}}$. This is the desired result in disguise. □

9.5. *Convergence of* AFEM-DG-TS

Algorithm 9.1 (AFEM-DG-TS) relies on two modules: GALERKIN-DG and DATA. We have analysed the performance of GALERKIN-DG in the previous section and showed in Section 7 that the output $[\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k] = \mathsf{DATA}(\mathcal{T}_k, \mathcal{D}, \omega\varepsilon_k)$ satisfies

$$\|\mathcal{D} - \widehat{\mathcal{D}}_k\|_{D(\Omega)} \le \omega\varepsilon_k \quad \Rightarrow \quad |u - \widehat{u}_k|_{H_0^1(\Omega)} \le C_D \omega \varepsilon_k. \tag{9.57}$$

Recall that $u = u(\mathcal{D}), \widehat{u}_k = u(\widehat{\mathcal{D}}_k) \in H_0^1(\Omega)$ are the exact solutions to (2.7) with exact data $\mathcal{D}$ and discrete data $\widehat{\mathcal{D}}_k$, respectively. We also recall that $\widehat{\mathcal{D}}_k$ satisfies the structural assumption (9.20) uniformly in k and thus C_D does not depend on k.

We start with a result guaranteeing that the cost of GALERKIN-DG does not depend on the iteration counter k within AFEM-DG-TS.

Lemma 9.25 (computational cost of GALERKIN-DG). *For any $\kappa \ge \overline{\kappa}_{\mathrm{conv}}$ and any $k \in \mathbb{N}$, the number of sub-iterations J_k inside a call of* GALERKIN-DG *at iteration k of Algorithm 9.1 (*AFEM-DG-TS*) is bounded independently of k.*

Proof. We proceed as in the proof of Proposition 5.27 (computational cost of GALERKIN) for the conforming case, and focus on the essential differences. We fix the iteration counter $k \ge 1$, recall that the output of the $(k-1)$th loop of AFEM-DG-TS is $[\mathcal{T}_k, u_k] = \mathsf{GALERKIN\text{-}DG}(\widehat{\mathcal{T}}_{k-1}, \widehat{\mathcal{D}}_{k-1}, \varepsilon_{k-1})$, and let $\mathcal{T}_{k,j}$ and $\widehat{u}_{k,j} \in \mathbb{V}_{\mathcal{T}_{k,j}}^{-1}$ denote the jth mesh and Galerkin solution to (9.21) with data $\widehat{\mathcal{D}}_k$ in the kth loop of AFEM-DG-TS. The exact solution to the perturbed problem (5.5) with discrete coefficient $\widehat{\mathcal{D}}_k$ is $\widehat{u}_k = u(\widehat{\mathcal{D}}_k)$.

We recall that $\mathcal{T}_{k,0} = \widehat{\mathcal{T}}_k$ is the mesh produced by DATA, and assume that $u_{k,0} \in \mathbb{V}_{k,0}$ satisfies $\eta_{\mathcal{T}_{k,0}}(u_{k,0}) > \varepsilon_k$, because otherwise $J_k = 0$ and there is nothing to prove. In view of Corollary 9.23 (linear convergence), all we need to prove is that the error $\|u_{k,0} - \widehat{u}_k\|_{\kappa,\mathcal{T}_{k,0}}$ entering GALERKIN-DG is bounded by ε_k. We resort to Corollary 9.15 (quasi-monotonicity with different data) to write

$$\|u_{k,0} - \widehat{u}_k\|_{\kappa,\mathcal{T}_{k,0}} \le C_{\mathrm{Mo}}\big(\|u_k - \widehat{u}_{k-1}\|_{\kappa,\mathcal{T}_k} + |\widehat{u}_k - \widehat{u}_{k-1}|_{H_0^1(\Omega)}\big).$$

The appearance of the last term is the only difference with respect to Proposition 5.27. However, in view of property (9.57) of DATA, we infer that

$$|\widehat{u}_k - \widehat{u}_{k-1}|_{H_0^1(\Omega)} \le |\widehat{u}_k - u|_{H_0^1(\Omega)} + |\widehat{u}_{k-1} - u|_{H_0^1(\Omega)} \le C_D\omega(\varepsilon_k + \varepsilon_{k-1}) = 3C_D\omega\varepsilon_k.$$

Moreover, the stabilization-free upper bound (9.44) implies

$$\|u_k - \widehat{u}_{k-1}\|_{\kappa,\mathcal{T}_k} \le C_U\, \eta_{\mathcal{T}_k}(u_k) \le C_U\varepsilon_{k-1} = 2C_U\varepsilon_k,$$

which, combined with the lower bound (9.40), further yields

$$\|u_{k,0} - \widehat{u}_k\|_{\kappa,\mathcal{T}_{k,0}} \le (3C_D\omega + 2C_U)\varepsilon_k = \Lambda\varepsilon_k \quad \Rightarrow \quad \eta_{\mathcal{T}_{k,0}} \le C_L^{-1}\Lambda\varepsilon_k.$$

This is the requisite estimate. In fact, recalling Corollary 9.23, we see that

$$\eta_{\mathcal{T}_{k,j}}(u_{k,j}) \le C_L^{-1}\|u_{k,j} - \widehat{u}_k\|_{\kappa,\mathcal{T}_{k,j}} \le C_L^{-1}C_*\alpha^j\|u_{k,0} - \widehat{u}_k\|_{\kappa,\mathcal{T}_{k,0}} \le C_L^{-1}C_*\Lambda\varepsilon_k\alpha^j.$$

Since GALERKIN-DG stops when $\eta_{\mathcal{T}_{J_k,k}}(u_{J_k,k}) \le \varepsilon_k$, we finally conclude as in the proof of Proposition 5.27 that J_k is independent of k. □

The proof of convergence of AFEM-DG-TS is identical to the proof of Proposition 5.29 upon replacing the seminorm $|.|_{H_0^1(\Omega)}$ with the appropriate dG norm. It is therefore not repeated here.

Proposition 9.26 (convergence of AFEM-DG-TS). *For any $\kappa \ge \overline{\kappa}_{\mathrm{conv}}$ and $k \ge 0$, the $(k+1)$th iteration of* AFEM-DG-TS *terminates and requires a finite number of inner iterations of* GALERKIN-DG *independent of k. Moreover, if $u \in H_0^1(\Omega)$ denotes the solution to (2.7), there exists a constant C_* such that the output of $[\mathcal{T}_{k+1}, u_{k+1}] = \mathsf{GALERKIN\text{-}DG}(\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k, \varepsilon_k)$ satisfies*

$$\|u - u_{k+1}\|_{\kappa,\mathcal{T}_{k+1}} \le C_*\varepsilon_k \quad \textit{for all } k \ge 0.$$

Therefore AFEM-DG-TS *stops after*

$$K < 2 + \frac{\log \varepsilon_0/\mathtt{tol}}{\log 2}$$

iterations and delivers

$$\|u - u_{K+1}\|_{\kappa,\mathcal{T}_K} \le C_*\,\mathtt{tol}\,.$$

9.6. *Rate-optimality of* AFEM-DG-TS

To derive rates of convergence for the discontinuous Galerkin method, we proceed similarly to Section 6 for the conforming case. Recall that in the kth step of Algorithm 9.1 (AFEM-DG-TS), the output of $[\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k] = \mathsf{DATA}(\mathcal{T}_k, \mathcal{D}, \omega\varepsilon_k)$ is fed to $[\mathcal{T}_{k+1}, u_{k+1}] = \mathsf{GALERKIN\text{-}DG}(\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k, \varepsilon_k)$, which in turn iterates J_k times. Lemma 9.25 shows that J_k is uniformly bounded in k, and we assume that $J_k \ge 1$, for otherwise the module GALERKIN-DG is skipped altogether. We let $(\mathcal{T}_{k,j}, \mathcal{M}_{k,j}, u_{\mathcal{T}_{k,j}})$ denote the triplets of grids, marked sets and discrete solutions computed within $\mathsf{GALERKIN\text{-}DG}(\widehat{\mathcal{T}}_k, \widehat{\mathcal{D}}_k, \varepsilon_k)$ for $0 \le j < J_k$. Note that

$$\widehat{\varepsilon}_{k,j} := \eta_{\mathcal{T}_{k,j}}(u_{\mathcal{T}_{k,j}}, \widehat{\mathcal{D}}_k) > \varepsilon_k, \quad 0 \le j < J_k$$

so that together with the lower *a posteriori* error estimate (9.40), we infer that

$$\|\widehat{u}_k - u_{\mathcal{T}_{k,j}}\|_{\kappa,\mathcal{T}_{k,j}} \ge C_L\widehat{\varepsilon}_{k,j} > C_L\varepsilon_k,$$

where $\widehat{u}_k = u(\widehat{\mathcal{D}}_k) \in H_0^1(\Omega)$ is the exact solution with approximate data $\widehat{\mathcal{D}}_k$.

The module DATA guarantees (9.57), and the parameter ω modulates the discrepancy between u and $\widehat{u}_k$ relative to ε_k. The error due to data approximation can be made small relative to the finite element approximation by choosing ω much smaller than 1. In addition, we have established Lemma 9.24 (Dörfler marking) for $\theta_0 < 1$, which implies a Dörfler property for any $0 < \theta \le \theta_0$. The restrictions on the parameters κ, ω and θ are gathered in the following assumption.

Assumption 9.27 (restrictions on κ, ω and θ). Assume that $\kappa > \max(\overline{\kappa}_D, \overline{\kappa}_{\mathrm{conv}})$, that $0 < \omega \le \frac{1}{4} C_{\mathrm{Mo}}^{-1} C_L C_D^{-1}$ and that $0 < \theta \le \theta_0(\kappa)$, where κ_D and θ_0 are defined in Lemma 9.24 (Dörfler marking).

Note that if Assumption 9.27 is valid then

$$|u - \widehat{u}_k|_{H_0^1(\Omega)} \le \frac{1}{4} C_{\mathrm{Mo}}^{-1} C_L \varepsilon_k. \tag{9.58}$$

The next results rely on Assumption 6.3 (approximability of u) and Assumption 6.10 (approximability of data). They are stated and proved for conforming meshes and continuous approximations of u. However, Proposition 9.4 (equivalence of classes for u) and Remark 9.5 (equivalence of classes for $\mathcal{D}$) show that these classes coincide with the conforming case.

Proposition 9.28 (cardinality of marked sets). *Let Assumptions 6.3 (approximability of u), 6.17 (cardinality of $\mathcal{M}$) and 9.27 (restrictions on κ, ω and θ) hold. If $\widehat{\varepsilon}_{k,0} > \varepsilon_k$, then* GALERKIN-DG *at iteration k of* AFEM-DG-TS *is called and the cardinality $N_{k,j}(u)$ of the marked set $\mathcal{M}_{k,j}$ satisfies*

$$N_{k,j}(u) \lesssim |u|_s^{1/s} \, \|u - u_{\mathcal{T}_{k,j}}\|_{\kappa,\mathcal{T}_{k,j}}^{-1/s} \quad \textit{for all } 0 \le j < J_k. \tag{9.59}$$

Proof. Fix $0 \le j < J_k$ and set

$$\delta := \frac{1}{2} C_{\mathrm{Mo}}^{-1} C_L \, \eta_{\mathcal{T}_{k,j}}(u_{\mathcal{T}_{k,j}}) \ge \frac{1}{2} C_{\mathrm{Mo}}^{-1} C_L \varepsilon_k,$$

because $\eta_{\mathcal{T}_{k,j}}(u_{\mathcal{T}_{k,j}}) > \varepsilon_k$ for $j < J_k$. Thanks to (9.58), $\widehat{u}_k$ is an $(\frac{1}{2} C_{\mathrm{Mo}}^{-1} C_L \varepsilon_k)$-approximation of order s to u according to Lemma 6.13 (ε-approximation of u of order s). Hence there exists a conforming mesh $\mathcal{T}_\delta \in \mathbb{T}^\Lambda$ and $u^0_{\mathcal{T}_\delta} \in \mathbb{V}_{\mathcal{T}_\delta}$ such that

$$\|\widehat{u}_k - u^0_{\mathcal{T}_\delta}\|_{\kappa,\mathcal{T}_\delta} = |\widehat{u}_k - u^0_{\mathcal{T}_\delta}|_{H_0^1(\Omega)} \le \delta, \quad \#\mathcal{T}_\delta \lesssim |u|_{\mathbb{A}_s}^{1/s} \delta^{-1/s}.$$

To compare $\mathcal{T}_\delta$ with $\mathcal{T}_{k,j}$ we consider the overlay $\mathcal{T}_* = \mathcal{T}_{k,j} \oplus \mathcal{T}_\delta$, which satisfies

$$\#\mathcal{T}_* \le \#\mathcal{T}_{k,j} + \#\mathcal{T}_\delta - \#\mathcal{T}_0;$$

see Proposition 8.15 (mesh overlay is Λ-admissible). Let $u_{\mathcal{T}_*} \in \mathbb{V}_{\mathcal{T}_*}^{-1}$ be the Galerkin solution on the subspace $\mathbb{V}_{\mathcal{T}_*}^{-1}$ and invoke Corollary 9.13 (Céa's lemma), to write

$$\eta_{\mathcal{T}_*}(u_{\mathcal{T}_*}) \le C_L^{-1} \|\widehat{u}_k - u_{\mathcal{T}_*}\|_{\kappa,\mathcal{T}_*} \le C_L^{-1} C_{\mathrm{Mo}} \|\widehat{u}_k - u^0_{\mathcal{T}_\delta}\|_{\kappa,\mathcal{T}_*} = C_L^{-1} C_{\mathrm{Mo}} |\widehat{u}_k - u^0_{\mathcal{T}_\delta}|_{H_0^1(\Omega)},$$

whence $\eta_{\mathcal{T}_*}(u_{\mathcal{T}_*}) \le C_L^{-1} C_{\mathrm{Mo}} \delta$ and

$$\eta_{\mathcal{T}_*}(u_{\mathcal{T}_*}) \le \frac{1}{2} \eta_{\mathcal{T}_{k,j}}(u_{\mathcal{T}_{k,j}}).$$

Applying Lemma 9.24 (Dörfler marking) to $\mathcal{T}_*$ and $\mathcal{T}_{k,j}$, we infer that the enlarged refined set $\omega(\mathcal{T}_{k,j} \setminus \mathcal{T}_*)$ satisfies the Dörfler marking property

$$\eta_{\mathcal{T}_{k,j}}(u_{\mathcal{T}_{k,j}}, \omega(\mathcal{T}_{k,j} \setminus \mathcal{T}_*)) \ge \theta\, \eta_{\mathcal{T}_*}(u_{\mathcal{T}_*})$$

since $0 < \theta \le \theta_0$ by Assumption 9.27. The Dörfler marking involves a minimal set $\mathcal{M}_{k,j}$ according to Assumption 6.17, which thus implies

$$N_{k,j}(u) \le \#\omega(\mathcal{T}_{k,j} \setminus \mathcal{T}_*) \lesssim \#(\mathcal{T}_{k,j} \setminus \mathcal{T}_*) \le \#\mathcal{T}_\delta - \#\mathcal{T}_0 \lesssim |u|_{\mathbb{A}_s}^{1/s} \delta^{-1/s} \lesssim |u|_{\mathbb{A}_s}^{1/s} \varepsilon_k^{-1/s},$$

because $\#(\mathcal{T}_{k,j} \setminus \mathcal{T}_*) \le \#\mathcal{T}_* - \#\mathcal{T}_{k,j}$. This concludes the proof. □

Corollary 9.29 (quasi-optimality of GALERKIN-DG). *Let Assumptions 6.3 (approximability of u), 6.17 (cardinality of $\mathcal{M}$) and 9.27 (restrictions on κ, ω and θ) hold. Assume $\kappa \ge \max(\overline{\kappa}_{\mathrm{conv}}, \overline{\kappa}_{\mathrm{D}})$. Then the total number of marked elements $N_k(u)$ within a call to* GALERKIN-DG *satisfies*

$$N_k(u) \le J C_0 |u|_{\mathbb{A}_s}^{1/s} \varepsilon_k^{-1/s},$$

where $J \ge J_k$ is a uniform upper bound for the number of iterations required by GALERKIN-DG *according to Lemma 9.25 (computational cost of* GALERKIN-DG*).*

Proof. Use the fact that $N_k(u) = \sum_{j=0}^{J_k-1} N_{k,j}(u)$ and combine Propositions 9.28 (cardinality of marked sets) and 9.25 (computational cost of GALERKIN-DG). □

We finally address the rate-optimality of the two-step algorithm AFEM-DG-TS.

Theorem 9.30 (rate-optimality of AFEM-DG-TS). *Let Assumptions 6.3 (approximability of u), 6.10 (approximability of data), 6.11 (quasi-optimality of* DATA*), 6.17 (cardinality of $\mathcal{M}$), 6.19 (initial labelling) and 9.27 (restrictions on κ, ω and θ) hold. Then* AFEM-DG-TS *gives rise to a sequence $\left(\mathcal{T}_k, \mathbb{V}_{\mathcal{T}_k}^{-1}, u_{\mathcal{T}_k}\right)_{k=0}^{K+1}$ such that*

$$\|u - u_{\mathcal{T}_k}\|_{\kappa,\mathcal{T}_k} \le C(u,\mathcal{D})(\#\mathcal{T}_k)^{-s}, \quad 1 \le k \le K+1,$$

where $0 < s = \min\{s_u, s_{\mathcal{D}}\} = \min\{s_u, s_A, s_c, s_f\} \le n/d$ and

$$C(u,\mathcal{D}) = C_* \left(|u|_{\mathbb{A}_{s_u}}^{1/s_u} + |\boldsymbol{A}|_{\mathbb{M}_{s_A}}^{1/s_A} + |c|_{\mathbb{C}_{s_c}}^{1/s_c} + |f|_{\mathbb{F}_{s_f}}^{1/s_f} \right)^s$$

with constant $C_* > 0$ independent of u and $\mathcal{D}$.

Proof. Assumptions 6.3, 6.17 and 9.27 combined with Corollary 9.29 for u, and Assumptions 6.10 and 6.11 for $\mathcal{D}$, imply the existence of a constant $C_\#$ such that the total number of marked elements within one loop of AFEM-DG-TS is

$$N_k(u) + N_k(\mathcal{D}) \le C_\# \left(|u|_{\mathbb{A}_{s_u}}^{1/s_u} + |\mathcal{D}|_{\mathbb{A}_{s_{\mathcal{D}}}}^{1/s_{\mathcal{D}}} \right) \varepsilon_k^{-1/s},$$

with $s_u, s_{\mathcal{D}} \le n/d$. Moreover, upon termination DATA and GALERKIN-DG give

$$|u - \widehat{u}_k|_{H_0^1(\Omega)} \le \frac{1}{4} C_{\mathrm{Mo}}^{-1} C_L \varepsilon_k,$$
$$\|\widehat{u}_k - u_{\mathcal{T}_{k+1}}\|_{\kappa,\mathcal{T}_{k+1}} \le C_U \eta_{\mathcal{T}_{k+1}}(u_{\mathcal{T}_{k+1}}) \le C_U \varepsilon_k,$$

because of (9.58) and (9.44). This implies, by the triangle inequality,

$$\|u - u_{\mathcal{T}_{k+1}}\|_{\kappa,\mathcal{T}_{k+1}} \le \left(\frac{1}{4} C_{\mathrm{Mo}}^{-1} C_L + C_U\right) \varepsilon_k.$$

We finally conclude as in Theorem 6.24 (rate-optimality of AFEM-TS). □

Remark 6.25 on the role of ω, θ^* and Remark 6.26 on the optimality of the result, written after Theorem 6.24 for the conforming case, remain valid for the non-conforming case and are not repeated here.

9.7. *Operator $P_{\mathcal{T}}$ and routine* DATA *on Λ-admissible partitions*

In this section we have extensively used the notion of Λ-admissible meshes for the design and study of dG methods, including forcing $f \in H^{-1}(\Omega)$. To this end, as well as for the design of the two-step AFEM for dG, namely AFEM-DG-TS, the construction of the local projection $P_{\mathcal{T}} f \in \mathbb{F}_{\mathcal{T}}$ is critical. We discuss this now.

Recall that for a conforming partition $\mathcal{T} \in \mathbb{T}$, $P_{\mathcal{T}} f$ is defined as a projection to $\mathbb{F}_{\mathcal{T}}$; see Definition 4.24 (projection onto discrete functionals). The definition and subsequent properties of $P_{\mathcal{T}}$ hinge on extensions E_F for $F \in \mathcal{F}$, studied in Lemma 4.20 (extending from faces), as well as on bubble functions ϕ_T, $T \in \mathcal{T}$, and ϕ_F, $F \in \mathcal{F}$ satisfying Assumption 4.21 (abstract cut-off).

The definition of the element bubble functions ϕ_T in (4.14) is local to T and is thus unchanged on non-conforming subdivisions. The situation is different for faces. If F is a conforming face, we have the conforming definitions of E_F and ϕ_F as in (4.17). Instead, if F is a non-conforming face, $F = T \cap T_*$ with $g(T_*) > g(T)$, we use a virtual conforming refinement of ω_F to define E_F and ϕ_F as in (4.17). Recall that $g(T)$ is the generation of $T \in \mathcal{T}$, and $\mathcal{T} \in \mathbb{T}$ is a uniform refinement of $\mathcal{T}_0$ if and only if g is constant on $\mathcal{T}$. Let $\overline{\mathcal{T}}$ be the uniform refinement of $\mathcal{T}_0$ containing T_*, whence $g(T) = g(T_*)$ for all $T \in \overline{\mathcal{T}}$; $\overline{\mathcal{T}}$ is conforming thanks to Assumption 6.19 (initial labelling) on $\mathcal{T}_0$. Let $\overline{T} \in \overline{\mathcal{T}}$ be the element sharing F with T_* (and thus contained in T) and let $\overline{\omega}_F := T_* \cup \overline{T}$ be the virtual conforming patch around F. We now proceed by defining E_F via (4.4) with ω_F replaced by $\overline{\omega}_F$ and ϕ_F as in (4.17) using the basis functions ϕ_z, $z \in \mathcal{V} \cap F$, associated with $\prod_{T \subset \overline{\omega}_F} \mathbb{P}_n(T) \cap H_0^1(\overline{\omega}_F)$. Note that because $\mathcal{T}$ is Λ-admissible, Proposition 3.27 guarantees that the diameters of $\overline{T}$, T, T_*, ω_F and $\overline{\omega}_F$ are all comparable with constants depending on the initial mesh $\mathcal{T}_0$ and Λ.

Assumption 4.21 is an important ingredient in the analysis of $P_{\mathcal{T}}$ and it holds true with ω_F replaced by $\overline{\omega}_F$ when F is a non-conforming face. Therefore Remark 4.26 (local computation) and Corollary 4.31 (local near-best approximation)

are valid for Λ-admissible partitions as well. Consequently, all the algorithms and results presented in Section 7 (data approximation) readily extend to Λ-admissible subdivisions as well. We do not dwell on this matter any further.

10. AFEMs for inf-sup stable problems

We go back to the functional framework introduced in Section 2.4. Precisely, let the bilinear form $\mathcal{B}\colon \mathbb{V} \times \mathbb{W} \to \mathbb{R}$ be continuous and *inf-sup stable* (i.e. it satisfies one of the equivalent conditions stated in Theorem 2.8 (Nečas)). Given $f \in \mathbb{W}^*$, let $u \in \mathbb{V}$ be the unique solution of the variational problem

$$u \in \mathbb{V}: \quad \mathcal{B}[u, w] = \langle f, w \rangle \quad \text{for all } w \in \mathbb{W}. \tag{10.1}$$

Let $\mathbb{V}_j \subset \mathbb{V}$, $\mathbb{W}_j \subset \mathbb{W}$ be finite-dimensional subspaces depending on an integer parameter $j \geq 0$, such that

$$\dim \mathbb{V}_j = \dim \mathbb{W}_j = n_j, \quad \mathbb{V}_j \subset \mathbb{V}_{j+1}, \quad \mathbb{W}_j \subset \mathbb{W}_{j+1}.$$

(Note that the notation has changed with respect to Section 3.1, where $\mathbb{V}_N$ was a subspace of dimension N. Here $\mathbb{V}_j$ may stand for $\mathbb{V}_{\mathcal{T}_j}$, where $\mathcal{T}_j$ is the jth mesh generated by an adaptive algorithm.)

We assume $\mathcal{B}$ satisfies a *uniform* discrete inf-sup condition on any product of subspaces $\mathbb{V}_j \times \mathbb{W}_j$, that is, there exists a constant $\beta > 0$ such that for all j

$$\inf_{v \in \mathbb{V}_j} \sup_{w \in \mathbb{W}_j} \frac{\mathcal{B}[v, w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}}} \geq \beta. \tag{10.2}$$

Let $u_j \in \mathbb{V}_j$ be the solution of the (Petrov–)Galerkin problem

$$u_j \in \mathbb{V}_j: \quad \mathcal{B}[u_j, w] = \langle f, w \rangle \quad \text{for all } w \in \mathbb{W}_j. \tag{10.3}$$

The first part of this section, which is mostly based on the recent work by Feischl (2022), is devoted to studying the convergence of this approximation. Convergence and rate-optimality of different AFEMs will be discussed next in Section 10.3. Applications will be given to the Stokes problem (see Section 10.4) and the mixed formulation of a scalar diffusion problem (see Section 10.5).

10.1. Linear convergence of inf-sup stable methods

We make the following key assumptions that guarantee the convergence of the sequence u_j to u in the $\mathbb{V}$-norm, and comment about them afterwards. The first assumption is a relaxed form of the *general quasi-orthogonality* property introduced in Carstensen *et al.* (2014) as part of an abstract set of axioms of adaptivity.

