PERMUTABLE SUBGROUPS OF SOME FINITE p-GROUPS

Dedicated to the memory of Hanna Neumann

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1. Introduction

A subgroup H of a group G is said to be a *permutable* subgroup of G if $HK = KH = \langle H,K \rangle$ for all subgroups K of G. It is known that a core-free permutable subgroup H of a finite group G is always nilpotent [5]; and even when G is not finite, H is always a subdirect product of finite nilpotent groups [11]. Thus nilpotency is a measure of the extent to which a permutable subgroup differs from being normal. Examples of non-abelian, core-free, permutable subgroups are rare and difficult to construct. The first, due to Thompson [12], had class 2. Further examples of class 2 appeared in [8]. More recently Bradway, Gross and Scott [1] have constructed corresponding to each positive integer c and each prime $p > c$, a finite p-group possessing a core-free permutable subgroup of class c. In [3] Gross succeeded in dispensing with the requirement $p > c$.

The present work is to be compared with that of Gross in that we shall exhibit, for each prime p, finite p-groups with core-free permutable subgroups of arbitrarily large class. However, whereas the Gross groups were constructed via ingenious extensions, ours are ready-made. They appeared in Philip Hall's Cambridge lectures during the 1960s and occur as finite quotients of some remarkable groups constructed by Hall in [4]. In fact we consider the (upper or lower) triangular subgroups H of certain 2×2 matrix groups M over Z_{p^n}. It is curious that the subgroups H are permutable in M. Moreover, the task of establishing this property might have been formidable but for the fact that there are certain natural embeddings among the groups M which enable us to argue by induction on group order.

It is also worth remarking that our subgroups H all have rank 2; indeed they are metacyclic. In previous examples this rank has increased with p. Thus our main result is

THEOREM A. *For each positive integer c and prime p, there exists a finite*

p-group possessing a metacyclic, core-free, permutable subgroup of nilpotency class at least c.

It is a fact that all known examples of core-free permutable subgroups are metabelian. We leave the obvious question open.

It is perhaps of interest to indicate how the properties of the groups of Theorem A came to be discovered. It appeared to be an open question (see [11]) whether a group G generated by two soluble subgroups, each permutable in G, was necessarily soluble. It turned out that if this was not the case, then there were tricyclic finite p-groups (i.e. the product of three cyclic subgroups) of the above form having arbitrarily large derived length. Hall's groups were tricyclic and had arbitrarily large derived length. Thus it was natural to look for permutable subgroups there. As a consequence we can prove

THEOREM B. *There exists a non-soluble group G generated by two metabelian subgroups, each permutable in G.*

We recall that a permutable subgroup is always ascendant ([11] Theorem A). Therefore a weaker version of Theorem B is

THEOREM B*. *There exists a non-soluble group which is the product of two ascendant metabelian subgroups.*

So far our results claim the existence of counterexamples to possible conjectures. Our original investigation concerned a join G of soluble subgroups, each permutable in G. In this connection we have

THEOREM C. *A group G which is generated by any number of soluble subgroups, each permutable in G, is locally soluble.*

In proving this result we were able to improve Theorem B of [11] which says that *a permutable subgroup of a finitely generated group is always subnormal.* Thus, following Rae [10], we call a subgroup H of a group G *locally subnormal* in G if H is subnormal in each subgroup of G of the form $\langle H, g_1, g_2, \cdots, g_n\rangle$, for all choices of finitely many elements $g_1, g_2, \cdots, g_n$ in G. Then we have

THEOREM D. *A permutable subgroup is locally subnormal.*

We remark that this result is independent of Theorem A of [11] since ascendant subgroups need not be locally subnormal and conversely Example 1 of Kargapolov [1] shows that locally subnormal subgroups need not be ascendant. Also we remark that subpermutable subgroups (as defined in [11]) need not be locally subnormal. For, there exist countable groups with permutable subgroups which are not subnormal (see [6]); and any countable group can be subnormally embedded in a finitely generated group according to Dark [2].

