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ON GROUP UNIFORMITIES ON SQUARE OF A SPACE AND EXTENDING PSEUDOMETRICS II

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We find topological conditions on a space X under which the left (right, or two-sided) uniformity of the free topological group $F(X)$ induces the universal uniformity $\mathcal{U}_{X^2}$ or the product uniformity $\mathcal{U}_X \times \mathcal{U}_X$ on the square of X. Special attention is given to k_ω-spaces and metrisable spaces. The main technical tool in the paper is an extension of certain continuous pseudometrics from X^2 to $F(X)$ considered by the author in the previous volume of this journal.

0. Introduction

By a theorem of Graev [5], any continuous pseudometric d on a Tikhonov space X extends to an invariant pseudometric $\widehat{d}$ on the free topological group $F(X)$. This result was applied by Pestov [9] to prove the equality ${}^*\mathcal{V}^*|_X = \mathcal{U}_X$ for every Tikhonov space X, where ${}^*\mathcal{V}^*$ is the two-sided uniformity of $F(X)$ and $\mathcal{U}_X$ is the universal uniformity of X (that is, the finest uniformity on X compatible with the topology of X). A generalisation of the above equality for uniform free topological groups was obtained by Nummela [8].

Our aim is to study the uniformities on X^2 induced by ${}^*\mathcal{V}$, $\mathcal{V}^*$ and ${}^*\mathcal{V}^*$, the left, right and two-sided group uniformities of $F(X)$. In talking about induced uniformities on X^2, it is understood that we identify X^2 with a subspace of $F(X)$ under the embedding $(x,y) \mapsto x \cdot y$; $x, y \in X$. So, we can formulate the following three problems (see [17]).

Problem A. What are the relations between ${}^*\mathcal{V}|_{X^2}$, $\mathcal{V}^*|_{X^2}$ and ${}^*\mathcal{V}^*|_{X^2}$ on the one hand and $\mathcal{U}_X \times \mathcal{U}_X$, $\mathcal{U}_{X^2}$ on the other ($\mathcal{U}_{X^2}$ stands for the universal uniformity on X^2)?

This general problem can be specialised as follows.

Problem B. When does the equality ${}^*\mathcal{V}^*|_{X^2} = \mathcal{U}_X \times \mathcal{U}_X$ hold?

Problem C. For which spaces X does the equality ${}^*\mathcal{V}^*|_{X^2} = \mathcal{U}_{X^2}$ hold?

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One can as well replace ${}^*\mathcal{V}^*$ by ${}^*\mathcal{V}$ or $\mathcal{V}^*$ in Problems B and C, thus obtaining four more problems. The resulting problems will be denoted by the same letters. The majority of these problems is solved here by means of a simultaneous extension method that applies to certain pairs (d_1, d_2) of continuous pseudometrics d_1 and d_2 on X and X^2 respectively and produces continuous seminorms on $F(X)$ (see Theorems 1.4, 2.1 and 3.1 of [**17**]).

We start with general assertions about uniformities on topological groups. Then we show that both uniformities ${}^*\mathcal{V}|_{X^2}$ and $\mathcal{V}^*|_{X^2}$ are finer than $\mathcal{U}_X \times \mathcal{U}_X$ (Theorem 1.6), that contributes to Problem A. As an application of simple topological tools we prove that the equality ${}^*\mathcal{V}^*|_{X^2} = \mathcal{U}_X \times \mathcal{U}_X$ holds for every pseudocompact space X (Theorem 1.8), thus giving a partial answer to Problem B. A complete solution of Problem B is given in Theorem 2.1: the equality ${}^*\mathcal{V}^*|_{X^2} = \mathcal{U}_X \times \mathcal{U}_X$ holds if and only if there exists an infinite cardinal τ such that X is pseudo-τ-compact and a P_τ-space simultanuously. This characterisation remains valid if one replaces ${}^*\mathcal{V}^*$ by ${}^*\mathcal{V}$ or $\mathcal{V}^*$.

Problem C seems the most difficult among the others. First, we characterise spaces X satisfying the condition ${}^*\mathcal{V}|_{X^2} = \mathcal{U}_{X^2}$ (or equivalently, $\mathcal{V}^*|_{X^2} = \mathcal{U}_{X^2}$): if X is not a P-space then the above condition is equivalent to the requirement that the projection $p : X^2 \to X$ is z-closed, and for a P-space X it is equivalent to the condition that for every open cover γ of X^2 there exists a disjoint open cover μ of X such that $\mu \times \mu = \{U \times V : U, V \in \mu\}$ is finer than γ (Theorem 3.1).

We also show that the equality ${}^*\mathcal{V}^*|_{X^2} = \mathcal{U}_{X^2}$ holds for each k_ω-space X (Theorem 4.2) and characterise metrisable spaces satisfying it (Theorem 4.4): the criterion is that a metrisable space must be locally compact or the set X' of all non-isolated points of X must be compact.

All spaces considered are assumed completely regular. We say that X is a P-space if every G_δ-set in X is open. A space X is said to be pseudo-τ-compact if every locally finite family of open sets in X has cardinality strictly less than τ. The Čech-Stone compactification of a space X is denoted by βX.

The set of positive integers is denoted by N^+; $\mathbf{R}$ stands for the reals with the interval topology.

Every element g of the free topological group $F(X)$ on a space X has the form $x_1^{\varepsilon_1} \cdot \ldots \cdot x_n^{\varepsilon_n}$ for some $x_1, \ldots, x_n \in X$ and $\varepsilon_1, \ldots, \varepsilon_n = \pm 1$. We put $l_+(g) = \{i \leqslant n : \varepsilon_i = 1\}$ and $l_-(g) = \{i \leqslant n : \varepsilon_i = -1\}$. Then we define a subgroup $G(X)$ of $F(X)$ by

$$G(X) = \{g \in F(X) : l_+(g) = l_-(g)\}.$$

Note that $G(X)$ is an open subgroup of $F(X)$ [**17**].

All the necessary facts in uniform space theory are contained in [**2**, Chapter 8] or [**6**, Chapters 1–3]. An exposition of results on uniform structures on topological groups is given in [**10**].

1. Preliminary facts and results

Let G be a topological group with identity e and O a neighbourhood of e in G. We put

$$U_O^l = \{(g,h) \in G \times G : g^{-1} \cdot h \in O\}, \quad U_O^r = \{(g,h) \in G \times G : g \cdot h^{-1} \in O\},$$

and $U_O = U_O^l \bigcap U_O^r$. Recall that a base of the left (right, two-sided) uniformity ${}^*\mathcal{V}_G$ (respectively $\mathcal{V}_G^*$, ${}^*\mathcal{V}_G^*$) of the group G consists of the sets U_O^l (respectively U_O^r, U_O) where O runs through all neighbourhoods of e in G.

The following notion seems to be folklore.

Definition 1.1: Let τ be an infinite cardinal. A subset $Y \subseteq X$ is said to be *τ-bounded* in a uniform space $(X,\mathcal{U})$ if for each $U \in \mathcal{U}$ there exists a subset $A \subseteq X$, $|A| \leqslant \tau$, such that $Y \subseteq \bigcup_{a \in A} B(a,U)$, where $B(a,U) = \{x \in X : (a,x) \in U\}$.

If one puts $Y = X$ in Definition 1.1, the notion of a *τ-bounded uniform space* $(X,\mathcal{U})$ will be obtained. For the sake of completeness we present a proof of the following well-known result. Recall that $\mathcal{U}_X$ always stands for the universal uniformity of a space X.

Assertion 1.2. A uniform space $(X,\mathcal{U})$ is τ-bounded if and only if the space X is pseudo-τ^+-compact.

Proof: *The necessity.* Suppose that there exists a locally finite family γ of open sets in X, $|\gamma| = \tau^+$. For every $V \in \gamma$ define a continuous real-valued function f_V on X, $0 \leqslant f_V \leqslant 1$, such that f_V is equal to 1 at some point of V and vanishes outside of V. Put

$$d(x,y) = \sum_{V \in \gamma} |f_V(x) - f_V(y)|; \qquad x,y \in X.$$

Then d is a continuous pseudometric on X. The set $W = \{(x,y) \in X^2 : d(x,y) < 1\}$ is an open entourage of the diagonal in X^2 and $W \in \mathcal{U}_X$. It is easy to verify that no subset $A \subseteq X$ with $|A| \leqslant \tau$ satisfies the condition $X = \bigcup_{a \in A} B(a,W)$ of Definition 1.1, that is, the uniform space $(X,\mathcal{U}_X)$ is not τ-bounded.

Sufficiency. Suppose that $(X,\mathcal{U}_X)$ is not τ-bounded and choose an element $W \in \mathcal{U}_X$ witnessing that. By Corollary 8.1.11 of [**2**] there exists a continuous pseudometric ϱ on X such that $\{(x,y) \in X^2 : \varrho(x,y) < 1\} \subseteq W$. Let K be a maximal subset of X with the property that $\varrho(a,b) \geqslant 1$ for all distinct $a,b \in K$. Then $|K| > \tau$ by the choice of W and ϱ. Obviously, the family of all balls of radius $1/3$ with points of K as centers is discrete (hence locally finite) and has cardinality greater than τ. ∎

DEFINITION 1.3: A subset Y of a topological group G is called *left-τ-bounded in G* if for every neighbourhood V of the identity in G there exists a subset $A \subseteq G$ such that $|A| \leqslant \tau$ and $Y \subseteq A \cdot V$; analogously, the inclusion $Y \subseteq V \cdot A$ defines the notion of *right-τ-boundedness in G*.

If a subset Y of G is left- and right-τ-bounded in G, we shall simply say that Y is *τ-bounded in G*.

Note that Y is left-τ-bounded (right-τ-bounded) in G if and only if Y is a τ-bounded subset of $(G, {}^*\mathcal{V})$ (respectively $(G, \mathcal{V}^*)$). We also mention that the subset A of G in Definition 1.3 can be chosen to satisfy the condition $A \subseteq Y$.

ASSERTION 1.4. If Y is (right-) left-τ-bounded in a topological group G then $Y \cdot Y$ is (right-) left-τ-bounded in G.