Assumption 10.1 (relaxed quasi-orthogonality). For each $N \in \mathbb{N}$ there exists a non-decreasing constant $C = C(N)$ such that

$$\sum_{k=j}^{j+N} \|u_{k+1} - u_k\|_{\mathbb{V}}^2 \leq C(N) \|u - u_j\|_{\mathbb{V}}^2, \quad j \geq 0, \tag{10.4}$$

and

$$C(N) = o(N) \quad \text{as } N \to \infty.$$

Assumption 10.2 (equivalence of error and estimator). There exist constants $C_U \geq C_L > 0$ and, for each $j \geq 0$, an error estimator $\eta_j = \eta_j(u_j)$, such that

$$C_L \eta_j \leq \|u - u_j\|_{\mathbb{V}} \leq C_U \eta_j, \quad j \geq 0. \tag{10.5}$$

Assumption 10.3 (estimator reduction). There exist constants $0 < \rho_1 < 1$ and $C_1 > 0$ such that

$$\eta_{j+1}^2 \leq \rho_1 \eta_j^2 + C_1 \|u_{j+1} - u_j\|_{\mathbb{V}}^2, \quad j \geq 0. \tag{10.6}$$

Remark 10.4. Assumptions 10.2 and 10.3 are abstract and allow for a general convergence theory. In the context of our model problems of Section 2.3, they are valid for discrete data, that is, if the coefficients of the linear operator corresponding to the bilinear form $\mathcal{B}$ are piecewise polynomials on the adopted meshes, and if $f \in \mathbb{F}_{\mathcal{T}}$ (see Section 4.3). We make this concrete in Sections 10.4 and 10.5 below.

Remark 10.5. We comment on the significance of Assumption 10.1 upon considering two extreme cases.

(1) Assumption 10.1 with $C(N) = O(1)$ is precisely the general quasi-orthogonality property of Carstensen *et al.* (2014). It is valid with $C(N) = 1$ for $\mathbb{V} = \mathbb{W}$ and $\mathcal{B}$ symmetric and coercive. Indeed,

$$\mathcal{B}[u_{k+1} - u_k, u - u_{k+1}] = 0 \quad \text{(Galerkin orthogonality)},$$

whence

$$|||u_{k+1} - u_k|||_\Omega^2 + |||u - u_{k+1}|||_\Omega^2 = |||u - u_k|||_\Omega^2,$$

where $|||\cdot|||_\Omega$ is the energy norm induced by $\mathcal{B}$. Adding upon k and using telescopic cancellation yields

$$\begin{aligned}\sum_{k=j}^{j+N} |||u_{k+1} - u_k|||_\Omega^2 &= \sum_{k=j}^{j+N} |||u - u_k|||_\Omega^2 - |||u - u_{k+1}|||_\Omega^2 \\ &= |||u - u_j|||_\Omega^2 - |||u - u_{j+N+1}|||_\Omega^2 \leq |||u - u_j|||_\Omega^2.\end{aligned}$$

Finally, the equivalence (2.30) of the norms $\|\cdot\|_{\mathbb{V}}$ and $|||\cdot|||_\Omega$ yields the result.

(2) Assumption 10.1 trivially holds with $C(N) = O(N)$ for $\mathcal{B}$ continuous and inf-sup stable. Indeed, choosing in Corollary 3.3 (quasi-monotonicity) $\mathbb{V}_N = \mathbb{V}_k$ or $\mathbb{V}_{k+1}$ and $\mathbb{V}_M = \mathbb{V}_j$ for $j \leq k$, and using the triangle inequality gives

$$\|u_{k+1} - u_k\|_{\mathbb{V}}^2 \lesssim \|u_{k+1} - u\|_{\mathbb{V}}^2 + \|u_k - u\|_{\mathbb{V}}^2 \lesssim \|u - u_j\|_{\mathbb{V}}^2.$$

Adding, we get

$$\sum_{k=j}^{j+N} \|u_{k+1} - u_k\|_{\mathbb{V}}^2 \leq C \sum_{k=j}^{j+N} \|u - u_j\|_{\mathbb{V}}^2 = CN \|u - u_j\|_{\mathbb{V}}^2.$$

However, the relation $C(N) = O(N)$ is not enough for the subsequent analysis. In fact, we need $C(N) = o(N)$.

We now prove that the stated assumptions guarantee the linear convergence of the sequence of Petrov–Galerkin solutions (10.3). This result is similar to the convergence result for the estimators given in Feischl (2022), and exploits the equivalence (10.5) between errors and estimators.

Theorem 10.6 (linear convergence). *Under Assumptions 10.1, 10.2 and 10.3, the discretization (10.3) is convergent; precisely, there exist constants* $0 < \rho < 1$ *and* $c > 0$ *such that*

$$e_{j+i} \leq c\rho^i e_j \quad \textit{for all } i, j \in \mathbb{N}, \tag{10.7}$$

where $e_j := \|u - u_j\|_{\mathbb{V}}$.

Proof. The proof is divided into several steps. First, we set

$$E_k := \|u_k - u_{k-1}\|_{\mathbb{V}}.$$

1 We start by iterating (10.6) $1 \leq n \leq k$ times to obtain

$$\begin{aligned}
\eta_k^2 &\leq \rho_1 \eta_{k-1}^2 + C_1 E_k^2 \\
&\leq \rho_1 \left(\rho_1 \eta_{k-2}^2 + C_1 E_{k-1}^2\right) + C_1 E_k^2 \leq \rho_1^2 \eta_{k-2}^2 + C_1 \left(E_k^2 + E_{k-1}^2\right) \\
&\leq \rho_1^n \eta_{k-n}^2 + C_1 \sum_{\ell=k-n+1}^{k} E_\ell^2.
\end{aligned}$$

We now invoke Assumption 10.2 to state the upper bound

$$e_k^2 \leq c_1 \eta_k^2$$

and the lower bound

$$\eta_k^2 \leq c_2 e_k^2$$

(with $c_1 = C_U^2$ and $c_2 = C_L^{-2}$). This yields

$$e_k^2 \leq c_1 \eta_k^2 \leq c_1 c_2 \rho_1^n e_{k-n}^2 + c_1 C_1 \sum_{\ell=k-n+1}^{k} E_\ell^2. \tag{10.8}$$

Let $n \in \mathbb{N}$ be sufficiently large that

$$\rho_2 = c_1 c_2 \rho_1^n < 1,$$

and let us relabel $c_1 C_1$ as C_1 to get

$$e_k^2 \leq \rho_2 e_{k-n}^2 + C_1 \sum_{\ell=k-n+1}^{k} E_\ell^2. \tag{10.9}$$

This shows that the reduction property (10.6) of the estimator is valid for the error after n iterations. We cannot expect (10.9) to hold for e_k with $n = 1$, even in the

coercive case: see Example 5.7 (lack of strict error monotonicity) for $\boldsymbol{A} = \boldsymbol{I}$ and $f = 1$. It is thus convenient to rewrite (10.9) as follows:

$$e_{kn}^2 \le \rho_2 e_{(k-1)n}^2 + C_1 \sum_{\ell=(k-1)n+1}^{kn} E_\ell^2. \tag{10.10}$$

$\boxed{2}$ Sum up (10.10) from $k = j+1$ to $k = j+N$, to get

$$\sum_{k=j+1}^{j+N} e_{kn}^2 \le \rho_2 \sum_{k=j+1}^{j+N} e_{(k-1)n}^2 + C_1 \sum_{\ell=jn+1}^{(j+N)n} E_\ell^2.$$

Using (10.4) we see that

$$\sum_{\ell=jn+1}^{(j+N)n} E_\ell^2 = \sum_{\ell=jn}^{(j+N)n-1} E_{\ell+1}^2 \le C(Nn-1)\, e_{jn}^2 \le C(Nn)\, e_{jn}^2,$$

whence

$$\begin{aligned}\sum_{k=j+1}^{j+N} e_{kn}^2 &\le \rho_2 \sum_{k=j+1}^{j+N} e_{(k-1)n}^2 + C_1 C(Nn)\, e_{jn}^2 \\ &\le \rho_2 \left(\sum_{k=j+1}^{j+N} e_{kn}^2 + e_{jn}^2 \right) + C_1 C(Nn)\, e_{jn}^2.\end{aligned}$$

This implies

$$(1-\rho_2) \sum_{k=j+1}^{j+N} e_{kn}^2 \le (\rho_2 + C_1 C(Nn))\, e_{jn}^2,$$

or equivalently

$$\frac{1-\rho_2}{\rho_2 + C_1 C(Nn)} \sum_{k=j+1}^{j+N} e_{kn}^2 \le e_{jn}^2. \tag{10.11}$$

Let us add the quantity $\sum_{k=j+1}^{j+N} e_{kn}^2$ to both sides to arrive at

$$\left(1 + \frac{1-\rho_2}{\rho_2 + C_1 C(Nn)}\right) \sum_{k=j+1}^{j+N} e_{kn}^2 \le \sum_{k=j}^{j+N} e_{kn}^2.$$

We can rewrite this inequality as follows:

$$\sum_{k=j+1}^{j+N} e_{kn}^2 \le \rho(N) \sum_{k=j}^{j+N} e_{kn}^2, \tag{10.12}$$

where

$$\rho(N) = \frac{1}{1 + \frac{1-\rho_2}{\rho_2 + C_1 C(Nn)}} = \frac{\rho_2 + C_1 C(Nn)}{1 + C_1 C(Nn)}$$
$$= 1 - \frac{1-\rho_2}{1 + C_1 C(Nn)} = 1 - \frac{1}{D(N)}$$

with

$$D(N) = \frac{1 + C_1 C(Nn)}{1 - \rho_2} \to \infty, \quad N \to \infty,$$

whenever $C(N)$ diverges. Therefore (10.12) is a contraction for the quantity $\sum_{k=j+1}^{j+N} e_{kn}^2$ with a constant $\rho(N)$ uniform in j that may degenerate to 1 as $N \to \infty$.

3 We iterate (10.12) and exploit the fact that the left-hand side has one fewer term than the right-hand side. Take

$$j \to j+1, \quad N \to N-1,$$

to get

$$\sum_{k=j+2}^{j+N} e_{kn}^2 \le \rho(N-1) \sum_{k=j+1}^{j+N} e_{kn}^2,$$

whence

$$\sum_{k=j+2}^{j+N} e_{kn}^2 \le \rho(N-1)\rho(N) \sum_{k=j}^{j+N} e_{kn}^2.$$

Iterating, we get

$$e_{(j+N)n}^2 = \sum_{k=j+N}^{j+N} e_{kn}^2 \le \rho(1)\ldots\rho(N-1)\rho(N) \sum_{k=j}^{j+N} e_{kn}^2.$$

We now need to bound the sum on the right-hand side by a single term. To this end, we resort to (10.11), that is,

$$\sum_{k=j+1}^{j+N} e_{kn}^2 \le \frac{\rho_2 + C_1 C(Nn)}{1 - \rho_2} e_{jn}^2,$$

and add e_{jn}^2 to both sides:

$$\sum_{k=j}^{j+N} e_{kn}^2 \le \left(1 + \frac{\rho_2 + C_1 C(Nn)}{1 - \rho_2}\right) e_{jn}^2 = D(N)\, e_{jn}^2.$$

Altogether, we arrive at

$$e^2_{(j+N)n} \le \rho(1)\dots\rho(N)D(N)\, e^2_{jn} = D(N)\prod_{k=1}^{N}\left(1-\frac{1}{D(k)}\right)e^2_{jn}.$$

4 We estimate the factor on the right-hand side. For $N_0 > 0$ to be chosen later, set

$$\rho_0 := D(N_0)\prod_{k=1}^{N_0}\left(1-\frac{1}{D(k)}\right)$$

and compute the logarithm of ρ_0 via

$$\log(\rho_0) = \log(D(N_0)) + \sum_{k=1}^{N_0}\log\left(1-\frac{1}{D(k)}\right) \le \log(D(N_0)) - \sum_{k=1}^{N_0}\frac{1}{D(k)},$$

because the log is concave and $\log(1+x) \le x$. Since we assume in (10.4) that $D(k) \simeq C(kN_0) = o(k)$, the series diverges and we see that

$$\log(\rho_0) < 0$$

for N_0 sufficiently large. Summarizing, there exist $N_0 > 0$ and $0 < \rho_0 < 1$ such that

$$e^2_{(j+N_0)n} \le \rho_0\, e^2_{jn} \quad \text{for all } j \in \mathbb{N}. \tag{10.13}$$

5 For any $j, i \in \mathbb{N}$, we now find $c > 0$ and $0 < \rho < 1$ such that the inequality

$$e_{j+i} \le c\rho^i e_j$$

holds. We decompose j and $j+i$ in terms of integers k, m,

$$j = (k-1)n + \widehat{j}, \quad k \ge 1, \quad 0 \le \widehat{j} < n,$$
$$j+i = (k+m)n + \widehat{i}, \quad m \ge -1, \quad 0 \le \widehat{i} < n,$$

and first examine the case $m \ge 0$. We further decompose

$$m = aN_0 + b, \quad a, b \in \mathbb{N}, \quad 0 \le b < N_0 \quad \Rightarrow \quad a = \frac{m}{N_0} - \frac{b}{N_0}.$$

Note that

$$kn = j - \widehat{j} + n > j, \quad m = \frac{i}{n} - \left(\frac{\widehat{i}-\widehat{j}}{n} + 1\right), \quad a > \frac{i}{nN_0} - \frac{2+N_0}{N_0}.$$

Therefore, invoking Corollary 3.3 (quasi-monotonicity),

$$e_{j_2} \le \frac{\|\mathcal{B}\|}{\beta}\, e_{j_1} = C_* e_{j_1}, \quad j_2 \ge j_1 \ge 0,$$

in conjunction with (10.13), yields

$$e_{j+i} \le C_* e_{(k+aN_0)n} \le C_*\rho_0^{a/2} e_{kn} \le C_*^2\rho_0^{a/2} e_j < C_*^2\rho_0^{-(2+N_0)/(2N_0)}\left(\rho_0^{1/(2nN_0)}\right)^i e_j.$$

This is the desired estimate with $c = C_*^2 \rho_0^{-(2+N_0)/(2N_0)}$ and $\rho = \rho_0^{1/(2nN_0)}$ for $m \geq 0$. We finally consider $m = -1$ and again use the error quasi-monotonicity to write

$$e_{j+i} \leq C_* e_j = \frac{C_*}{\rho^i} \rho^i e_j < \frac{C_*}{\rho^n} \rho^i e_j.$$

This concludes the proof with

$$c = \max\left\{ C_*^2 \rho_0^{-(2+N_0)/(2N_0)}, \frac{C_*}{\rho^n} \right\}. \qquad \square$$

Remark 10.7 (improving on Assumption 10.1). It is worth emphasizing that while linear convergence (10.7) is established in Theorem 10.6 using Assumption 10.1, the same property combined with uniform inf-sup stability tells us *a posteriori* that the constant $C(N)$ in (10.4) can be made independent of N, i.e. $C(N) = O(1)$. To see this, we apply the linear convergence bound

$$\|u - u_k\|_{\mathbb{V}} \leq c\rho^{k-j} \|u - u_j\|_{\mathbb{V}}$$

in conjunction with the triangle inequality

$$\|u_{k+1} - u_k\|_{\mathbb{V}} \leq \|u - u_{k+1}\|_{\mathbb{V}} + \|u - u_k\|_{\mathbb{V}} \leq 2c\rho^{k-j} \|u - u_j\|_{\mathbb{V}};$$

summation of a geometric series gives

$$\sum_{k=j}^{j+N} \|u_{k+1} - u_k\|_{\mathbb{V}}^2 \leq C \|u - u_j\|_{\mathbb{V}}^2$$

with $C = 4c^2 \sum_{\ell=0}^{\infty} \rho^{2\ell} < +\infty$. This suggests that Assumption 10.1 might be too pessimistic.

10.2. Inf-sup stability implies quasi-orthogonality

We aim at proving the following key result in this section.

Theorem 10.8 (sufficient condition for Assumption 10.1). *Assumption 10.1 (relaxed quasi-orthogonality) is valid if the bilinear form $\mathcal{B}\colon \mathbb{V} \times \mathbb{W} \to \mathbb{R}$ is continuous and uniformly inf-sup stable on the sequence of subspaces $\mathbb{V}_j \times \mathbb{W}_j$, $j \geq 0$.*

To accomplish this task, we proceed in two steps. Using variational techniques, we first establish an intermediate result formally similar to (10.4) (see Corollary 10.14), but involving the norm of a matrix $\boldsymbol{U}$ related to the form $\mathcal{B}$. Next, we rely on algebraic techniques to estimate such a norm (see Theorem 10.15) and complete the proof of the desired result.

In order to perform the first step, we introduce orthonormal bases of the finite element spaces $\mathbb{V}_j$, $\mathbb{W}_j$, $0 \leq j \leq N$, and next we biorthogonalize them. This procedure turns out to be crucial.

We start with some notation. Let

$$n_j = \dim \mathbb{V}_j = \dim \mathbb{W}_j.$$

Let $\mathbb{V}_{j-1}^{\perp}$ and $\mathbb{W}_{j-1}^{\perp}$ denote the orthogonal complements of $\mathbb{V}_{j-1}$ and $\mathbb{W}_{j-1}$ within $\mathbb{V}_j$ and $\mathbb{W}_j$, respectively. Let

$$d_j = n_j - n_{j-1} = \dim \mathbb{V}_{j-1}^{\perp} = \dim \mathbb{W}_{j-1}^{\perp}$$

be the dimension of the Galerkin update to augment the space $\mathbb{V}_{j-1}$ into the next space $\mathbb{V}_j$, and likewise with the space $\mathbb{W}_{j-1}$ and $\mathbb{W}_j$.

We consider orthonormal bases

$$\boldsymbol{v} = \{\boldsymbol{v}(j)\}_{j=0}^{N} \subset \mathbb{V}_N, \quad \boldsymbol{w} = \{\boldsymbol{w}(i)\}_{i=0}^{N} \subset \mathbb{W}_N \tag{10.14}$$

partitioned into blocks for $1 \le j \le N$

$$\boldsymbol{v}(j) = (v_k)_{k=n_{j-1}+1}^{n_j} \subset \mathbb{V}_{j-1}^{\perp}, \quad \boldsymbol{w}(i) = (w_k)_{k=n_{i-1}+1}^{n_i} \subset \mathbb{W}_{i-1}^{\perp}, \tag{10.15}$$

and $\boldsymbol{v}(0) \subset \mathbb{V}_0$, $\boldsymbol{w}(0) \subset \mathbb{W}_0$. In other words, $(\boldsymbol{v}(j), \boldsymbol{w}(j))$ represent the d_j new directions added by Galerkin to the current spaces $(\mathbb{V}_{j-1}, \mathbb{W}_{j-1})$ for $1 \le j \le N$.

We recall that the bilinear form $\mathcal{B}\colon \mathbb{V}_N \times \mathbb{W}_N \to \mathbb{R}$ satisfies the following uniform properties for all $0 \le j \le N$.

(P1) Continuity:

$$\left|\mathcal{B}[v, w]\right| \le \|\mathcal{B}\| \|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}} \quad \text{for all } v \in \mathbb{V}_j,\ w \in \mathbb{W}_j. \tag{10.16}$$

(P2) Inf-sup condition:

$$\beta \|v\|_{\mathbb{V}} \le \sup_{w \in \mathbb{W}_j} \frac{\mathcal{B}[v, w]}{\|w\|_{\mathbb{W}}} \quad \text{for all } v \in \mathbb{V}_j. \tag{10.17}$$

The block bases $\boldsymbol{v}$ and $\boldsymbol{w}$ given in (10.14) induce a block matrix

$$\boldsymbol{B} := (\boldsymbol{B}(i, j))_{i,j=0}^{N} \in \mathbb{R}^{n_N \times n_N}$$

defined by

$$\boldsymbol{B}(i, j) = \mathcal{B}[\boldsymbol{v}(j), \boldsymbol{w}(i)]. \tag{10.18}$$

Note that the actual size of $\boldsymbol{B}$ is $n_N = \dim \mathbb{V}_N \gg N$, and that the following analysis entails expressing important quantities in terms of the number of blocks N rather than the dimension n_N.

We will use this block decomposition for a generic matrix

$$\boldsymbol{M} = (\boldsymbol{M}(i, j))_{i,j=0}^{N} \in \mathbb{R}^{n_N \times n_N},$$

and we let $\boldsymbol{M}[k] = (\boldsymbol{M}(i, j))_{i,j=0}^{k}$ denote the *principal kth block* of $\boldsymbol{M}$. Figure 10.1 shows schematically what this means.

We stress that (P2) implies that $\boldsymbol{B}[k]$ is uniformly invertible with

$$\|\boldsymbol{B}[k]^{-1}\|_2 \le \frac{1}{\beta} \quad \text{for all } 0 \le k \le N. \tag{10.19}$$

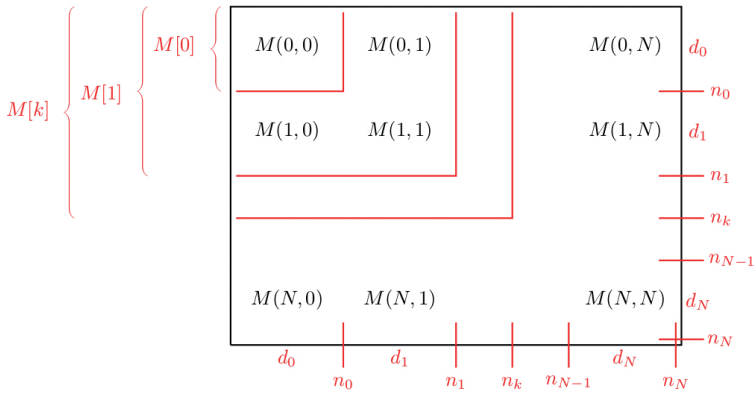


Figure 10.1. Block partition of a matrix $\boldsymbol{M} \in \mathbb{R}^{n_N \times n_N}$ with $(N+1)\times(N+1)$ blocks $\boldsymbol{M}(i,j) \in \mathbb{R}^{d_i \times d_j}$ and principal kth block $\boldsymbol{M}[k] \in \mathbb{R}^{n_k \times n_k}$ with $0 \le i, j, k \le N$.

In fact, (P2) with j replaced by k can be rephrased as follows in terms of the coordinates $\mathbf{v} \in \mathbb{R}^{n_k}$ relative to the orthonormal basis $\{\boldsymbol{v}(j)\}_{j=0}^{k}$ of $\mathbb{V}_k$ of a generic vector in $\mathbb{V}_k$:

$$\beta\|\mathbf{v}\|_2 \le \|\boldsymbol{B}[k]\mathbf{v}\|_2 \quad \text{for all } \mathbf{v} \in \mathbb{R}^{n_k},$$

that is, setting $\mathbf{z} = \boldsymbol{B}[k]\mathbf{v}$,

$$\beta\,\|\boldsymbol{B}[k]^{-1}\mathbf{z}\|_2 \le \|\mathbf{z}\|_2 \quad \text{for all } \mathbf{z} \in \mathbb{R}^{n_k},$$

which is precisely (10.19).

A fundamental linear algebra theorem of Gaussian elimination guarantees the existence of a unique normalized block $\boldsymbol{LU}$ decomposition of $\boldsymbol{B}$ without pivoting due to (10.19):

$$\boldsymbol{B} = \boldsymbol{LU}, \tag{10.20}$$

with block partitioning

$$\boldsymbol{L}(i,j) \in \mathbb{R}^{d_i \times d_j}, \quad \boldsymbol{L}(i,j) = \mathbf{0} \quad \text{for } j > i, \quad \boldsymbol{L}(i,i) = \boldsymbol{I}(i,i), \tag{10.21}$$

$$\boldsymbol{U}(i,j) \in \mathbb{R}^{d_i \times d_j}, \quad \boldsymbol{U}(i,j) = \mathbf{0} \quad \text{for } i > j. \tag{10.22}$$

10.2.1. Matrix representation

The kth Galerkin solution u_k satisfies

$$u_k \in \mathbb{V}_k: \quad \mathcal{B}[u_k, w] = \langle f, w\rangle \quad \text{for all } w \in \mathbb{W}_k.$$

Equivalently, if $\{\boldsymbol{\gamma}(j)\}_{j=0}^{k} \in \mathbb{R}^{n_k}$ are the coordinates of u_k with respect to the orthonormal basis $\{\boldsymbol{v}(j)\}_{j=0}^{k}$, that is,

$$u_k = \sum_{j=0}^{k} \boldsymbol{\gamma}(j) \cdot \boldsymbol{v}(j),$$

then

$$\sum_{j=0}^{k} \boldsymbol{\gamma}(j) \cdot \mathcal{B}[\boldsymbol{v}(j), \boldsymbol{w}(i)] = \langle f, \boldsymbol{w}(i) \rangle \quad \text{for all } 0 \le i \le k,$$

or using matrix notation

$$\sum_{j=0}^{k} \boldsymbol{B}(i,j)\, \boldsymbol{\gamma}(j) = \boldsymbol{f}(i) = \langle f, \boldsymbol{w}(i) \rangle \quad \text{for all } 0 \le i \le k. \tag{10.23}$$

If we further write

$$\boldsymbol{\gamma}_k = (\boldsymbol{\gamma}(j))_{j=0}^{k} \in \mathbb{R}^{n_k}, \quad \boldsymbol{f}_k = (\boldsymbol{f}(i))_{i=0}^{k} \in \mathbb{R}^{n_k},$$

then (10.23) reduces to

$$\boldsymbol{B}[k]\boldsymbol{\gamma}_k = \boldsymbol{f}_k. \tag{10.24}$$

In view of the definition of $\boldsymbol{f}_k$ we realize that the kth section $\boldsymbol{f}_N[k] \in \mathbb{R}^{n_k}$ of $\boldsymbol{f}_N$ coincides with $\boldsymbol{f}_k$:

$$\boldsymbol{f}_N[k] = (\boldsymbol{f}(i))_{i=0}^{k} = \boldsymbol{f}_k.$$

However, this statement is *not* true for the solution $\boldsymbol{\gamma}_k$ of (10.24), namely

$$\boldsymbol{\gamma}_N[k] \ne \boldsymbol{\gamma}_k.$$

10.2.2. Block biorthogonal bases

We define biorthogonal bases $\widetilde{\boldsymbol{v}} \subset \mathbb{V}_N$ and $\widetilde{\boldsymbol{w}} \subset \mathbb{W}_N$ as follows:

$$\widetilde{\boldsymbol{v}} := \boldsymbol{U}^{-\top}\boldsymbol{v} \quad \Rightarrow \quad \widetilde{\boldsymbol{v}}(j) = \sum_{m=0}^{j} \boldsymbol{U}^{-\top}(j,m)\boldsymbol{v}(m), \quad 0 \le j \le N, \tag{10.25}$$

$$\widetilde{\boldsymbol{w}} := \boldsymbol{L}^{-1}\boldsymbol{w} \quad \Rightarrow \quad \widetilde{\boldsymbol{w}}(i) = \sum_{m=0}^{i} \boldsymbol{L}^{-1}(i,m)\boldsymbol{w}(m), \quad 0 \le i \le N. \tag{10.26}$$

We will see below that these bases are convenient for representing the Galerkin solution $u_k \in \mathbb{V}_k$. We start with a list of properties.