2. Some subgroups of $SL\,(2, Z_{p^n})$

Let n be a positive integer and p be any prime. Let $A = Z_{p^n}$ be the ring of integers modulo p^n. Choose positive integers h, k, l and let

$$M = M_{h,k,l}$$

be the set of all 2×2 matrices over A of the form

$$\xi = \begin{pmatrix} q & r \\ s & t \end{pmatrix},$$

where $r \equiv 0(p^h)$, $s \equiv 0(p^k)$, $t \equiv 1(p^l)$ and $\det \xi = 1$.

Suppose that

$$h + k \geqq l. \tag{1}$$

Then it follows that M is closed with respect to multiplication and so M is a group. Let

$$\rho(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \ \sigma(y) = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, \ \tau(u) = \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}$$

be elements of M. Then it is straightforward to verify that

$$\xi = \rho(rt^{-1})\sigma(st)\tau(t)$$

Thus if

$$\begin{aligned} R_h &= \text{all } \rho(x) \text{ with } x \equiv 0(p^h) \\ S_k &= \text{all } \sigma(y) \text{ with } y \equiv 0(p^k) \\ T_l &= \text{all } \tau(u) \text{ with } u \equiv 1(p^l), \end{aligned}$$

then

$$M = R_h S_k T_l. \tag{2}$$

R_h and S_k are clearly cyclic of orders p^{n-h} and p^{n-k} respectively. Also it is easy to check that T_l is cyclic of order p^{n-l} provided

$$l \geqq 2 \tag{3}$$

in case $p = 2$.

The following relations hold between the elements of R_h, S_k and T_l:

$$\tau(u)\rho(x) = \rho(xu^{-2})\tau(u) \tag{4}$$

and hence $R_h \lhd R_h T_l$;

$$\tau(u)\sigma(y) = \sigma(yu^2)\tau(u) \tag{5}$$

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and hence $S_k \lhd S_k T_l$;

(6) $$\sigma(y)\rho(x) = \rho(xz^{-1})\sigma(yz)\tau(z)$$

where $z = 1 + xy$. Thus we have

(7) $$[\rho(x), \tau(u)] = \rho(x(u^2 - 1)),$$

(8) $$[\sigma(y), \tau(u)] = \sigma(y(u^{-2} - 1)),$$

(9) $$[\rho(x), \sigma(y)] = \rho(x^2 y\omega^{-1})\sigma(-xy^2\omega)\tau(\omega),$$

where $\omega = 1 - xy$. So by (7)

$$[R_h, T_l] = \begin{cases} R_{h+l} & \text{if } p \text{ is odd} \\ R_{h+l+1} & \text{if } p = 2. \end{cases}$$

In both cases

(10) $$[R_h, T_l] \geqq R_{h+l+1}.$$

Similarly by (8)

(11) $$[S_k, T_l] \geqq S_{k+l+1}.$$

Thus provided

(12) $$h + k \geqq l + 1$$

(superseding (1)), we see from (9), (10) and (11) that $M' \geqq T_{h+k}$. Therefore

(13) $$M' = M'_{h,k,l} \geqq M_{h+l+1\ k+l+1,h+k}$$

and the right-hand side of this inequality is trivial if and only if $n \leqq \min\{h + l + 1, k + l + 1, h + k\}$. It follows that we can make the derived length of M as large as we please by choosing n sufficiently large and h, k, l consistent with (3) and (12), e.g. $h = 1$, $k = l = 2$. So we have proved

LEMMA 2.1. *Provided $l \geqq 2$ and $h + k \geqq l + 1$, the derived length of M tends to infinity with n.*

For simplicity we write $R = R_h$, $S = S_k$, $T = T_l$.

