A FORMAL SOLUTION OF $\sum_{i=1}^{\infty} A_i\, e^{B_i X} = X$

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1. Introduction. We consider a combinatorial enumeration problem involving certain collections of labelled rooted trees having coloured nodes and edges. Notations and definitions are introduced in §§2 and 3, and the problem is described in §4. We give recursion formulas for its solution in §5. Then, by using a modification of a method of Prüfer, we obtain a direct solution in terms of multinomial coefficients and power products in §§6 and 7. These results are combined in §8. Working in a formal power series algebra we find a formal solution of the equation

$$X = \sum_{i=1}^{\infty} A_i \exp(B_i X),$$

which expresses the unknown X as a multiple power series in the A_i and the B_i.

2. Definitions and notations. We let N_0 be the set of natural numbers including zero and N_1 the set of non-zero natural numbers. We let P be the set of infinite sequences of natural numbers (functions from N_1 into N_0) in which all but a finite number of terms are zero. For

$$k \in N_0, \quad \bar{p}, \bar{q} \in P, \quad \bar{p} = (p_1, p_2, \ldots), \quad \bar{q} = (q_1, q_2, \ldots),$$

we make the following definitions:

(1) $|\bar{p}| = \sum_{i=1}^{\infty} p_i$;

(2) $k\bar{p} = (kp_1, kp_2, \ldots)$;

(3) $\bar{p} + \bar{q} = (p_1 + q_1, p_2 + q_2, \ldots)$;

(4) $\bar{p} \geqslant \bar{q}$ if and only if $p_i \geqslant q_i$ for all $i \in N_1$;

(5) when $\bar{p} \geqslant \bar{q}$, $\bar{p} - \bar{q} = (p_1 - q_1, p_2 - q_2, \ldots)$;

(6) $\bar{p}^{\bar{q}} = p_1^{q_1} p_2^{q_2} \ldots$, where $0^0 = 1$ whenever it occurs;

(7) $\bar{0} = (0, 0, \ldots)$;

(8) for each $i \in N_1$, $\bar{e}_i$ is the sequence all of whose terms are 0 except the ith term, which is 1.

If $\bar{p} = (p_1, p_2, \ldots) \in P$ and X is a set having $|\bar{p}|$ elements, then we call a function $g: X \to N_1$ a *$\bar{p}$-colouring of X* if for each $i \in N_1$ there are exactly p_i elements $x \in X$ such that $g(x) = i$. We denote the set of all $\bar{p}$-colourings

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of X by $M(\bar{p}, X)$. The number of elements of $M(\bar{p}, X)$ is the multinomial coefficient

$$M(\bar{p}) = |\bar{p}|! \Big/ \left(\prod_{i=1}^{\infty} p_i!\right).$$

3. Cycle-free functions. If X is a set and f is a function which maps X into itself, then for each $k \in N_0$ we define f^k, the kth iterate of f, recursively as follows: for every $x \in X$, $f^0(x) = x$, $f^{k+1}(x) = f(f^k(x))$. A sequence x, $f(x), f^2(x), \ldots, f^m(x)$ in which $f^i(x) \neq x$ for $1 \leqslant i < m$ and $f^m(x) = x$ will be called a *cycle of* f if $m > 1$, and an element $x \in X$ such that $f(x) = x$ is called a *fixed point of* f. A function $f : X \rightarrow X$ having no cycles will be called *cycle-free.*

Each cycle-free function $f : X \rightarrow X$ may be pictured as a collection of rooted trees having the elements of the set X as nodes. The fixed points of f are the roots of the trees and an edge descends from the node x to the node y if and only if $y \neq x$ and $y = f(x)$.

If A is a set and B is a subset of A, then we let $F(A, B)$ denote the set of all cycle-free functions $f : A \rightarrow A$ whose set of fixed points is B.