Lemma 10.9 (span of new bases). *The vectors $\widetilde{\boldsymbol{v}}$ and $\widetilde{\boldsymbol{w}}$ are bases of $\mathbb{V}_N$ and $\mathbb{W}_N$, respectively, and satisfy*

$$\operatorname{span}\{\widetilde{\boldsymbol{v}}(j)\}_{j=0}^{k} = \operatorname{span}\{\boldsymbol{v}(j)\}_{j=0}^{k},$$
$$\operatorname{span}\{\widetilde{\boldsymbol{w}}(i)\}_{i=0}^{k} = \operatorname{span}\{\boldsymbol{w}(i)\}_{i=0}^{k}.$$

Proof. This relies on the fact that $\boldsymbol{U}^{-\top}$ and $\boldsymbol{L}^{-1}$ are lower triangular and the diagonal blocks are non-singular (i.e. both $\boldsymbol{L}$ and $\boldsymbol{U}$ are invertible). □

Now consider the matrix $\widetilde{\boldsymbol{B}}$ induced by $(\widetilde{\boldsymbol{v}}, \widetilde{\boldsymbol{w}})$, namely

$$\widetilde{\boldsymbol{B}} := \mathcal{B}[\widetilde{\boldsymbol{v}}, \widetilde{\boldsymbol{w}}] \in \mathbb{R}^{n_N \times n_N}. \tag{10.27}$$

Lemma 10.10 (biorthogonality). *The block matrix* $\widetilde{\boldsymbol{B}}$ *is equal to the identity, namely*

$$\widetilde{\boldsymbol{B}}(i,j) = \boldsymbol{I}(i,j) \quad \textit{for all } 0 \le i, j \le N.$$

Proof. We simply combine the definition (10.27) with (10.25) and (10.26) to deduce, for all $0 \le i, j \le N$, that

$$\begin{aligned}
\widetilde{\boldsymbol{B}}(i,j) &= \mathcal{B}[\widetilde{\boldsymbol{v}}(j), \widetilde{\boldsymbol{w}}(i)] \\
&= \mathcal{B}\left[\sum_{m=0}^{j} \boldsymbol{U}^{-\top}(j,m)\boldsymbol{v}(m), \sum_{k=0}^{i} \boldsymbol{L}^{-1}(i,k)\boldsymbol{w}(k)\right] \\
&= \sum_{m=0}^{j}\sum_{k=0}^{i} \boldsymbol{L}^{-1}(i,k)\mathcal{B}[\boldsymbol{v}(m), \boldsymbol{w}(k)]\boldsymbol{U}^{-\top}(j,m) \\
&= \sum_{m=0}^{j}\sum_{k=0}^{i} \boldsymbol{L}^{-1}(i,k)\boldsymbol{B}(k,m)\boldsymbol{U}^{-1}(m,j) \\
&= (\boldsymbol{L}^{-1}\boldsymbol{B}\boldsymbol{U}^{-\top})(i,j) \\
&= (\boldsymbol{L}^{-1}(\boldsymbol{L}\boldsymbol{U})\boldsymbol{U}^{-1})(i,j) \\
&= \boldsymbol{I}(i,j),
\end{aligned}$$

as asserted. □

Generic functions $v \in \mathbb{V}_N$ and $w \in \mathbb{W}_N$ can be represented as follows in terms of the old and new bases:

$$v = \sum_{j=0}^{N} \boldsymbol{\gamma}(j) \cdot \boldsymbol{v}(j) = \sum_{j=0}^{N} \widetilde{\boldsymbol{\gamma}}(j) \cdot \widetilde{\boldsymbol{v}}(j), \tag{10.28}$$

$$w = \sum_{i=0}^{N} \boldsymbol{\alpha}(i) \cdot \boldsymbol{w}(i) = \sum_{i=0}^{N} \widetilde{\boldsymbol{\alpha}}(i) \cdot \widetilde{\boldsymbol{w}}(i). \tag{10.29}$$

The following lemma relates the coordinates in the two systems.

Lemma 10.11 (change of basis). *The coordinates*

$$\boldsymbol{\gamma} = (\boldsymbol{\gamma}(j))_{j=0}^{N}, \quad \boldsymbol{\alpha} = (\boldsymbol{\alpha}(i))_{i=0}^{N}$$

satisfy

$$\boldsymbol{\gamma} = \boldsymbol{U}^{-1}\widetilde{\boldsymbol{\gamma}}, \quad \boldsymbol{\alpha} = \boldsymbol{L}^{-\top}\widetilde{\boldsymbol{\alpha}}. \tag{10.30}$$

Proof. Write (10.28) in vector form and use (10.25) to obtain

$$v = \widetilde{\boldsymbol{\gamma}}^{\top}\widetilde{\boldsymbol{v}} = \widetilde{\boldsymbol{\gamma}}^{\top}(\boldsymbol{U}^{-\top}\boldsymbol{v}) = (\boldsymbol{U}^{-1}\widetilde{\boldsymbol{\gamma}})^{\top}\boldsymbol{v} = \boldsymbol{\gamma}^{\top}\boldsymbol{v},$$

whence

$$\boldsymbol{\gamma} = \boldsymbol{U}^{-1}\widetilde{\boldsymbol{\gamma}}.$$

Similarly, combining (10.29) and (10.26) yields

$$w = \widetilde{\boldsymbol{\alpha}}^\top \widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{\alpha}}^\top (\boldsymbol{L}^{-1}\boldsymbol{w}) = (\boldsymbol{L}^{-\top}\widetilde{\boldsymbol{\alpha}})^\top \boldsymbol{w} = \boldsymbol{\alpha}^\top \boldsymbol{w}$$

and

$$\boldsymbol{\alpha} = \boldsymbol{L}^{-\top}\widetilde{\boldsymbol{\alpha}}.$$

This concludes the proof. □

10.2.3. An intermediate inequality

We intend to prove the following crucial estimate. This result is in Feischl (2022), but we give a different proof based on variational arguments.

Proposition 10.12 (quasi-orthogonality I). *If $u_k \in \mathbb{V}_k$ denotes the kth Galerkin solution of* (10.3)*, then*

$$\sum_{k=0}^{N-1} \|u_{k+1} - u_k\|_{\mathbb{V}}^2 \le \frac{\|\boldsymbol{U}\|_2^2}{\beta^2} \|u_N - u_0\|_{\mathbb{V}}^2. \tag{10.31}$$

Proof. We proceed in several steps.

$\boxed{1}$ *Estimate of* $\|u_{k-1} - u_k\|_{\mathbb{V}}$. Galerkin orthogonality yields

$$\mathcal{B}[u_{k+1} - u_k, w] = 0 \quad \text{for all } w \in \mathbb{W}_k. \tag{10.32}$$

The uniform discrete inf-sup property (P2) implies the existence of $\overline{w} \in \mathbb{W}_{k+1}$ with $\|\overline{w}\|_{\mathbb{W}} = 1$ such that

$$\beta \|u_{k+1} - u_k\|_{\mathbb{V}} \le \mathcal{B}[u_{k+1} - u_k, \overline{w}]. \tag{10.33}$$

We decompose $\overline{w}$ orthogonally as follows:

$$\overline{w} = \overline{w}_k + \overline{w}_k^\perp, \quad \overline{w}_k \in \mathbb{W}_k, \quad \overline{w}_k^\perp \in \mathbb{W}_k^\perp,$$

with $\|\overline{w}_k^\perp\|_{\mathbb{W}} \le 1$. In view of (10.32), (10.33) also reads

$$\beta \|u_{k+1} - u_k\|_{\mathbb{V}} \le \mathcal{B}[u_{k+1} - u_k, \overline{w}_k^\perp] \le \mathcal{B}\left[u_{k+1} - u_k, \frac{\overline{w}_k^\perp}{\|\overline{w}_k^\perp\|_{\mathbb{W}}}\right].$$

We now let

$$\widehat{w}_{k+1} := \frac{\overline{w}_k^\perp}{\|\overline{w}_k^\perp\|_{\mathbb{W}}} \in \mathbb{W}_k^\perp \subset \mathbb{W}_{k+1},$$

and decompose it along the oblique subspaces $\mathbb{W}_k = \operatorname{span}\{\widetilde{\boldsymbol{w}}(j)\}_{j=0}^k$ and $\operatorname{span}\{\widetilde{\boldsymbol{w}}(k+1)\}$, as illustrated in Figure 10.2. Since $\mathbb{W}_k^\perp = \operatorname{span}\{\boldsymbol{w}(k+1)\}$ and $\boldsymbol{w}(k+1) = (w_j)_{j=n_k+1}^{n_{k+1}}$ is an orthonormal basis, the function $\widehat{w}_{k+1} \in \mathbb{W}_k^\perp$ can be written uniquely as

$$\widehat{w}_{k+1} = \boldsymbol{\alpha}(k+1) \cdot \boldsymbol{w}(k+1),$$

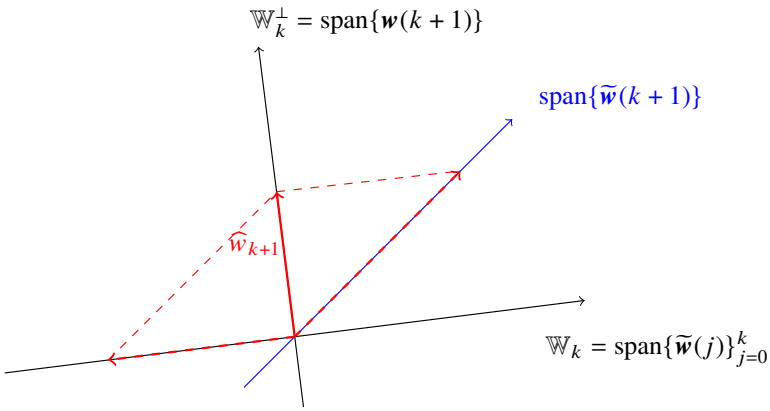


Figure 10.2. Oblique decomposition of the space $\mathbb{W}_{k+1}$ into the subspaces $\mathbb{W}_k = \mathrm{span}\{\widetilde{\boldsymbol{w}}(j)\}_{j=0}^k$ and $\mathrm{span}\{\widetilde{\boldsymbol{w}}(k+1)\}$.

with $\boldsymbol{\alpha}(k+1) \in \mathbb{R}^{d_{k+1}}$ satisfying

$$\|\boldsymbol{\alpha}(k+1)\|_2 = 1 = \|\widehat{w}_{k+1}\|_{\mathbb{W}}.$$

Invoking (10.26), we can express $\boldsymbol{w}(k+1)$ in terms of $\{\widetilde{\boldsymbol{w}}(j)\}_{j=0}^k$ as

$$\begin{aligned}\boldsymbol{w}(k+1) &= \sum_{j=0}^{k+1} \boldsymbol{L}(k+1,j)\,\widetilde{\boldsymbol{w}}(j)\\ &= \widetilde{\boldsymbol{w}}(k+1) + \sum_{j=0}^{k} \boldsymbol{L}(k+1,j)\,\widetilde{\boldsymbol{w}}(j)\end{aligned}$$

because $\boldsymbol{L}(k+1,k+1) = \boldsymbol{I}(k+1,k+1) \in \mathbb{R}^{d_{k+1}\times d_{k+1}}$. Consequently

$$\mathcal{B}[u_{k+1} - u_k, \widehat{w}_{k+1}] = \mathcal{B}[u_{k+1} - u_k, \boldsymbol{\alpha}(k+1)\cdot\widetilde{\boldsymbol{w}}(k+1)]$$

because (10.32) implies

$$\mathcal{B}[u_{k+1} - u_k, \widetilde{\boldsymbol{w}}(j)] = 0 \quad \text{for all } 0 \le j \le k.$$

In addition, the biorthogonality of $\widetilde{\boldsymbol{w}}(k+1)$ with respect to $\widetilde{\boldsymbol{v}}(j)$ for $0 \le j \le k$ translates into

$$\mathcal{B}[u_k, \widetilde{\boldsymbol{w}}(k+1)] = 0 = \mathcal{B}[u_0, \widetilde{\boldsymbol{w}}(k+1)].$$

Moreover, Galerkin orthogonality yields

$$\mathcal{B}[u_{k+1}, \widetilde{\boldsymbol{w}}_{k+1}] = \langle f, \widetilde{\boldsymbol{w}}_{k+1}\rangle = \mathcal{B}[u_N, \widetilde{\boldsymbol{w}}_{k+1}],$$

and collecting the preceding expressions we obtain

$$\|u_{k+1} - u_k\| \le \frac{1}{\beta}\boldsymbol{\alpha}(k+1)\cdot\mathcal{B}[u_N - u_0, \widetilde{\boldsymbol{w}}(k+1)]. \tag{10.34}$$

$\boxed{2}$ *Estimate of* $\mathcal{B}[u_N - u_0, \widetilde{\boldsymbol{w}}(k+1)]$. We exploit the biorthogonality between $\{\widetilde{\boldsymbol{v}}(j)\}_{j=0}^N$ and $\{\widetilde{\boldsymbol{w}}(j)\}_{j=0}^N$. In fact we write

$$u_N - u_0 = \sum_{j=0}^N \widetilde{\boldsymbol{\gamma}}(j) \cdot \widetilde{\boldsymbol{v}}(j),$$

and substitute into the right-hand side of (10.34) to arrive at

$$\mathcal{B}[u_N - u_0, \widetilde{\boldsymbol{w}}(k+1)] = \sum_{j=0}^N \widetilde{\boldsymbol{\gamma}}(j) \cdot \mathcal{B}[\widetilde{\boldsymbol{v}}(j), \widetilde{\boldsymbol{w}}(k+1)] = \widetilde{\boldsymbol{\gamma}}(k+1).$$

Therefore (10.34) gives

$$\|u_{k+1} - u_k\|_{\mathbb{V}} \le \frac{1}{\beta} \boldsymbol{\alpha}(k+1) \cdot \widetilde{\boldsymbol{\gamma}}(k+1),$$

whence

$$\|u_{k+1} - u_k\|_{\mathbb{V}} \le \frac{1}{\beta} \|\widetilde{\boldsymbol{\gamma}}(k+1)\|_2$$

because $\|\boldsymbol{\alpha}(k+1)\|_2 = 1$.

$\boxed{3}$ *Final estimate.* Compute

$$\sum_{k=0}^{N-1} \|u_{k+1} - u_k\|_{\mathbb{V}}^2 \le \frac{1}{\beta^2} \sum_{k=0}^{N-1} \|\widetilde{\boldsymbol{\gamma}}(k+1)\|_2^2 \le \frac{1}{\beta^2} \|\widetilde{\boldsymbol{\gamma}}\|_2^2 \le \frac{\|\boldsymbol{U}\|_2^2}{\beta^2} \|\boldsymbol{\gamma}\|_2^2,$$

according to (10.30). Since $\{\boldsymbol{v}(j)\}_{j=0}^N$ are orthonormal, we get

$$u_N - u_0 = \sum_{j=0}^N \boldsymbol{\gamma}(j) \cdot \boldsymbol{v}(j) \quad \Rightarrow \quad \|u_N - u_0\|_{\mathbb{V}} = \|\boldsymbol{\gamma}\|_2$$

and

$$\sum_{k=0}^{N-1} \|u_{k+1} - u_k\|_{\mathbb{V}}^2 \le \frac{\|\boldsymbol{U}\|_2^2}{\beta^2} \|u_N - u_0\|_{\mathbb{V}}^2,$$

as asserted. This concludes the proof. □

In order to get the quasi-orthogonality estimate, we still need to compare the errors $\|u_N - u_0\|_{\mathbb{V}}$ and $\|u - u_0\|_{\mathbb{V}}$. The following is a variant of (3.3).

Lemma 10.13 (stability). *We have*

$$\|u_N - u_0\|_{\mathbb{V}} \le \frac{\|\mathcal{B}\|}{\beta} \|u - u_0\|_{\mathbb{V}}.$$

Proof. We use (10.17) and (10.16), in this order, together with Galerkin orthogonality, to deduce

$$\begin{aligned}\beta\|u_N - u_0\|_{\mathbb{V}} &\le \sup_{w\in\mathbb{W}_N} \frac{\mathcal{B}[u_N - u_0, w]}{\|w\|_{\mathbb{W}}} \\ &= \sup_{w\in\mathbb{W}_N} \frac{\mathcal{B}[u - u_0, w]}{\|w\|_{\mathbb{W}}} \le \|\mathcal{B}\|\|u - u_0\|_{\mathbb{V}}.\end{aligned}$$

This completes the proof. □

Corollary 10.14 (quasi-orthogonality II). *Let $u_k \in \mathbb{V}_k$ be the kth Galerkin solution of* (10.3). *Then, for all $0 \le j \le N$, we have*

$$\sum_{k=j}^{j+N-1} \|u_{k+1} - u_k\|_{\mathbb{V}}^2 \le \frac{\|\mathcal{B}\|^2}{\beta^4}\|\boldsymbol{U}\|_2^2\|u - u_j\|_{\mathbb{V}}^2.$$

Proof. Combining Proposition 10.12 with Lemma 10.13 yields

$$\sum_{k=0}^{N-1} \|u_{k+1} - u_k\|_{\mathbb{V}}^2 \le \frac{\|\mathcal{B}\|^2}{\beta^4}\|\boldsymbol{U}\|_2^2\|u - u_0\|_{\mathbb{V}}^2.$$

Finally, replacing $u_0 \in \mathbb{V}_0$ with the jth Galerkin solution $u_j \in \mathbb{V}_j$, we obtain the desired estimate. □

This corollary says that in order to prove Theorem 10.8, i.e. to check the validity of Assumption 10.1, it is enough to investigate the growth of the block triangular factor $\boldsymbol{U}$ introduced in (10.20), and more precisely to prove that

$$\|\boldsymbol{U}\|_2 = o(N^{1/2}).$$

This is the second step of our analysis. In fact we prove something more, which is expressed by the following result.

Theorem 10.15 (bound of block matrices $\boldsymbol{L}$ and $\boldsymbol{U}$). *There are constants $C_{LU} > 0$ and $p > 2$ such that the block $\boldsymbol{LU}$ factors of $\boldsymbol{B}$ satisfy*

$$\|\boldsymbol{U}\|_2 + \|\boldsymbol{L}\|_2 + \|\boldsymbol{U}^{-1}\|_2 + \|\boldsymbol{L}^{-1}\|_2 \le C_{LU}\, N^{1/p}. \tag{10.35}$$

The proof of this theorem is lengthy and very technical; it involves subtle linear algebra arguments, which may not be familiar to many readers. For such reasons, we prefer to postpone it to the end of this section (see Section 10.6).

10.3. Convergence rates of AFEMs for inf-sup stable methods

In this section we discuss AFEMs to solve a boundary value problem admitting a variational formulation of the form

$$u \in \mathbb{V}: \quad \mathcal{B}[u, w] = \langle f, w\rangle \quad \text{for all } w \in \mathbb{W}, \tag{10.36}$$

in which the bilinear form $\mathcal{B}$ on $\mathbb{V} \times \mathbb{W}$ is continuous and inf-sup stable, with inf-sup constant $\beta > 0$, and $f \in \mathbb{W}^*$. We will consider the one-step AFEM given by Algorithm 5.4 (GALERKIN) when all data are discrete, the one-step AFEM with switch given by Algorithm 5.16 (AFEM-SW) when the operator coefficients are discrete but the forcing term is not (as in the Stokes problem), and the general two-step AFEM given by Algorithm 5.1 (AFEM-TS).

10.3.1. Algorithm 5.4 (GALERKIN)
For $j \geq 0$, let $(\mathcal{T}_j, \mathbb{V}_j, u_j)$, with $u_j \in \mathbb{V}_j = \mathbb{V}_{\mathcal{T}_j}$, denote the sequence of meshes, subspaces and Galerkin approximations to (10.36) generated by GALERKIN. Let

$$\eta_{\mathcal{T}_j}(v) = \eta_{\mathcal{T}_j}(v, f) = \Big(\sum_{T \in \mathcal{T}_j} \eta_{\mathcal{T}}(v, T)^2 \Big)^{1/2} \tag{10.37}$$

be the PDE error estimators used in the loop. If such estimators fulfil Assumptions 10.2 (equivalence of error and estimator) and 10.3 (estimator reduction), then Theorem 10.6 (linear convergence) applies and the following result holds.

Proposition 10.16 (convergence and termination of GALERKIN). *The module* GALERKIN *produces a sequence $\{u_j\}$ converging linearly to $u \in \mathbb{V}$,*

$$\|u - u_{j+i}\|_{\mathbb{V}} \leq C\, \rho^i \|u - u_j\|_{\mathbb{V}} \quad \textit{for all } j, i \geq 0, \quad 0 < \rho < 1,$$

thereby reaching any prescribed accuracy $\|u - u_j\|_{\mathbb{V}} \leq \varepsilon$ in a finite number of iterations.

10.3.2. Algorithm 5.16 (AFEM-SW)
This algorithm applies to the situation in which the operator coefficients are discrete, whereas the forcing $f \in \mathbb{W}^*$ is not. Then the PDE estimator $\eta_{\mathcal{T}}(v, f)$ depends on f via a projection $P_{\mathcal{T}} f$ upon a finite-dimensional subspace of $\mathbb{W}^*$. Inspired by Lemma 4.5 (localization of H^{-1}-norm), we let $\mathbb{W}^*_{\mathcal{T}}$ denote a suitable decomposition of $\mathbb{W}^*$ subordinate to $\mathcal{T}$ with norm $\|f\|_{\mathbb{W}^*_{\mathcal{T}}}$. In this part of the discussion, we prefer to make the dependence of $\eta_{\mathcal{T}}$ upon $P_{\mathcal{T}} f$ explicit to avoid confusion, so we will write $\eta_{\mathcal{T}}(v, P_{\mathcal{T}} f)$ rather than $\eta_{\mathcal{T}}(v, f)$ as usual.

Let us begin by stating two assumptions on the estimator (10.37) to be used below.

Assumption 10.17 (Lipschitz continuity of estimator). There exists a constant $C_{\mathrm{Lip}} > 0$ such that for any $\mathcal{T} \in \mathbb{T}$, any $v, w \in \mathbb{V}_{\mathcal{T}}$ and any $f, g \in \mathbb{W}^*$, we have

$$|\eta_{\mathcal{T}}(v, P_{\mathcal{T}} f) - \eta_{\mathcal{T}}(w, P_{\mathcal{T}} g)| \leq C_{\mathrm{Lip}} \big(\|v - w\|_{\mathbb{V}} + \|P_{\mathcal{T}} f - P_{\mathcal{T}} g\|_{\mathbb{W}^*_{\mathcal{T}}} \big).$$

Assumption 10.18 (monotonicity of estimator). If $\mathcal{T} \in \mathbb{T}$ and $\mathcal{T}_*$ is a refinement of $\mathcal{T}$, then the projection operator satisfies $P_{\mathcal{T}_*} P_{\mathcal{T}} = P_{\mathcal{T}}$ and

$$\eta_{\mathcal{T}_*}(v, P_{\mathcal{T}} f, T) \leq \eta_{\mathcal{T}}(v, P_{\mathcal{T}} f, T) \quad \text{for all } T \in \mathcal{T} \text{ and } v \in \mathbb{V}_{\mathcal{T}},\ f \in \mathbb{W}^*.$$

It is useful for the subsequent applications to have explicit criteria that guarantee the fulfilment of Assumption 10.3. This is the purpose of the following result.