We establish now

LEMMA 2.2. *RT and ST are both permutable subgroups of M.*

PROOF. Let $\xi = \begin{pmatrix} q & r \\ s & t \end{pmatrix} \in M$. We must show that

(14) $$RT\langle\xi\rangle = \langle\xi\rangle RT.$$

We proceed by induction on $n - k$. If $n - k = 0$, then $RT = M$ and the result is

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trivial. If $n-k=1$, then S has order p. Thus $|M:RT| = p$, by (2), and so $RT \lhd M$. In this case (14) clearly holds and so we may suppose that $n-k \geqq 2$.

If $s \equiv 0(p^{k+1})$, then $\xi \in M_{h,k+1,l}$ and so (14) holds by induction. Therefore we may assume that

(15) $$s \not\equiv 0(p^{k+1}).$$

We claim that, for each $m \geqq 1$,

(16) $$\xi^m = \rho(\cdots)\sigma(su_m(m+\lambda_m p^2))\tau(\cdots),$$

where $u_m \equiv 1(p^2)$ and $\lambda_m \in A$. To see this, we proceed by induction on m. When $m=1$, (16) is certainly true. Thus suppose that (16) holds for some $m \geqq 1$. Then, for some $t_m \in A$,

$$\begin{aligned}\xi^{m+1} &= \xi^m\xi = \rho(\cdots)\sigma(su_m(m+\lambda_m p^2))\tau(t_m)\rho(rt^{-1})\sigma(st)\tau(t)\\ &= \rho(\cdots)\sigma(su_m(m+\lambda_m p^2))\rho(rt^{-1}t_m^{-2})\tau(t_m)\sigma(st)\tau(t),\end{aligned}$$

by (4). Therefore by (6)

$$\xi^{m+1} = \rho(\cdots)\sigma(su_m(m+\lambda_m p^2)z)\tau(z)\tau(t_m)\sigma(st)\tau(t),$$

where $z = 1 + rsu_m t^{-1}t_m^{-2}\ (m+\lambda_m p^2)$. Hence by (5)

$$\begin{aligned}\xi^{m+1} &= \rho(\cdots)\sigma(su_m(m+\lambda_m p^2)z + stz^2t_m^2)\tau(zt_m t)\\ &= \rho(\cdots)\sigma(su_m z(m+\lambda_m p^2 + u_m^{-1}tzt_m^2))\tau(zt_m t).\end{aligned}$$

Put $u_{m+1} = u_m z$. Since u_m, t, z, t_m are all congruent to 1 modulo p^2 (recalling (3)), we see that (16) holds for $m+1$. Therefore (16) is valid for all $m \geqq 1$.

Now consider

$$\xi^p = \rho(\cdots)\sigma(su_p(p+\lambda_p p^2))\tau(\cdots).$$

Clearly $\xi^p \in M_{h,k+1,l}$ and hence, by induction on $n-k$,

(17) $$RT\langle\xi^p\rangle = \langle\xi^p\rangle RT.$$

On the other hand, $su_p(p+\lambda_p p^2) \not\equiv 0(p^{k+2})$, by (15), and so

(18) $$\xi^p \notin M_{h,k+2,l}.$$

Since the subgroups between $RT = M_{h,n,l}$ and $M = M_{h,k,l}$ are nested, it follows from (17) and (18) that

$$RT\langle\xi^p\rangle = M_{h,k+1,l}.$$

But, according to (2), $|M_{h,k,l}: M_{h,k+1,l}| = p$ and so $M_{h,k+1,l} \lhd M_{h,k,l}$. Therefore since $\xi \notin M_{h,k+1,l}$,

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$$M = M_{h\,k+1\,l}\langle\xi\rangle = RT\langle\xi^p\rangle\langle\xi\rangle = RT\langle\xi\rangle,$$

and (14) follows.

In the same way we see that ST is a permutable subgroup of M.