4. The combinatorial problem. Let

$$m \in N_0, \quad \bar{r} \in P, \quad \bar{u} \in P, \quad \bar{r} = (r_1, r_2, \ldots), \quad \bar{u} = (u_1, u_2, \ldots).$$

Let R be a set having m elements. Let A be a set disjoint from R and having $|\bar{u}|$ elements. We consider the problem of determining the number of ordered pairs (f, g) satisfying the three conditions

(1) $f \in F(A \cup R, R)$, (2) $g \in M(\bar{r}, A \cup R)$, (3) $gf|A \in M(\bar{u}, A)$,

where $gf|A$ is the restriction of the composite function gf to the set A. We denote the set of these ordered pairs by $D(m; \bar{r}, \bar{u}, R, A)$ and the number of elements of this set by $D(m; \bar{r}, \bar{u})$. We emphasize that owing to Condition (3) the functions f and g may not be chosen independently.

The elements (f, g) of the set $D(m; \bar{r}, \bar{u}, R, A)$ may be pictured as certain collections of labelled rooted trees having coloured nodes and edges. The m elements of the set R, the fixed points of f, are the roots of the trees, and the $|\bar{u}|$ elements of the set A are the higher nodes. For each node $x \in A, f(x) \neq x$ and there is an edge descending from the node x to the node $f(x)$. The number of edges is $|\bar{u}|$. Each node $x \in A \cup R$ is "coloured" with the non-zero natural number $g(x)$ and it is required that for each $i \in N_1$ the number of nodes coloured i is r_i. Each edge is given the colour of the node to which it descends and it is required that for each $i \in N_1$ the number of edges coloured i is u_i.

5. Recursion formulas for $D(m; \bar{r}, \bar{u})$. In what follows $C(m, n)$ denotes the binomial coefficient $m!/(n!(m - n)!)$.

THEOREM 1. *If* $\bar{r}, \bar{u} \in P$ *satisfy* $|\bar{r}| = 1 + |\bar{u}|$, *then*

$$D(1; \bar{r}, \bar{u}) = \sum C(|\bar{u}|, k) D(k; \bar{r} - \bar{e}_i, \bar{u} - k\bar{e}_i), \tag{I}$$

where the summation extends over all pairs $(i, k) \in N_1 \times N_0$ *for which* $r_i \geqslant 1$ *and* $u_i \geqslant k$.

Proof. If one of our objects (f, g) belongs to $D(1; \bar{r}, \bar{u}, R, A)$, then the set R of its roots has just one element. Call it b. This root b is coloured i for some $i \in N_1$ for which $r_i \geqslant 1$ and the number of edges descending to the root b is some $k \in N_0$ satisfying $u_i \geqslant k$. Let R' be the set of nodes $x \in A$ which are the upper nodes of the edges descending to the root b. The set R' is one of the $C(|\bar{u}|, k)$ k-element subsets of the $|\bar{u}|$-element set A. If the root b and the edges descending to it are deleted, then the resulting object corresponds to one of the $D(k; \bar{r} - \bar{e}_i, \bar{u} - k\bar{e}_i)$ elements of the set

$$D(k; \bar{r} - \bar{e}_i, \bar{u} - k\bar{e}_i, R', A - R').$$

We leave it to the reader to complete the proof by showing that this process can be reversed.

THEOREM 2. *Let* $m, n \in N_0$, $\bar{t}, \bar{w} \in P$. *Then*

$$D(m + n; \bar{t}, \bar{w}) = \sum C(|\bar{w}|, |\bar{u}|) D(m; \bar{r}, \bar{u}) D(n; \bar{s}, \bar{v}), \tag{II}$$

where the summation extends over all $\bar{r}, \bar{s}, \bar{u}, \bar{v}$ *such that* $\bar{r} + \bar{s} = \bar{t}$ *and* $\bar{u} + \bar{v} = \bar{w}$.