Proposition 10.19 (estimator reduction under Dörfler marking). *Let the estimator $\eta_{\mathcal{T}}(v, P_{\mathcal{T}} f)$ in (10.37) be used in* GALERKIN. *Let Assumptions 10.17 and 10.18 be valid. Let $\mathcal{T}_*$ be a refinement of $\mathcal{T}$, with estimator $\eta_{\mathcal{T}_*}(v, P_{\mathcal{T}_*} f)$, obtained by bisecting the elements $T \in \mathcal{M}$ marked in* MARK, *using a Dörfler condition on the estimator $\eta_{\mathcal{T}}(u_{\mathcal{T}}, P_{\mathcal{T}} f)$ for the Galerkin solution $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$.*

Suppose that there exists $\lambda \in (0, 1)$ such that

$$\eta_{\mathcal{T}_*}(u_{\mathcal{T}}, P_{\mathcal{T}} f, T)^2 \le \lambda\, \eta_{\mathcal{T}}(u_{\mathcal{T}}, P_{\mathcal{T}} f, T)^2 \quad \text{for all } T \in \mathcal{M}. \tag{10.38}$$

Then there exists $0 < \rho < 1$ and $C > 0$ such that, for all $v_{\mathcal{T}_*} \in \mathbb{V}_{\mathcal{T}_*}$,

$$\eta_{\mathcal{T}_*}(v_{\mathcal{T}_*}, P_{\mathcal{T}_*} f)^2 \le \rho\, \eta_{\mathcal{T}}(u_{\mathcal{T}}, P_{\mathcal{T}} f)^2 + C\big(\|v_{\mathcal{T}_*} - u_{\mathcal{T}}\|_{\mathbb{V}}^2 + \|P_{\mathcal{T}_*} f - P_{\mathcal{T}} f\|_{\mathbb{W}^*_{\mathcal{T}_*}}^2\big).$$

Proof. By Assumption 10.17 applied to $\mathcal{T}_*$, we have

$$\eta_{\mathcal{T}_*}(v_{\mathcal{T}_*}, P_{\mathcal{T}_*} f) \le \eta_{\mathcal{T}_*}(u_{\mathcal{T}}, P_{\mathcal{T}} f) + C_{\mathrm{Lip}}(\|v_{\mathcal{T}_*} - u_{\mathcal{T}}\|_{\mathbb{V}} + \|P_{\mathcal{T}_*} f - P_{\mathcal{T}} f\|_{\mathbb{W}^*_{\mathcal{T}_*}}).$$

Using Assumption 10.18 while extending Proposition 4.56 to the current abstract setting, we have for any $\delta > 0$

$$\begin{aligned}\eta_{\mathcal{T}_*}(v_{\mathcal{T}_*}, P_{\mathcal{T}_*} f)^2 \le{}& (1+\delta)\big(\eta_{\mathcal{T}}(u_{\mathcal{T}}, P_{\mathcal{T}} f)^2 - (1-\lambda)\, \eta_{\mathcal{T}}(u_{\mathcal{T}}, P_{\mathcal{T}} f, \mathcal{M})^2\big)\\ &+ 2(1+\delta^{-1})\, C_{\mathrm{Lip}}^2\big(\|v_{\mathcal{T}_*} - u_{\mathcal{T}}\|_{\mathbb{V}}^2 + \|P_{\mathcal{T}_*} f - P_{\mathcal{T}} f\|_{\mathbb{W}^*_{\mathcal{T}_*}}^2\big).\end{aligned}$$

We conclude using Dörfler condition $\eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{M}) \ge \theta \eta_{\mathcal{T}}(u_{\mathcal{T}})$ and choosing δ small enough. □

Before proceeding further, let us introduce the quantity

$$\mathrm{osc}_{\mathcal{T}}(f) := \|f - P_{\mathcal{T}} f\|_{\mathbb{W}^*_{\mathcal{T}}},$$

which is a measure of the oscillation of the data f. If $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$ is the solution of AFEM-SW, we let $\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)$ indicate the full estimator defined by

$$\mathcal{E}_{\mathcal{T}}(u_{\mathcal{T}}, f)^2 := \eta_{\mathcal{T}}(u_{\mathcal{T}}, P_{\mathcal{T}} f)^2 + \mathrm{osc}_{\mathcal{T}}(f)^2. \tag{10.39}$$

We formulate the following assumption on the data oscillation.

Assumption 10.20 (quasi-monotonicity of oscillation). There exists a constant $C_{\mathrm{osc}} > 0$ such that, for any $\mathcal{T} \in \mathbb{T}$ and any admissible refinement $\mathcal{T}_* \ge \mathcal{T}$, we have

$$\mathrm{osc}_{\mathcal{T}_*}(f) \le C_{\mathrm{osc}}\, \mathrm{osc}_{\mathcal{T}}(f).$$

A consequence of this assumption is the bound

$$\|P_{\mathcal{T}_*} f - P_{\mathcal{T}} f\|_{\mathbb{W}^*_{\mathcal{T}_*}} \le \|f - P_{\mathcal{T}_*} f\|_{\mathbb{W}^*_{\mathcal{T}_*}} + \|f - P_{\mathcal{T}} f\|_{\mathbb{W}^*_{\mathcal{T}_*}} \le (1 + C_{\mathrm{osc}})\, \mathrm{osc}_{\mathcal{T}}(f),$$

which, inserted into the reduction estimate of Proposition 10.19, gives the existence of $0 < \rho_0 < 1$ and $C_0 > 0$ independent of $\mathcal{T}$ such that

$$\eta_{\mathcal{T}_*}(u_{\mathcal{T}_*}, P_{\mathcal{T}_*} f)^2 \le \rho_0\, \eta_{\mathcal{T}}(u_{\mathcal{T}}, P_{\mathcal{T}} f)^2 + C_0\big(\|u_{\mathcal{T}_*} - u_{\mathcal{T}}\|_{\mathbb{V}}^2 + \mathrm{osc}_{\mathcal{T}}(f)^2\big). \tag{10.40}$$

We aim at establishing a linear convergence result similar to Theorem 10.6 for the sequence $\{u_j\}_{j=0}^{\infty}$ generated by AFEM-SW. To this end, we introduce as usual the short-hand notation $e_j = \|u - u_j\|_{\mathbb{V}}$, $E_{j+1} = \|u_{j+1} - u_j\|_{\mathbb{V}}$, $\eta_j = \eta_{\mathcal{T}_j}(u_j, P_{\mathcal{T}_j} f)$, $\operatorname{osc}_j = \operatorname{osc}_{\mathcal{T}_j}(f)$, $\mathcal{E}_j = \mathcal{E}_{\mathcal{T}_j}(u_j, f)$, and we also introduce the scaled estimator

$$\zeta_j^2 := \eta_j^2 + \gamma \operatorname{osc}_j^2, \tag{10.41}$$

where the parameter $\gamma > 0$ is to be found. Note that at this point we have three parameters $\omega \in (0,1)$, $\xi \in (0,1)$ and $\gamma > 0$ to play with, and the idea is to find conditions on them such that an inequality similar to (10.9) in the proof of Theorem 10.6 holds true. The following result is an intermediate step.

Lemma 10.21 (linear estimator reduction). *Let Assumptions 10.3 (estimator reduction), 10.17 (Lipschitz continuity of estimator), 10.18 (monotonicity of estimator) and 10.20 (quasi-monotonicity of oscillation) be valid. There exists $\omega_0 > 0$ such that, for any choice of parameters $0 < \omega \le \omega_0$ and $0 < \xi \le 1/\sqrt{2}$ in* AFEM-SW*, there exist constants $0 < \rho < 1$, $\Lambda > 0$, $\gamma \ge 1$ for which*

$$\zeta_k^2 \le \rho^{k-j} \zeta_j^2 + \Lambda \sum_{\ell=j+1}^{k} E_\ell^2, \quad k \ge j \ge 0. \tag{10.42}$$

Proof. We discuss the two alternatives in Algorithm 5.16 (AFEM-SW) separately.

$\boxed{1}$ *Case* $\operatorname{osc}_j \le \omega \mathcal{E}_j$. We use (10.40) to get

$$\eta_{j+1}^2 \le \rho_0 \eta_j^2 + C_0 E_{j+1}^2 + C_0 \operatorname{osc}_j^2$$

and Assumption 10.20 to write

$$\operatorname{osc}_{j+1} \le C_{\operatorname{osc}} \operatorname{osc}_j \,.$$

From

$$\operatorname{osc}_j^2 \le \omega^2 \mathcal{E}_j^2 = \omega^2(\eta_j^2 + \operatorname{osc}_j^2) \le \omega^2(\eta_j^2 + \gamma \operatorname{osc}_j^2) = \omega^2 \zeta_j^2,$$

provided $\gamma \ge 1$, we deduce

$$\begin{aligned}
\zeta_{j+1}^2 &= \eta_{j+1}^2 + \gamma \operatorname{osc}_{j+1}^2 \le \rho_0 \eta_j^2 + C_0 E_{j+1}^2 + \left(C_0 + \gamma C_{\operatorname{osc}}^2\right) \operatorname{osc}_j^2 \\
&\le \rho_0 \eta_j^2 + C_0 E_{j+1}^2 + \left(C_0 + \gamma C_{\operatorname{osc}}^2\right) \omega^2 \left(\eta_j^2 + \gamma \operatorname{osc}_j^2\right) \\
&= \left[\rho_0 + \left(C_0 + \gamma C_{\operatorname{osc}}^2\right)\omega^2\right] \eta_j^2 + \left[\left(C_0 + \gamma C_{\operatorname{osc}}^2\right)\omega^2\right] \gamma \operatorname{osc}_j^2 + C_0 E_{j+1}^2 \\
&\le \left[\rho_0 + \left(C_0 + \gamma C_{\operatorname{osc}}^2\right)\omega^2\right] \zeta_j^2 + C_0 E_{j+1}^2.
\end{aligned} \tag{10.43}$$

Below we will impose

$$\rho_1 := \rho_0 + \left(C_0 + \gamma C_{\operatorname{osc}}^2\right)\omega^2 < 1, \tag{10.44}$$

which will yield the desired bound

$$\zeta_{j+1}^2 \le \rho_1 \zeta_j^2 + C_0 E_{j+1}^2. \tag{10.45}$$

$\boxed{2}$ *Case* $\operatorname{osc}_j > \omega\mathcal{E}_j$. We use $\mathcal{E}_j^2 = \eta_j^2 + \operatorname{osc}_j^2 > \eta_j^2$ to get

$$\eta_j^2 < \frac{1}{\omega^2}\operatorname{osc}_j^2 .$$

Proceeding as in the proofs of Proposition 10.19 (now with $\mathcal{M} = \emptyset$) and (10.40), we obtain for any $\delta > 0$

$$\eta_{j+1}^2 \le (1+\delta)\eta_j^2 + C_\delta\left(E_{j+1}^2 + \operatorname{osc}_j^2\right) \le (1-\delta)\eta_j^2 + C_\delta E_{j+1}^2 + \left(\frac{2\delta}{\omega^2} + C_\delta\right)\operatorname{osc}_j^2$$

with $C_\delta = C_0^2(1+\delta^{-1})$. On the other hand, since osc_{j+1} is computed after a call to DATA, it satisfies

$$\operatorname{osc}_{j+1}^2 \le \xi^2\sigma_j^2 = \xi^2\omega^2\mathcal{E}_j^2 < \xi^2\operatorname{osc}_j^2 = \frac{1+\xi^2}{2}\operatorname{osc}_j^2 - \frac{1-\xi^2}{2}\operatorname{osc}_j^2 .$$

Combining the last two equations, we obtain

$$\begin{aligned}\zeta_{j+1}^2 &= \eta_{j+1}^2 + \gamma\operatorname{osc}_{j+1}^2\\ &\le (1-\delta)\eta_j^2 + \gamma\frac{1+\xi^2}{2}\operatorname{osc}_j^2 + C_\delta E_{j+1}^2 + \left(\frac{2\delta}{\omega^2} + C_\delta - \gamma\frac{1-\xi^2}{2}\right)\operatorname{osc}_j^2 .\end{aligned}$$

Below we will enforce

$$\Gamma := \frac{2\delta}{\omega^2} + C_\delta - \gamma\frac{1-\xi^2}{2} \le 0, \tag{10.46}$$

which will guarantee

$$\zeta_{j+1}^2 \le \rho_2\zeta_j^2 + C_\delta E_{j+1}^2, \tag{10.47}$$

with $\rho_2 := \max(1-\delta, (1+\xi^2)/2) < 1$.

$\boxed{3}$ *Choice of parameters.* Summarizing, in both cases $\boxed{1}$ and $\boxed{2}$ we have obtained

$$\zeta_{j+1}^2 \le \rho\zeta_j^2 + \Lambda E_{j+1}^2, \tag{10.48}$$

with $\rho := \max(\rho_1, \rho_2) < 1$ and $\Lambda := \max(C_0, C_\delta)$, which holds under the conditions (10.44) and (10.46). Iterating (10.48), we obtain the desired bound (10.42).

To fulfil (10.44), we write ω^2 in the form $\omega^2 = \sigma_0/\gamma$, which gives

$$\rho_1 = \rho_0 + \left(C_0 + \gamma C_{\operatorname{osc}}^2\right)\frac{\sigma_0}{\gamma} = \rho_0 + \left(\frac{C_0}{\gamma} + C_{\operatorname{osc}}^2\right)\sigma_0 \le \rho_0 + \left(C_0 + C_{\operatorname{osc}}^2\right)\sigma_0$$

since $\gamma \ge 1$, and we pick a $\sigma_0 > 0$ small enough to make $\rho_0 + \left(C_0 + C_{\operatorname{osc}}^2\right)\sigma_0 < 1$.

To fulfil (10.46), we use $\xi \le 1/\sqrt{2}$ and again $\omega^2 = \sigma_0/\gamma$ to write

$$\Gamma = C_\delta + \left(\frac{2\delta}{\sigma_0} - \frac{1-\xi^2}{2}\right)\gamma \le C_\delta + \left(\frac{2\delta}{\sigma_0} - \frac{1}{4}\right)\gamma.$$

Choosing $\delta = \delta_0 = \sigma_0/16$ yields

$$\Gamma \le C_{\delta_0} - \frac{1}{8}\gamma \le 0 \quad \text{provided} \quad \gamma \ge 8C_{\delta_0}.$$

In conclusion, setting $\gamma_0 = \max(1, 8C_{\delta_0})$ and $\omega_0 = \sqrt{\sigma_0/\gamma_0}$, we fulfil both conditions (10.44) and (10.46) for any $0 < \omega \le \omega_0$, by choosing the scaling parameter $\gamma = \sigma_0/\omega^2 \ge \gamma_0$. This completes the proof. □

Before establishing the linear convergence result for Algorithm 5.16 (AFEM-SW), we need to extend Assumption 10.2 (equivalence of error and estimator) to the present situation, in which the estimator $\eta_{\mathcal{T}}$ is replaced by the full estimator $\mathcal{E}_{\mathcal{T}}$ defined in (10.39); see Theorem 4.45 (modified residual estimator).

Assumption 10.22 (equivalence of error and full estimator). There exist constants $C_U \ge C_L > 0$ such that

$$C_L \mathcal{E}_j \le \|u - u_j\|_{\mathbb{V}} \le C_U \mathcal{E}_j, \quad j \ge 0, \tag{10.49}$$

where $\mathcal{E}_j = \mathcal{E}_{\mathcal{T}_j}(u_{\mathcal{T}_j}, f)$.

Theorem 10.23 (linear convergence for AFEM-SW). *Suppose Assumptions 10.22 (equivalence of error and full estimator), 10.3 (estimator reduction), 10.17 (Lipschitz continuity of estimator) and 10.20 (quasi-monotonicity of oscillation) are valid. There exists $\omega_0 \in (0, 1]$ such that, for any choice of parameters $0 < \omega < \omega_0$ and $0 < \xi, \theta < 1$ in* AFEM-SW*, constants $0 < \rho < 1$ and $c > 0$ exist for which*

$$e_{j+1} \le c\rho^i e_j \quad \textit{for all } i, j \in \mathbb{N}, \tag{10.50}$$

where $e_j := \|u - u_j\|_{\mathbb{V}}$.

Proof. By Assumption 10.22 and $\gamma \ge 1$ in (10.41), we get the equivalence of error and scaled estimator

$$\frac{C_L^2}{\gamma}\zeta_j^2 = \frac{C_L^2}{\gamma}(\eta_j^2 + \gamma\,\mathrm{osc}_j^2) \le C_L^2\mathcal{E}_j^2 \le e_j^2 \le C_U^2\mathcal{E}_j^2 \le C_U^2(\eta_j^2 + \gamma\,\mathrm{osc}_j^2) = C_U^2\zeta_j^2.$$

Invoking (10.42) yields

$$\begin{aligned} e_k^2 \le C_U^2\zeta_k^2 &\le C_U^2\rho^{k-j}\zeta_j^2 + C_U^2\Lambda\sum_{\ell=j+1}^{k} E_\ell^2 \\ &\le \gamma\frac{C_U^2}{C_L^2}\rho^{k-j}e_j^2 + C_U^2\Lambda\sum_{\ell=j+1}^{k} E_\ell^2. \end{aligned}$$

This inequality is similar to the expression (10.8) obtained in step $\boxed{1}$ in the proof of Theorem 10.6. Therefore we can finally proceed as in that proof and obtain the desired result. □

10.3.3. Algorithm 5.28 (AFEM-TS)

As usual, DATA produces discrete data $\widehat{\mathcal{D}}_k$ on a mesh $\widehat{\mathcal{T}}_k \ge \mathcal{T}_k$, whereas GALERKIN produces an approximation u_{k+1} on a mesh $\mathcal{T}_{k+1} \ge \widehat{\mathcal{T}}_k$ to the exact solution $\widehat{u}_k$ of the boundary value problem of interest with data $\widehat{\mathcal{D}}_k$. Its kernel is given in Algorithm 5.4 (GALERKIN).

In order to proceed, we need some notation and some assumptions. Let $D(\Omega)$ denote the space of admissible data $\mathcal{D}$ for the boundary value problem at hand; let $\|\mathcal{D}\|_{D(\Omega)}$ be a (quasi-)norm on $D(\Omega)$. If $\mathcal{D}$ collects all data of problem (10.36), we write $\mathcal{B} = \mathcal{B}(\mathcal{D})$ and $\mathcal{F} = \mathcal{F}(\mathcal{D}) = \langle f(\mathcal{D}), \cdot\rangle$ to highlight the dependence of the bilinear and linear forms on the chosen data; similarly, we write $u = u(\mathcal{D})$ for the corresponding solution. A perturbation $\widehat{\mathcal{D}}$ of $\mathcal{D}$ generates perturbed bilinear and linear forms $\widehat{\mathcal{B}} = \mathcal{B}(\widehat{\mathcal{D}})$ and $\widehat{\mathcal{F}} = \mathcal{F}(\widehat{\mathcal{D}}) = \langle f(\widehat{\mathcal{D}}), \cdot\rangle$, and a perturbation $\widehat{u} = u(\widehat{\mathcal{D}})$ of u, which satisfies

$$\widehat{u} \in \mathbb{V}: \quad \widehat{\mathcal{B}}[\widehat{u}, w] = \widehat{\mathcal{F}}[w] \quad \text{for all } w \in \mathbb{W}. \tag{10.51}$$

We assume, as in Section 5.4.2, that a call $[\widehat{\mathcal{T}}, \widehat{\mathcal{D}}] = \mathsf{DATA}(\mathcal{T}, \mathcal{D}, \tau)$ generates an admissible refinement $\widehat{\mathcal{T}}$ of $\mathcal{T}$ and discrete data $\widehat{\mathcal{D}}$ over $\widehat{\mathcal{T}}$, such that

$$\|\mathcal{D} - \widehat{\mathcal{D}}\|_{D(\Omega)} \le C_{\text{data}}\tau, \tag{10.52}$$

where $C_{\text{data}} > 0$ depends on data (see Section 7.2). Finally, we associate to any admissible refinement $\mathcal{T}$ of $\mathcal{T}_0$ two finite-dimensional spaces $\mathbb{V}_{\mathcal{T}} \subset \mathbb{V}$ and $\mathbb{W}_{\mathcal{T}} \subset \mathbb{W}$ of equal dimension, made of piecewise polynomial functions on $\mathcal{T}$ (typically this is accomplished by choosing a type of finite element compatible with the pair $(\mathbb{V}, \mathbb{W})$ and adopting it in any $\mathcal{T} \in \mathbb{T}$).

We are ready to state the assumptions which will rule our forthcoming analysis of AFEM-TS.

Assumption 10.24 (perturbation estimate). For any $\widetilde{\mathcal{T}} \in \mathbb{T}$ and $\varepsilon \le \varepsilon_0$, let $[\widehat{\mathcal{T}}, \widehat{\mathcal{D}}] = \mathsf{DATA}(\widetilde{\mathcal{T}}, \mathcal{D}, \varepsilon)$ and let $\widehat{u} = u(\widehat{\mathcal{D}})$ be the solution of (10.51). There exists a constant $C_D > 0$, independent of $\widetilde{\mathcal{T}}$ and ε, such that

$$\|u - \widehat{u}\|_{\mathbb{V}} \le C_D \|\mathcal{D} - \widehat{\mathcal{D}}\|_{D(\Omega)}. \tag{10.53}$$

Note that concatenating this inequality with (10.52) for $\tau = \varepsilon$, we can quantify the effect of a call to DATA on the perturbation of the exact solution; we indeed have

$$\|u - \widehat{u}\|_{\mathbb{V}} \le C_D C_{\text{data}} \varepsilon. \tag{10.54}$$

Assumption 10.25 (uniform continuity constant). For any $\widetilde{\mathcal{T}} \in \mathbb{T}$ and $\varepsilon \le \varepsilon_0$, let $[\widehat{\mathcal{T}}, \widehat{\mathcal{D}}] = \mathsf{DATA}(\widetilde{\mathcal{T}}, \mathcal{D}, \varepsilon)$ and let $\widehat{\mathcal{B}} = \mathcal{B}(\widehat{\mathcal{D}})$ be the associated bilinear form. There exists a constant $C_B \ge \|\mathcal{B}\|$, independent of $\widetilde{\mathcal{T}}$ and ε, such that

$$\|\widehat{\mathcal{B}}\| \le C_B. \tag{10.55}$$

Assumption 10.26 (uniform inf-sup constant). For any $\widetilde{\mathcal{T}} \in \mathbb{T}$ and $\varepsilon \le \varepsilon_0$, let $[\widehat{\mathcal{T}}, \widehat{\mathcal{D}}] = \mathsf{DATA}(\widetilde{\mathcal{T}}, \mathcal{D}, \varepsilon)$, let $\widehat{\mathcal{B}} = \mathcal{B}(\widehat{\mathcal{D}})$ be the associated bilinear form, let $\mathcal{T}$ be either $\widehat{\mathcal{T}}$ or an admissible refinement of $\widehat{\mathcal{T}}$, and finally let $\mathbb{V}_{\mathcal{T}} \subset \mathbb{V}$, $\mathbb{W}_{\mathcal{T}} \subset \mathbb{W}$ be the subspaces built on $\mathcal{T}$ as above. There exists a constant $0 < \bar{\beta} \le \beta$, independent

of $\widetilde{\mathcal{T}}$, ε and $\mathcal{T}$, such that

$$\inf_{w \in \mathbb{W}_{\mathcal{T}}} \sup_{v \in \mathbb{V}_{\mathcal{T}}} \frac{\widehat{\mathcal{B}}[v, w]}{\|v\|_{\mathbb{V}} \|w\|_{\mathbb{W}}} \geq \bar{\beta}. \tag{10.56}$$

The last assumption guarantees the well-posedness of all the discrete variational problems

$$u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}: \quad \widehat{\mathcal{B}}[u_{\mathcal{T}}, w_{\mathcal{T}}] = \widehat{\mathcal{F}}[w_{\mathcal{T}}] \quad \text{for all } w_{\mathcal{T}} \in \mathbb{W}_{\mathcal{T}}, \tag{10.57}$$

associated with the successive refinements of the initial mesh $\mathcal{T}_0$ performed by AFEM-TS.

We want to prove, as in the coercive case (see Proposition 5.27), that the number of iterations performed in any call to GALERKIN inside AFEM-TS (which is finite by Proposition 10.16) is indeed uniformly bounded.

Proposition 10.27 (computational cost of GALERKIN). *Let Assumptions 10.2, 10.3, 10.24, 10.25 and 10.26 be valid. For any $k \in \mathbb{N}$, the number of subiterations J_k inside a call to* GALERKIN *at iteration k of* AFEM-TS *is bounded by a constant J independent of k.*

Proof. Let $\mathcal{T}_{k,j}$ denote the successive refinements of $\widehat{\mathcal{T}}_k$ defined in GALERKIN at iteration k, and let $u_{k,j} \in \mathbb{V}_{k,j} = \mathbb{V}_{\mathcal{T}_{k,j}}$ be the corresponding Galerkin solutions, which are approximations of the solution $\widehat{u}_k \in \mathbb{V}$ of the perturbed problem (10.51) with forms $\widehat{\mathcal{B}} = \widehat{\mathcal{B}}_k = \mathcal{B}(\widehat{\mathcal{D}}_k)$ and $\widehat{f} = \widehat{f}_k = f(\widehat{\mathcal{D}}_k)$. Note also that we use *a posteriori* estimators $\eta_{k,j} = \eta_{k,j}(v)$ defined on $\mathbb{V}_{k,j}$, which depend on $\widehat{\mathcal{D}}_k$ via the coefficients of the equation. However, in reference to Assumptions 10.2 and 10.3, we always suppose the constants in the bounds (10.5) and (10.6) to be independent of k and j.

Let us pick $j := J_k - 1$. By definition of stopping criterion in GALERKIN, and by (10.5) and (10.7), we get

$$\varepsilon_k < \eta_{k,j}(u_{k,j}) \leq \frac{1}{C_L} \|\widehat{u}_k - u_{k,j}\|_{\mathbb{V}} \leq \frac{c}{C_L} \rho^j \|\widehat{u}_k - u_{k,0}\|_{\mathbb{V}}. \tag{10.58}$$

The norm on the right-hand side can be bounded via Corollary 3.3 (quasi-monotonicity), applied to $\mathcal{B} := \widehat{\mathcal{B}}_k$, $u := \widehat{u}_k \in \mathbb{V}$, $u_N := u_{k,0} \in \mathbb{V}_N := \mathbb{V}_{\widehat{\mathcal{T}}_k}$ and $v := u_k \in \mathbb{V}_M := \mathbb{V}_{\mathcal{T}_k} \subseteq \mathbb{V}_{\widehat{\mathcal{T}}_k}$ (the output of GALERKIN at iteration $k-1$). Using Assumptions 10.25 (uniform continuity constant) and 10.26 (uniform inf-sup constant), we thus have

$$\|\widehat{u}_k - u_{k,0}\|_{\mathbb{V}} \leq \lambda \|\widehat{u}_k - u_k\|_{\mathbb{V}}, \tag{10.59}$$

with $\lambda = C_B/\bar{\beta}$.