PROOF: It suffices to consider the "left" case. Let V and V_1 be neighbourhoods of the identity in G, $V_1^2 \subseteq V$. There exists a subset B of G, $|B| \leqslant \tau$, such that $Y \subseteq B \cdot V_1$. For each $b \in B$ choose a neighbourhood W_b of the identity satisfying the condition $b^{-1} \cdot W_b \cdot b \subseteq V_1$ and find a subset C_b of G of cardinality $\leqslant \tau$ with $Y \subseteq C_b \cdot W_b$. Put $C = \bigcup\limits_{b \in B} C_b$ and $A = C \cdot B$. Obviously, $|A| \leqslant |C| \leqslant \tau$. We claim that $Y \cdot Y \subseteq A \cdot V$. Indeed, let $x, y \in Y$ be arbitrary. Then $y \in b \cdot V_1$ for some $b \in B$. Since $Y \subseteq C_b \cdot W_b$, there exists $c \in C_b$ such that $x \in c \cdot W_b$. We have

$$x \cdot y \in c \cdot W_b \cdot b \cdot V_1 = c \cdot b \cdot \left(b^{-1} \cdot W_b \cdot b\right) \cdot V_1 \subseteq c \cdot b \cdot V_1^2 \subseteq c \cdot b \cdot V,$$

where $c \cdot b \in C \cdot B = A$. Thus, $Y \cdot Y \subseteq A \cdot V$. ∎

ASSERTION 1.5. Suppose that Y is a τ-bounded set in a topological group G with the two-sided uniformity ${}^*\mathcal{V}^*$. Then Y is a τ-bounded subset of $(G, {}^*\mathcal{V}^*)$.

PROOF: Let V be a neighbourhood of the identity in G. It suffices to define a subset $A \subseteq G$ such that $|A| \leqslant \tau$ and $Y \subseteq \bigcup\{a \cdot V \bigcap V \cdot a : a \in A\}$. Choose a symmetric neighbourhood V_1 of the identity so that $V_1^3 \subseteq V$ and let the subset B of G satisfy $Y \subseteq B \cdot V_1$, $|B| \leqslant \tau$. For every $b \in B$ find a neighbourhood W_b of the identity such that $W_b \subseteq V$ and $b^{-1} \cdot W_b \cdot b \subseteq V_1$. For each $b \in B$ there exists a subset $A_b \subseteq G$ such that $Y \subseteq W_b \cdot A_b$ and $|A_b| \leqslant \tau$. We claim that the set $A = \bigcup\limits_{b \in B} A_b$ works. Indeed, let $y \in Y$ be arbitrary. Then $y \in b \cdot V_1$ for some $b \in B$ and $y \in W_b \cdot a$ for some $a \in A_b \subseteq A$. Therefore $y = b \cdot v = w \cdot a$ for some $v \in V_1$ and $w \in W_b$. This implies that $a^{-1} \cdot b = v^{-1} \cdot b^{-1} \cdot w \cdot b \in V_1^{-1} \cdot b^{-1} \cdot W_b \cdot b$. We have

$$y \in b \cdot V_1 = a \cdot \left(a^{-1} \cdot b\right) \cdot V_1 \subseteq a \cdot V_1^{-1} \cdot \left(b^{-1} \cdot W_b \cdot b\right) \cdot V_1 \subseteq a \cdot V_1^3 \subseteq a \cdot V.$$

Thus, $y \in a \cdot V$ and $y \in W_b \cdot a \subseteq V \cdot a$. This proves the lemma. ∎

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We start considering Problem A. For the sake of generality the induced uniformities on X^n for an arbitrary $n \in N^+$ are considered here. From now on the symbols ${}^*\mathcal{V}$, $\mathcal{V}^*$ and ${}^*\mathcal{V}^*$ always denote respectively the left, right and two-sided uniformities of the free topological group $F(X)$ on a given space X.

Theorem 1.6. *Both uniformities ${}^*\mathcal{V}|_{X^n}$ and $\mathcal{V}^*|_{X^n}$ are finer than $\mathcal{U}_X \times \ldots \times \mathcal{U}_X$ (n times) for every space X and each $n \in N^+$.*

Proof: Let V be an entourage of the diagonal Δ_n in X^{2n}, $V \in \mathcal{U}_X \times \ldots \times \mathcal{U}_X$ (n times). It suffices to find $W^l \in {}^*\mathcal{V}$ and $W^r \in \mathcal{V}^*$ such that $W^l \bigcap X^{2n} \subseteq V$ and $W^r \bigcap X^{2n} \subseteq V$.

By the definition of a uniform product (see [**2**, Chapter 8] or [**6**]), one can find an entourage U of the diagonal Δ_1 in X^2 such that $(\overline{x}, \overline{y}) \in V$ for all points $\overline{x} = (x_1, \ldots, x_n) \in X^n$ and $\overline{y} = (y_1, \ldots, y_n) \in X^n$ satisfying $(x_i, y_i) \in U$ for each $i \leqslant n$. Use Corollary 8.1.11 of [**2**] to define a continuous pseudometric d on X such that $\{(x,y) \in X : d(x,y) < 1\} \subseteq U$. Denote by $\widehat{d}$ the Graev extension of d to a continuous invariant pseudometric on $G(X)$ and put $O = \{g \in G(X) : \widehat{d}(g,e) < 1\}$, where e is the identity of $G(X)$. Then define the sets W^l and W^r by

$$W^l = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in X^{2n} : x_n^{-1} \cdot \ldots \cdot x_1^{-1} \cdot y_1 \cdot \ldots \cdot y_n \in O\},$$

$$W^r = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in X^{2n} : x_1 \cdot \ldots \cdot x_n \cdot y_n^{-1} \cdot \ldots \cdot y_1^{-1} \in O\}.$$

It is clear that $W^l \in {}^*\mathcal{V}|_{X^n}$ and $W^r \in \mathcal{V}^*|_{X^n}$. We claim that $W^l \subseteq V$ and $W^r \subseteq V$. It suffices to verify the first of these inclusions. Assume that $x_n^{-1} \cdot \ldots \cdot x_1^{-1} \cdot y_1 \cdot \ldots \cdot y_n \in O$ where $x_i, y_i \in X$ for each $i \leqslant n$. Then we have

$$1 > \widehat{d}\big(x_n^{-1} \cdot \ldots \cdot x_1^{-1} \cdot y_1 \cdot \ldots \cdot y_n, e\big) = \sum_{i=1}^{n} d(x_i, y_i),$$

which readily follows from the definition of $\widehat{d}$ (see [**5**, **12**]). In particular, $d(x_i, y_i) < 1$ for each $i \leqslant n$, and the choice of d and U implies that $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in V$. This proves the inclusion $W^l \subseteq V$. An analogous argument shows that $W^r \subseteq V$. ∎

A compact space admits only one uniformity compatible with the topology of the space. Therefore, ${}^*\mathcal{V}|_{X^n} = \mathcal{V}^*|_{X^n} = {}^*\mathcal{V}^*|_{X^n} = \mathcal{U}_X \times \ldots \times \mathcal{U}_X$ (n times) for any compact space X; $n \in N^+$. Theorem 1.8 below generalises this obvious fact. In its proof we shall use one auxiliary result, which follows from [**18**, Theorem 2].

Lemma 1.7. *Every pseudo-ω_1-compact, C-embedded subset Z of a space T is P-embedded in T, that is, every continuous pseudometric on Z extends to a continuous pseudometric on T.*

Theorem 1.8. *For a pseudocompact space X, the equalities ${}^*\mathcal{V}|_{X^n} = \mathcal{V}^*|_{X^n} = {}^*\mathcal{V}^*|_{X^n} = \mathcal{U}_X \times \ldots \times \mathcal{U}_X$ (n times) hold for each $n \in N^+$.*

Proof: Let the space X be pseudocompact and $n \in N^+$ be arbitrary. Since ${}^*\mathcal{V}^*$ is finer than ${}^*\mathcal{V}$ and $\mathcal{V}^*$ and both uniformities ${}^*\mathcal{V}|_{X^n}$, $\mathcal{V}^*|_{X^n}$ are finer than $\mathcal{U}_n = \mathcal{U}_X \times \ldots \times \mathcal{U}_X$ (n times) (Theorem 1.6), it suffices to show that ${}^*\mathcal{V}^*|_{X^n} = \mathcal{U}_n$. We shall prove a more general result: the universal uniformity $\mathcal{W}$ of $F(X)$ restricted to X^n coincides with $\mathcal{U}_n$. Since $\mathcal{W}$ is finer than ${}^*\mathcal{V}^*$, it suffices to verify that $\mathcal{U}_n$ is finer than $\mathcal{W}|_{X^n}$.

The uniformity $\mathcal{W}$ is generated by the family of all continuous pseudometrics on $F(X)$. Let d be one of them. It is necessary to verify that the restriction $\varrho = d|_{X^n}$ of d to the subspace X^n of $F(X)$ is uniformly continuous with respect to $\mathcal{U}_n$. Since X is pseudocompact, the natural monomorphism of $F(X)$ to $F(\beta X)$ is a homeomorphic embedding by a theorem of Pestov [**9**], where βX is the Čech-Stone compactification of X. So, we can identify $F(X)$ with the corresponding subgroup of $F(\beta X)$ generated by the set X. Again, since X is pseudocompact, Theorem 3 of [**16**] implies that $F(X)$ is C-embedded into $F(\beta X)$, that is, every continuous real-valued function on $F(X)$ extends to a continuous function on $F(\beta X)$. Furthermore, the group $F(\beta X)$ is σ-compact, and hence has countable cellularity by Corollary 2 of [**14**]. Being dense in $F(\beta X)$, the group $F(X)$ has countable cellularity as well. In particular, $F(X)$ is pseudo-ω_1-compact. Applying Lemma 1.7, we conclude that $F(X)$ is P-embedded in $F(\beta X)$. Let $\tilde{d}$ be a continuous pseudometric on $F(\beta X)$ which extends d. The restriction of $\tilde{d}$ to the subspace $Y = \beta X \times \ldots \times \beta X$ (n times) of $F(\beta X)$ is an extension of the pseudometric ϱ. Since every continuous pseudometric on $F(X)$ extends to a continuous pseudometric on $F(\beta X)$, the universal uniformity $\widetilde{\mathcal{W}}$ of $F(\beta X)$ induces the universal uniformity $\mathcal{W}$ on $F(X)$, that is, $\widetilde{\mathcal{W}}|_{F(X)} = \mathcal{W}$. In particular, $\widetilde{\mathcal{W}}|_{X^n} = \mathcal{W}|_{X^n}$. Denote by $\widetilde{\mathcal{U}}_1$ the (unique) uniformity of the space βX compatible with its topology. Note that $\widetilde{\mathcal{U}}_1|_X = \mathcal{U}_X$. Obviously, $\widetilde{\mathcal{W}}$ induces the uniformity $\widetilde{\mathcal{U}}_n = \widetilde{\mathcal{U}}_1 \times \ldots \times \widetilde{\mathcal{U}}_1$ (n times) on the compact space Y, and $\widetilde{\mathcal{U}}_n|_{X^n} = \mathcal{U}_n$. Therefore, the uniform continuity of $\tilde{d}|_Y$ with respect to $\widetilde{\mathcal{U}}_n$ implies that $\varrho = \tilde{d}|_{X^n}$ is a uniformly continuous pseudometric with respect to $\mathcal{U}_n$. This proves that $\mathcal{U}_n$ is finer than $\widetilde{\mathcal{W}}|_{X^n} = \mathcal{W}|_{X^n}$. ∎

We generalise the above theorem in the next section by means of more subtle methods.