Proof. If one of our objects (f, g) belongs to $D(m + n; \bar{t}, \bar{w}, R, A)$, then the set R of its roots has $m + n$ elements. Let $R = R' \cup R''$, where R' and R'' are disjoint sets, $R' = \{b_1', \ldots, b_m'\}$, $R'' = \{b_1'', \ldots, b_n''\}$. Let A' (let A'') be the set of nodes from which there is a chain of edges descending to one of the roots in R' (in R''). For each $i \in N_1$, let r_i (let u_i) be the number of nodes (of edges) in $A' \cup R'$ coloured i, and let s_i (let v_i) be the number of nodes (of edges) in $A'' \cup R''$ coloured i. Then $r_1 + s_i = t_i$ and $u_i + v_i = w_i$. The object (f', g') (the object (f'', g'')) obtained by restricting (f, g) to $A' \cup R'$ (to $A'' \cup R''$) is one of the $D(m; \bar{r}, \bar{u})$ (of the $D(n; \bar{s}, \bar{v})$) elements of $D(m; \bar{r}, \bar{u}, R', A')$ (of $D(n; \bar{s}, \bar{v}, R'', A'')$). The splitting of A into disjoint sets A' and A'' is one of the $C(|\bar{w}|, |\bar{u}|)$ possible splittings of A into two disjoint sets having $|\bar{u}|$ and $|\bar{v}|$ members respectively. Again we leave it to the reader to complete the proof.

By writing

$$X(m; \bar{r}, \bar{u}) = D(m; \bar{r}, \bar{u}) / |\bar{u}|! \tag{*}$$

the formulas (I) and (II) may be put into the simpler forms

$$X(1; \bar{r}, \bar{u}) = \sum (k!)^{-1} X(k; \bar{r} - \bar{e}_i, \bar{u} - k\bar{e}_i), \tag{I$'$}$$

$$X(m + n; \bar{t}, \bar{w}) = \sum X(m; \bar{r}, \bar{u}) X(n; \bar{s}, \bar{v}); \tag{II$'$}$$

but it should now be noted that $X(m; \bar{r}, \bar{u})$ need not be an integer.

6. The Prüfer correspondence. The correspondence to be described in this section is a modification of one due to Prüfer **(3)** and more recently studied by Neville **(2)**. In §7 we will make use of it in obtaining a direct evaluation of $D(m; \bar{r}, \bar{u})$.

Let A be a finite set having n elements and let B be a subset of A having m elements. Let $H(A, B)$ be the set of all sequences of $n - m$ terms chosen from the set A such that if $n - m > 0$, the last term is chosen from B. We shall describe a method by which the various cycle-free functions $f \in F(A, B)$ can be put into one-to-one correspondence with the sequences $h \in H(A, B)$.

Let the n elements of the set A be arranged in an order $a_1, a_2, \ldots, a_n$ in such a way that the m elements of B are $a_{n-m+1}, a_{n-m+2}, \ldots, a_n$. Let $f \in F(A, B)$ be given. If $n = m$, then every element $x \in A$ is a fixed point of f and we take the corresponding $h \in H(A, B)$ to be the empty sequence. If $n > m$, then we build the sequence h in the following manner. There is an $x \in A$ which is not a value of the function f. Let x_1 be the first such element in A and let $h_1 = f(x_1)$. Let $A_1 = A - \{x_1\}$ and let $f|A_1$ be the restriction of f to A_1. Then A_1 has $n - 1$ elements, $B \subseteq A_1$, $h_1 \in A_1$, and $f|A_1 \in F(A_1, B)$. Now proceed recursively. Suppose that for $1 \leqslant k < n - m$, x_k, h_k, and A_k have been defined and that A_k has $n - k$ elements, $B \subseteq A_k$, $h_k \in A_k$, and $f|A_k \in F(A_k, B)$. Then there is an $x \in A_k$ which is not a value of the function $f|A_k$. Let x_{k+1} be the first such element in A_k and let $h_{k+1} = f(x_{k+1})$. Then $A_{k+1} = A_k - \{x_{k+1}\}$ has $n - (k + 1)$ elements, $B \subseteq A_{k+1}$, $h_{k+1} \in A_{k+1}$, and $f|A_{k+1} \in F(A_{k+1}, B)$. Continue until x_{n-m}, h_{n-m}, and A_{n-m} have been defined. Then A_{n-m} has m elements, $B \subseteq A_{n-m}$, $h_{n-m} \in A_{n-m}$, and $f|A_{n-m} \in F(A_{n-m}, B)$. Since B has m elements, $B = A_{n-m}$ and hence $h_{n-m} \in B$. Therefore, the sequence $h = (h_1, \ldots, h_{n-m})$ is in $H(A, B)$.