Finally, we again use the triangle inequality to get

$$\begin{aligned} \|\widehat{u}_k - u_k\|_{\mathbb{V}} &\leq \|\widehat{u}_k - \widehat{u}_{k-1}\|_{\mathbb{V}} + \|\widehat{u}_{k-1} - u_k\|_{\mathbb{V}} \\ &\leq \|u - \widehat{u}_k\|_{\mathbb{V}} + \|u - \widehat{u}_{k-1}\|_{\mathbb{V}} + \|\widehat{u}_{k-1} - u_k\|_{\mathbb{V}}; \end{aligned}$$

then, Assumption 10.24 (perturbation estimate) yields $\|u - \widehat{u}_k\|_{\mathbb{V}} \le C_D C_{\text{data}}\, \omega \varepsilon_k$ and $\|u - \widehat{u}_{k-1}\|_{\mathbb{V}} \le C_D C_{\text{data}}\, \omega \varepsilon_{k-1}$, whereas the termination test for GALERKIN at iteration $k-1$ yields $\|\widehat{u}_{k-1} - u_k\|_{\mathbb{V}} \le C_U\, \varepsilon_{k-1}$. Hence, recalling $\varepsilon_{k-1} = 2\varepsilon_k$ and $\omega \le 1$, we get

$$\|\widehat{u}_k - u_k\|_{\mathbb{V}} \le \sigma\, \varepsilon_k \tag{10.60}$$

with $\sigma = 3C_D C_{\text{data}} + 2C_U$. Finally, concatenating (10.58), (10.59) and (10.60), we obtain

$$\rho^j \ge \frac{C_L}{c\lambda\sigma},$$

which implies

$$J_k \le 1 + \left(\log \frac{C_L}{c\lambda\sigma}\right)(\log \rho)^{-1} =: J. \qquad \square$$

The remainder of this section is devoted to investigating the rate-optimality of AFEMs for inf-sup stable problems. Precisely, we aim at establishing the analogue of bound (6.1) for such problems, that is,

$$\|u - u_{\mathcal{T}}\|_{\mathbb{V}} \le C(u, \mathcal{D})\, (\#\mathcal{T})^{-s}. \tag{10.61}$$

To this end, we have to introduce approximation classes for the solution and the data, and to study the quasi-optimality properties of mesh refinement and GALERKIN.

10.3.4. Nonlinear approximation classes

The definition of the approximation class $\mathbb{A}_s = \mathbb{A}_s(\mathbb{V}; \mathcal{T}_0)$ for functions in $\mathbb{V}$ is identical to that given in Section 6.1.1 for functions in $H^1_0(\Omega)$ (see Definition 6.1), provided the norm $|v|_{H^1_0(\Omega)}$ is replaced by the norm $\|v\|_{\mathbb{V}}$ at all occurrences.

In the rest of the section we will make the following regularity assumption.

Assumption 10.28 (approximability of u). The exact solution $u \in \mathbb{V}$ of problem (10.36) belongs to the approximation class $\mathbb{A}_s(\mathbb{V}; \mathcal{T}_0)$ for some $s = s_u \in (0, n/d]$.

The approximation classes of data $\mathcal{D} \in D(\Omega)$ are defined via discrete approximations $\mathcal{D}_{\mathcal{T}} \in \mathbb{D}_{\mathcal{T}}$ subordinate to a partition $\mathcal{T} \in \mathbb{T}$, which produce the oscillation

$$\operatorname{osc}_{\mathcal{T}}(\mathcal{D}) = \inf_{\mathcal{D}_{\mathcal{T}} \in \mathbb{D}_{\mathcal{T}}} \|\mathcal{D} - \mathcal{D}_{\mathcal{T}}\|_{D(\Omega)}.$$

Definition 10.29 (approximation classes of $\mathcal{D}$). Let $\mathbb{D}_s := \mathbb{D}_s(D(\Omega); \mathcal{T}_0)$ be the set of data $\mathcal{D} \in D(\Omega)$ satisfying

$$|\mathcal{D}|_{\mathbb{D}_s} := \sup_{N \ge \#\mathcal{T}_0} \left(N^s \inf_{\mathcal{T} \in \mathbb{T}_N} \operatorname{osc}_{\mathcal{T}}(\mathcal{D})\right) < \infty \;\Rightarrow\; \inf_{\mathcal{T} \in \mathbb{T}_N} \operatorname{osc}_{\mathcal{T}}(\mathcal{D}) \le |\mathcal{D}|_{\mathbb{D}_s} N^{-s}. \tag{10.62}$$

The following assumptions on the data of our boundary value problem will be valid in the rest of the section.

Assumption 10.30 (approximability of $\mathcal{D}$). The data $\mathcal{D} \in D(\Omega)$ of problem (10.36) belongs to the approximation class $\mathbb{D}_s(D(\Omega); \mathcal{T}_0)$ for some $s = s_{\mathcal{D}} \in (0, n/d]$.

Assumption 10.31 (quasi-optimality of DATA). A call $[\widehat{\mathcal{T}}, \widehat{\mathcal{D}}] = \mathsf{DATA}(\mathcal{T}, \mathcal{D}, \varepsilon)$ marks a set of elements $\mathcal{M}_{\mathcal{D}}$ whose cardinality $N(\mathcal{D}) = \#\mathcal{M}_{\mathcal{D}}$ obeys

$$N(\mathcal{D}) \lesssim |\mathcal{D}|_{\mathbb{D}_s}^{1/s_{\mathcal{D}}} \, \varepsilon^{-1/s_{\mathcal{D}}}. \tag{10.63}$$

The concept of *ε-approximation of order s* of $u \in \mathbb{A}_s(\mathbb{V}; \mathcal{T}_0)$ is identical to the one given in Definition 6.12, and so is the proof of the following result.

Lemma 10.32 (ε-approximation of u of order s). *Let $u \in \mathbb{A}_s(\mathbb{V}; \mathcal{T}_0)$ and $v \in \mathbb{V}$ satisfy $\|u - v\|_{\mathbb{V}} \le \varepsilon$ for some $0 < \varepsilon \le \varepsilon_0$. Then v is a 2ε-approximation of order s to u.*

10.3.5. Rate-optimality of GALERKIN

To estimate the growth of the cardinality of the meshes produced inside a call to GALERKIN, which always deals with discrete data, and to relate it to the approximation class of the exact solution u, we need an additional assumption of the estimators $\eta_{\mathcal{T}}$. Henceforth, for any subset $\mathcal{S} \subset \mathcal{T}$, we define $\eta_{\mathcal{T}}(v, \mathcal{S})$ by

$$\eta_{\mathcal{T}}(v, \mathcal{S})^2 = \sum_{T \in \mathcal{S}} \eta_{\mathcal{T}}(v, T)^2.$$

Assumption 10.33 (discrete reliability of the estimator). There exists a constant $c_2 > 0$ such that for any $\mathcal{T} \in \mathbb{T}$ and any refinement $\mathcal{T}_* \ge \mathcal{T}$, if $\mathcal{R} = \mathcal{R}_{\mathcal{T} \to \mathcal{T}_*} = \mathcal{T} \setminus \mathcal{T}_*$ is the set of refined elements of $\mathcal{T}$, then

$$\|u_{\mathcal{T}_*} - u_{\mathcal{T}}\|_{\mathbb{V}} \le c_2 \eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{R}),$$

where $u_{\mathcal{T}}$ (resp. $u_{\mathcal{T}_*}$) are the Galerkin solutions in $\mathbb{V}_{\mathcal{T}}$ (resp. $\mathbb{V}_{\mathcal{T}_*}$).

We recall that the module MARK in GALERKIN implements Dörfler's strategy, that is, for a fixed $\theta \in (0, 1]$, it identifies a subset $\mathcal{M} \subseteq \mathcal{T}$ of elements undergoing bisection by the condition

$$\eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{M}) \ge \theta \, \eta_{\mathcal{T}}(u_{\mathcal{T}}). \tag{10.64}$$

The following property is the analogue of the one stated in Lemma 6.16 for coercive problems. Since the proof is similar, we omit it.

Lemma 10.34 (Dörfler marking). *Let Assumptions 10.17 and 10.33 be valid. For all $0 < \mu < \frac{1}{2}$ there exists $0 < \theta_0 < 1$ such that, if $\mathcal{T} \in \mathbb{T}$ and $\mathcal{T}_*$ is a refinement of $\mathcal{T}$ with refined set $\mathcal{R} = \mathcal{T} \setminus \mathcal{T}_*$, and if the Galerkin solutions $u_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}$ and $u_{\mathcal{T}_*} \in \mathbb{V}_{\mathcal{T}_*}$ satisfy*

$$\eta_{\mathcal{T}_*}(u_{\mathcal{T}_*}) \le \mu \, \eta_{\mathcal{T}}(u_{\mathcal{T}}),$$

then

$$\theta_0 \, \eta_{\mathcal{T}}(u_{\mathcal{T}}) \le \eta_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{R}).$$

We are ready to investigate the rate-optimality of the kth call to GALERKIN in the two-step AFEM (see Definition 5.1). We let $\mathcal{M}_{k,j} \subseteq \mathcal{T}_{k,j}$ denote the marked set at the jth iteration inside GALERKIN (hereafter we refer to the notation in the proof of Proposition 10.27). To achieve quasi-optimality, the following assumption is fundamental.

Assumption 10.35 (minimality of marked sets). The module MARK selects a set $\mathcal{M}_{k,j}$ with *minimal* cardinality among those satisfying Dörfler's condition

$$\eta_{k,j}(u_{k,j}, \mathcal{M}) \geq \theta\, \eta_{k,j}(u_{k,j}) \quad \text{for all } k, j.$$

Proposition 10.36 (cardinality of marked sets). *Let Assumptions 10.2, 10.24, 10.25, 10.26, 10.28, 10.17, 10.33 and 10.35 hold true. There exists a constant $C_0 > 0$ independent of k and j such that the cardinality $N_{k,j}(u)$ of $\mathcal{M}_{k,j}$ satisfies*

$$N_{k,j}(u) \leq C_0\, |u|_{\mathbb{A}_s}^{1/s} \varepsilon_k^{-1/s} \tag{10.65}$$

and

$$N_{k,j}(u) \leq C_0\, |u|_{\mathbb{A}_s}^{1/s} \|u - u_{k,j}\|_{\mathbb{V}}^{-1/s}. \tag{10.66}$$

Proof. The proof can be easily obtained by slightly adapting to the current abstract setting the proof of Corollary 6.22, also taking into account Proposition 6.18. □

Let $\mathcal{M}_k$ denote the set of marked elements in GALERKIN at iteration k of AFEM. Since the cardinality $N_k(u) = \#\mathcal{M}_k$ of $\mathcal{M}_k$ satisfies $N_k(u) = \sum_{j=0}^{J_k-1} N_{k,j}(u)$, we can estimate its cardinality by combining Propositions 10.27 and 10.36.

Corollary 10.37 (rate-optimality of GALERKIN). *Under the assumptions of Propositions 10.27 and 10.36, the total number of marked elements $\mathcal{M}_k$ in* GALERKIN *at iteration k of* AFEM *satisfies*

$$N_k(u) \leq J\, C_0\, |u|_{\mathbb{A}_s}^{1/s} \varepsilon_k^{-1/s}.$$

10.3.6. Rate-optimality of AFEM-TS

At last, we focus on the two-step AFEM in Definition 5.1 (AFEM-TS), and prove its rate-optimality, in relation to the nonlinear approximation classes of the solution u and the problem data $\mathcal{D}$.

Theorem 10.38 (rate-optimality of AFEM-TS). *Under the same assumptions as Proposition 10.36, plus Assumptions 10.30 and 10.31, there exists a constant C_* independent of u and $\mathcal{D}$ such that the sequence $(\mathcal{T}_k, \mathbb{V}_{\mathcal{T}_k}, u_{\mathcal{T}_k})$, $k \geq 0$, produced by* AFEM-TS *satisfies*

$$\|u - u_{\mathcal{T}_k}\|_{\mathbb{V}} \leq C_* \Big(|u|_{\mathbb{A}_{s_u}}^{1/s_u} + |\mathcal{D}|_{\mathbb{D}_{s_{\mathcal{D}}}}^{1/s_{\mathcal{D}}} \Big)^s (\#\mathcal{T}_k)^{-s},$$

with $0 < s = \min(s_u, s_{\mathcal{D}}) \leq n/d$.

Proof. Let $\mathcal{M}_\ell^u$ (resp. $\mathcal{M}_\ell^{\mathcal{D}}$) denote the set of elements marked by GALERKIN (resp. DATA) at iteration ℓ of AFEM. By Corollary 10.37 and Assumption 10.31, there exist constants D_1, D_2 independent of u, $\mathcal{D}$ and k such that

$$N_\ell(u) \le D_1\, |u|_{\mathbb{A}_{s_u}}^{1/s_u} \varepsilon_\ell^{-1/s_u}, \quad N_\ell(\mathcal{D}) \le D_2\, |\mathcal{D}|_{\mathbb{D}_{s_{\mathcal{D}}}}^{1/s_{\mathcal{D}}} \varepsilon_\ell^{-1/s_{\mathcal{D}}}.$$

Then we conclude as in the proof of Theorem 6.24. □

10.4. The Stokes problem

Here we consider the Stokes problem

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{10.67}$$

already introduced in Section 2.3. Assuming $\boldsymbol{f} \in H^{-1}(\Omega;\mathbb{R}^d) = \mathbb{V}^*$, its weak formulation is given in (2.15) or, equivalently, in (2.16), where the bilinear form $\mathcal{B}$ is continuous and inf-sup stable, as a consequence of Theorem 2.11 (Brezzi); see Section 2.4.

A Galerkin discretization of this problem, based on finite-dimensional subspaces $\mathbb{V}_\mathcal{T} \subset \mathbb{V} = H_0^1(\Omega;\mathbb{R}^d)$ and $\mathbb{Q}_\mathcal{T} \subset \mathbb{Q} = L_0^2(\Omega)$, reads as follows: find $(\boldsymbol{u}_\mathcal{T}, p_\mathcal{T}) \in \mathbb{V}_\mathcal{T} \times \mathbb{Q}_\mathcal{T}$ such that

$$\begin{aligned} a[\boldsymbol{u}_\mathcal{T}, \boldsymbol{v}] + b[p_\mathcal{T}, \boldsymbol{v}] &= \langle \boldsymbol{f}, \boldsymbol{v}\rangle \quad \text{for all } \boldsymbol{v} \in \mathbb{V}_\mathcal{T}, \\ b[q, \boldsymbol{u}_\mathcal{T}] &= 0 \qquad\quad\ \ \text{for all } q \in \mathbb{Q}_\mathcal{T}, \end{aligned} \tag{10.68}$$

or equivalently

$$(\boldsymbol{u}_\mathcal{T}, p_\mathcal{T}) \in \mathbb{V}_\mathcal{T} \times \mathbb{Q}_\mathcal{T}: \quad \mathcal{B}[(\boldsymbol{u}_\mathcal{T}, p_\mathcal{T}), (\boldsymbol{v}, q)] = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \quad \text{for all } (\boldsymbol{v}, q) \in \mathbb{V}_\mathcal{T} \times \mathbb{Q}_\mathcal{T}.$$

We assume that the pair $(\mathbb{V}_\mathcal{T}, \mathbb{Q}_\mathcal{T})$ is *uniformly inf-sup stable* for the form b, that is, there exists a constant $\beta > 0$, independent of the refinement $\mathcal{T}$, such that

$$\inf_{q \in \mathbb{Q}_\mathcal{T}} \sup_{\boldsymbol{v} \in \mathbb{V}_\mathcal{T}} \frac{b[q, \boldsymbol{v}]}{\|\boldsymbol{v}\|_\mathbb{V} \|w\|_\mathbb{W}} \ge \beta. \tag{10.69}$$

This condition is equivalent to the uniform inf-sup stability of the bilinear form $\mathcal{B}$ on the product space $\mathbb{X}_\mathcal{T} := \mathbb{V}_\mathcal{T} \times \mathbb{Q}_\mathcal{T}$. Then, applying a discrete form of Brezzi's theorem, we obtain the existence and uniqueness of the solution of (10.68), which satisfies the stability bound

$$\|\boldsymbol{u}_\mathcal{T}\|_\mathbb{V} + \|p_\mathcal{T}\|_\mathbb{Q} \le C\, \|\boldsymbol{f}\|_{\mathbb{V}^*}, \tag{10.70}$$

where C depends only on the continuity constant $\|a\|$ and the coercivity constant α of the form a, and the inf-sup constant β. Furthermore, we have the quasi-best

approximation bounds (Boffi *et al.* 2013, Proposition 8.2.1)

$$\|\boldsymbol{u} - \boldsymbol{u}_{\mathcal{T}}\|_{\mathbb{V}} \le C_{11} \min_{\boldsymbol{v}\in\mathbb{V}_{\mathcal{T}}} \|\boldsymbol{u} - \boldsymbol{v}\|_{\mathbb{V}} + C_{12} \min_{q\in\mathbb{Q}_{\mathcal{T}}} \|p - q\|_{\mathbb{Q}}, \tag{10.71}$$

$$\|p - p_{\mathcal{T}}\|_{\mathbb{V}} \le C_{21} \min_{\boldsymbol{v}\in\mathbb{V}_{\mathcal{T}}} \|\boldsymbol{u} - \boldsymbol{v}\|_{\mathbb{V}} + C_{22} \min_{q\in\mathbb{Q}_{\mathcal{T}}} \|p - q\|_{\mathbb{Q}}, \tag{10.72}$$

where the constants C_{ij}, $1 \le i, j \le 2$, depend only on the quantities $\|a\|$, $\|b\|$, α and β.

There are many families of finite element spaces that are uniformly inf-sup stable for the Stokes problem; see Boffi *et al.* (2013, Chapter 8). Among them, we consider here the Taylor–Hood element (Taylor and Hood 1973) and its higher-order versions. They all use continuous discrete pressures, so they fit into the general form

$$\mathbb{V}_{\mathcal{T}} = \{\boldsymbol{v} \in H_0^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v}|_T \in \boldsymbol{V}_T,\ T \in \mathcal{T}\},$$
$$\mathbb{Q}_{\mathcal{T}} = \{q \in L_0^2(\Omega) \cap C^0(\bar{\Omega}) \mid q|_T \in Q_T,\ T \in \mathcal{T}\},$$

where $\boldsymbol{V}_T$ and Q_T are spaces of polynomials on the element T. Considering simplicial elements, we have for $n \ge 2$

$$\boldsymbol{V}_T = (\mathbb{P}_n(T))^d, \quad Q_T = \mathbb{P}_{n-1}(T). \tag{10.73}$$

The convergence and optimality of an adaptive algorithm for the Stokes problem based on Taylor–Hood elements was first established by Feischl (2019) (see also Feischl 2022, Section 6). We aim at deriving a similar result using the abstract framework presented in this section.

We start by developing the *a posteriori* error analysis, and for this we introduce the weak residual

$$\langle \boldsymbol{\mathcal{R}}_{\mathcal{T}}, (\boldsymbol{v}, q)\rangle := \langle \boldsymbol{f}, \boldsymbol{v}\rangle - \mathcal{B}[(\boldsymbol{u}_{\mathcal{T}}, p_{\mathcal{T}}), (\boldsymbol{v}, q)] \quad \text{for all } (\boldsymbol{v}, q) \in \mathbb{V}\times\mathbb{Q},$$

which we represent as $\boldsymbol{\mathcal{R}}_{\mathcal{T}} = (\boldsymbol{\mathcal{R}}^m_{\mathcal{T}}, \mathcal{R}^c_{\mathcal{T}})$ according to the momentum and continuity equations; note that $\mathcal{R}^c_{\mathcal{T}} = \operatorname{div} \boldsymbol{u}_{\mathcal{T}}$. The continuity and inf-sup stability properties of the exact Stokes form $\mathcal{B}$ yield the equivalence

$$\|\boldsymbol{u} - \boldsymbol{u}_{\mathcal{T}}\|_{\mathbb{V}} + \|p - p_{\mathcal{T}}\|_{\mathbb{Q}} \approx \|\boldsymbol{\mathcal{R}}_{\mathcal{T}}\|_{\mathbb{V}^*\times\mathbb{Q}^*} \approx \|\boldsymbol{\mathcal{R}}^m_{\mathcal{T}}\|_{\mathbb{V}^*} + \|\operatorname{div} \boldsymbol{u}_{\mathcal{T}}\|_{L^2(\Omega)}. \tag{10.74}$$

We now apply Corollary 4.6 (star localization of residual norm) to each component of the momentum residual $\boldsymbol{\mathcal{R}}^m_{\mathcal{T}}$, to get

$$\|\boldsymbol{\mathcal{R}}^m_{\mathcal{T}}\|^2_{\mathbb{V}^*} \approx \sum_{z\in\mathcal{V}} \|\boldsymbol{\mathcal{R}}^m_{\mathcal{T}}\|^2_{(H^{-1}(\omega_z))^d},$$

whereas Lemma 4.35 (splitting of local residual norm) yields the equivalence

$$\|\boldsymbol{\mathcal{R}}^m_{\mathcal{T}}\|^2_{(H^{-1}(\omega_z))^d} \approx \|P_{\mathcal{T}}\boldsymbol{f} + \Delta\boldsymbol{u}_{\mathcal{T}} - \nabla p_{\mathcal{T}}\|^2_{(H^{-1}(\omega_z))^d} + \|\boldsymbol{f} - P_{\mathcal{T}}\boldsymbol{f}\|^2_{(H^{-1}(\omega_z))^d}.$$

In view of mesh refinement, we recall Lemma 4.8 (localization re-indexing), and we express the error indicator in terms of elements $T \in \mathcal{T}$ rather than stars ω_z, in

analogy with the scalar case (4.52). To this end, define

$$\begin{aligned}\eta_{\mathcal{T}}^m(\boldsymbol{u}_{\mathcal{T}}, p_{\mathcal{T}}, T)^2 &:= h_T^2 \, \|P_T \boldsymbol{f} + \Delta \boldsymbol{u}_{\mathcal{T}} - \nabla p_{\mathcal{T}}\|^2_{(L^2(T))^d} \\ &\quad + h_T \sum_{F \subset \partial T \setminus \partial\Omega} \|[\![(\nabla \boldsymbol{u}_{\mathcal{T}})\boldsymbol{n}_F]\!] - P_F \boldsymbol{f}\|^2_{(L^2(F))^d}, \qquad (10.75)\\ \operatorname{osc}_{\mathcal{T}}(\boldsymbol{f}, T)^2_{-1} &:= \|\boldsymbol{f} - P_{\mathcal{T}} \boldsymbol{f}\|^2_{(H^{-1}(\omega_T))^d}.\end{aligned}$$

Note that the jump term $[\![(\nabla \boldsymbol{u}_{\mathcal{T}})\boldsymbol{n}_F]\!]$ does not contain the pressure contribution, since discrete pressures in $\mathbb{Q}_{\mathcal{T}}$ are globally continuous. We thus have

$$\|\boldsymbol{\mathcal{R}}_{\mathcal{T}}^m\|^2_{\mathbb{V}^*} \approx \sum_{T \in \mathcal{T}} \big(\eta_{\mathcal{T}}^m(\boldsymbol{u}_{\mathcal{T}}, p_{\mathcal{T}}, T)^2 + \operatorname{osc}_{\mathcal{T}}(\boldsymbol{f}, T)^2_{-1}\big). \qquad (10.76)$$

Recalling (10.74), the full local PDE residual indicator could be defined as

$$\eta_{\mathcal{T}}(\boldsymbol{u}_{\mathcal{T}}, p_{\mathcal{T}}, T)^2 := \eta_{\mathcal{T}}^m(\boldsymbol{u}_{\mathcal{T}}, p_{\mathcal{T}}, T)^2 + \|\operatorname{div} \boldsymbol{u}_{\mathcal{T}}\|^2_{L^2(T)}.$$

However, such a quantity is not guaranteed to strictly reduce under mesh refinement, due to the presence of the divergence term, which is not scaled by a positive power of the mesh size. The following result provides an equivalent expression of the last term, which does reduce. We recall the definition (9.1) of jumps across faces.

Lemma 10.39 (norm equivalence for divergence). *We have*

$$\|\operatorname{div} \boldsymbol{u}_{\mathcal{T}}\|^2_{L^2(\Omega)} \approx \sum_{T \in \mathcal{T}} \sum_{F \subset \partial T \setminus \partial\Omega} h_F \, \|[\![\operatorname{div} \boldsymbol{u}_{\mathcal{T}}]\!]\|^2_{L^2(F)}.$$

Proof. The result follows from applying to $\varphi = \operatorname{div} \boldsymbol{u}_{\mathcal{T}}$ the equivalence

$$\|\varphi - \Pi_{\mathcal{T}} \varphi\|^2_{L^2(\Omega)} \approx \sum_{T \in \mathcal{T}} \sum_{F \subset \partial T \setminus \partial\Omega} h_F \, \|[\![\varphi]\!]\|^2_{L^2(F)} \quad \text{for all } \varphi \in \mathbb{S}_{\mathcal{T}}^{n-1,-1}$$

(where $\Pi_{\mathcal{T}}$ is the L^2-orthogonal projection upon $\mathbb{S}_{\mathcal{T}}^{n-1,0}$), after observing that $\Pi_{\mathcal{T}} \varphi = 0$ since $\boldsymbol{u}_{\mathcal{T}}$ is discretely divergence-free, that is, it satisfies the second set of equations in (10.68)). To prove the equivalence for arbitrary $\varphi \in \mathbb{S}_{\mathcal{T}}^{n-1,-1}$, we use the quasi-interpolation operator $\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}}$ introduced in Section 9.1.2, which leaves $\mathbb{S}_{\mathcal{T}}^{n-1,0}$ invariant. Then it is easily seen that

$$\|\varphi - \Pi_{\mathcal{T}} \varphi\|^2_{L^2(\Omega)} \approx \|\varphi - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} \varphi\|^2_{L^2(\Omega)},$$

so it is enough to prove the equivalence with $\Pi_{\mathcal{T}}$ replaced by $\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}}$. But this calculation can be done on patches ω_T since $\mathcal{I}_{\mathcal{T}}^{\mathrm{dG}}$ is quasi-local:

$$\begin{aligned}\|\varphi - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} \varphi\|^2_{L^2(T)} &\lesssim \sum_{F \subset \omega_T} h_F \, \|[\![\varphi]\!]\|^2_{L^2(F)} \\ &\lesssim \|\varphi - \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}} \varphi\|^2_{L^2(\omega_T)} \quad \text{for all } \varphi \in \mathbb{S}_{\mathcal{T}}^{n-1,-1}.\end{aligned}$$

The first inequality follows from (9.10) (see also Bänsch, Morin and Nochetto 2002,

Proposition 5.4). The second inequality results from the fact that if the rightmost term vanishes on ω_T then $\varphi = \mathcal{I}_{\mathcal{T}}^{\mathrm{dG}}\varphi$, whence φ is continuous in ω_T. This yields $[[\varphi]]|_F = 0$ for all internal faces F of ω_T and the middle term vanishes. □

Applying Lemma 10.39, we are led to define the elemental residual indicator

$$\eta_{\mathcal{T}}(\boldsymbol{u}_{\mathcal{T}}, p_{\mathcal{T}}, T)^2 := \eta_{\mathcal{T}}^m(\boldsymbol{u}_{\mathcal{T}}, p_{\mathcal{T}}, T)^2 + h_T \sum_{F \subset \partial T \setminus \partial\Omega} \|[[\operatorname{div} \boldsymbol{u}_{\mathcal{T}}]]\|_{L^2(F)}^2. \tag{10.77}$$

Concatenating (10.74), (10.76) and Lemma 10.39, we fulfil Assumption 10.22 (equivalence of error and full estimator). The precise result is as follows.