2. Solution of Problem B

The following theorem completely characterises those spaces X satisfying the equality ${}^*\mathcal{V}^*|_{X^2} = \mathcal{U}_X \times \mathcal{U}_X$. Recall [15] that a subset X of a topological group G is said to be *thin* in G if for every neighbourhood U of the identity in G there exists a neighbourhood V of the identity such that $x \cdot V \cdot x^{-1} \subseteq U$ for each $x \in X$.

Theorem 2.1. *The following conditions are equivalent for a Tikhonov space X:*

(1) $^*\mathcal{V}|_{X^2} = \mathcal{U}_X \times \mathcal{U}_X$;
(1′) $^*\mathcal{V}|_{X^n} = \mathcal{U}_X \times \ldots \times \mathcal{U}_X$ *(n times) for each $n \geqslant 2$;*
(2) $\mathcal{V}^*|_{X^2} = \mathcal{U}_X \times \mathcal{U}_X$;
(2′) $\mathcal{V}^*|_{X^n} = \mathcal{U}_X \times \ldots \times \mathcal{U}_X$ *(n times) for each $n \geqslant 2$;*
(3) $^*\mathcal{V}^*|_{X^2} = \mathcal{U}_X \times \mathcal{U}_X$;
(3′) $^*\mathcal{V}^*|_{X^n} = \mathcal{U}_X \times \ldots \times \mathcal{U}_X$ *(n times) for each $n \geqslant 2$;*
(4) $^*\mathcal{V}|_{X^2} = \mathcal{V}^*|_{X^2}$;
(4′) $^*\mathcal{V}|_{X^n} = \mathcal{V}^*|_{X^n}$ *for each $n \geqslant 2$;*
(5) *X is thin in $F(X)$;*
(6) *there exists an infinite cardinal τ such that X is a P_τ-space and pseudo-$\tau-$compact.*

Proof: Obviously, (1′) implies (1),..., (4′) implies (4). By Theorem 1.6, the implications (3) $\Rightarrow$ (1), (3) $\Rightarrow$ (2) and (3′) $\Rightarrow$ (1′), (3′) $\Rightarrow$ (2′) are valid. It is also clear that (1)&(2) $\Rightarrow$ (4) and (1′)&(2′) $\Rightarrow$ (4′); so (3) implies (4) and (3′) implies (4′). The equivalence of (5) and (6) follows from [**15**, Theorem 3].

(5) $\Rightarrow$ (3′). Let n be any positive integer and O be a neighbourhood of the identity in $F(X)$. Put

$$U_O = \{(x_1,\ldots,x_n,y_1,\ldots y_n) \in X^{2n} \quad x_1\ldots x_n\cdot y_n^{-1}\ldots y_1^{-1} \in O,\ x_n^{-1}\ldots x_1^{-1}\cdot y_1\ldots y_n \in O\}.$$

Then $U_O \in {}^*\mathcal{V}^*|_{X^n}$ and we have to find $U \in \mathcal{U}_X$ such that the uniform product $U^n = U \times \ldots \times U$ (n times) is contained in U_O. To this end, choose a symmetric neighbourhood V of the identity such that $V^n \subseteq O$. Since X is thin in $F(X)$, one can define a decreasing sequence $V = V_0 \supseteq V_1 \supseteq \ldots \supseteq V_n$ of open neighbourhoods of the identity in $F(X)$ such that $x^\varepsilon \cdot V_{i+1} \cdot x^{-\varepsilon} \subseteq V_i$ for all $x \in X$, $i \leqslant n-1$ and $\varepsilon = \pm 1$. Put $U = \{(x,y) \in X^2 : x^{-1}\cdot y \in V_n,\ x\cdot y^{-1} \in V_n\}$. Then $U \in \mathcal{U}_X$ and we claim that U works. Indeed, let $p = (x_1,\ldots,x_n,y_1,\ldots,y_n)$ be any point of X^{2n} satisfying $(x_i,y_i) \in U$ for each $i \leqslant n$, that is, $p \in U^n$. For every i with $2 \leqslant i \leqslant n$ put $g_i = x_1\cdot\ldots\cdot x_{i-1}$. We have

(*)
$$x_1\cdot\ldots\cdot x_n\cdot y_n^{-1}\cdot\ldots\cdot y_1^{-1} = \left(g_n\cdot x_n\cdot y_n^{-1}\cdot g_n^{-1}\right)\cdot\left(g_{n-1}\cdot x_{n-1}\cdot y_{n-1}^{-1}\cdot g_{n-1}^{-1}\right)\cdot\ldots\cdot\left(x_1\cdot y_1^{-1}\right).$$

By the choice of p and the sets V_i we also have $x_1\cdot y_1^{-1} \in V_n$, $g_2\cdot x_2\cdot y_2^{-1}\cdot g_2^{-1} \in g_2\cdot V_n\cdot g_2^{-1} \subseteq V_{n-1},\ldots\ldots,\ g_n\cdot x_n\cdot y_n^{-1}\cdot g_n^{-1} \subseteq g_n\cdot V_n\cdot g_n^{-1} \subseteq V_1$. In its turn, (*) implies that

$$x_1\cdot\ldots\cdot x_n\cdot y_n^{-1}\cdot\ldots\cdot y_1^{-1} \in V_1\cdot\ldots\cdot V_{n-1}\cdot V_n \subseteq V^n \subseteq O.$$

An analogous argument shows that $x_n^{-1}\cdot\ldots\cdot x_1^{-1}\cdot y_1\cdot\ldots\cdot y_n\in V^n\subseteq O$. Thus, the inclusion $U^n\subseteq U_O$ is proved.

(2) $\Rightarrow$ (6). Let $\mathcal{T}$ be the topology of X. Denote by τ the minimal cardinality of a subfamily $\theta\subseteq\mathcal{T}$ such that $\bigcap\theta\notin\mathcal{T}$. If the space X is a counterexample to the implication, there must be a locally finite family $\gamma=\{U_\alpha:\alpha<\tau\}$ of open sets in X with $|\gamma|=\tau$. We can assume that γ is discrete (see Lemma 1 of [11]). The definition of τ implies that there exist a point $x^*\in X$ and a decreasing sequence $\{V_\alpha:\alpha<\tau\}$ of open neighbourhoods of x^* such that x^* does not belong to the interior of the intersection $\bigcap\limits_{\alpha<\tau}V_\alpha$. For every $\alpha<\tau$ pick a point $a_\alpha\in U_\alpha$. Let continuous functions f_α and g_α on X with values in [0,1] be such that $f_\alpha(a_\alpha)=1$, $g_\alpha(x^*)=1$, f_α vanishes outside of U_α and g_α vanishes outside of V_α. Define continuous pseudometrics $d_{1,\alpha}$ and ϱ_α on X by

$$d_{1,\alpha}(x,y)=|f_\alpha(x)-f_\alpha(y)|\text{ and }\varrho_\alpha(x,y)=|g_\alpha(x)-g_\alpha(y)|\text{ for all }x,y\in X.$$

Apply Theorem 2.1 of [17] to obtain a continuous pseudometric $d_{2,\alpha}$ on X^2 such that $d_{1,\alpha}$ and $d_{2,\alpha}$ are right-concordant in the sense of Definition 1.3 of [17] and $d_{2,\alpha}$ satisfies the condition

$$\text{(RP)}\qquad d_{2,\alpha}((a,x),(a,y))=f_\alpha(a)\cdot\varrho_\alpha(x,y)\text{ for all }a,x,y\in X.$$

Put $d_1=\sum\limits_{\alpha<\tau}d_{1,\alpha}$ and $d_2=\sum\limits_{\alpha<\tau}d_{2,\alpha}$. Clearly, d_1 and d_2 are continuous right-concordant pseudometrics on X and X^2 respectively. By Theorem 1.4 of [17] there exists a continuous seminorm N on $G(X)$ satisfying the properties

$$\text{(P1)}\qquad N(a\cdot b^{-1})=d_1(a,b)\text{ for all }a,b\in X;$$

$$\text{(P2)}\qquad N(a\cdot x\cdot y^{-1}\cdot a^{-1})=d_2((a,x),(a,y))\text{ for all }a,x,y\in X.$$

Put $O=\{g\in G(X):N(g)<1\}$. Then O is open in $G(X)$ and, a fortiori, in $F(X)$. Finally, define an open entourage U_O^r of the diagonal $\triangle_2$ in X^4 by

$$U_O^r=\{(x,y,z,t)\in X^4:x\cdot y\cdot t^{-1}\cdot z^{-1}\in O\}.$$

Clearly, $U_O^r\in\mathcal{V}^*|_{X^2}$, and we claim that for each continuous pseudometric ϱ on X there exist an ordinal $\alpha<\tau$ and a point $x\in X$ such that $\varrho(x^*,x)<1$ and $a_\alpha\cdot x^*\cdot x^{-1}\cdot a_\alpha^{-1}\notin O$, that is, $(a_\alpha,x^*,a_\alpha,x)\notin U_O^r$. The latter will obviously contradict (2).