It is readily checked that distinct cycle-free functions $f \in F(A, B)$ lead to distinct sequences $h \in H(A, B)$ and that every $h \in H(A, B)$ is obtainable from some $f \in F(A, B)$. One sees that the function f consists of the ordered pairs (x, x) for $x \in B$ together with the ordered pairs (x_i, h_i) for $1 \leqslant i \leqslant n - m$. In terms of our picture of a cycle-free function $f \in F(A, B)$ as a collection of rooted trees, the elements of the set B are the roots of the trees and each ordered pair (x_i, h_i) corresponds to an edge, the edge which descends from the node x_i to the node $h_i = f(x_i)$. We shall call the sequence of pairs $(x_1, h_1), \ldots, (x_{n-m}, h_{n-m})$ the *Prüfer sequence of edges* for f and the sequence $h_1, \ldots, h_{n-m}$ the *Prüfer sequence of nodes* for f.

7. Direct evaluation of $D(m; \bar{r}, \bar{u})$.

THEOREM 3. *Let $m \in N_0$, $\bar{r}, \bar{u} \in P$. Then*

$$\text{(III)} \qquad D(m; \bar{r}, \bar{u}) = \begin{cases} 0, & \text{if } |\bar{r}| \neq m + |\bar{u}|; \\ 1, & \text{if } |\bar{r}| = m + |\bar{u}| = 0; \\ m|\bar{r}|^{-1} M(\bar{r}) M(\bar{u}) \bar{r}^{\bar{u}}, & \text{if } |\bar{r}| = m + |\bar{u}| \neq 0. \end{cases}$$

Proof. Unless $|\bar{r}| = m + |\bar{u}|$, the set $M(\bar{r}, A \cup R)$ will be empty, and we conclude that $D(m; \bar{r}, \bar{u}) = 0$ whenever $|\bar{r}| \neq m + |\bar{u}|$.

If $|\bar{r}| = m + |\bar{u}| = 0$, then $m = 0$ and $|\bar{u}| = 0$, so that $A \cup R$ is empty. In this case both f and g must be the empty function and we have $D(m; \bar{r}, \bar{u}) = 1$.

If $|\bar{r}| = m + |\bar{u}| \neq 0$ and $m = 0$, then the right member in (III) reduces to 0. The set R is empty, but $A \cup R$ is not empty. Since a cycle-free function on a non-empty finite set must have at least one fixed point, $F(A \cup R, R)$ is empty. Consequently, $D(m; \bar{r}, \bar{u}) = 0$ and (III) is satisfied.

If $|\bar{r}| = m + |\bar{u}| \neq 0$ and $|\bar{u}| = 0$, then the right member in (III) reduces to $M(\bar{r})$. The set A is empty and the condition $gf|A \in M(\bar{u}, A)$ is trivially satisfied. The set $F(A \cup R, R) = F(R, R)$ has just one element f and $M(\bar{r}, A \cup R) = M(\bar{r}, R)$ has $M(\bar{r})$ elements g. Consequently, $D(m; \bar{r}, \bar{u}) = M(\bar{r})$, and (III) is satisfied.

