Can. J. Math., Vol. XXIX, No. 4, 1977, pp. 696–700

SYSTEMS THAT ARE PURELY SIMPLE AND PURE INJECTIVE

FRANK OKOH

Introduction. There has been a lot of progress made on the finite-dimensional representations of species. In [**3**] and [**11**] the finite-dimensional representations of tame species are classified and in [**13**] it is shown that if S is a species of finite type, then every representation of S is a direct sum of finite-dimensional ones. However, comparatively little is known about infinite-dimensional representations. This is, perhaps, in the nature of things; see [**11**, p. 302]. The study of infinite-dimensional representations of particular species is, therefore, not without interest. We remark that long before the above developments, Aronszajn and Fixman had studied in [**1**] the representations of the species

$$K \xrightarrow{{}_K K_K^2} K$$

where K is an algebraically closed field, in particular the field of complex numbers. They called the representations "systems". Arongszajn was led to this study by his investigations of finite-dimensional perturbations of spectral problems. In such a context, infinite-dimensional representations beg for consideration.

All the facts known for infinite-dimensional representations of general K-species do not give us any extra information for systems. For instance, the locally indecomposable representations defined in [**11**, p. 302] give precisely the indecomposable systems of rank less than or equal to one. A similar remark applies to the infinite-dimensional representations studied in [**12**]. A perusal of [**1**], [**4**], or [**10**] shows that a lot is known about some classes of infinite-dimensional systems.

In this paper, we determine the purely simple and pure injective systems. The terminology for systems is as in [**6**]. Let (S, T), (V, W) be systems. In the case when (S, T) is purely simple and pure injective, it is shown in [**4**] that there is an invariant of (V, W) corresponding to its isomorphism type. If (S, T) is also finite-dimensional, this invariant corresponds to the dimension of a vector space described in [**1**, Theorem 6.7]. We became interested, therefore, in finding the systems that are purely simple and pure injective. It turns out that the class of such systems is quite sparse. It consists of systems of type I^m, $II_\theta{}^m$, $II_\theta{}^\infty$, III^m or $\mathscr{R}$, all of which are already accounted for in [**1**]. This is

Received January 14, 1976, and in revised form, February 7, 1977.

https://doi.org/10.4153/CJM-1977-073-4 Published online by Cambridge University Press

surprising in view of the fact that there exist purely simple systems of any finite rank, [**9**].

It is possible that the results on systems are prototypes of results that can be proved for a larger class of tame species. We have not been able to determine this. However, in Section 1, the necessary definitions have been formulated for general K-species. For concreteness, we have stuck to the field of complex numbers $\mathbf{C}$, but this can be replaced by any algebraically-closed field.

1. Preliminaries. Let $S = (F_i, {}_iM_j)_{1 \leq i, j \leq n}$ be a connected K-species without oriented cycles, see [**3**] for a definition. It is convenient to identify some representations of S. For example if $F_i = F_j = \mathbf{C}$, ${}_iM_j = {}_{\mathbf{C}}\mathbf{C}^2{}_{\mathbf{C}}$ and $(V_i, {}_j\phi_i)$ a representation, then the linear mapping

$$\phi: V_i \otimes_{\mathbf{C}} \mathbf{C}^2 \to V_j$$

can be considered as a pair of linear mappings ϕ_1, ϕ_2 from V_i to V_j. For any pair (α_1, α_2) in $\mathbf{C}^2$ we get another linear map $\alpha_1\phi_1 + \alpha_2\phi_2$ from V_i to V_j. For a given ϕ all such representations are identified, yielding the following:

Definition 1.1. A system is a pair of complex vector spaces (V, W) together with a *system operation* which is a $\mathbf{C}$-bilinear map $(e, v) \mapsto ev$ of $\mathbf{C}^2 \times V$ into W.

Our terminology for species and systems will be as in [**5**] and [**11**] respectively. For any species S, let $L(S)$ be the category of all representations of S and let $l(S)$ be the category of finite-dimensional representations of S.

Definition 1.2. A subobject A of B in $L(S)$ is said to be *pure in B* provided that for every intermediate subobject, C, $A \subset C \subset B$ such that C/A is in $l(S)$, A is a direct summand of C.

This definition already in [**1**] for systems has all the desirable properties. For details, we refer to [**14**]. In the language of that paper, 1.2 defines $l(S)$-purity in $L(S)$. We have the following.

PROPOSITION 1.3. ([**7**, p. 129] or [**14**]) (i) *A direct summand of an object is a pure subobject in the object.*

(ii) *If A is pure in B then it is pure in every subobject between A and B.*

(iii) *If A is pure in B and B is pure in D then A is pure in D.*

Definition 1.4 (a) An object in $L(S)$ is said to be *purely simple* if it has no proper pure subobjects.

(b) An object in $L(S)$ is said to be *pure injective* if it is a direct summand of any object containing it as a pure subobject.