Indeed, let ϱ be a continuous pseudometric on X. Since $x^*\notin\operatorname{Int}\bigcap\limits_{\alpha<\tau}V_\alpha$, one can find an ordinal $\alpha<\tau$ and a point $x\in X\setminus V_\alpha$ such that $\varrho(x^*,x)<1$. We have

$$\begin{aligned}N(a_\alpha\cdot x^*\cdot x^{-1}\cdot a_\alpha^{-1})&\overset{(P2)}{=}d_2((a_\alpha,x^*),(a_\alpha,x))\\&\geqslant d_{2,\alpha}((a_\alpha,x^*),(a_\alpha,x))\overset{(RP)}{=}f_\alpha(a_\alpha)\cdot|g_\alpha(x)-g_\alpha(x^*)|=1,\end{aligned}$$

because $f_\alpha(a_\alpha) = g_\alpha(x^*) = 1$ and $g_\alpha(x) = 0$ (for $x \notin V_\alpha$). Thus, $N(a_\alpha \cdot x^* \cdot x^{-1} \cdot a_\alpha^{-1}) \geqslant 1$, and hence $a_\alpha \cdot x^* \cdot x^{-1} \cdot a_\alpha^{-1} \notin O$.

(1) $\Rightarrow$ (6). Use the above argument along with Theorems 1.5 and 2.3 of [17].

(4) $\Rightarrow$ (6). This is the last implication to be proved. However, it has almost been shown in the proof of the implication (2) $\Rightarrow$ (6). Indeed, if X does not satisfy (6), define a continuous seminorm N on $G(X)$ and an open neighbourhood O of the identity in $F(X)$ as above. It was shown that for every continuous pseudometric ϱ on X there exist an ordinal $\alpha < \tau$ and a point $x \in X$ such that $\varrho(x^*, x) < 1$ and $a_\alpha \cdot x^* \cdot x^{-1} \cdot a_\alpha^{-1} \notin O$. We claim that for each neighbourhood W of the identity in $F(X)$, the set $U_W^l \setminus U_O^r$ is not empty, where $U_W^l = \{(x,y,z,t) \in X^4 : y^{-1} \cdot x^{-1} \cdot z \cdot t \in W\}$. To see this, for a given neighbourhood W of the identity choose a continuous pseudometric ϱ on X such that $\varrho(x,y) < 1$ implies $x^{-1} \cdot y \in W$ for all $x, y \in X$. One can find $\alpha < \tau$ and $x \in X$ satisfying the conditions $\varrho(x^*, x) < 1$ and $a_\alpha \cdot x^* \cdot x^{-1} \cdot a_\alpha^{-1} \notin O$. We have $(x^*)^{-1} \cdot a_\alpha^{-1} \cdot a_\alpha \cdot x = (x^*)^{-1} \cdot x \in W$, for $\varrho(x^*, x) < 1$. Thus, $(a_\alpha, x^*, a_\alpha, x) \in U_W^l \setminus U_O^r$, which proves our claim. However, this contradicts (4). The theorem is completely proved. $\square$

REMARK 2.2. Theorem 2.1 remains valid if one replaces "each" by "some" in conditions $(1') - (4')$.

3. Solution of Problem C for ${}^*\mathcal{V}$ and $\mathcal{V}^*$

The following theorem is an almost complete solution (modulo Question 3.2 below) of Problem C in the case of the left and right group uniformities ${}^*\mathcal{V}$ and $\mathcal{V}^*$ of $F(X)$.

THEOREM 3.1. *The following implications are valid for every space X:*
(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5), (1) $\Rightarrow$ (6) *and* (5)&(6) $\Rightarrow$ (1), *where*

(1) ${}^*\mathcal{V}|_{X^2} = \mathcal{U}_{X^2}$;
(2) $\mathcal{V}^*|_{X^2} = \mathcal{U}_{X^2}$;
(3) $\mathcal{U}_{X^2} = \mathcal{U}_X \times \mathcal{U}_X$;
(4) *there exists an infinite cardinal τ such that X is a P_τ-space and X^2 is pseudo-τ-compact;*
(5) *the projection $\pi : X \times X \to X$ is z-closed (that is, π takes zero-sets in X^2 to closed sets in X);*
(6) *for every disjoint open cover γ of X^2 there exists a disjoint open cover μ of X such that $\mu \times \mu = \{U \times V : U, V \in \mu\}$ is finer than γ.*

PROOF: (3) $\Rightarrow$ (1). Obviously, $\mathcal{U}_{X^2}$ is finer than ${}^*\mathcal{V}|_{X^2}$. From Theorem 1.6 it follows that ${}^*\mathcal{V}|_{X^2}$ is finer than $\mathcal{U}_X \times \mathcal{U}_X$. This proves the implication.

(3) $\Rightarrow$ (2). Just replace ${}^*\mathcal{V}$ by $\mathcal{V}^*$ in the above two lines.

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(1) $\Rightarrow$ (4). Suppose that (1) holds. We claim that if there exists a locally finite family of open sets in X^2 of cardinality τ^+ for some τ, then the same is true for X. Indeed, if X^2 contains such a family, then the uniform space $(X^2, \mathcal{U}_{X^2})$ is not τ-bounded (Assertion 1.2), and by (1), $(X^2, {}^*\mathcal{V}|_{X^2})$ is not τ-bounded either. However, ${}^*\mathcal{V}$ induces the maximal uniformity $\mathcal{U}_X$ on X; hence by Assertion 1.4, $(X, \mathcal{U}_X)$ fails to be τ-bounded. By Assertion 1.2, this means that X contains a locally finite family μ of open sets with $|\mu| > \tau$.

Now suppose that (4) does not hold. Let $\mathcal{T}$ be the topology of X. Denote by τ the minimal cardinality of a subfamily γ of $\mathcal{T}$ such that $\bigcap\gamma$ is not open in X, and choose a family $\gamma \subseteq \mathcal{T}$ with a non-open intersection satisfying $|\gamma| = \tau$. Pick a point $a \in \bigcap\gamma \setminus \operatorname{Int}(\bigcap\gamma)$. Since (4) fails, X^2 contains a locally finite family of open sets of cardinality τ. The above assertion gives us a locally finite family μ of open sets in X of the same cardinality τ. By Lemma 1 of [11] the family μ can even be chosen discrete. Let $\mu = \{U_\alpha : \alpha < \tau\}$ and $\gamma = \{V_\alpha : \alpha < \tau\}$. For each $\alpha < \tau$ define continuous real-valued functions f_α and g_α on X with values in $[0,1]$ such that f_α is equal to one at some point of U_α and vanishes outside of U_α, $g_\alpha(a) = 1$ and g_α vanishes outside V_α. For any points $x, y, z, t \in X$ put

$$d((x,y),(z,t)) = \sum_{\alpha<\tau} |f_\alpha(x)\cdot g_\alpha(y) - f_\alpha(z)\cdot g_\alpha(t)|.$$

Since the family $\{U_\alpha \times V_\alpha : \alpha < \tau\}$ is discrete in X^2, d is a continuous pseudometric on X^2. Obviously, $d \leqslant 1$. Define an entourage W of the diagonal in X^4 by $W = \{(x,y,z,t) \in X^4 : d((x,y),(z,t)) < 1\}$. Then $W \in \mathcal{U}_{X^2}$. We claim that the existence of W contradicts (1).

Let O be any open neighbourhood of the identity in $F(X)$. Put $V(a) = X \bigcap a\cdot O$; $V(a)$ is an open neighbourhood of a in X. Since $a \notin \operatorname{Int}\bigcap\gamma$, one can find an ordinal $\alpha < \tau$ such that $V(a) \setminus V_\alpha \neq \emptyset$. Pick points $b \in V(a)\setminus V_\alpha$ and $x \in U_\alpha$ with $f_\alpha(x) = 1$. We have

$$d((x,a),(x,b)) \geqslant |f_\alpha(x)\cdot g_\alpha(a) - f_\alpha(x)\cdot g_\alpha(b)| = |g_\alpha(a) - g_\alpha(b)| = 1,$$

because $b \notin V_\alpha$ and $g_\alpha(b) = 0$. So, $(x,a,x,b) \notin W$. On the other hand, $(x\cdot a)^{-1}\cdot(x\cdot b) = a^{-1}\cdot b \in O$, for $b \in V(a) \subseteq a\cdot O$. Thus, we have proved that $W_O \setminus W \neq \emptyset$ for every neighbourhood O of the identity in $F(X)$, where $W_O = \{(x,y,z,t) \in X^4 : y^{-1}\cdot x^{-1}\cdot z\cdot t \in O\}$. This contradicts (1).

(2) $\Rightarrow$ (4). Apply the above reasoning to the pseudometric ϱ on X^2 defined by $\varrho((x,y),(z,t)) = d((y,x),(t,z))$.

(1) $\Rightarrow$ (3). Let (1) hold. Since (1) implies (4), there exists an infinite cardinal τ such that X is a P_τ-space and X^2 (and, a fortiori, X) is pseudo-τ-compact. In its

turn, this and Theorem 2.1 together imply that ${}^*\mathcal{V}|_{X^2} = \mathcal{U}_X \times \mathcal{U}_X$. This equality and (1) imply that $\mathcal{U}_{X^2} = \mathcal{U}_X \times \mathcal{U}_X$.

(2) $\Rightarrow$ (3). Just repeat the above argument with $\mathcal{V}^*$ instead of ${}^*\mathcal{V}$ and apply (2) instead of (1).

Thus, we have now proved the equivalence of (1), (2) and (3).