These trivial cases aside, we now turn to the main case, where $|\bar{r}| = m + |\bar{u}|$, $m > 0$ and $|\bar{u}| > 0$. For each $n \in N_0$, let $S(n)$ denote the set of natural numbers k such that $1 \leqslant k \leqslant n$. We shall establish that (III) holds by describing a one-to-one correspondence from the cartesian product $D(m; \bar{r}, \bar{u}, R, A) \times S(|\bar{r}|)$ to the cartesian product

$$R \times M(\bar{r}, S(|\bar{r}|)) \times M(\bar{u}, S(|\bar{u}|)) \times S(r_1)^{S(u_1)} \times S(r_2)^{S(u_2)} \times \ldots .$$

To given $(f, g) \in D(m; \bar{r}, \bar{u}, R, A)$ and $s \in S(|\bar{r}|)$ this correspondence assigns a sequence $(\rho, \psi, \phi, \theta_1, \theta_2, \ldots)$ such that $\rho \in R$, $\psi \in M(\bar{r}, S(|\bar{r}|))$, $\phi \in M(\bar{u}, S(|\bar{u}|))$, and for each $i \in N_1$, $\theta_i \in S(r_i)^{S(u_i)}$.

We list the elements of the set A in order $a_1, a_2, \ldots, a_{|\bar{u}|}$ and list the elements of the set R in order $b_1, b_2, \ldots, b_m$. These lists are combined to form the list $a_1, a_2, \ldots, a_{|\bar{u}|}, b_1, b_2, \ldots, b_m$ of the elements of $A \cup R$, and the elements of this list are renamed $c_1, c_2, \ldots, c_{|\bar{r}|}$. Now, using the given $s \in S(|\bar{r}|)$, we apply the cyclic permutation which puts c_s into the first position, obtaining the sequence $d_1, d_2, \ldots, d_{|\bar{r}|}$, where $d_i = c_{s+i-1}$ for $s + i - 1 \leqslant |\bar{r}|$ and $d_i = c_{s+i-1-|\bar{r}|}$ for $s + i - 1 > |\bar{r}|$. The elements $d_1, d_2, \ldots, d_{|\bar{r}|}$ are just the elements of $A \cup R$ in a certain order and the given $g \in M(\bar{r}, A \cup R)$ assigns to each of them a "colour." We transfer this colouring to the subscripts in the sequence of d's. The resulting $\bar{r}$-colouring of the natural numbers $1, 2, \ldots, |\bar{r}|$ is the assigned member ψ of $M(\bar{r}, S(|\bar{r}|))$.

Relative to the ordering $c_1, c_2, \ldots, c_{|\bar{r}|}$ of the elements of $A \cup R$, in which the roots occur last, the given $f \in F(A \cup R, R)$ determines a Prüfer sequence of nodes $h_1, h_2, \ldots, h_{|\bar{u}|}$ in which the last term is a root. This last term $h_{|\bar{u}|}$ is the assigned member ρ of R.

Although the nodes $h_1, h_2, \ldots, h_{|\bar{u}|}$ need not all be distinct, the edges (x_1, h_1), (x_2, h_2), $\ldots$, $(x_{|\bar{u}|}, h_{|\bar{u}|})$ in the Prüfer sequence of edges for f are all distinct. In fact, the nodes $x_1, x_2, \ldots, x_{|\bar{u}|}$ are just the elements of the set A in some order. Since $gf|A \in M(\bar{u}, A)$, if each edge (x_j, h_j) is coloured with the colour of the node h_j to which it descends, then a $\bar{u}$-colouring of the edges is obtained. We transfer this colouring to the subscripts in the sequence of h's. The resulting $\bar{u}$-colouring of the natural numbers $1, 2, \ldots, |\bar{u}|$ is the assigned member ϕ of $M(\bar{u}, S(|\bar{u}|))$.