It is natural to ask for the purely simple objects in $L(S)$—these being even more basic than the indecomposable objects, (1.3(i)). In the category of $K[t]$-modules, where $K[t]$ is the polynomial ring over a field K, the purely simple modules are easily determined (see [**7**, p. 119]). However, in the category of systems, the purely simple objects are a lot more complex. If the category of A-modules, where A is any tame algebra, contains a pure-closed full subcategory which is representation equivalent to the category of $K[t]$-modules, then the purely simple $K[t]$-modules yield purely simple A-modules. (Every known tame algebra has the above property minus the pure-closed. See [**2**]. "Pure-closed" means that any pure subobject of an object in the subcategory is again in the subcategory).

Definition 1.5. Let (V, W) be a system. (V, W) is said to be *torsion-free* if for every $0 \neq e \in \mathbf{C}^2$, the map $v \mapsto ev$ is injective.

Let $\{w_i\}_{i \in I} \subset W$. There is a smallest subsystem (X, Y) of (V, W) such that $\{w_i\}_{i \in I} \subset Y$ and $(V, W)/(X, Y)$ is torsion-free. (X, Y) is said to be the *subsystem of* (V, W) *generated by the set* $\{w_i\}_{i \in I}$. (X, Y) is also called the *torsion-closure of* $(\emptyset, \{w_i\}_{i \in I})$ *in* (V, W) denoted by $\mathrm{tc}_{(V,W)}(\emptyset, \{w_i\}_{i \in I})$.

Definition 1.6 [**4**]. A system (V, W) is said to be *of rank n*, not necessarily finite, if $(V, W) = \mathrm{tc}_{(V,W)}(\emptyset, \{w_i\}_{i \in I})$ and $\mathrm{card}\,(I) = n$ but $(V, W) \neq \mathrm{tc}_{(V,W)}(\emptyset, \{w_i\}_{i \in I})$ for any $J \subsetneqq I$.

It is shown in [**4**] that the rank of a system is well-defined. The torsion-free systems of rank one are also completely classified there, using so-called height functions. Let $V = W = \mathbf{C}(\xi)$—the complex rational functions. (V, W) is made into a system in the following way:

$$af(\xi) = f(\xi), \quad bf(\xi) = \xi f(\xi), \quad a, b \text{ a fixed basis of } \mathbf{C}^2.$$

The isomorphism type of (V, W) is denoted by $\mathscr{R}$ and it is of rank 1. For this and a description of systems of type I^m, $II_\theta{}^m$, III^m, we refer to [**1**] and [**4**].

The next proposition is crucial to the proof of the main result.

PROPOSITION 1.7 (a) [**5**]. *If* (V, W) *is of rank* 1 *and not of type* $\mathscr{R}$ *then* Ext $(\mathscr{R}, (V, W)) \neq 0$.

(b) [**10**]. *If* (V, W) *is of type* III^m *then* Ext $(\mathscr{R}, III^m)$ *is of dimension* 2^c *as a right vector space over* $\mathbf{C}(\xi)$.

2.

THEOREM 2.1. *A system* (V, W) *is purely simple and pure injective if and only if it is of one of the following types:* I^m, $II_\theta{}^m$, $II_\theta{}^\infty$, III^m *or* $\mathscr{R}$.

Proof. The fact that a system of any of the above types is both purely simple and pure injective is readily deduced from [**1**, Propositions 2.2, 5.5, 8.4 and 9.15] and [**4**, p. 433].

Suppose (V, W) is both purely simple and pure injective. Since a direct summand of a system is a pure subsystem, by 1.3(i), (V, W) must be indecomposable. So by [**1**, Corollary 9.16(b)] it is either torsion or torsion-free, and in the former case, it is of one of the types I^m, $II_\theta{}^m$, or $II_\theta{}^\infty$ again by [**1**, 9.16(b)]. So we may assume that (V, W) is torsion-free. If it is finite-dimensional, then it must be of type III^m by [**1**, Theorem 4.3] and the assumption that (V, W) is torsion-free and indecomposable. If (V, W) is infinite-dimensional of rank 1, then it must be of type $\mathscr{R}$ because otherwise Ext $(\mathscr{R}, (V, W)) \neq 0$ by 1.7(a). But (V, W) purely simple and pure injective implies that Ext $(\mathscr{R}, (V, W)) = 0$ by [**6**, Theorem 1], contradiction.

We are left with the case in which (V, W) is infinite-dimensional, torsion-free and of rank larger than 1. Let $\{w_i\}_{i \in I}$ be a basis of (V, W) with respect to generation and (V^1, W^1) a subsystem of (V, W) generated by all but one element of $\{w_i\}_{i \in I}$. $(V, W)/(V^1, W^1)$ is a torsion-free rank 1 system and we have the following exact sequence

$$0 \rightarrow (V^1, W^1) \rightarrow (V, W) \rightarrow (V, W)/(V^1, W^1) \rightarrow 0.$$