3 Proofs of Theorems A and B

PROOF OF THEOREM A. We consider the group M introduced in § 2. According to Lemma 2.2, the subgroups $H = RT$ and $K = ST$ are both permutable in M. Also, by (4) and (5), we see that H and K are metacyclic. Let $\bar{H}$, $\bar{K}$ be the cores of H, K respectively in M. Suppose, for a contradiction, that there exists a bound c for the nilpotency class of a core-free permutable subgroup of any group. Then $H/\bar{H}$, $K/\bar{K}$ would be nilpotent of class at most c. Thus $H\bar{K}/\bar{H}\bar{K}$ and $K\bar{H}/\bar{H}\bar{K}$ would be nilpotent of class at most c and both of these subgroups are permutable in $M/\bar{H}\bar{K}$. It would then follow, according to Theorem E of [11], that $HK/\bar{H}\bar{K} = M/\bar{H}\bar{K}$ was soluble with derived length bounded by some function of c. Since $\bar{H}\bar{K}$ has derived length at most 4, this would bound the derived length of M, contradicting Lemma 2. This proves Theorem A.

PROOF OF THEOREM B. If p is the ith prime, let M_i be one of the p-groups M, defined in § 2, with derived length $\geqq i$. Let $H_i = RT$ and $K_i = ST$. Then define (writing Dr for the restricted direct product)

$$G = \operatorname*{Dr}_{i \geqq 1} M_i,$$

$$H = \operatorname*{Dr}_{i \geqq 1} H_i, \qquad K = \operatorname*{Dr}_{i \geqq 1} K_i.$$

So $G = HK$; and H, K are metabelian. However, since H_i and K_i are both permutable subgroups of M_i, it is straightforward to show (see [1]) that H and K are both permutable in G. Finally, G cannot be soluble since it has subgroups of arbitrarily large derived length.

4. Proofs of Theorems C and D

We need

LEMMA 4.1. *Let H be a permutable subgroup of $G = \langle H, g_1, g_2, \cdots, g_n\rangle$ with $g_i^{m_i} \in H$, for some $m_i \geqq 1$, $1 \leqq i \leqq n$. Then $|H^G : H|$ is finite.*

PROOF. We argue by induction on

$$\sum_{i=1}^{n} |H\langle g_i\rangle : H|.$$

The method is an easy adaptation of the proof of Theorem B of [11] and may be omitted.

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We recall the following result (Corollary 2.2) of [11].

LEMMA 4.2. *Let H be a permutable subgroup of G and let N be the subgroup of G generated by all those elements no positive power of which lies in H. Then*

(i) $N \lhd G$;

(ii) $H \lhd HN$;

(iii) *HN/N is a permutable subgroup of G/N and every element of G/N has some positive power in HN/N.*

PROOF OF THEOREM D. We suppose that H is a permutable subgroup of $G = \langle H, g_1, g_2, \cdots, g_n \rangle$ and show that H is subnormal in G. Thus let N be defined as in Lemma 4.2. Since $N \lhd G$ and $H \lhd HN$, we may suppose, without loss of generality, that $N = 1$. Then $|H^G : H|$ is finite, by Lemmas 4.1 and 4.2. Since a permutable subgroup of a finite group is always subnormal [9], it follows that H is subnormal in H^G. Therefore H is subnormal in G.

LEMMA 4.3. *Let $G = HK$, where H is a soluble permutable subgroup of G and K is a locally soluble subgroup of G. Then G is locally soluble.*

PROOF. Without loss of generality we may suppose that K is finitely generated and therefore soluble. In this case we show that G is soluble.

Let N be as defined in Lemma 4.2. Then

$$H \lhd HN = HN \cap HK = H(HN \cap K).$$

Thus HN is soluble and so N is soluble. Therefore we may suppose that $N = 1$. Then, by Lemma 4.1, $|H^G : H|$ is finite and so it is easy to see that H^G is soluble. Hence G is soluble.

PROOF OF THEOREM C. We have to show that a group G generated by soluble subgroups, each permutable in G, is locally soluble. Clearly we may suppose that G is generated by finitely many such subgroups. Then an induction argument on this number, together with Lemma 4.3, establishes the Theorem.

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