Now let $i \in N_1$ be any colour. On one hand, there are u_i subscripts n in the Prüfer sequence $h_1, \ldots, h_{|\bar{u}|}$ for which the node h_n is coloured i. We let these subscripts in increasing order be $n_1, n_2, \ldots, n_{u_i}$. On the other hand, there are r_i nodes in the sequence $d_1, d_2, \ldots, d_{|\bar{r}|}$ which have the colour i. We list these in order of increasing subscripts and rename them $\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_{r_i}$. Now for each j satisfying $1 \leqslant j \leqslant u_i$, let $\theta_i(j)$ be defined by $h_{n_j} = \bar{d}_{\theta_i(j)}$. The resulting function θ_i is the assigned member of $S(r_i)^{S(u_i)}$.

Conversely, if we are given a sequence $(\rho, \psi, \phi, \theta_1, \theta_2, \ldots)$ of the specified type we can reconstruct the pair $((f, g), s)$. We begin by reconstructing the Prüfer sequence $h_1, h_2, \ldots, h_{|\bar{u}|}$. Given the function ϕ, we know what colour is to be assigned to each of its subscripts. We also know that the last element $h_{|\bar{u}|}$ is the given element $\rho \in R$. Thus, the colour i of the element ρ is known, and if the subscripts n in the Prüfer sequence where the node h_n is coloured i are in increasing order $n_1, n_2, \ldots, n_{u_i}$, then we know that the node having subscript n_{u_i} is ρ. Given the function ψ we know what colour is to be assigned to each subscript in the sequence $d_1, d_2, \ldots, d_{|\bar{r}|}$. Let the nodes coloured with i, the colour of ρ, be $\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_{r_i}$. Given the function θ_i and using $h_{n_j} = \bar{d}_{\theta_i(j)}$ with $j = u_i$, we can determine the position of the root ρ in the sequence $\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_{r_i}$, and hence also its position in the sequence $d_1, d_2, \ldots, d_{|\bar{r}|}$. We are now able to determine the natural number s associated with the cyclic permutation which carries $c_1, c_2, \ldots, c_{|\bar{r}|}$ into $d_1, d_2, \ldots, d_{|\bar{r}|}$. Now all nodes in the sequence $d_1, d_2, \ldots, d_{|\bar{r}|}$ are known, and therefore the function g is known. Finally, by using the various functions θ we can determine the nodes occurring in the Prüfer sequence. When the Prüfer sequence is known, the function f can then be reconstructed.

This completes the proof of Theorem 3. Working with the special case where there is only one "colour," we obtain the corollary that the number of cycle-free functions from an n-element set A to itself having a specified m-element subset B as the set of fixed points is mn^{n-m-1}. (This may also be obtained directly from the Prüfer correspondence.) Then, summing over all choices of the subset B, we obtain the well-known result that the total number of cycle-free functions from A to itself is $(n + 1)^{n-1}$. Also included as a special case (involving two colours) is a recent result of Clarke **(1)**.

8. Formal solution of $X = \sum A_i \exp(B_i X)$. It is interesting to restate the preceding results in terms of formal series. We suppose that R is a commutative ring which contains the rational numbers, and we consider the formal multiple power series algebra with coefficients in R which consists of the functions $f : P \times P \rightarrow R$. Using indeterminates $A_1, A_2, \ldots ;\ B_1, B_2, \ldots,$ and letting $\bar{A}^{\bar{r}}\bar{B}^{\bar{u}}$ stand for

$$A_1^{r_1} A_2^{r_2} \ldots B_1^{u_1} B_2^{u_2} \ldots,$$

a typical element f of this power series algebra takes the form $\sum f(\bar{r}, \bar{u})\bar{A}^{\bar{r}}\bar{B}^{\bar{u}}$, where the sum extends over all pairs $(\bar{r}, \bar{u}) \in \bar{P} \times \bar{P}$. Sums, scalar multiples, and products in this algebra are defined in the usual way; in particular, for products we have

$$\left(\sum f(\bar{r}, \bar{u})\bar{A}^{\bar{r}}\bar{B}^{\bar{u}}\right)\left(\sum g(\bar{s}, \bar{v})\bar{A}^{\bar{s}}\bar{B}^{\bar{v}}\right) = \sum h(\bar{t}, \bar{w})\bar{A}^{\bar{t}}\bar{B}^{\bar{w}},$$

where $h(\bar{t}, \bar{w}) = \sum f(\bar{r}, \bar{u})g(\bar{s}, \bar{v})$, this last sum extending over all $\bar{r}, \bar{s}, \bar{u}, \bar{v}$ such that $\bar{r} + \bar{s} = \bar{t}$ and $\bar{u} + \bar{v} = \bar{w}$.