(4) $\Rightarrow$ (5). We use the argument from [**1**]. Let Z be a zero-set in $X \times X$ and $f : X \times X \to [0,1]$ a continuous function such that $Z = f^{-1}(0)$. Assume that $\pi(Z)$ is not closed in X and choose a point $a \in \operatorname{cl}\pi(Z) \setminus \pi(Z)$, where π is the projection of $X \times X$ onto the first factor. We can assume that $f(a,x) = 1$ for each $x \in X$ — otherwise consider continuous function g on $X \times X$ defined by $g(x,y) = \min\{f(x,y)/f(a,y), 1\}$. Since X is a P_τ-space, we can define by induction a sequence $\{(x_\alpha, y_\alpha) : \alpha < \tau\}$ of points of Z and two sequences $\{W_\alpha : \alpha < \tau\}$ and $\{W'_\alpha : \alpha < \tau\}$ of open subsets of $X \times X$ such that $W_\alpha = U_\alpha \times V_\alpha$ is a neighbourhood of (x_α, y_α) satisfying $f(W_\alpha) \subseteq [0,1/3]$, $W'_\alpha = U'_\alpha \times V_\alpha$ is a neighbourhood of (a, y_α) satisfying $f(W'_\alpha) \subseteq [2/3,1]$, and $\operatorname{cl}U_\alpha \subseteq U_\beta$, $U_{\alpha+1} \bigcup U'_{\alpha+1} \subseteq U_\alpha$ whenever $\beta < \alpha < \tau$. It is easy to verify that the family $\{W_\alpha : \alpha < \tau\}$ of open sets in $X \times X$ is locally finite, contradicting the pseudo-τ-compactness of $X \times X$.

(1) $\Rightarrow$ (6). It suffices to show that (3) implies (6). Let (3) hold. Consider two cases.

(a) X is not a P-space. Since (3) implies (4), X^2 is pseudocompact. By a theorem of Glicksberg [4], $\beta(X^2) \cong \beta X \times \beta X$. Hence, to prove (6), one can assume that X is compact. To this end, use the well-known facts that a quasicomponent of any point in a compact space coincides with its component (see Theorem 6.1.23 of [**2**]) and that a product of two connected sets is connected.

(b) X is a P-space. Let γ be a disjoint open cover of X^2. Define a continuous pseudometric d on X^2 by $d(a,b) = 0$ if both a and b lie in the same element of γ and $d(a,b) = 1$ otherwise. By (3), there exists a continuous pseudometric ϱ on X such that the conditions $\varrho(x,z) < 1$ and $\varrho(y,t) < 1$ imply $d((x,y),(z,t)) < 1$ for all $x, y, z, t \in X$. Consider the equivalence relation $\sim$ on X defined by $x \sim y$ if and only if $\varrho(x,y) = 0$. The relation $\sim$ defines a partition μ of X to disjoint open sets, because X is a P-space. The definition of ϱ implies that the cover $\{U \times V : U, V \in \mu\}$ refines γ.

(5)&(6) $\Rightarrow$ (1). Assume that both (5) and (6) hold and deduce (3). Consider two cases.

(a) X is not a P-space. Then (5) implies that X is pseudocompact (otherwise one can define a zero-set in $X \times X$ with a non-closed projection). By a theorem of Tamano (see Problem 3.12.20(b) of [**2**]), pseudocompactness of X and (5) together imply that X^2 is pseudocompact. Therefore, $\beta(X^2) \cong \beta X \times \beta X$, which gives us (3).

(b) X is a P-space. Then (6) implies (3). Indeed, let d be any continuous pseudometric on X^2 and $\sim$ be an equivalence relation on X^2 defined by $a \sim b$ if and only if $d(a,b) = 0$. Since X and X^2 are P-spaces, the relation $\sim$ defines a partition γ of X^2 into open subsets. By (6), there exists a disjoint open cover μ of X such that $\{U \times V : U, V \in \mu\}$ refines γ. Define a continuous pseudometric ϱ on X by $\varrho(x,y) = 0$ if both x and y lie in some element of μ, and $\varrho(x,y) = 1$ otherwise. It is clear that $\varrho(x,z) < 1$ and $\varrho(y,t) < 1$ imply $d((x,y),(z,t)) = 0$ for all $x,y,z,t \in X$. This proves (3), and hence (1). ∎

QUESTION 3.2. Does the condition (4) of Theorem 3.1 imply (1)?

4. Treating Problem C

We prove that every k_ω-space X satisfies the equality ${}^*\mathcal{V}^*|_{X^2} = \mathcal{U}_{X^2}$ and characterise metrisable spaces satisfying this equality. However, we still have not a complete solution of Problem C. The case of a k_ω-space X is considered first. Recall that X is a k_ω-space (see [**3**, **7**]) if there exists a countable increasing sequence $\{K_n : n \in N\}$ of compact sets in X such that $X = \bigcup_{n \in N} K_n$ and a subset F of X is closed if and only if $F \cap K_n$ is closed for each $n \in N^+$.

Denote by $C_b(X)$ the linear space of all continuous real-valued bounded functions on X with the sup-norm defined by $\|f\| = \sup_{x \in X} |f(x)|$.

LEMMA 4.1. *Let X be a k_ω-space and U an open neighbourhood of the diagonal $\triangle_2$ in X^4, $\triangle_2 = \{(x,y,x,y) \in X^4 : x,y \in X\}$. Then there exists a continuous mapping $f : X \to C_b(X)$ such that for any $x,y,z,t \in X$ the inequalities*

(IL) $$\|f(x)\| \cdot \|f(y) - f(t)\| < 1, \ \|f(z)\| \cdot \|f(y) - f(t)\| < 1;$$

(IR) $$\|f(y)\| \cdot \|f(x) - f(z)\| < 1, \ \|f(t)\| \cdot \|f(x) - f(z)\| < 1$$

imply that $(x,y,z,t) \in U$.

PROOF: There exists an increasing sequence $\{X_n : n \in N\}$ of compact subsets of $X = \bigcup_{n \in N} X_n$ which determines the topology of X according to the definition of a k_ω-space. Being σ-compact and, a fortiori, paracompact, the space X^2 admits a continuous pseudometric d such that $\{(x,y,z,t) \in X^4 : d((x,y),(z,t)) < 1\} \subseteq U$. (Use Corollary 8.1.11 of [**2**].) Let $\mathcal{M}$ be the family of all continuous mappings of X onto second-countable spaces. The family $\mathcal{M}$ is $\aleph_0$-complete, that is, a diagonal product of any countable subfamily of $\mathcal{M}$ (considered as a mapping onto its image) belongs to $\mathcal{M}$. Put $\mathcal{N} = \{\varphi^4 : \varphi \in \mathcal{M}\}$. Then the family $\mathcal{N}$ is $\aleph_0$-complete and generates the topology of X^4. Since X^4 is σ-compact, and hence Lindelöf, Corollary 1 of [**13**] implies that

$\mathcal{N}$ has the factorisation property, that is, for every continuous mapping $p : X^4 \to \mathbf{R}$ one can find $\psi \in \mathcal{N}$ and a continuous mapping $q : \psi(X^4) \to \mathbf{R}$ such that $p = q \circ \psi$. In particular, there exist $\varphi \in \mathcal{M}$ and a continuous pseudometric d_1 on $\varphi(X) \times \varphi(X)$ such that $d((x,y),(z,t)) = d_1((\varphi(x),\varphi(y)),(\varphi(z),\varphi(t)))$ for all $x,y,z,t \in X$. Denote $Y = \varphi(X)$. Let ϱ_0 be a metric on Y generating the topology of Y. Of course, we can assume that $|Y| > 1$. One can find $a, b \in Y$ and an integer k such that $k \cdot \varrho_0(a,b) > 2$. Put $\varrho = \min\{1, k\varrho_0\}$. Then ϱ is a continuous pseudometric on Y with the following property:

(*) for any point $x \in Y$ there exists $y \in Y$ such that $\varrho(x,y) = 1$.

The function ϱ_2 defined by $\varrho_2((x,y),(z,t)) = \varrho(x,z) + \varrho(y,t)$ for all $x,y,z,t \in Y$ is a continuous metric on Y^2. For every point $(x,y) \in Y^2$ denote

$$B(x,y) = \{(z,t) \in Y^2 : d_1((x,y),(z,t)) < 1\} \text{ and } \varepsilon(x,y) = \varrho_2\big((x,y), Y^2 \setminus B(x,y)\big).$$

Obviously, $\varepsilon(x,y) > 0$ for all $x, y \in Y$. Since $Y_n = \varphi(X_n)$ is compact, for every $n \in \mathbb{N}$ there exists $\varepsilon_n > 0$ such that $\varrho_2(\overline{x}, \overline{y}) < \varepsilon_n$ implies $d_1(\overline{x}, \overline{y}) < 1$ for all $\overline{x}, \overline{y} \in Y_n^2$. Denote $V_n = \{(x, y) \in Y^2 : \varrho_2\big((x, y), Y_n^2\big) < \varepsilon_n/2\}$, an open neighbourhood of Y_n^2 in Y^2. By compactness of Y_n, there exists an open subset O_n of Y such that $Y_n \subseteq O_n$ and $O_n \times O_n \subseteq V_n$. For every $n \in N$ define a continuous real-valued function h_n on Y such that $h_n(Y_n) = 0$, $h_n(Y \setminus O_n) = 4/\varepsilon_{n+1}$ and $h_n \geqslant 0$. Then put $h = \sum_{n=1}^{\infty} h_n$, where $h_0 \equiv 4 + 4/\varepsilon_0$. Obviously, $4 < h(y) < \infty$ for each $y \in Y$, but h can be discontinuous. We claim that the function $\widetilde{h} = h \circ \varphi$ on X is continuous. This readily follows from the fact that X is a k_ω-space with the decomposition $X = \bigcup_{n=0}^{\infty} X_n$ and the choice of the functions h_n; $n \in N$. Put $\widetilde{\varrho}(x,y) = \varrho(\varphi(x),\varphi(y))$ for $x,y \in X$. Clearly, $\widetilde{\varrho}$ is a bounded continuous pseudometric on X.