Since (V, W) is purely simple, (V^1, W^1) must have a direct summand, (V^2, W^2) of type III^m by [**6**, Theorem 1]. Let (X, Y) be a system of type $\mathscr{R}$. We have the long exact sequence (see [**8**])

$$\begin{aligned} (*)\ &\mathrm{Hom}\ ((X, Y), (V, W)) \rightarrow \mathrm{Hom}\ ((X, Y), (V, W)/(V^1, W^1)) \\ &\rightarrow \mathrm{Ext}\ ((X, Y), (V^2, W^2) \dot{+} (V^3, W^3)) \rightarrow \mathrm{Ext}\ ((X, Y), (V, W)) \\ &\rightarrow \mathrm{Ext}\ ((X, Y), (V, W)/(V^1, W^1)) \rightarrow 0 \end{aligned}$$

Suppose that $(V, W)/(V^1, W^1)$ is not of type $\mathscr{R}$; then Hom $((X, Y), (V, W)/(V^1, W^1)) = 0$ by [**4**, Lemma 3.1]. As above, since (V,W) is infinite-dimensional, purely simple and pure injective, we have Ext $((X, Y), (V, W)) = 0$. Substituting these values in (*) gives Ext $((X, Y), (V^2, W^2) \dot{+} (V^3, W^3)) = 0$. However, Ext $((X, Y), (V^2, W^2) \dot{+} (V^3, W^3))$ is isomorphic to Ext$((X, Y), (V^2, W^2)) \oplus$ Ext $(X, Y), (V^3, W^3))$. (V^2, W^2) is of type III^m and so by 1.7(a), Ext $((X, Y), (V^2, W^2)) \neq 0$, a contradiction. Hence, we must suppose that $(V, W)/(V^1, W^1)$ is of type $\mathscr{R}$. From (*) we get

$$\mathrm{Hom}\ ((X, Y), (V, W)/(V^1, W^1)) \rightarrow \mathrm{Ext}\ ((X, Y), (V^1, W^1)) \rightarrow 0$$

The first entry in this sequence is isomorphic to $\mathbf{C}(\xi)$, the complex rational functions by [**4**, Corollary 3.7], so it is a one-dimensional vector space over End $(\mathscr{R}) = \mathbf{C}(\xi)$. But by Proposition 1.7(b), the second entry is infinite-dimensional over $\mathbf{C}(\xi)$. Therefore it cannot be a homomorphic image of a one-dimensional vector space over $\mathbf{C}(\xi)$. So if (V, W) is of rank greater than or equal to two, it cannot be both purely simple and pure injective. We are done with the proof of Theorem 2.1.

Remarks. Theorem 2.1 generalizes [**4**, Theorem 5.8] which states that a torsion-free system of rank 1 which is not of type III^m or $\mathscr{R}$ is not pure-injective. The proof there exhibits a short exact sequence which is claimed to be non-splitting.

However, there is an error which can be corrected as follows: Instead of $bv_k = w_{k+1} + 1$, for $k = 0, 1, 2, \ldots$ Let $bv_k = w_{k+1} + \alpha_k$ where $\sum_{j=1}^{\infty} \alpha_{j-1}/\xi^j$ is not the expansion of a rational function in $\mathbf{C}(\xi)$. The proof then proceeds as in [**4**].

For a given species, the main problem in finding the purely simple and pure injective representations lies in finding the infinite-dimensional ones.

Acknowledgement. Theorem 2.1 is taken from the author's Ph.D. thesis written at Queen's University, Kingston, Ontario under the helpful supervision of Professor Uri Fixman.

I also wish to thank the referee whose comments influenced the final version of the paper.

References

1. N. Aronszajn and U. Fixman, *Algebraic spectral problems*, Studia Math *30* (1968), 273–338.
2. S. Brenner and C. M. Ringel, *Pathological modules over tame rings*, to appear.
3. V. Dlab and C. M. Ringel, *Representations of graphs and algebras*, Mem. Amer. Math. Soc. *173* (1976).
4. U. Fixman, *On algebraic equivalence between pairs of linear transformations*, Trans. Amer. Math. Soc. *113* (1964), 424–453.
5. U. Fixman and F. Okoh, *Extensions of pairs of linear transformations between infinite-dimensional vector spaces*, to appear, Linear Algebra and Its Applications.
6. U. Fixman and F. Zorzitto, *A purity criterion for pairs of linear transformations*, Can. J. Math. *26* (1973), 734–745.
7. L. Fuchs, *Infinite abelian groups*, Vol. 1 (Academic Press, 1970).
8. S. Maclane, *Homology* (Springer Verlag, New York, 1967).
9. F. Okoh, *A bound on the rank of purely simple systems*, to appear, Trans. Amer. Math. Soc.
10. ——— *Torsion-free modules over a non-commutative hereditary ring*, Ph.D. thesis, Queen's University Kingston, Ontario, 1975.
11. C. M. Ringel, *Representations of K-species and bimodules*, J. Algebra *41* (1976), 269–302.
12. ——— *Unions of chains of indecomposable modules*. Comm. in Algebra *3* (12) (1975), 1121–1144.
13. C. M. Ringel and H. Tachikawa, *QF-3 rings*, J. Reine Ang. Math. *272* (1975), 49–72.
14. C. P. Walker, *Relative homological algebra and abelian groups*, Illinois J. Math. *10* (1966), 186–209.

University of Nigeria,
Nsukka, Nigeria