Proposition 10.40 (*a posteriori* error analysis for Stokes). *There exist constants $C_U \geq C_L > 0$ such that*

$$C_L \mathcal{E}_{\mathcal{T}}(\boldsymbol{u}_{\mathcal{T}}, p_{\mathcal{T}}, \boldsymbol{f}) \leq \|\boldsymbol{u} - \boldsymbol{u}_{\mathcal{T}}\|_{\mathbb{V}} + \|p - p_{\mathcal{T}}\|_{\mathbb{Q}} \leq C_U \mathcal{E}_{\mathcal{T}}(\boldsymbol{u}_{\mathcal{T}}, p_{\mathcal{T}}, \boldsymbol{f}),$$

where the full estimator is defined by

$$\mathcal{E}_{\mathcal{T}}(\boldsymbol{u}_{\mathcal{T}}, p_{\mathcal{T}}, \boldsymbol{f})^2 := \sum_{T \in \mathcal{T}} \mathcal{E}_{\mathcal{T}}(\boldsymbol{u}_{\mathcal{T}}, p_{\mathcal{T}}, \boldsymbol{f}, T)^2,$$

with $\mathcal{E}_{\mathcal{T}}(\boldsymbol{u}_{\mathcal{T}}, p_{\mathcal{T}}, \boldsymbol{f}, T)^2 := \eta_{\mathcal{T}}(\boldsymbol{u}_{\mathcal{T}}, p_{\mathcal{T}}, T)^2 + \operatorname{osc}_{\mathcal{T}}(\boldsymbol{f}, T)_{-1}^2$ introduced in (10.77) *and* (10.75) *and* $\operatorname{osc}_{\mathcal{T}}(f, T)_{-1} = \|f - P_{\mathcal{T}} f\|_{H^{-1}(\omega_T)}$.

Since the Stokes problem has constant coefficients but variable forcing, it is natural to resort to Algorithm 5.16 (AFEM-SW), the one-step AFEM with switch, for its adaptive discretization. With respect to the functional setting of Section 10.3.2, the ambient space $\mathbb{W}$ is $\mathbb{V} \times \mathbb{Q}$ and the data projection operator $P_{\mathcal{T}}$ is

$$\boldsymbol{P}_{\mathcal{T}} := \left((P_{\mathcal{T}})^d, \Pi_{\mathcal{T}}^{n-1}\right) : \mathbb{W}^* \to (\mathbb{F}_{\mathcal{T}})^d \times \mathbb{S}^{n-1,-1},$$

where $\mathbb{F}_{\mathcal{T}}$ is the scalar discrete space introduced in Definition 4.17, $P_{\mathcal{T}}$ is here the scalar projection operator introduced in Definition 4.24, and $\Pi_{\mathcal{T}}^{n-1}$ is the L^2-orthogonal projection upon $\mathbb{S}^{n-1,-1}$. Furthermore, the norm used to measure data perturbations is

$$\|(\boldsymbol{f}, g)\|_{\mathbb{W}_{\mathcal{T}}^*}^2 = \sum_{T \in \mathcal{T}} \left(\|\boldsymbol{f}\|_{(H^{-1}(\omega_T))^d}^2 + \|g\|_{L^2(T)}^2\right).$$

It is easily seen that $\eta_{\mathcal{T}}(\boldsymbol{u}_{\mathcal{T}}, p_{\mathcal{T}}, T)$ satisfies Assumptions 10.17 (Lipschitz continuity of estimator) and 10.18 (monotonicity of estimator) as well as the hypotheses of Proposition 10.19 (estimator reduction under Dörfler marking): the estimator is clearly Lipschitz-continuous and monotone, and it satisfies condition (10.38) since all its addends appear multiplied by a positive power of the mesh size. In addition, the oscillation $\operatorname{osc}_{\mathcal{T}}((\boldsymbol{f}, g)) = \|(\boldsymbol{f}, g) - \boldsymbol{P}_{\mathcal{T}}(\boldsymbol{f}, g)\|_{\mathbb{W}_{\mathcal{T}}^*}$ fulfils Assumption 10.20 (quasi-monotonicity of oscillation).

Theorem 10.23 provides sufficient conditions for the linear convergence of the algorithm, and these conditions have been verified according to the previous discussion. Therefore we may summarize our findings in the following theorem.

Theorem 10.41 (linear convergence for Stokes). *Consider the Galerkin discretization* (10.68) *of the Stokes problem which uses Taylor–Hood elements of order* $n \geq 2$, *and let the* a posteriori *estimator be given in Proposition 10.40. Then Theorem 10.6 guarantees the linear convergence of Algorithm 5.16 (*AFEM-SW*) applied to this problem, that is, for some* $c > 0$ *and* $0 \leq \rho < 1$,

$$e_{j+i} \leq c\rho^i e_j \quad \text{for all } i, j \in \mathbb{N},$$

with $e_j := \|\nabla(\boldsymbol{u} - \boldsymbol{u}_j)\|_\Omega + \|p - p_j\|_\Omega$.

In order to assess the optimality of the discretization, we specify the definition of approximation classes for the solution of the Stokes problem. Precisely, given $(\boldsymbol{v}, q) \in \mathbb{V} \times \mathbb{Q}$, we let $\sigma_N(\boldsymbol{v}, q)$ be the smallest approximation error incurred on $(\boldsymbol{v}, q)$ with elements in $\mathbb{V}_\mathcal{T} \times \mathbb{Q}_\mathcal{T}$ over meshes belonging to $\mathbb{T}_N$:

$$\sigma_N(\boldsymbol{v}, q) := \inf_{\mathcal{T} \in \mathbb{T}_N} \inf_{(\boldsymbol{v}_\mathcal{T}, q_\mathcal{T}) \in \mathbb{V}_\mathcal{T} \times \mathbb{Q}_\mathcal{T}} \left(\|\nabla(\boldsymbol{v} - \boldsymbol{v}_\mathcal{T})\|_\Omega^2 + \|q - q_\mathcal{T}\|_\Omega^2 \right)^{1/2}. \tag{10.78}$$

For $0 < s \leq n/d$, the class $\mathbb{A}_s = \mathbb{A}_s(\mathbb{V} \times \mathbb{Q}; \mathcal{T}_0)$, relative to the partition $\mathcal{T}_0$ is the set of functions $(\boldsymbol{v}, q) \in \mathbb{V} \times \mathbb{Q}$ such that

$$|(\boldsymbol{v}, q)|_{\mathbb{A}_s} := \sup_{N \geq \#\mathcal{T}_0} (N^s \sigma_N(\boldsymbol{v}, q)) < \infty. \tag{10.79}$$

By adapting the arguments used in the proof of Theorem 6.20 (rate-optimality of one-step AFEMs), we can prove the following result.

Theorem 10.42 (rate-optimality of AFEM-SW for Stokes). *Let the assumptions of Theorem 10.41 be valid. If* $(\boldsymbol{u}, p) \in \mathbb{A}_s$, *then the sequence* $\{\mathcal{T}_k, \mathbb{V}_k, (\boldsymbol{u}_k, p_k)\}_{k \geq 0}$ *generated by* AFEM-SW *satisfies*

$$\|\nabla(\boldsymbol{u} - \boldsymbol{u}_k)\|_{L^2(\Omega)} + \|p - p_k\|_{L^2(\Omega)} \lesssim |(\boldsymbol{u}, p)|_{\mathbb{A}_s} (\#\mathcal{T}_k)^{-s}, \quad k \geq 0. \tag{10.80}$$

Remark 10.43 (limits of the analysis). Other inf-sup stable elements, such as the Mini element or the Crouzeix–Raviart element (see e.g. Boffi *et al.* 2013), do not fit into the present setting of the analysis, since their velocities contain element-wise bubble components (which are indeed responsible for the stability of the elements). Unfortunately, a bubble on an element does not restrict to two bubbles when the element is bisected, preventing the nestedness condition $\mathbb{V}_\mathcal{T} \subset \mathbb{V}_{\mathcal{T}_*}$ from being satisfied when $\mathcal{T}_*$ is a refinement of $\mathcal{T}$.

10.5. Mixed FEMs for scalar elliptic PDEs

The diffusion–reaction problem (2.5) can be formulated in mixed form as follows:

$$\begin{aligned} \boldsymbol{A}^{-1}\boldsymbol{\sigma} &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} \boldsymbol{\sigma} + cu &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{10.81}$$

Introducing the porosity matrix $\boldsymbol{K} := \boldsymbol{A}^{-1}$, we assume hereafter that data

$$\mathcal{D} = (\boldsymbol{K}, c, f) \in D(\Omega) := M(\alpha_1, \alpha_2) \times R(c_1, c_2) \times L^2(\Omega), \tag{10.82}$$

where $M(\alpha_1, \alpha_2)$ and $R(c_1, c_2)$ are defined in (5.48) and (5.49), respectively. Note that the current parameters α_1, α_2 are the reciprocals of α_2, α_1 in (5.48), but to avoid complicating the notation further, we relabel them hereafter.

Weak formulation. To write the weak formulation of these equations, we introduce the Hilbert space

$$H(\mathrm{div}; \Omega) := \{\boldsymbol{\tau} \in L^2(\Omega; \mathbb{R}^d) \mid \mathrm{div}\, \boldsymbol{\tau} \in L^2(\Omega)\} \tag{10.83}$$

equipped with the norm $\|\boldsymbol{\tau}\|^2_{H(\mathrm{div};\Omega)} := \|\boldsymbol{\tau}\|^2_\Omega + \|\mathrm{div}\, \boldsymbol{\tau}\|^2_\Omega$. Then we multiply the first equation in (10.81) by $\boldsymbol{\tau} \in H(\mathrm{div}; \Omega)$ and the second equation by $v \in L^2(\Omega)$, we integrate over Ω and apply partial integration to the term containing ∇u, taking into account the Dirichlet boundary condition. In this way we obtain the following variational problem: find $(\boldsymbol{\sigma}, u) \in \mathbb{V} := H(\mathrm{div}; \Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega \boldsymbol{K}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} + \int_\Omega u\, \mathrm{div}\, \boldsymbol{\tau} &= 0 && \text{for all } \boldsymbol{\tau} \in H(\mathrm{div}; \Omega), \\ \int_\Omega v\, \mathrm{div}\, \boldsymbol{\sigma} - \int_\Omega c\, u\, v &= -\int_\Omega f\, v && \text{for all } v \in L^2(\Omega). \end{aligned} \tag{10.84}$$

This can be written as follows: find $(\boldsymbol{\sigma}, u) \in V \times Q$ such that

$$\begin{aligned} a[\boldsymbol{\sigma}, \boldsymbol{\tau}] + b[u, \boldsymbol{\tau}] &= 0 && \text{for all } \boldsymbol{\sigma} \in V, \\ b[v, \boldsymbol{\sigma}] + m[u, v] &= -\langle f, v\rangle && \text{for all } v \in Q, \end{aligned} \tag{10.85}$$

if we set $V := H(\mathrm{div}; \Omega)$, $Q := L^2(\Omega)$, and we introduce the continuous bilinear forms $a\colon V \times V \to \mathbb{R}$, $b\colon Q \times V \to \mathbb{R}$ and $m\colon Q \times Q \to \mathbb{R}$ by

$$a[\boldsymbol{\sigma}, \boldsymbol{\tau}] = \int_\Omega \boldsymbol{K}\boldsymbol{\sigma} \cdot \boldsymbol{\tau}, \quad b[v, \boldsymbol{\tau}] = \int_\Omega v\, \mathrm{div}\, \boldsymbol{\tau}, \quad m[u, v] = -\int_\Omega c\, u\, v,$$

and the linear form $\langle f, v\rangle = \int_\Omega f v$. An equivalent formulation, similar to (2.16), is as follows:

$$(\boldsymbol{\sigma}, u) \in V \times Q\colon \quad \mathcal{B}[(\boldsymbol{\sigma}, u), (\boldsymbol{\tau}, v)] = -\langle f, v\rangle \quad \text{for all } (\boldsymbol{\tau}, v) \in V \times Q, \tag{10.86}$$

with

$$\mathcal{B}[(\boldsymbol{\sigma}, u), (\boldsymbol{\tau}, v)] := a[\boldsymbol{\sigma}, \boldsymbol{\tau}] + b[u, \boldsymbol{\tau}] + b[v, \boldsymbol{\sigma}] + m[u, v].$$

Formulation (10.85) is a generalization of the classical saddle point problem considered in Section 2.4, given by the presence of the third bilinear form m. According to Boffi *et al.* (2013, Theorem 4.3.1), the well-posedness of such a problem can be derived from the following three conditions:

(i) the form a is coercive on $V_0 = \{\boldsymbol{\tau} \in V \mid b[v, \boldsymbol{\tau}] = 0 \text{ for all } v \in Q\}$,

(ii) the form b satisfies an inf-sup condition on $V \times Q$,

(iii) the form m is non-positive on Q, i.e. $m[v,v] \le 0$ for all $v \in Q$.

These conditions are easily checked for our mixed formulation of the Dirichlet problem.

Discretization. To define a finite element discretization of this problem, we consider partitions $\mathcal{T} \in \mathbb{T}$ obtained by conforming bisection refinements of an initial partition $\mathcal{T}_0$, and let $V_\mathcal{T} \subset V$ and $Q_\mathcal{T} \subset Q$ be finite-dimensional subspaces made of piecewise polynomial functions on $\mathcal{T}$. Among the families of *uniformly inf-sup stable* finite element spaces for this problem, we consider the Raviart–Thomas–Nédélec family (Raviart and Thomas 1977, Nédélec 1980), and the Brezzi–Douglas–Marini family (Brezzi, Douglas Jr and Marini 1985) on simplicial elements. They fit into the general definition

$$V_\mathcal{T} = \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) \mid \boldsymbol{\tau}|_T \in \boldsymbol{V}_T,\ T \in \mathcal{T}\},$$
$$Q_\mathcal{T} = \{q \in L^2(\Omega) \mid q|_T \in Q_T,\ T \in \mathcal{T}\}.$$

For the Raviart–Thomas–Nédélec (RTN) family we have

$$\boldsymbol{V}_T = (\mathbb{P}_{n-1}(T))^d + \boldsymbol{x}\,\mathbb{P}_{n-1}(T), \quad Q_T = \mathbb{P}_{n-1}(T), \quad n \ge 1,$$

where $\boldsymbol{x} = (x_1, \dots, x_d)$ is the coordinate vector, whereas for the Brezzi–Douglas–Marini (BDM) family we have

$$\boldsymbol{V}_T = (\mathbb{P}_n(T))^d, \quad Q_T = \mathbb{P}_{n-1}(T), \quad n \ge 1.$$

Note that for any face F of the triangulation we have $\boldsymbol{\tau}|_F \cdot \boldsymbol{n}_F \in \mathbb{P}_{n-1}(F)$ for the RTN family, and $\boldsymbol{\tau}|_F \cdot \boldsymbol{n}_F \in \mathbb{P}_n(F)$ for the BDM family; furthermore, $\mathrm{div}\,\boldsymbol{V}_T = Q_T$. We refer to Boffi *et al.* (2013, Sections 2.3.1, 7.1.2) for more details.

Due to the presence of variable data, it is natural to perform the adaptive discretization of the problem by adopting Algorithm 5.28 (AFEM-TS), the two-step AFEM. The procedure $[\widehat{\mathcal{T}}, \widehat{\mathcal{D}}] = \mathsf{DATA}(\mathcal{T}, \mathcal{D}, \tau)$ generates an admissible refinement $\widehat{\mathcal{T}}$ of $\mathcal{T}$ and discrete data

$$\widehat{\mathcal{D}} = (\widehat{\boldsymbol{K}}, \widehat{c}, \widehat{f}) \in \mathbb{D}_{\widehat{\mathcal{T}}} := \left[\mathbb{S}^{n-1,-1}_{\widehat{\mathcal{T}}}\right]^{d\times d} \times \mathbb{S}^{n-1,-1}_{\widehat{\mathcal{T}}} \times \mathbb{S}^{n-1,-1}_{\widehat{\mathcal{T}}}$$

over $\widehat{\mathcal{T}}$, such that $\widehat{\boldsymbol{K}} \in M(\widehat{\alpha}_1, \widehat{\alpha}_2)$, $\widehat{c} \in R(\widehat{c}_1, \widehat{c}_2)$ (see Sections 7.2.2 and 7.2.3), and

$$\|\mathcal{D} - \widehat{\mathcal{D}}\|_{\widehat{D}(\Omega)} \le C_{\mathrm{data}}\tau,$$

where the space $\widehat{D}(\Omega)$ is defined in (5.55).

The Galerkin discretization with these discrete data reads: find $(\boldsymbol{\sigma}_\mathcal{T}, u_\mathcal{T}) \in V_\mathcal{T} \times Q_\mathcal{T}$ such that

$$\begin{aligned} \widehat{a}[\boldsymbol{\sigma}_\mathcal{T}, \boldsymbol{\tau}] + b[u_\mathcal{T}, \boldsymbol{\tau}] &= 0 && \text{for all } \boldsymbol{\tau} \in V_\mathcal{T}, \\ b[v, \boldsymbol{\sigma}_\mathcal{T}] + \widehat{m}[u_\mathcal{T}, v] &= -\langle \widehat{f}, v\rangle && \text{for all } v \in Q_\mathcal{T}, \end{aligned} \tag{10.87}$$

with

$$\widehat{a}[\boldsymbol{\sigma},\boldsymbol{\tau}] = \int_\Omega \widehat{\boldsymbol{K}}\boldsymbol{\sigma}\cdot\boldsymbol{\tau}, \quad \widehat{m}[u,v] = -\int_\Omega \widehat{c}\,u\,v, \quad \langle \widehat{f}, v\rangle = \int_\Omega \widehat{f}\,v,$$

or equivalently

$$(\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}) \in V_{\mathcal{T}}\times Q_{\mathcal{T}}: \quad \widehat{\mathcal{B}}[(\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}),(\boldsymbol{\tau}, v)] = -\langle \widehat{f},\, v\rangle \quad \text{for all } (\boldsymbol{\tau}, v) \in V_{\mathcal{T}}\times Q_{\mathcal{T}}.$$

A posteriori *error estimator.* Let $(\widehat{\boldsymbol{\sigma}}, \widehat{u}) \in V \times Q$ denote the exact solution of the perturbed problem

$$\widehat{\mathcal{B}}[(\widehat{\boldsymbol{\sigma}}, \widehat{u}),(\boldsymbol{\tau}, v)] = -\langle \widehat{f},\, v\rangle \quad \text{for all } (\boldsymbol{\tau}, v) \in V \times Q;$$

note that the forcing $\widehat{f}$ appears with a negative sign. Then, by continuity and uniform inf-sup stability of the form $\widehat{\mathcal{B}}$, we know that the error

$$\|\widehat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_{\mathcal{T}}\|_{H(\mathrm{div};\Omega)} + \|\widehat{u} - u_{\mathcal{T}}\|_{L^2(\Omega)}$$

is equivalent to the quantity

$$\sup_{(\boldsymbol{\tau},v)\in V\times Q} \frac{\widehat{\mathcal{B}}[(\widehat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_{\mathcal{T}}, \widehat{u} - u_{\mathcal{T}}),(\boldsymbol{\tau}, v)]}{\|\boldsymbol{\tau}\|_{H(\mathrm{div};\Omega)} + \|v\|_{L^2(\Omega)}} = \sup_{(\boldsymbol{\tau},v)\in V\times Q} \frac{\langle \widehat{f}, v\rangle + \widehat{\mathcal{B}}[(\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}),(\boldsymbol{\tau}, v)]}{\|\boldsymbol{\tau}\|_{H(\mathrm{div};\Omega)} + \|v\|_{L^2(\Omega)}}.$$

By Galerkin orthogonality, the numerator is equal to

$$\langle \widehat{f}, v - v_{\mathcal{T}}\rangle + \widehat{\mathcal{B}}[(\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}),(\boldsymbol{\tau} - \boldsymbol{\tau}_{\mathcal{T}}, v - v_{\mathcal{T}})] \quad \text{for all } (\boldsymbol{\tau}_{\mathcal{T}}, v_{\mathcal{T}}) \in V_{\mathcal{T}} \times Q_{\mathcal{T}},$$

which we now proceed to estimate. The term

$$\widehat{\mathcal{B}}[(\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}),(\boldsymbol{\tau} - \boldsymbol{\tau}_{\mathcal{T}}, 0)]$$

can be analysed as in Carstensen (1997) (see also Verfürth 2013, Section 4.8), by resorting to a stable decomposition of $H(\mathrm{div};\Omega)$: precisely, given $\boldsymbol{\tau} \in H(\mathrm{div};\Omega)$, there exist $\boldsymbol{\Phi} \in (H^1(\Omega))^d$ and $\boldsymbol{u} \in (H^1(\Omega))^d$ such that

$$\boldsymbol{\tau} = \boldsymbol{\Phi} + \mathrm{curl}\ \boldsymbol{u} \tag{10.88}$$

with $\|\boldsymbol{\Phi}\|_{(H^1(\Omega))^d} + \|\boldsymbol{u}\|_{(H^1(\Omega))^d} \lesssim \|\boldsymbol{\tau}\|_{H(\mathrm{div};\Omega)}$ (see Xu, Chen and Nochetto 2009, Section 5.1.3). Note that if Ω is convex, then (10.88) is the Helmholtz decomposition of $\boldsymbol{\tau}$, with $\boldsymbol{\Phi} = \nabla G$ for some $G \in (H^2(\Omega))^d$. Using (10.88) and a suitable choice of $\boldsymbol{\tau}_{\mathcal{T}}$, one can show that

$$|\widehat{\mathcal{B}}[(\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}),(\boldsymbol{\tau} - \boldsymbol{\tau}_{\mathcal{T}}, 0)]| \lesssim \eta_{\mathcal{T},1}((\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}))\, \|\boldsymbol{\tau}\|_{H(\mathrm{div};\Omega)}, \tag{10.89}$$

with

$$\eta_{\mathcal{T},1}((\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}))^2 = \sum_{T\in\mathcal{T}} \eta_{\mathcal{T},1}((\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}), T)^2$$

and

$$\begin{aligned}
&\eta_{\mathcal{T},1}((\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}), T)^2 \\
&\quad := h_T^2 \|\widehat{\boldsymbol{K}}\boldsymbol{\sigma}_{\mathcal{T}} - \nabla u_{\mathcal{T}}\|^2_{L^2(T)} + h_T^2 \|\operatorname{curl}(\widehat{\boldsymbol{K}}\boldsymbol{\sigma}_{\mathcal{T}})\|^2_{L^2(T)} \\
&\qquad + h_T \sum_{F \subset \partial T \setminus \partial\Omega} \|[\![(\widehat{\boldsymbol{K}}\boldsymbol{\sigma}_{\mathcal{T}})_t]\!]\|^2_{L^2(F)} + h_T \sum_{F \subset \partial T \cap \partial\Omega} \|(\widehat{\boldsymbol{K}}\boldsymbol{\sigma}_{\mathcal{T}})_t\|^2_{L^2(F)},
\end{aligned} \tag{10.90}$$

where $\boldsymbol{\phi}_t = \boldsymbol{\phi} - (\boldsymbol{\phi} \cdot \boldsymbol{n}_F)\boldsymbol{n}_F$ denotes the tangential component of the vector field $\boldsymbol{\phi}$ on F. On the other hand, the term

$$\langle \widehat{f}, v - v_{\mathcal{T}} \rangle + \widehat{\mathcal{B}}[(\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}), (0, v - v_{\mathcal{T}})] = \int_\Omega (\widehat{f} + \operatorname{div} \boldsymbol{\sigma}_{\mathcal{T}} - \widehat{c}\, u_{\mathcal{T}})(v - v_{\mathcal{T}})$$

can be bounded as follows. For any $T \in \mathcal{T}$, let $\Pi_T = \Pi_T^{n-1}$ be the L^2-orthogonal projection upon $Q_T = \mathbb{P}_{n-1}(T)$, and let us choose $(v_{\mathcal{T}})|_T = \Pi_T v$. Then, noticing that $\widehat{f} + \operatorname{div} \boldsymbol{\sigma}_{\mathcal{T}} \in Q_T$, we have

$$\begin{aligned}
\int_\Omega (\widehat{f} + \operatorname{div} \boldsymbol{\sigma}_{\mathcal{T}} - \widehat{c}\, u_{\mathcal{T}})(v - v_{\mathcal{T}}) &= \sum_{T \in \mathcal{T}} \int_T (\widehat{f} + \operatorname{div} \boldsymbol{\sigma}_{\mathcal{T}} - \Pi_T(\widehat{c}\, u_{\mathcal{T}}))(v - \Pi_T v) \\
&\quad - \sum_{T \in \mathcal{T}} \int_T (\widehat{c}\, u_{\mathcal{T}} - \Pi_T(\widehat{c}\, u_{\mathcal{T}}))(v - \Pi_T v) \\
&= - \sum_{T \in \mathcal{T}} \int_T (\widehat{c}\, u_{\mathcal{T}} - \Pi_T(\widehat{c}\, u_{\mathcal{T}}))v,
\end{aligned} \tag{10.91}$$

whence

$$|\langle \widehat{f}, v - v_{\mathcal{T}} \rangle + \widehat{\mathcal{B}}[(\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}), (0, v - v_{\mathcal{T}})]| \le \sum_{T \in \mathcal{T}} \|\widehat{c}\, u_{\mathcal{T}} - \Pi_T(\widehat{c}\, u_{\mathcal{T}})\|_{L^2(T)} \|v\|_{L^2(T)}.$$

Conversely, it is easily checked that (10.91) implies the bound

$$\|\widehat{c}\, u_{\mathcal{T}} - \Pi_T(\widehat{c}\, u_{\mathcal{T}})\|_{L^2(T)} \lesssim \|\operatorname{div} \boldsymbol{\sigma} - \operatorname{div} \boldsymbol{\sigma}_{\mathcal{T}}\|_{L^2(T)} + \|\widehat{c}\|_{L^\infty(T)} \|\widehat{u} - u_{\mathcal{T}}\|_{L^2(T)}.$$

The choice $n = 1$ yields $\widehat{c}\, u_{\mathcal{T}} \in \mathbb{P}_0(T)$, hence $\widehat{c}\, u_{\mathcal{T}} - \Pi_T(\widehat{c}\, u_{\mathcal{T}}) = 0$. For $n \ge 2$, we could define as a (squared) local error indicator the quantity

$$\eta_{\mathcal{T},1}((\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}), T)^2 + \|\widehat{c}\, u_{\mathcal{T}} - \Pi_T(\widehat{c}\, u_{\mathcal{T}})\|^2_{L^2(T)},$$

but the second addend is not guaranteed to reduce under refinement, since it is not scaled by a positive power of the mesh size. However, there is an equivalent quantity which does reduce, as stated in the following result.