Consider the mapping $f : X \to C_b(X)$ defined by $[f(x)](y) = \widetilde{h}(x) \cdot \widetilde{\varrho}(x,y)$; $x,y \in X$. Obviously, $\|f(x)\| \leqslant \widetilde{h}(x)$, because $\widetilde{\varrho} \leqslant 1$. On the other hand, (*) implies that there exists a point $y \in Y$ such that $\varrho(x,y) = 1$; hence $\|f(x)\| = \widetilde{h}(x)$ for each $x \in X$. Let us check the continuity of f. For any $x_1, x_2, y \in X$ we have

$$\begin{aligned}\left|\widetilde{h}(x_1)\widetilde{\varrho}(x_1,y) - \widetilde{h}(x_2)\widetilde{\varrho}(x_2,y)\right| &\leqslant \left|\widetilde{h}(x_1)\widetilde{\varrho}(x_1,y) - \widetilde{h}(x_1)\widetilde{\varrho}(x_2,y)\right| \\ &+ \left|\widetilde{h}(x_1)\widetilde{\varrho}(x_2,y) - \widetilde{h}(x_2)\widetilde{\varrho}(x_2,y)\right| \leqslant \widetilde{h}(x_1)\widetilde{\varrho}(x_1,x_2) + \left|\widetilde{h}(x_1) - \widetilde{h}(x_2)\right|.\end{aligned}$$

Taking the supremum over $y \in Y$ in both left and right parts of the above inequalities, we get $\|f(x_1) - f(x_2)\| \leqslant \widetilde{h}(x_1) \cdot \varrho(x_1,x_2) + \left|\widetilde{h}(x_1) - \widetilde{h}(x_2)\right|$. The latter inequality implies the continuity of f.

We claim that the mapping f is as required. To this end, one auxiliary assertion will be useful.

CLAIM. $\|f(a)-f(b)\| \geqslant \widetilde{\varrho}(a,b)$ for all $a,b \in X$. Moreover, if $\varphi(a) \notin O_n$ for some $n \in N$ then $\|f(a)-f(b)\| \geqslant \widetilde{\varrho}(a,b)/\varepsilon_{n+1}$.

Indeed, by the definition of the sup-norm and the function f we have $\|f(a)-f(b)\| \geqslant \left|\widetilde{h}(a)\widetilde{\varrho}(a,y) - \widetilde{h}(b)\widetilde{\varrho}(b,y)\right|$ for each $y \in Y$. Putting $y = b$ in the above inequality, we get $\|f(a)-f(b)\| \geqslant \widetilde{h}(a)\widetilde{\varrho}(a,b) \geqslant \widetilde{\varrho}(a,b)$, because $\widetilde{h}(a) = h(\varphi(a)) \geqslant 1$ for each $a \in X$. Furthermore, if $\varphi(a) \notin O_n$ then the definition of the function h implies $h(\varphi(a)) \geqslant 1/\varepsilon_{n+1}$, and hence $\|f(a)-f(b)\| \geqslant \widetilde{h}(a)\widetilde{\varrho}(a,b) \geqslant \widetilde{\varrho}(a,b)/\varepsilon_{n+1}$. This proves the claim.

Let x,y,z,t be arbitrary points of X satisfying (IL) and (IR). Denote $x' = \varphi(x), \ldots, t' = \varphi(t)$. First, we show that $d_1((x',y'),(z',t')) < 1$, or equivalently, $d((x,y),(z,t)) < 1$. Consider two cases.

CASE 1. $\{x',y',z',t'\} \subseteq O_0$. By the definition of V_0 and O_0, we have $\varepsilon(x',y') > \varepsilon_0/2$. By the definition of $\varepsilon(x',y')$, the latter inequality means that $d_1((x',y'),(z',t')) < 1$ whenever the point $(z',t') \in Y^2$ satisfies the condition $\varrho_2((x',y'),(z',t')) \leqslant \varepsilon_0/2$. So, it suffices to check the inequality $\varrho_2((x',y'),(z',t')) \leqslant \varepsilon_0/2$. Using (IL), (IR) and the Claim, we have

$$\varrho(x',z') = \widetilde{\varrho}(x,z) \leqslant \|f(x)-f(z)\| \leqslant 1/\|f(y)\| = 1/\widetilde{h}(y) \leqslant \varepsilon_0/4,$$

because $\widetilde{h} = \varphi \circ h$ and $h \geqslant 4/\varepsilon_0$. The same argument shows that $\varrho(y',t') \leqslant \varepsilon_0/4$. Thus, $\varrho_2((x',y'),(z',t')) = \varrho(x',z') + \varrho(y',t') \leqslant \varepsilon_0/2$, which implies the inequality $d_1((x',y'),(z',t')) < 1$.

CASE 2. $\{x',y',z',t'\} \setminus O_0 \neq \emptyset$. Let n be the maximal integer with the property $\{x',y',z',t'\} \setminus O_n \neq \emptyset$. Without loss of generality we can assume that $x' \notin O_n$. From the definition of n it follows that $(x',y') \in O_{n+1} \times O_{n+1} \subseteq V_{n+1}$, and hence $\varepsilon(x',y') > \varepsilon_{n+1}/2$. The definition of h and the fact that $x' \notin O_n$ imply $h(x') \geqslant 4/\varepsilon_{n+1}$. Apply Claim and (IL), (IR) to get the following estimate:

$$\varrho(y',t') = \widetilde{\varrho}(y,t) \leqslant \|f(y)-f(t)\| \leqslant 1/\|f(x)\| = 1/h(x') \leqslant \varepsilon_{n+1}/4.$$

Then apply the "$\varphi(a) \notin O_n$" part of the Claim and (IL), (IR) to the points x,y,z,t:

$$\varrho(x',z') = \widetilde{\varrho}(x,z) \leqslant \varepsilon_{n+1} \cdot \|f(x)-f(z)\| \leqslant \varepsilon_{n+1}/\|f(y)\| \leqslant \varepsilon_{n+1}/4,$$

because $\|f(y)\| = h(y')$ and $h \geqslant 4$. This implies the estimate

$$\varrho_2((x',y'),(z',t')) = \varrho(x',z') + \varrho(y',t') < \varepsilon_{n+1}/4 + \varepsilon_{n+1}/4 = \varepsilon_{n+1}/2.$$

Since $\varepsilon(x',y') > \varepsilon_{n+1}/2$, the above inequality implies that $d_1((x',y'),(z',t')) < 1$. (The argument here is just the same as in Case 1.)

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So, the inequality $d((x,y),(z,t)) < 1$ is proved. From the definition of the pseudometric d it readily follows that $(x,y,z,t) \in U$. This completes the proof. ∎

THEOREM 4.2. *If X is a k_ω-space then ${}^*\mathcal{V}^*|_{X^2} = \mathcal{U}_{X^2}$.*

PROOF: Let U be an open entourage of the diagonal Δ_2 in X^4. Define a continuous mapping $f : X \to C_b(X)$ as in Lemma 4.1. Put $g(x) = \|f(x)\|$ and $\varrho(x,y) = \|f(x) - f(y)\|$ for all $x,y \in X$. Then $\varrho_1 = \min\{\varrho, 1\}$ is a continuous bounded pseudometric on X, and we can apply Theorem 2.1 of [17] to the function f and the pseudometric ϱ_1 to get two continuous seminorms N_l and N_r on the subgroup $G(X)$ of $F(X)$ satisfying the following conditions for all $x,y,z,t \in X$:

(1) $N_l(y^{-1} \cdot x^{-1} \cdot z \cdot y) = g(y) \cdot \varrho_1(x,z)$ and $N_r(x \cdot y \cdot z^{-1} \cdot x^{-1}) = g(x) \cdot \varrho_1(y,z)$;

(2) $N_l(y^{-1} \cdot x^{-1} \cdot z \cdot y) \leqslant N_l(y^{-1} \cdot x^{-1} \cdot z \cdot t)$ and $N_r(x \cdot y \cdot t^{-1} \cdot x^{-1}) \leqslant N_r(x \cdot y \cdot t^{-1} \cdot z^{-1})$.

Put $N = N_l + N_r$ and define an open neighbourhood O of the identity in $F(X)$ by $O = \{g \in G(X) : N(g) < 1\}$. We claim that the element $V_O = \{(g,h) \in F(X) \times F(X) : g^{-1} \cdot h \in O,\ g \cdot h^{-1} \in O\}$ of the uniformity ${}^*\mathcal{V}^*$ on $F(X)$ satisfies the condition $V_O \cap (X^2 \times X^2) \subseteq U$. (Recall that we identify X^2 with the image $i_2(X^2) \subseteq F(X)$ under the homeomorphic embedding i_2, where $i_2(x,y) = x \cdot y$.) Indeed, let x,y,z,t be arbitrary points of X and suppose that $((x,y),(z,t)) \in V_O$. By the definition of O and V_O, we have

$$N_l(y^{-1} \cdot x^{-1} \cdot z \cdot t) = N_l(t^{-1} \cdot z^{-1} \cdot x \cdot y) < 1$$

and

$$N_r(x \cdot y \cdot t^{-1} \cdot z^{-1}) = N_r(z \cdot t \cdot y^{-1} \cdot x^{-1}) < 1.$$

Use (1) and (2) to conclude that

$$\|f(y)\| \cdot \|f(x) - f(z)\| < 1, \qquad \|f(t)\| \cdot \|f(x) - f(z)\| < 1,$$
$$\|f(x)\| \cdot \|f(y) - f(t)\| < 1, \qquad \|f(z)\| \cdot \|f(y) - f(t)\| < 1.$$

These inequalities and the choice of the mapping f together imply that $((x,y),(z,t)) \in U$. Therefore, the inclusion $V_O \cap (X^2 \times X^2) \subseteq U$ is proved. ∎

Theorem 4.4 below gives a solution to Problem C for metrisable spaces. We start with an auxiliary result.

Let γ be a cover of a space Y and $y_1, y_2 \in Y$. We say that y_1 and y_2 are γ-neighbours and write $y_1 \overset{\gamma}{\sim} y_2$ if there exists $U \in \gamma$ such that $y_1, y_2 \in U$. Again, $C_b(X)$ stands for the linear space of continuous real-valued bounded functions on X with the sup-norm.