Now working with the $X(m; \bar{r}, \bar{u})$ defined in (*) at the end of §5, let us look at the particular formal power series

$$X = \sum X(1; \bar{r}, \bar{u})\bar{A}^{\bar{r}}\bar{B}^{\bar{u}}.$$

First, by using the convolution formula (II′), we can prove inductively that the powers of X satisfy

$$X^m = \sum X(m; \bar{r}, \bar{u})\bar{A}^{\bar{r}}\bar{B}^{\bar{u}}.$$

Then by applying (I′) we have

$$X = \sum_{(\bar{r},\bar{u})} X(1; \bar{r}, \bar{u})\bar{A}^{\bar{r}}\bar{B}^{\bar{u}} = \sum_{(\bar{r},\bar{u})} \left(\sum_{(i,k)} (k!)^{-1} X(k; \bar{r} - \bar{e}_i, \bar{u} - k\bar{e}_i)\right)\bar{A}^{\bar{r}}\bar{B}^{\bar{u}},$$

where $(\bar{r}, \bar{u}) \in P \times P$ and $(i, k) \in N_1 \times N_0$ must satisfy $\bar{r} \geqslant \bar{e}_i$ and $\bar{u} \geqslant k\bar{e}_i$. Inverting the order of summation, we may write

$$X = \sum_{(i,k)} (k!)^{-1} A_i B_i^k \left(\sum_{(\bar{r},\bar{u})} X(k; \bar{r} - \bar{e}_i, \bar{u} - k\bar{e}_i)\bar{A}^{\bar{r}-\bar{e}_i}\bar{B}^{\bar{u}-k\bar{e}_i}\right),$$

where $(i, k) \in N_1 \times N_0$ and $(\bar{r}, \bar{u}) \in P \times P$ must satisfy $\bar{r} - \bar{e}_i \geqslant \bar{0}$ and $\bar{u} - k\bar{e}_i \geqslant \bar{0}$. Since $(\bar{r} - \bar{e}_i, \bar{u} - k\bar{e}_i)$ may take any value in $P \times P$, we may simplify this to

$$X = \sum_{(i,k)} (k!)^{-1} A_i B_i^k X^k = \sum_{i=1}^{\infty} A_i \sum_{k=0}^{\infty} (k!)^{-1} (B_i X)^k = \sum_{i=1}^{\infty} A_i \exp(B_i X).$$

These observations are summarized in the following theorem.

THEOREM 4. *Let $X(m; \bar{r}, \bar{u})$ be defined by* (*) *of* §5 *and* (III) *of* §7. *Then the formal power series $X = \sum X(1; \bar{r}, \bar{u})\bar{A}^{\bar{r}}\bar{B}^{\bar{u}}$ is a formal solution of the equation*

$$\sum_{i=1}^{\infty} A_i \exp(B_i X) = X.$$

In the special case where there is just one colour, this theorem yields the familiar result that the solution of $z = we^z$ which vanishes when $w = 0$ is given by the series

$$z = \sum_{n=1}^{\infty} (n^{n-1}/n!)w^n.$$

REFERENCES

1. L. E. Clarke, *On Cayley's formula for counting trees*, J. Lond. Math. Soc., *33* (1958), 471–474.
2. E. H. Neville, *The codifying of tree-structure*, Proc. Camb. Phil. Soc., *49* (1953), 381–385.
3. H. Prüfer, *Neuer Beweis eines Satzes über Permutationen*, Arch. Math. u. Phys., *27* (1918). 142–144.

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