Lemma 10.44 (equivalence of local error indicators). *Assume $\widehat{c}, u_{\mathcal{T}} \in \mathbb{P}_{n-1}(T)$, for $n \ge 2$. Let Π_T^j be the L^2-orthogonal projection upon $\mathbb{P}_j(T)$. Then*

$$\|\widehat{c}\, u_{\mathcal{T}} - \Pi_T(\widehat{c}\, u_{\mathcal{T}})\|_{L^2(T)} \approx h_T \sum_{j=1}^{n} \|u_{\mathcal{T}} - \Pi_T^{n-1-j} u_{\mathcal{T}}\|_{L^2(T)} \|\nabla \widehat{c} - \Pi_T^{j-2} \nabla \widehat{c}\|_{L^\infty(T)}, \tag{10.92}$$

where the constants hidden in the symbol $\approx$ are independent of $\widehat{c}$, $u_{\mathcal{T}}$ and T.

Proof. By the Bramble–Hilbert theorem,

$$\|\widehat{c}\,u_{\mathcal{T}} - \Pi_T(\widehat{c}\,u_{\mathcal{T}})\|_{L^2(T)} \lesssim h_T^n |\widehat{c}\,u_{\mathcal{T}}|_{H^n(T)}$$

and

$$|\widehat{c}\,u_{\mathcal{T}}|_{H^n(T)} \lesssim \sum_{j=1}^{n-1} |\widehat{c}|_{W^j_\infty(T)} |u_{\mathcal{T}}|_{H^{n-j}(T)}.$$

Moreover,

$$|\widehat{c}|_{W^j_\infty(T)} = |\nabla\widehat{c}|_{W^{j-1}_\infty(T)} = |\nabla\widehat{c} - \Pi_T^{j-2}\nabla\widehat{c}|_{W^{j-1}_\infty(T)}$$

and

$$|u_{\mathcal{T}}|_{H^{n-j}(T)} = |u_{\mathcal{T}} - \Pi_T^{n-1-j} u_{\mathcal{T}}|_{H^{n-j}(T)}.$$

Applying inverse estimates for seminorms, we obtain the $\lesssim$ inequality in (10.92).

To get the opposite inequality, it is enough to check that the vanishing of the left-hand side implies the vanishing of the right-hand side, since both quantities are defined on finite-dimensional spaces and their scaling with respect to the element size is the same. Now, $\widehat{c}\,u_{\mathcal{T}} = \Pi_T(\widehat{c}\,u_{\mathcal{T}})$ implies $\widehat{c}\,u_{\mathcal{T}} \in \mathbb{P}_{n-1}(T)$. Let us assume that $u_{\mathcal{T}} \in \mathbb{P}_{n-1-k}(T)$ for some $0 \le k \le n-1$, and consequently $\widehat{c} \in \mathbb{P}_k(T)$, i.e. $\nabla\widehat{c} \in \mathbb{P}_{k-1}(T)$. Then $\Pi_T^{n-1-j} u_{\mathcal{T}} = u_{\mathcal{T}}$ for any $j \le k$, and hence the corresponding differences in the summation on the right-hand side of (10.92) vanish. Conversely, for $j > k$ we have $j - 2 \ge k - 1$, which implies $\Pi_T^{j-2}\nabla\widehat{c} = \nabla\widehat{c}$, that is, the corresponding differences in the summation on the right-hand side vanish. In conclusion, all terms in the summation in (10.92) vanish, and the thesis is proved. □

Summarizing, we have obtained the following result.

Proposition 10.45 (*a posteriori* error estimator for mixed methods). *For every $T \in \mathcal{T}$, the local quantity*

$$\begin{aligned}
&\eta_{\mathcal{T}}((\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}}), T)^2 \\
&\quad := h_T^2 \|\widehat{\boldsymbol{K}}\boldsymbol{\sigma}_{\mathcal{T}} - \nabla u_{\mathcal{T}}\|^2_{L^2(T)} + h_T^2 \|\operatorname{curl}(\widehat{\boldsymbol{K}}\boldsymbol{\sigma}_{\mathcal{T}})\|^2_{L^2(T)} \\
&\qquad + h_T \sum_{F \subset \partial T \setminus \partial\Omega} \|[(\widehat{\boldsymbol{K}}\boldsymbol{\sigma}_{\mathcal{T}})_t]\|^2_{L^2(F)} + h_T \sum_{F \subset \partial T \cap \partial\Omega} \|(\widehat{\boldsymbol{K}}\boldsymbol{\sigma}_{\mathcal{T}})_t\|^2_{L^2(F)} \\
&\qquad + h_T^2 \sum_{j=1}^{n} \|u_{\mathcal{T}} - \Pi_T^{n-1-j} u_{\mathcal{T}}\|^2_{L^2(T)} \|\nabla\widehat{c} - \Pi_T^{j-2}\nabla\widehat{c}\|^2_{L^\infty(T)}
\end{aligned} \tag{10.93}$$

is a (squared) a posteriori *error indicator for the mixed problem* (10.87)*, which gives rise to a global* a posteriori *error estimator $\eta_{\mathcal{T}}(\boldsymbol{\sigma}_{\mathcal{T}}, u_{\mathcal{T}})$ that satisfies Assumption 10.2.*

Finally, Assumption 10.3 follows from Proposition 10.19 (estimator reduction under Dörfler marking), since the estimator is clearly Lipschitz-continuous and monotone, and it satisfies condition (10.38) since all its addends are scaled by positive powers of the mesh size.

As a consequence, the GALERKIN step in AFEM-TS converges linearly by Theorem 10.6, and the number of sub-iterations in the kth call to GALERKIN is bounded by a constant J independent of k (Proposition 10.27). Furthermore, Theorem 10.38 guarantees the quasi-optimality of the two-step AFEM.

Theorem 10.46 (quasi-optimality of AFEM-TS for mixed methods). *Let the exact solution $(\boldsymbol{\sigma}, u)$ of the mixed problem (10.84) belong to the approximation class $\mathbb{A}_{s_u}(V; \mathcal{T}_0)$, and let the data $(\boldsymbol{K}, c, f)$ belong to the approximation class $\mathbb{D}_{s_\mathcal{D}}(D(\Omega); \mathcal{T}_0)$. Let Assumptions 6.14 (marking parameter), 6.21 (size of ω) and 6.19 (initial labelling) be valid. Consider the Galerkin discretization (10.87) based on one of the Raviart–Thomas–Nédélec or Brezzi–Douglas–Marini finite element pairs. There exists a constant C_* independent of the exact solution $(\boldsymbol{\sigma}, u)$ and the data $\mathcal{D} = (\boldsymbol{K}, c, f)$ such that the sequence $\{(\mathcal{T}_k, \mathbb{V}_{\mathcal{T}_k} \times \mathbb{Q}_{\mathcal{T}_k}, (\boldsymbol{\sigma}_{\mathcal{T}_k}, u_{\mathcal{T}_k}))\}_{k\geq 0}$ produced by* AFEM-TS *satisfies for $k \geq 0$*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathcal{T}_k}\|_{H(\mathrm{div};\Omega)} + \|u - u_{\mathcal{T}_k}\|_{L^2(\Omega)} \leq C_* \left(|(\boldsymbol{\sigma}, u)|_{\mathbb{A}_{s_u}}^{1/s_u} + |\mathcal{D}|_{\mathbb{D}_{s_\mathcal{D}}}^{1/s_\mathcal{D}} \right)^s (\#\mathcal{T}_k)^{-s},$$

with $0 < s = \min(s_u, s_\mathcal{D}) \leq n/d$.

Remark 10.47. Another family of uniformly inf-sup stable spaces is that of Brezzi, Douglas Jr, Fortin and Marini (1987), where $\mathbb{V}_T = \{\boldsymbol{\tau} \in (\mathbb{P}_n(T))^d \mid \boldsymbol{\tau} \cdot \boldsymbol{n}_F \in \mathbb{P}_{n-1}(F) \text{ for all } F \in \mathcal{F} \cap \partial T\}$ and $Q_T = \mathbb{P}_n(T)$, $n \geq 1$. However, the imposed condition on the normal component of vector fields on each face of $\mathcal{T}$ prevents the inclusion of $\mathbb{V}_\mathcal{T}$ into $\mathbb{V}_{\mathcal{T}_*}$ from holding if $\mathcal{T}_*$ is a bisection refinement of $\mathcal{T}$.

10.6. Proof of Theorem 10.15

This section is devoted to establishing Theorem 10.15, which in turn contributes with Corollary 10.14 to the proof of Theorem 10.6.

It is important to notice that the growth of $\|\boldsymbol{U}\|_2$ is dictated by the number of blocks N rather than the actual dimension $n_N \gg N$ of $\boldsymbol{U}$. Therefore we again use the block notation from Section 10.2,

$$\boldsymbol{B} = (\boldsymbol{B}(i,j))_{i,j=0}^N \in \mathbb{R}^{n_N \times n_N},$$

with lower and upper triangular factors

$$\boldsymbol{L} = (\boldsymbol{L}(i,j))_{i,j=0}^N \in \mathbb{R}^{n_N \times n_N}, \quad \boldsymbol{U} = (\boldsymbol{U}(i,j))_{i,j=0}^N \in \mathbb{R}^{n_N \times n_N}.$$

We also set

$$\boldsymbol{A} = (\boldsymbol{A}(i,j))_{i,j=0}^N := \sqrt{\alpha}\boldsymbol{B}$$

for a suitable parameter $\alpha > 0$ defined below.

10.6.1. Representation of block inverse matrices
We first show that it suffices to derive the estimates

$$\|\boldsymbol{U}^{-1}\|_2 \lesssim N^{-1/p}, \quad p > 2, \tag{10.94}$$

$$\|\tilde{\boldsymbol{U}}^{-1}\|_2 \lesssim N^{-1/p}, \quad p > 2, \tag{10.95}$$

$$\|\boldsymbol{D}\|_2 \lesssim 1, \tag{10.96}$$

where $\boldsymbol{B}^\top = \tilde{\boldsymbol{L}}\tilde{\boldsymbol{U}}$ is the normalized block triangular decomposition of $\boldsymbol{B}^\top$, and $\boldsymbol{D} \in \mathbb{R}^{n_N \times n_N}$ stands for the block diagonal part of $\boldsymbol{U}$,

In fact, in view of property (P1) (continuity of $\mathcal{B}$), we see that

$$\|\boldsymbol{B}\|_2 = \|\boldsymbol{B}^\top\|_2 \le \|\mathcal{B}\|,$$

whence

$$\boldsymbol{L} = \boldsymbol{B}\boldsymbol{U}^{-1} \Rightarrow \|\boldsymbol{L}\|_2 \le \|\boldsymbol{B}\|_2\|\boldsymbol{U}^{-1}\|_2 \le \|\mathcal{B}\|\|\boldsymbol{U}^{-1}\|_2$$

and, similarly,

$$\|\tilde{\boldsymbol{L}}\|_2 \le \|\mathcal{B}\|\|\tilde{\boldsymbol{U}}^{-1}\|_2.$$

On the other hand, from

$$\boldsymbol{B} = \boldsymbol{L}\boldsymbol{D}(\boldsymbol{D}^{-1}\boldsymbol{U}) \quad \Rightarrow \quad \boldsymbol{B}^\top = (\boldsymbol{U}^\top\boldsymbol{D}^{-1})\boldsymbol{D}\boldsymbol{L}^\top$$

we infer that

$$\begin{aligned} \tilde{\boldsymbol{L}} = \boldsymbol{U}^\top\boldsymbol{D}^{-1} \quad &\Rightarrow \quad \boldsymbol{U} = \boldsymbol{D}\tilde{\boldsymbol{L}}^\top, \\ \tilde{\boldsymbol{U}} = \boldsymbol{D}\boldsymbol{L}^\top \quad &\Rightarrow \quad \boldsymbol{L}^{-1} = \boldsymbol{D}\tilde{\boldsymbol{U}}^{-\top}, \end{aligned} \tag{10.97}$$

which implies

$$\begin{aligned} \|\boldsymbol{U}\|_2 &\le \|\boldsymbol{D}\|_2\|\tilde{\boldsymbol{L}}^\top\|_2, \\ \|\boldsymbol{L}^{-1}\|_2 &\le \|\boldsymbol{D}\|_2\|\tilde{\boldsymbol{U}}^{-\top}\|_2. \end{aligned}$$

Therefore we can focus on proving (10.94) and (10.96), since the proof of (10.95) is identical to that of (10.94). We proceed in several steps. The most delicate estimate is (10.94).

$\boxed{1}$ *jth column of $\boldsymbol{U}^{-1}$.* To prove (10.94), it turns out to be convenient to first get an explicit expression for the jth column of $\boldsymbol{U}^{-1}$. We achieve this next.

Lemma 10.48 (representation of the jth column of $\boldsymbol{U}^{-1}$). *We have*

$$\boldsymbol{U}^{-1}(i,j) = \boldsymbol{B}[j]^{-1}(i,j) \quad \textit{for all } 0 \le i \le j \le N. \tag{10.98}$$

Proof. We compute the (i,j) block of $\boldsymbol{B}[j]^{-1} = \boldsymbol{U}[j]^{-1}\boldsymbol{L}[j]^{-1}$,

$$\boldsymbol{B}[j]^{-1}(i,j) = \sum_{k=0}^{j} \boldsymbol{U}^{-1}[j](i,k)\boldsymbol{L}^{-1}[j](k,j) = \boldsymbol{U}[j]^{-1}(i,j),$$

because $\boldsymbol{L}^{-1}(k,j) = \mathbf{0}$ for $k < j$ and $\boldsymbol{L}^{-1}(j,j) = \boldsymbol{I}(j,j)$. Moreover, we claim that

$$\boldsymbol{U}^{-1}[j](i,j) = \boldsymbol{U}^{-1}(i,j), \quad i \le j,$$

because $\boldsymbol{U}^{-1}$ is block upper triangular. To see this, let

$$\boldsymbol{x}(:,j) = \boldsymbol{U}^{-1}(:,j) \in \mathbb{R}^{n_N \times d_j}$$

be the jth block column of $\boldsymbol{U}^{-1}$, which satisfies

$$\boldsymbol{U}\boldsymbol{x}(:,j) = \boldsymbol{I}(:,j) \in \mathbb{R}^{n_N \times d_j}.$$

Since $\boldsymbol{I}(i,j) = \mathbf{0}$ for $i > j$ and $\boldsymbol{U}$ is block upper triangular, we have $\boldsymbol{x}(i,j) = \mathbf{0}$ for $i > j$. Therefore the matrix

$$\tilde{\boldsymbol{x}}(:,j) = (\boldsymbol{x}(i,j))_{i=0}^{j} \in \mathbb{R}^{n_j \times d_j},$$

with the first j blocks of $\boldsymbol{x}(:,j)$, satisfies the reduced system

$$\boldsymbol{U}[j](j,j)\tilde{\boldsymbol{x}}(j,j) = \boldsymbol{I}(j,j),$$
$$\sum_{k=i}^{j} \boldsymbol{U}[j](i,k)\tilde{\boldsymbol{x}}(k,j) = \mathbf{0}, \quad 0 \le i \le j.$$

We thus deduce that $\tilde{\boldsymbol{x}}(:,j) = \boldsymbol{U}[j]^{-1}(:,j)$, as asserted. □

This lemma justifies dealing with $\boldsymbol{B}[j]^{-1}$.

$\boxed{2}$ *Representation of* $\boldsymbol{B}[j]^{-1}$. We resort to the Neumann series expansion. We first consider the uniform SPD matrix

$$\boldsymbol{B}[j]\boldsymbol{B}[j]^{\top} \in \mathbb{R}^{n_j \times n_j},$$

for which there exists $\alpha > 0$ such that

$$\|\boldsymbol{I}[j] - \alpha \boldsymbol{B}[j]\boldsymbol{B}[j]^{\top}\|_2 < 1$$

uniformly in j. In fact, note that for $\boldsymbol{x} \in \mathbb{R}^{n_j \times n_j}$

$$\|\boldsymbol{x} - \alpha \boldsymbol{B}[j]\boldsymbol{B}[j]^{\top}\boldsymbol{x}\|_2^2 = \|\boldsymbol{x}\|_2^2 - 2\alpha\langle \boldsymbol{x},\, \boldsymbol{B}[j]\boldsymbol{B}[j]^{\top}\boldsymbol{x}\rangle + \alpha^2\|\boldsymbol{B}[j]\boldsymbol{B}[j]^{\top}\boldsymbol{x}\|_2^2,$$

as well as

$$\langle \boldsymbol{x},\, \boldsymbol{B}[j]\boldsymbol{B}[j]^{\top}\boldsymbol{x}\rangle = \|\boldsymbol{B}[j]^{\top}\boldsymbol{x}\|_2^2 \ge \beta^2\|\boldsymbol{x}\|_2^2$$

in view of property (P2) (discrete inf-sup) and (3.2), and

$$\|\boldsymbol{B}[j]\boldsymbol{B}[j]^{\top}\boldsymbol{x}\|_2 \le \|\boldsymbol{B}[j]\|_2^2\|\boldsymbol{x}\|_2 \le \|\mathcal{B}\|^2\|\boldsymbol{x}\|_2.$$

Consequently

$$\|\boldsymbol{x} - \alpha \boldsymbol{B}[j]\boldsymbol{B}[j]^{\top}\boldsymbol{x}\|_2^2 \le (1 - 2\alpha\beta^2 + \alpha^2\|\mathcal{B}\|^4)\|\boldsymbol{x}\|_2^2.$$

The quadratic polynomial in α on the right-hand side is minimized by $\alpha = \beta^2/\|\mathcal{B}\|^4$, and gives

$$\|\boldsymbol{I}[j] - \alpha \boldsymbol{B}[j]\boldsymbol{B}[j]^\top\|_2^2 \le 1 - \frac{\beta^4}{\|\mathcal{B}\|^4} =: \rho^2. \tag{10.99}$$

From now on, we fix this value of α and assume the uniform bound (10.99). Let

$$\boldsymbol{A}[j] = \sqrt{\alpha}\boldsymbol{B}[j] \in \mathbb{R}^{n_j \times n_j}$$

be the jth principal section of the matrix $\boldsymbol{A}$ introduced previously, and let

$$\boldsymbol{G}[j] := \boldsymbol{I}[j] - \boldsymbol{A}[j]\boldsymbol{A}[j]^\top \in \mathbb{R}^{n_j \times n_j}. \tag{10.100}$$

Lemma 10.49 (representation of $\boldsymbol{B}[j]^{-1}$). *The following expression is valid:*

$$\boldsymbol{B}[j]^{-1} = \alpha \boldsymbol{B}[j]^\top \sum_{m=0}^{\infty} \boldsymbol{G}[j]^m \quad \textit{for all } 0 \le j \le N. \tag{10.101}$$

Proof. Since $\|\boldsymbol{G}[j]\|_2 \le \rho < 1$ according to (10.99), the Neumann series theorem guarantees that

$$\boldsymbol{A}[j]\boldsymbol{A}[j]^\top = \boldsymbol{I}[j] - \boldsymbol{G}[j]$$

is invertible, and the inverse reads

$$\boldsymbol{A}[j]^{-\top}\boldsymbol{A}[j]^{-1} = \sum_{m=0}^{\infty} \boldsymbol{G}[j]^m,$$

where $\boldsymbol{G}[j]^0 = \boldsymbol{I}[j]$. Multiplying on the left by $\boldsymbol{A}[j]^\top$, we obtain

$$\frac{1}{\sqrt{\alpha}}\boldsymbol{B}[j]^{-1} = \sqrt{\alpha}\boldsymbol{B}[j] \sum_{m=0}^{\infty} \boldsymbol{G}[j]^m,$$

which yields the assertion. □

3 *Representation of $\boldsymbol{U}^{-1}$.* In order to obtain a representation of $\boldsymbol{U}^{-1}$, we now build on (10.98), which gives a formula for the jth column of $\boldsymbol{U}^{-1}$ in terms of $\boldsymbol{B}[j]^{-1}$, and (10.101), which provides a series representation of $\boldsymbol{B}[j]^{-1}$. To this end, we introduce the block upper triangular matrix $\boldsymbol{G}_m \in \mathbb{R}^{n_N \times n_N}$ given by

$$\boldsymbol{G}_m(i,j) := \begin{cases} \boldsymbol{G}[j]^m(i,j), & i \le j, \\ 0, & i > j, \end{cases}$$

for $m \ge 1$ and $\boldsymbol{G}_0 = \boldsymbol{I}$. Hence

$$\boldsymbol{U}^{-1}(i,j) = \begin{cases} \alpha\left(\boldsymbol{B}[j]^\top \sum_{m=0}^{\infty} \boldsymbol{G}_m[j]\right)(i,j), & i \le j, \\ 0, & i > j. \end{cases}$$

To write this expression in compact form, it is convenient to introduce the *block upper triangular truncation operator* $\mathcal{U}\colon \mathbb{R}^{n_N\times n_N} \to \mathbb{R}^{n_N\times n_N}$ defined by

$$\mathcal{U}(\boldsymbol{M})(i,j) := \begin{cases} \boldsymbol{M}(i,j), & i \le j, \\ 0, & i > j, \end{cases} \quad \text{for all } \boldsymbol{M} \in \mathbb{R}^{n_N\times n_N}.$$

Lemma 10.50 (representation of U^{-1}). *We have*

$$\boldsymbol{U}^{-1} = \alpha\, \mathcal{U}\Big(\boldsymbol{B}^\top \sum_{m=0}^{\infty} \boldsymbol{G}_m\Big). \tag{10.102}$$

Proof. Since $\boldsymbol{G}_m$ is block upper triangular for all $m \ge 0$, so is the series $\sum_{m=0}^{\infty} \boldsymbol{G}_m$. It thus suffices to check that

$$\Big(\boldsymbol{B}^\top \sum_{m=0}^{\infty} \boldsymbol{G}_m\Big)(i,j) = \Big(\boldsymbol{B}[j]^\top \sum_{m=0}^{\infty} \boldsymbol{G}_m[j]\Big)(i,j), \quad i \le j.$$

This shows the desired relation (10.102). □

$\boxed{4}$ *Recursion.* In order to estimate $\boldsymbol{G}_m$, it is useful to relate $\boldsymbol{G}_m$ to $\boldsymbol{G}_{m-1}$. We start with a simple property of the operator $\mathcal{U}$: for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ and $1 \le i \le j \le n$, we have

$$(\boldsymbol{A}\,\mathcal{U}(\boldsymbol{B}))_{ij} = \sum_{k=1}^{n} \boldsymbol{A}_{ik}\,\mathcal{U}(\boldsymbol{B})_{kj} = \sum_{k=1}^{j} \boldsymbol{A}_{ik}\,\boldsymbol{B}_{kj} = (\boldsymbol{A}[j]\boldsymbol{B}[j])(i,j).$$

Lemma 10.51 (recursion). *The following is valid for all $m \ge 1$:*

$$\boldsymbol{G}_m = \boldsymbol{G}_{m-1} - \mathcal{U}(\boldsymbol{A}\,\mathcal{U}(\boldsymbol{A}^\top \boldsymbol{G}_{m-1})), \tag{10.103}$$

with $\boldsymbol{G}_0 = \boldsymbol{I}$. Therefore the jth column of $\boldsymbol{G}_m$ reads

$$\boldsymbol{G}_m(0\colon j, j) = \boldsymbol{G}[j]\boldsymbol{G}_{m-1}(0\colon j, j), \quad 0 \le j \le N. \tag{10.104}$$

Proof. First take $m = 1$ and apply the proceeding relation for $0 \le i \le j \le N$, to obtain

$$\begin{aligned}(\boldsymbol{G}_0 - \mathcal{U}(\boldsymbol{A}\,\mathcal{U}(\boldsymbol{A}^\top \boldsymbol{G}_0)))(i,j) &= \boldsymbol{I}(i,j) - (\boldsymbol{A}\,\mathcal{U}(\boldsymbol{A}^\top))(i,j) \\ &= \boldsymbol{I}(i,j) - \boldsymbol{A}[j]\boldsymbol{A}[j]^\top(i,j) \\ &= \boldsymbol{G}[j](i,j) = \boldsymbol{G}_1(i,j),\end{aligned}$$

in light of (10.100). Then take $m > 1$ and $0 \le i \le j$, to arrive at

$$\begin{aligned}&(\boldsymbol{G}_{m-1} - \mathcal{U}(\boldsymbol{A}\,\mathcal{U}(\boldsymbol{A}^\top \boldsymbol{G}_{m-1})))(i,j) \\ &\quad = \boldsymbol{G}_{m-1}(i,j) - \sum_{k,\ell=1}^{j} \boldsymbol{A}(i,k)\,\boldsymbol{A}^\top(k,\ell)\,\boldsymbol{G}_{m-1}(\ell,j) \\ &\quad = \sum_{\ell=1}^{j} (\boldsymbol{I}[j] - \boldsymbol{A}[j]\boldsymbol{A}[j]^\top)(i,\ell)\,\boldsymbol{G}_{m-1}(\ell,j)\end{aligned}$$

$$= \sum_{\ell=1}^{j} \boldsymbol{G}[j](i,\ell)\,\boldsymbol{G}[j]^{m-1}(\ell,j)$$
$$= \boldsymbol{G}[j]^m(i,j) = \boldsymbol{G}_m(i,j).$$

This is the asserted equality (10.103). The remaining relation (10.104) follows from the last equality upon realizing that

$$\boldsymbol{G}_m(0\colon j,j) = (\boldsymbol{G}_m(i,j))_{i=0}^{j} = \boldsymbol{G}[j]\boldsymbol{G}[j]^{m-1}(0\colon j,j) = \boldsymbol{G}[j]\boldsymbol{G}_{m-1}(0\colon j,j).$$

This completes the proof. □

5 *Schatten norms.* In the view of Lemmas 10.51 (recursion) and 10.50 (representation of $\boldsymbol{U}^{-1}$), we intend to estimate $\|\boldsymbol{U}^{-1}\|_2$ in terms of suitable norms of $\boldsymbol{G}_m$ that depend on the number N of blocks rather than the dimension n_N, because $n_N \gg N$. These special norms are called *block Schatten norms.*

However, for the sake of clarity, we start with the definition and properties of the usual Schatten norms. They include the operator 2-norm, the Frobenius norm, and satisfy a Hölder inequality.