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Lemma 4.3. *Let K be a compact space with a metric ϱ_K and X a locally compact metrisable space. Then for any open covers γ_1 of $X \times K$ and γ_2 of $K \times X$ there exists a continuous mapping $f : X \to C_b(X)$ such that for all $a, b \in K$ and $x \in X$ the inequality $\|f(x)\| \cdot \varrho_K(a,b) < 1$ implies that $(x,a) \overset{\gamma_1}{\sim} (x,b)$ and $(a,x) \overset{\gamma_2}{\sim} (b,x)$.*

Proof: Every locally compact metrisable space is a free topological sum of its open σ-compact subspaces [**2**, Theorem 5.1.27], so we can assume that X is σ-compact. Represent X as a union $\bigcup_{i=1}^{\infty} X_i$, where X_i is open in X, $\mathrm{cl}\, X_i \subseteq X_{i+1}$ and $\mathrm{cl}\, X_i$ is compact for each $i \in N^+$. Let d be a metric on X, $d \leqslant 1$. Denote by i_0 the standard embedding of X to $C_b(X)$, $[i_0(a)](x) = d(a,x)$ for all $a, x \in X$. Obviously, we have $\|i_0(a) - i_0(b)\| = d(a,b)$ for all $a, b \in X$, so i_0 is a homeomorphic embedding.

Define metrics κ_1 and κ_2 respectively on $X \times K$ and $K \times X$ by $\kappa_1((x,a),(y,b)) = d(x,y) + \varrho_K(a,b) = \kappa_2((a,x),(b,y))$ for all $a, b \in K$ and $x, y \in X$. For every $j \in N^+$, let $\varepsilon_{j,1}$ be a Lebesgue number of the cover $\{V \cap (X_j \times K) : V \in \gamma_1\}$ of the compact space $\mathrm{cl}\, X_j \times K$ with respect to κ_1, that is, a positive real number with the property that every pair of points $\overline{x}, \overline{y}$ of $\mathrm{cl}\, X_j \times K$ satisfying $\kappa_1(\overline{x}, \overline{y}) < \varepsilon_{j,1}$ is contained in some element of γ_1. Analogously, we define $\varepsilon_{j,2}$ as a Lebesgue number of the cover $\{V \cap (K \times \mathrm{cl}\, X_j) : V \in \gamma_2\}$ with respect to κ_2.

Denote by n_j a positive integer with $1/n_j \leqslant \min\{\varepsilon_{j,1}, \varepsilon_{j,2}\}$; $j \in N^+$. There exists a continuous mapping $g : X \to \mathbf{R}$ such that $g(x) \geqslant n_1$ for each $x \in X_1$ and $g(x) \geqslant n_{j+1}$ for each $x \in X_{j+1} \setminus X_j$, $j \in N^+$. Indeed, for every $j \geqslant 2$ we can find a continuous function $g_j : X \to \mathbf{R}$ such that $g_j(x) = n_{j+1}$ whenever $x \in \mathrm{cl}\, X_{j+1} \setminus X_j$ and $g_j(x) = 0$ for each $x \in \mathrm{cl}\, X_{j-1} \bigcup (X \setminus X_{j+2})$. Let also g_1 be a continuous function such that $g_1(x) = \max\{n_1, n_2\}$ for all $x \in \mathrm{cl}\, X_2$ and $g_1(x) = 0$ for each $x \in X \setminus X_3$. Then the function $g = \sum_{j=1}^{\infty} g_j$ is continuous and has the above property.

Put $f(x) = g(x) \cdot i_0(x)$ for every $x \in X$. Obviously, the mapping $f : X \to C_b(X)$ is continuous. Let points $a, b \in K$ and $x \in X$ be arbitrary and suppose that $\|f(x)\| \cdot \varrho_K(a,b) < 1$. Denote by j the minimal integer with $x \in X_j$. Then $\|f(x)\| = |g(x)| \cdot \|i_0(x)\| = g(x) \geqslant n_j$. It is clear that $\kappa_1((x,a),(x,b)) = d(x,x) + \varrho_K(a,b) = \varrho_K(a,b)$, and hence

$$1 > \|f(x)\| \cdot \varrho_K(a,b) \geqslant n_j \cdot \varrho_K(a,b) = n_j \cdot \kappa_1((x,a),(x,b)).$$

So, the choice of n_j and ε_j implies that $(x,a) \overset{\gamma_1}{\sim} (x,b)$. The same argument shows that $(a,x) \overset{\gamma_2}{\sim} (b,x)$. ∎

Theorem 4.4. *The following conditions are equivalent for any metrisable space X:*

(a) $\quad {}^*\mathcal{V}^*|_{X^2} = \mathcal{U}_{X^2}$;

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(b) *either X is locally compact or the set X' of non-isolated points of X is compact.*

PROOF: $(a) \Rightarrow (b)$. Assume the contrary. Let X be not locally compact and X' be not compact. By our assumption, there exists a point $a \in X$ the closure of any neighbourhood of which is not compact. Using the non-compactness of X', we can choose a base $\{U_n : n \in N^+\}$ of X at a such that $\operatorname{cl} V_{n+1} \subseteq V_n$, $\operatorname{cl} V_n \setminus V_{n+1}$ is not compact for each $n \in N^+$ and $X' \setminus V_1$ is not compact as well. For every $n \in N^+$ choose an infinite closed discrete subset $\{x_{n,k} : k \in N^+\}$ of X lying in $V_n \setminus V_{n+1}$. Let also $\{y_n : n \in N^+\}$ be an infinite closed discrete subset of $X' \setminus V_1$. For every $n \in N^+$ choose a sequence $\{z_{n,l} : l \in N^+\} \subseteq X \setminus (V_1 \cup \{y_n\})$ converging to y_n. It is easy to see that the set $F = \{(x_{n,k}, z_{n,k}, x_{n,k}, y_n) : n, k \in N^+\}$ is closed in X^4. Let $\triangle_2 = \{(x,y,x,y) : x, y \in X\}$ be the diagonal in X^4. Obviously, F and $\triangle_2$ are disjoint, so $U = X^4 \setminus F$ is an open neighbourhood of $\triangle_2$ in X^4. The metrisability of X implies that $U \in \mathcal{U}_{X^2}$.

Let O be an arbitrary neighbourhood of the identity e in $F(X)$. There exists a neighbourhood O_1 of e such that $O_1^{-1} = O_1$ and $O_1^3 \subseteq O$. Choose a neighbourhood O_2 of e so that $O_2^{-1} = O_2 \subseteq O$ and $a \cdot O_2 \cdot a^{-1} \subseteq O_1$. Put $W = X \cap (O_1 \cdot a)$. Since the sets U_n, $n \in N^+$, form a base of X at a and W is a neighbourhood of the point a in X, there exists an integer $p \in N^+$ such that $x_{n,k} \in W$ for all $n \geqslant p$ and $k \in N^+$ (it suffices to choose p with $U_p \subseteq W$). Choose $l \in N^+$ so that $z_{p,k} \in O_2 \cdot y_p$ for all $k \geqslant l$. The choice of l implies that

$$(1) \qquad z_{p,l} \cdot y_p^{-1} \in O_2 \text{ and } y_p \cdot z_{p,l}^{-1} \in O_2^{-1} = O_2.$$

We have also

$$(2) \quad x_{p,l} \cdot \left(z_{p,l} \cdot y_p^{-1}\right) \cdot x_{p,l}^{-1} \in (O_1 \cdot a) \cdot O_2 \cdot \left(a^{-1} \cdot O_1^{-1}\right) = O_1 \cdot \left(a \cdot O_2 \cdot a^{-1}\right) \cdot O_1 \subseteq O_1^3 \subseteq O.$$

Consider the entourage U_O of the diagonal in X^4 generated by O,

$$U_O = \{(x,y,z,t) \in X^4 : x \cdot y \cdot t^{-1} \cdot z^{-1} \in O \text{ and } y^{-1} \cdot x^{-1} \cdot z \cdot t \in O\}.$$

Then $U_O \in {}^*\mathcal{V}^*|_{X^2}$ and from (1), (2) and the inclusion $O_2 \subseteq O$ it follows that $(x_{p,l}, z_{p,l}, x_{p,l}, y_p) \in U_O \cap F$. Thus, we have proved that $U_O \setminus U \neq \emptyset$ for each neighbourhood O of the identity in $F(X)$, that is, ${}^*\mathcal{V}^*|_{X^2} \neq \mathcal{U}_{X^2}$.

$(b) \Rightarrow (a)$. We consider two cases.

I. *X' is compact.* Let U be an open entourage of the diagonal $\triangle_2$ in X^4. Since X^2 is paracompact, there exists an open symmetric neighbourhood U_1 of $\triangle_2$ in X^4 such that $U_1 \circ U_1 \subseteq U$. Denote by ϱ_0 a bounded metric on X that induces the

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topology of X. Define another metric ϱ on X by $\varrho(x,y) = 0$ if $x = y$ and $\varrho(x,y) = \inf\{\varrho_0(x,a) + \varrho_0(a,y) : a \in X'\}$ otherwise. We omit an easy proof of the fact that ϱ satisfies the inequality of triangle, is continuous bounded and generates the same topology of X. From the definition of ϱ it also follows that

$$\varrho|_{X'} = \varrho_0|_{X'} \text{ and } \varrho(a,b) \geqslant \varrho(a,X') \text{ for all } a,b \in X,\ a \neq b. \tag{3}$$

Since X' is compact, there exists $\varepsilon > 0$ such that $(x,y,z,t) \in U_1$ whenever $x,y \in X'$, $z,t \in X$ and $\varrho(x,z) < \varepsilon$, $\varrho(y,t) < \varepsilon$. Put $V = \{x \in X : \varrho(x,X') < \varepsilon/2\}$. Then V is a clopen neighbourhood of X' in X. Using the compactness of X' once again, for each $x \in X \setminus V$ choose $\varepsilon_x > 0$ so that all the points (x,t,x,y), (x,y,x,t), (t,x,y,x), (y,x,t,x) belong to U_1 for all $y \in X'$ and $t \in X$ satisfying $\varrho(y,t) < \varepsilon_x$.

Denote by $C_b(X)$ the space of all continuous real-valued bounded functions on X with the sup-norm $\|\cdot\|$. Let $h : X \to \mathbb{R}$ be a continuous function such that $h(y) \geqslant 1 + 2/\varepsilon$ for all $y \in X$ and $h(x) \geqslant 1 + 2/\varepsilon_x$ for all $x \in X \setminus V$. Define a mapping $f : X \to C_b(X)$ by $[f(x)](y) = h(x) \cdot \varrho(x,y)$; $x,y \in X$. Apply the argument in the proof of Lemma 4.1 to verify that f is continuous and has the following properties:

$$\|f(x) - f(y)\| \geqslant (1 + 2/\varepsilon) \cdot \varrho(x,y) \text{ for all } x,y \in X; \tag{4}$$

$$\|f(x) - f(y)\| \geqslant (1 + 2/\varepsilon_x) \cdot \varrho(x,y) \text{ whenever } x \in X \setminus V \text{ and } y \in X. \tag{5}$$

In particular, the inequality $\|f(x) - f(y)\| \geqslant \varrho(x,y)$ holds for all $x,y \in X$.

Put $\varrho_1(x,y) = \|f(x) - f(y)\|$ for all $x,y \in X$ and apply Theorems 2.1 and 2.3 of [17] to the metric ϱ_1 and the function f on X to obtain continuous seminorms N_l and N_r on $G(X)$ satisfying the following conditions for all $x,y,z,t \in X$:

(i) $N_l(y^{-1} \cdot x^{-1} \cdot z \cdot y) = \|f(y)\| \cdot \varrho_1(x,z)$ and $N_r(x \cdot y \cdot t^{-1} \cdot x^{-1}) = \|f(x)\| \cdot \varrho_1(y,t)$;

(ii) $N_l(y^{-1} \cdot x^{-1} \cdot z \cdot t) \geqslant \varrho_1(x,z) + \varrho_1(y,t)$ and $N_r(x \cdot y \cdot t^{-1} \cdot z^{-1}) \geqslant \varrho_1(x,z) + \varrho_1(y,t)$.

We put $N = N_l + N_r$ and $O = \{g \in G(X) : N(g) < 1\}$. Then O is an open neighbourhood of the identity in $F(X)$ and we claim that the entourage $V_O = \{(x,y,z,t) \in X^4 : y^{-1} \cdot x^{-1} \cdot z \cdot t \in O,\ x \cdot y \cdot t^{-1} \cdot z^{-1} \in O\}$ of the diagonal $\triangle_2$ in X^4 is contained in U. Indeed, if $(x,y,z,t) \in V_O$ then $N_l(y^{-1} \cdot x^{-1} \cdot z \cdot t) < 1$ and $N_r(x \cdot y \cdot t^{-1} \cdot z^{-1}) < 1$. Suppose that $\{x,y,z,t\} \subseteq V$. There exist points $x_1, y_1 \in X'$ such that $\varrho(x,x_1) < \varepsilon/2$ and $\varrho(y,y_1) < \varepsilon/2$. Applying (4) and (ii), we get

$$2/\varepsilon \cdot \varrho(x,z) \leqslant \varrho_1(x,z) \leqslant N_r(x \cdot y \cdot t^{-1} \cdot z^{-1}) < 1, \text{ that is, } \varrho(x,z) < \varepsilon/2.$$

From the choice of the point x_1 it follows that $\varrho(x,x_1) < \varepsilon/2$; therefore $\varrho(x_1,z) \leqslant \varrho(x_1,x)+\varrho(x,z) < \varepsilon/2+\varepsilon/2 = \varepsilon$. Analogously, one can show that $\varrho(y_1,t) < \varepsilon$. These two inequalities and the choice of ε together imply that $(x_1,y_1,z,t) \in U_1$. By the same reasoning we also have $(x_1,y_1,x,y) \in U_1$. It remains to note that U_1 is symmetric, and hence $(x,y,z,t) \in U_1 \circ U_1 \subseteq U$.

We can now assume that $\{x,y,z,t\}\setminus V \neq \emptyset$. It suffices to consider the case $x \notin V$. We claim that $z = x$. Indeed, by (3), $\varrho(x,z) \geqslant \varrho(x,X') \geqslant \varepsilon x/2$ if $z \neq x$. So, (ii) and (5) imply that

$$N_r(x\cdot y\cdot t^{-1}\cdot z^{-1}) \geqslant \varrho_1(x,z) \geqslant 2/\varepsilon_x \cdot \varrho(x,z) \geqslant 1,$$

which contradicts the inequality $N_r(x\cdot y\cdot t^{-1}\cdot z^{-1}) < 1$. Thus, $z = x$. Further, by (i) and (5), we have

$$1 > N_r(x\cdot y\cdot t^{-1}\cdot x^{-1}) = \|f(x)\|\cdot\varrho_1(y,t) \geqslant 2/\varepsilon_x\cdot\varrho(y,t), \text{ that is, } \varrho(y,t) < \varepsilon_x/2.$$

We claim that either $y = t$ or $\varrho(y,X') < \varepsilon_x/2$. Suppose not, let $y \neq t$ and $\varrho(y,X') \geqslant \varepsilon_x/2$. Then $\varrho(y,t) \geqslant \varrho(y,X') \geqslant \varepsilon_x/2$ by (3), which contradicts the above inequality. The case $y = t$ is trivial: $(x,y,z,t) = (x,y,x,y) \in \triangle_2 \subseteq U$; hence we assume that $y \neq t$. There exists a point $y_1 \in X'$ such that $\varrho(y,y_1) < \varepsilon_x/2$, and we have $\varrho(y_1,t) \leqslant \varrho(y_1,y)+\varrho(y,t) < \varepsilon_x/2+\varepsilon_x/2 = \varepsilon_x$, that is, $\varrho(y_1,t) < \varepsilon_x$. Since $y_1 \in X'$, these two inequalities together with the definition of ε_x imply that $(x,y,x,y_1) \in U_1$ and $(x,y_1,x,t) \in U_1$. Therefore, $(x,y,z,t) = (x,y,x,t) \in U_1 \circ U_1 \subseteq U$. This completes the proof of the inclusion $V_O \subseteq U$. Since the entourage U of the diagonal $\triangle_2$ in X^4 was chosen arbitrarily and $V_O \in {}^*\mathcal{V}^*|_{X^2}$, the equality $\mathcal{U}_{X^2} = {}^*\mathcal{V}^*|_{X^2}$ is proved.

II. *X is locally compact.* Suppose we are given an entourage U of the diagonal $\triangle_2$ in X^4, $U \in \mathcal{U}_{X^2}$. Choose an entourage V of $\triangle_2$ in X^4 with $V\circ V \subseteq U$. There exists an open cover γ of X^2 such that $\bigcup\{A\times A : A \in \gamma\} \subseteq V$. Since X is paracompact, there exists an open locally finite cover μ of X such that cl W is compact for each $W \in \mu$. Let ϱ be a bounded continuous metric on X such that $\{(x,y) \in X^2 : \varrho(x,y) < 1\} \subseteq \bigcup\{W\times W : W \in \mu\}$. By Lemma 4.3, for every $W \in \mu$ there exists a continuous mapping $f_W : X \to C_b(X)$ such that for all $a,b \in \mathrm{cl}\,W$ and $x \in X$ the inequality $\|f_W(x)\|\cdot\varrho(a,b) < 1$ implies that $(x,a) \overset{\gamma}{\sim} (x,b)$ and $(a,x) \overset{\gamma}{\sim} (b,x)$. Apply Theorem 3.1 of [17] to define a continuous seminorm N on $G(X)$ such that $N(g) \geqslant \widehat{\varrho}(g,e)$ for each $g \in G(X)$ ($\widehat{\varrho}$ is the Graev extension of ϱ to an invariant pseudometric on $G(X)$) and $N(a^\varepsilon\cdot x^\varepsilon\cdot y^{-\varepsilon}\cdot b^{-\varepsilon}) \geqslant \max\{\|f_W(a)\|,\|f_W(b)\|\}\cdot\varrho(x,y)$ whenever the points $a,b,x,y \in X$ satisfy the conditions $N(a^\varepsilon\cdot x^\varepsilon\cdot y^{-\varepsilon}\cdot b^{-\varepsilon}) < 1$, $\varepsilon = \pm1$ and $x,y \in W \in \mu$. Then $O = \{g \in G(X) : N(g) < 1\}$ is an open neighbourhood of the identity in $F(X)$, and we claim that the entourage

$$U_O = \{(a,x,b,y) \in X^4 : a\cdot x\cdot y^{-1}\cdot b^{-1} \in O,\ x^{-1}\cdot a^{-1}\cdot b\cdot y \in O\}$$

of the diagonal $\triangle_2$ in X^4 is contained in U. Indeed, let $(a,x,b,y) \in U_O$. Then $N(a \cdot x \cdot y^{-1} \cdot b^{-1}) < 1$ and $N(x^{-1} \cdot a^{-1} \cdot b \cdot y) < 1$. Since $N(g) \geqslant \widehat{\varrho}(g,e)$ for each $g \in G(X)$, we have

$$\varrho(a,b) + \varrho(x,y) = \widehat{\varrho}(a \cdot x \cdot y^{-1} \cdot b^{-1}, e) \leqslant N(a \cdot x \cdot y^{-1} \cdot b^{-1}) < 1.$$

In particular, $\varrho(a,b) < 1$ and $\varrho(x,y) < 1$. The choice of ϱ implies that there exist elements W, W' of μ such that $a,b \in W$ and $x,y \in W'$. By the choice of N, we have $\|f_W(a)\| \cdot \varrho(x,y) \leqslant N(a \cdot x \cdot y^{-1} \cdot b^{-1}) < 1$ and $\|f_W(y)\| \cdot \varrho(a,b) \leqslant N(x^{-1} \cdot a^{-1} \cdot b \cdot y) < 1$. In its turn, the choice of the functions f_W and $f_{W'}$ implies that $(a,x) \overset{\gamma}{\sim} (a,y)$ and $(a,y) \overset{\gamma}{\sim} (b,y)$. Since $\bigcup\{A \times A : A \in \gamma\} \subseteq V$, we conclude that $(a,x,a,y) \in V$ and $(a,y,b,y) \in V$. It remains to note that $V \circ V \subseteq U$, and hence $(a,x,b,y) \in U$. This proves the inclusion $U_O \subseteq U$. □

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