Definition 10.52 (Schatten norms). Given $\boldsymbol{M} \in \mathbb{R}^{n\times n}$, let

$$\sigma_1(\boldsymbol{M}) \geq \sigma_2(\boldsymbol{M}) \geq \cdots \geq \sigma_n(\boldsymbol{M}) \geq 0$$

be the singular values of $\boldsymbol{M}$. Given $1 \leq p \leq \infty$, let the p-Schatten norm be

$$|\boldsymbol{M}|_p := \left(\sum_{m=1}^{n} \sigma_m(\boldsymbol{M})^p\right)^{1/p}.$$

Remark 10.53. Note that if $p = \infty$, the Schatten norm reduces to the 2-norm, that is,

$$|\boldsymbol{M}|_\infty = \sigma_1(\boldsymbol{M}) = \|\boldsymbol{M}\|_2,$$

and if $p = 2$ it is equivalent to the Frobenius norm,

$$|\boldsymbol{M}|_2 = \left(\sum_{m=1}^{n} \sigma_m(\boldsymbol{M})^2\right)^{1/2} = \left(\sum_{m=1}^{n} \boldsymbol{M}_{ij}^2\right)^{1/2} = \|\boldsymbol{M}\|_F.$$

We now list a number of useful properties of these norms.

Lemma 10.54 (properties of $|\cdot|_p$). *The following properties hold for* $1 \leq p \leq \infty$*:*

(i) $\sigma_i(\boldsymbol{M}^\top\boldsymbol{M}) = \sigma_i(\boldsymbol{M})^2 \quad\Rightarrow\quad |\boldsymbol{M}^\top\boldsymbol{M}|_p = |\boldsymbol{M}|_{2p}^2,$

(ii) $\sigma_i(\boldsymbol{M}) = \sigma_i(\boldsymbol{M}^\top) \quad\Rightarrow\quad |\boldsymbol{M}|_p = |\boldsymbol{M}^\top|_p,$

(iii) *Hölder inequality: for* $1/r = 1/p + 1/q$*, with* $r,p,q \in [1,\infty]$,

$$|\boldsymbol{M}_1\,\boldsymbol{M}_2|_r \leq |\boldsymbol{M}_1|_p\,|\boldsymbol{M}_2|_q,$$

(iv) $|\mathcal{U}(\boldsymbol{M})|_\infty \lesssim \log(n)\, |\boldsymbol{M}|_\infty$,

(v) $|\mathcal{U}(\boldsymbol{M})|_{2^j} \le 2^{j-1} |\boldsymbol{M}|_{2^j}$.

Remark 10.55. Properties (i) and (ii) are trivial. We refer to Dunford and Schwartz (1988, Lemma XI.9.20) for property (iii), to Bhatia (2000, (15)) for property (iv), and to Davies (1988) and Feischl (2022, Lemma 17) for property (v).

To define the block Schatten norms, we consider the subspace $\mathcal{D}_b$ of $\mathbb{R}^{n_N \times (N+1)}$ of matrices of the form

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_0 & & & \\ & \boldsymbol{X}_1 & & \\ & & \ddots & \\ & & & \boldsymbol{X}_N \end{bmatrix}, \quad \boldsymbol{X}_j \in \mathbb{R}^{d_j}, \quad 0 \le j \le N,$$

or equivalently

$$\boldsymbol{X} \in \mathcal{D}_b \quad \Longleftrightarrow \quad \boldsymbol{X}_{ij} = \boldsymbol{0} \quad \text{for all } i \ne n_{j-1}+1, \dots, n_j.$$

We can represent $\boldsymbol{X}$ using block notation as follows:

$$\boldsymbol{X} = (\boldsymbol{X}(i,j))_{i,j=0}^N, \quad \boldsymbol{X}(i,j) \in \mathbb{R}^{d_j \times 1},$$

where

$$\boldsymbol{X}(i,j) = \begin{cases} \boldsymbol{X}_j, & i = j, \\ \boldsymbol{0}, & i \ne j. \end{cases}$$

Given a block matrix $\boldsymbol{M} = (\boldsymbol{M}(i,j))_{i,j=0}^N \in \mathbb{R}^{n_N \times n}$, we consider

$$\boldsymbol{M}\boldsymbol{X} = (\boldsymbol{M}(i,j)\boldsymbol{X}_j)_{i,j=0}^N \in \mathbb{R}^{n_N \times (N+1)},$$

namely the jth block column of $\boldsymbol{M}\boldsymbol{X}$ is

$$(\boldsymbol{M}(i,j)\boldsymbol{X}_j)_{i=0}^N \in \mathbb{R}^{n_N}.$$

Definition 10.56 (block Schatten norms). For $1 \le p \le \infty$, let

$$|\boldsymbol{M}|_{b,p} := \sup_{\boldsymbol{X} \in \mathcal{D}_b,\ |\boldsymbol{X}|_\infty \le 1} |\boldsymbol{M}\boldsymbol{X}|_p \quad \text{for all } \boldsymbol{M} \in \mathbb{R}^{n_N \times n_N}.$$

Note the unusual norm $|\boldsymbol{X}|_\infty$ instead of $|\boldsymbol{X}|_p$ in this definition of operator norm $|\boldsymbol{M}|_{b,p}$. This choice is deliberate and will be useful later; see Remark 10.58. We now list important properties of the block Schatten norms; see Feischl (2022, Lemmas 15, 16, 17) for proofs.

Lemma 10.57 (properties of $|\cdot|_{b,p}$). *The following properties hold for all $\boldsymbol{M}$, $\boldsymbol{M}_1$, $\boldsymbol{M}_2 \in \mathbb{R}^{n_N \times n_N}$ and $1 \le p \le \infty$:*

(i) $|\boldsymbol{M}|_{b,p} \le (N+1)^{1/p} |\boldsymbol{M}|_\infty = (N+1)^{1/p} \|\boldsymbol{M}\|_2$,

(ii) $|\boldsymbol{M}_1 \boldsymbol{M}_2|_{b,p} \le |\boldsymbol{M}_1|_\infty |\boldsymbol{M}_2|_{b,p}$,

(iii) $|\boldsymbol{M}|_\infty \le |\boldsymbol{M}|_{b,p}$, $|\boldsymbol{M}|_{b,\infty} = |\boldsymbol{M}|_\infty$,

(iv) *if* $\boldsymbol{M}_1 \in \mathbb{R}^{n_N \times n_N}$ *is block triangular with* j*th block column*

$$\boldsymbol{M}_1(0\colon j, j) = \boldsymbol{P}_j \boldsymbol{M}_2(0\colon j, j), \quad \boldsymbol{P}_j \in \mathbb{R}^{n_j \times n_j}$$

for $0 \le j \le N$, *then*

$$|\boldsymbol{M}_1|_{b,2} \le \max_{0 \le j \le N} |\boldsymbol{P}_j|_\infty \, |\boldsymbol{M}_2|_{b,2},$$

(v) $|\mathcal{U}(\boldsymbol{M})|_{b,2^k} \le 2^{k-1} |\boldsymbol{M}|_{b,2^k}$, $k = 1, 2$,

(vi) $|\mathcal{U}(\boldsymbol{M})|_\infty \le (\lceil \log_2(N) \rceil + 1) |\boldsymbol{M}|_\infty$.

Remark 10.58. To understand the significance of Definition 10.56, we examine the growth of the usual and block p-Schatten norm relative to the ∞-Schatten norm for $1 \le p < \infty$. Given $\boldsymbol{M} \in \mathbb{R}^{n_N \times n_N}$, we have for the usual p-norm

$$|\boldsymbol{M}|_p = \left(\sum_{i=1}^{n_N} \sigma_i(\boldsymbol{M})^p \right)^{1/p} \le n_N^{1/p} \sigma_1(\boldsymbol{M}) = n_N^{1/p} |\boldsymbol{M}|_\infty = n_N^{1/p} \|\boldsymbol{M}\|_2,$$

whereas for the block p-norm we get

$$|\boldsymbol{M}|_{b,p} \le (N+1)^{1/p} |\boldsymbol{M}|_\infty = (N+1)^{1/p} \|\boldsymbol{M}\|_2,$$

according to Lemma 10.57(i). In fact, given $\boldsymbol{X} \in \mathcal{D}_b$ with $|\boldsymbol{X}|_\infty = \|\boldsymbol{X}\|_2 = 1$, we first note that

$$|\boldsymbol{M}\boldsymbol{X}|_p = \left(\sum_{j=0}^{N} \sigma_j(\boldsymbol{M}\boldsymbol{X})^p \right)^{1/p} \le \sigma_0(\boldsymbol{M}\boldsymbol{X})(N+1)^{1/p} = \|\boldsymbol{M}\boldsymbol{X}\|_2 (N+1)^{1/p},$$

and also that

$$\|\boldsymbol{M}\boldsymbol{X}\|_2 = \sup_{\boldsymbol{x} \in \mathbb{R}^{N+1}} \frac{\|\boldsymbol{M}\boldsymbol{X}\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} \le \|\boldsymbol{M}\|_2 \sup_{\boldsymbol{x} \in \mathbb{R}^{N+1}} \frac{\|\boldsymbol{X}\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} \le \|\boldsymbol{M}\|_2,$$

because $\|\boldsymbol{M}\|_2 = |\boldsymbol{M}|_\infty = 1$. On the one hand, this explains why it is convenient to have the norm $|\boldsymbol{X}|_\infty$ rather than $|\boldsymbol{X}|_p$ in Definition 10.56. On the other hand, this calculation reveals the key point that

$$|\boldsymbol{M}|_{b,p} \ll |\boldsymbol{M}|_p,$$

because the growth of $|\boldsymbol{M}|_{b,p}$ is dictated by the number of blocks $N+1$ whereas that of $|\boldsymbol{M}|_p$ is proportional to the dimension n_N of $\boldsymbol{M}$ and $n_N \gg N$. This property is essential to the estimate of $\|\boldsymbol{U}^{-1}\|_2$ below.

6 *Estimate of* $\|\boldsymbol{U}^{-1}\|_2$. We are now in a position to prove the desired bound (10.94).

Proposition 10.59 (estimate of $\|\boldsymbol{U}^{-1}\|_2$). *Let* $\boldsymbol{B} \in \mathbb{R}^{n_N \times n_N}$ *be a block matrix such that*

$$\|\boldsymbol{B}\|_2 \le \|\mathcal{B}\|, \quad \max_{0 \le j \le N} \|\boldsymbol{B}[j]^{-1}\|_2 \le \frac{1}{\beta}.$$

Then there exist constants C_{LU} and $p > 2$ such that the block upper triangular factor $\boldsymbol{U}$ of $\boldsymbol{B}$ satisfies

$$\|\boldsymbol{U}^{-1}\|_2 \le C_{LU} N^{1/p}. \tag{10.105}$$

Proof. We recall (10.102) of Lemma 10.50 (representation of $\boldsymbol{U}^{-1}$),

$$\boldsymbol{U}^{-1} = \alpha\, \mathcal{U}\left(\boldsymbol{B}^\top \sum_{m=0}^{\infty} \boldsymbol{G}_m\right),$$

along with (10.103) of Lemma 10.51 (recursion),

$$\boldsymbol{G}_m = \boldsymbol{G}_{m-1} - \mathcal{U}(\boldsymbol{A}\, \mathcal{U}(\boldsymbol{A}^\top \boldsymbol{G}_{m-1})), \quad m \ge 1,$$

and (10.104) of Lemma 10.51,

$$\boldsymbol{G}(0\colon j, j) = \boldsymbol{G}[j]\boldsymbol{G}_{m-1}(0\colon j, j), \quad 0 \le j \le N,$$

with $\boldsymbol{G}_0 = \boldsymbol{I}$. We use these expressions in conjunction with Lemma 10.57 (properties of $|\cdot|_{b,p}$) to prove (10.105). We proceed in several steps.

(i) *Bound for* $|\boldsymbol{G}_m|_{b,2}$. In light of (10.99) and (10.100),

$$|\boldsymbol{G}[j]|_\infty = \|\boldsymbol{G}[j]\|_2 \le \rho = \sqrt{1 - \frac{\beta^4}{\|\mathcal{B}\|^4}} < 1, \quad 0 \le j \le N.$$

Applying Lemma 10.57(iv) to $\boldsymbol{G}_m$ yields

$$\begin{aligned} |\boldsymbol{G}_m|_{b,2} &\le \max_{0\le j\le N} |\boldsymbol{G}[j]|_\infty\, |\boldsymbol{G}_{m-1}|_{b,2} \\ &\le \rho\, |\boldsymbol{G}_{m-1}|_{b,2} \le \rho^m\, |\boldsymbol{I}|_{b,2}. \end{aligned}$$

Recalling Lemma 10.57(i),

$$|\boldsymbol{I}|_{b,2} \le (N+1)^{1/2} \|\boldsymbol{I}\|_2 = (N+1)^{1/2},$$

whence

$$|\boldsymbol{G}_m|_{b,2} \le \rho^m\, (N+1)^{1/2}.$$

We observe that this bound is not good enough for our purposes because it scales like $N^{1/2}$ instead of $N^{1/p}$ for $p > 2$. We next improve upon this bound.

(ii) *Bound for* $|\boldsymbol{G}_m|_{b,4}$. We take $k = 2$ in Lemma 10.57(v) and use the triangle inequality to arrive at

$$\begin{aligned} |\boldsymbol{G}_m|_{b,4} &\le |\boldsymbol{G}_{m-1}|_{b,4} + |\mathcal{U}(\boldsymbol{A}\, \mathcal{U}(\boldsymbol{A}^\top \boldsymbol{G}_{m-1}))|_{b,4} \\ &\le |\boldsymbol{G}_{m-1}|_{b,4} + 2|\boldsymbol{A}\, \mathcal{U}(\boldsymbol{A}^\top \boldsymbol{G}_{m-1})|_{b,4}. \end{aligned}$$

We further apply parts (ii) and (v) of Lemma 10.57 to obtain

$$|\boldsymbol{A}\, \mathcal{U}(\boldsymbol{A}^\top \boldsymbol{G}_{m-1})|_{b,4} \le 2|\boldsymbol{A}|_\infty\, |\boldsymbol{A}^\top \boldsymbol{G}_{m-1}|_{b,4} \le 2|\boldsymbol{A}|_\infty^2\, |\boldsymbol{G}_{m-1}|_{b,4}.$$

Therefore

$$|\boldsymbol{G}_m|_{b,4} \le (1 + 4|\boldsymbol{A}|_\infty^2)|\boldsymbol{G}_{m-1}|_{b,4},$$

but the prefactor on the right-hand side is greater than 1 and thus not suitable for iteration. We still have

$$|\boldsymbol{G}_m|_{b,4} \le (1 + 4|\boldsymbol{A}|_\infty^2)^m |\boldsymbol{I}|_{b,4}.$$

(iii) *Bound for* $|\boldsymbol{G}_m|_\infty$. We combine the estimates from steps (i) and (ii) to exploit their relative merits. Recall from Lemma 10.57(iii) that

$$|\boldsymbol{G}_m|_\infty \le |\boldsymbol{G}_m|_{b,p} \quad \text{for all } 1 \le p \le \infty.$$

Take $p = 2, 4$ and $0 < t < 1$ to be chosen later, and write

$$\begin{aligned} |\boldsymbol{G}_m|_\infty &\le |\boldsymbol{G}_m|_{b,2}^{1-t} |\boldsymbol{G}_m|_{b,4}^t \\ &\le \left[\rho^{1-t}\left(1 + 4|\boldsymbol{A}|_\infty^2\right)^t\right]^m |\boldsymbol{I}|_{b,2}^{1-t} |\boldsymbol{I}|_{b,4}^t. \end{aligned}$$

Consequently, there exists $0 < t_0 < 1$ such that

$$q := \rho^{1-t}\left(1 + 4|\boldsymbol{A}|_\infty^2\right)^t < 1, \quad 0 < t < t_0$$

and

$$|\boldsymbol{G}_m|_\infty \le q^m |\boldsymbol{I}|_{b,2}^{1-t} |\boldsymbol{I}|_{b,4}^t.$$

We now estimate the two terms on the right-hand side via Lemma 10.57(i), namely

$$\begin{aligned} |\boldsymbol{I}|_{b,2} &\le (N+1)^{1/2} \|\boldsymbol{I}\|_2 = (N+1)^{1/2}, \\ |\boldsymbol{I}|_{b,4} &\le (N+1)^{1/4} \|\boldsymbol{I}\|_2 = (N+1)^{1/4}. \end{aligned}$$

We thus obtain

$$|\boldsymbol{G}_m|_\infty \le q^m (N+1)^{1/\widetilde{p}},$$

with $1/\widetilde{p} = (1-t)/2 + t/4 < 1/2$ for $0 < t < t_0$.

(iv) *Estimate of* $\|\boldsymbol{U}^{-1}\|_2$. Recalling the expression

$$\boldsymbol{U}^{-1} = \alpha\, \mathcal{U}\left(\boldsymbol{B}^\top \sum_{m=0}^{\infty} \boldsymbol{G}_m\right),$$

and applying parts (vi), (ii) and (iii) of Lemma 10.57, we see that

$$\begin{aligned} \|\boldsymbol{U}^{-1}\|_2 = |\boldsymbol{U}^{-1}|_\infty &\lesssim \log(N)\, |\boldsymbol{B}|_\infty \sum_{m=0}^{\infty} |\boldsymbol{G}_m|_\infty \\ &\lesssim |\boldsymbol{B}|_\infty (N+1)^{1/\widetilde{p}} \log(N) \sum_{m=0}^{\infty} q^m \\ &\lesssim \|\boldsymbol{B}\|_2 (N+1)^{1/\widetilde{p}} \log(N). \end{aligned}$$

Finally, for any $2 < p < \widetilde{p}$, we can absorb the logarithm, thereby getting

$$\|\boldsymbol{U}^{-1}\|_2 \lesssim \|\boldsymbol{B}\|_2 (N+1)^{1/p},$$

which is the desired estimate (10.105).

This concludes the proof. □

7 *Estimate of block diagonal* $\boldsymbol{D}$. We recall that $\boldsymbol{D} = \operatorname{diag} \boldsymbol{U} \in \mathbb{R}^{n_N \times n_N}$ is the block diagonal of $\boldsymbol{U}$. We consider the block partitioning of $\boldsymbol{B}[j]$,

$$\boldsymbol{B}[j] = \begin{bmatrix} \boldsymbol{B}[j-1] & \boldsymbol{R}_1 \\ \boldsymbol{R}_2^\top & \boldsymbol{R}_3 \end{bmatrix} \in \mathbb{R}^{n_j \times n_j},$$

where

$$\begin{aligned} \boldsymbol{R}_1 &= \boldsymbol{B}[j](1\colon j-1, j) \in \boldsymbol{R}^{n_{j-1} \times d_j}, \\ \boldsymbol{R}_2^\top &= \boldsymbol{B}[j](j, 1\colon j-1) \in \boldsymbol{R}^{d_j \times n_{j-1}}, \\ \boldsymbol{R}_3 &= \boldsymbol{B}[j](j, j) \in \mathbb{R}^{d_j \times d_j}. \end{aligned}$$

Lemma 10.60 (bound of $\|\boldsymbol{D}\|_2$). *We have*

$$\|\boldsymbol{D}\|_2 \le \|\mathcal{B}\| + \frac{\|\mathcal{B}\|^2}{\beta} = C_D. \tag{10.106}$$

Proof. Compute the $\boldsymbol{LU}$ factorization of $\boldsymbol{B}[j]$,

$$\boldsymbol{B}[j] = \begin{bmatrix} \boldsymbol{I}[j-1] & 0 \\ \boldsymbol{R}_2^\top \boldsymbol{B}[j-1]^{-1} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{B}[j-1] & \boldsymbol{R}_1 \\ 0 & \boldsymbol{R}_3 - \boldsymbol{R}_2^\top \boldsymbol{B}[j-1]^{-1} \boldsymbol{R}_1 \end{bmatrix},$$

and realize that

$$\boldsymbol{U}(j,j) = \boldsymbol{D}(j,j) = \boldsymbol{R}_3 - \boldsymbol{R}_2^\top \boldsymbol{B}[j-1]^{-1} \boldsymbol{R}_1 \in \mathbb{R}^{d_j \times d_j}.$$

Since

$$\begin{aligned} |\boldsymbol{R}_i|_\infty &= \|\boldsymbol{R}_i\|_2 \le \|\boldsymbol{B}[j]\|_2 \le \|\boldsymbol{B}\|_2 = \|\mathcal{B}\|, \quad i = 1, 2, \\ |\boldsymbol{R}_3|_\infty &= \|\boldsymbol{R}_3\|_2 \le \|\boldsymbol{B}[j]\|_2 \le \|\mathcal{B}\|, \end{aligned}$$

and

$$|\boldsymbol{B}[j-1]^{-1}|_\infty = \|\boldsymbol{B}[j-1]\|_2 \le \frac{1}{\beta},$$

according to properties (P1) and (P2) of the bilinear form $\mathcal{B}$, we deduce

$$|\boldsymbol{D}(j,j)|_\infty = \|\boldsymbol{D}(j,j)\|_2 \le \|\mathcal{B}\| + \frac{\|\mathcal{B}\|^2}{\beta}$$

as asserted. □

8 *Bound of* $\boldsymbol{LU}$ *factors.* We are finally in the position to prove Theorem 10.15. We combine Proposition 10.59 (estimate of $\|\boldsymbol{U}^{-1}\|_2$) and $\boldsymbol{L} = \boldsymbol{B}\boldsymbol{U}^{-1}$ to obtain

$$\|\boldsymbol{L}\|_2 \le \|\boldsymbol{B}\|_2 \|\boldsymbol{U}^{-1}\|_2 \le \|\mathcal{B}\| C_{LU} N^{1/p}.$$

Then, invoking (10.97) in conjunction with Proposition 10.59 and Lemma 10.60

(bound of $\|\boldsymbol{G}\|_2$), as well as the bounds of $\|\boldsymbol{U}^{-1}\|_2$ and $\|\boldsymbol{L}\|_2$, yields

$$\begin{aligned} \|\boldsymbol{U}\|_2 &\le \|\boldsymbol{D}\|_2 \|\tilde{\boldsymbol{L}}^\top\|_2 \le C_D \|\mathcal{B}\| C_{LU} N^{1/p}, \\ \|\boldsymbol{L}^{-1}\|_2 &\le \|\boldsymbol{D}\|_2 \|\tilde{\boldsymbol{U}}^{-\top}\|_2 \le C_D C_{LU} N^{1/p}, \end{aligned} \tag{10.107}$$

with C_D being the constant in (10.106). This completes the proof. □

Acknowledgements

We are grateful to R. DeVore, P. Morin and A. Salgado for discussions about regularity classes, and to Ch. Kreuzer for pointing out a gap in an earlier version of the proof of Theorem 5.17. We are also thankful to G. Vacca for providing simulations and D. Fassino for assistance with the manuscript.

This work was partially supported by NSF grants DMS-2110811 (AB) and DMS-1908267 (RHN), and MUR-PRIN grants 20227K44ME (CC) and 201752HKM (AV), and INdAM research group GNCS (CC and AV). RHN was also partially supported by NSF grant DMS-1929284 while he was in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI, during the Numerical PDEs: Analysis, Algorithms, and Data Challenges semester programme.

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