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TAIL ASYMPTOTICS OF THE STATIONARY DISTRIBUTION OF A TWO-DIMENSIONAL REFLECTING RANDOM WALK WITH UNBOUNDED UPWARD JUMPS

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Abstract

We consider a two-dimensional reflecting random walk on the nonnegative integer quadrant. This random walk is assumed to be skip free in the direction to the boundary of the quadrant, but may have unbounded jumps in the opposite direction, which are referred to as upward jumps. We are interested in the tail asymptotic behavior of its stationary distribution, provided it exists. Assuming that the upward jump size distributions have light tails, we find the rough tail asymptotics of the marginal stationary distributions in all directions. This generalizes the corresponding results for the skip-free reflecting random walk in Miyazawa (2009). We exemplify these results for a two-node queueing network with exogenous batch arrivals.

Keywords: Two-dimensional reflecting random walk; stationary distribution; linear convex order; tail asymptotics; unbounded jump; two-node queueing network; batch arrival

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1. Introduction

We consider a two-dimensional reflecting random walk on the nonnegative integer quadrant. This random walk is assumed to be skip free toward the boundary of the quadrant but may have unbounded jumps in the opposite direction, which we call upward jumps. Here the boundary is composed of the origin and two half coordinate axes, which are called boundary faces. The transitions on each boundary face are assumed to be homogeneous. This reflecting process is referred to as a double M/G/1-type process. This process naturally arises in queueing networks with two-dimensional compound Poisson arrivals and exponentially distributed service times. Here customers simultaneously arrive in batches at different nodes. It is also of interest as a multidimensional reflecting random walk on the nonnegative integer quadrant.

A stationary distribution is one of the most important characteristics of this reflecting random walk in applications. However, deriving it analytically is difficult except in some special cases and so theoretical studies have focused on its tail asymptotic behaviors. Borovkov and Mogul´skiĭ [5] have made great contributions to this area of study. They proposed the so-called partially homogeneous chain, which includes the present random walk as a special case, and studied the tail asymptotic behavior of its stationary distribution. Their results are very general,

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but have the following limitations.

(a) The results are not very explicit. That is, it is hard to see how the modeling primitives, that is, the parameters which describe the model, influence the tail asymptotics.

(b) The tail asymptotics are obtained only for small rectangles, and no marginal distribution is considered. Furthermore, some extra technical conditions which seem to be unnecessary are assumed.

For the skip-free two-dimensional reflecting random walk, these issues have been addressed by Miyazawa [25], and its tail asymptotics have recently been studied for marginal distributions by the authors in [20], in which we considered the stationary equation using generating or moment generating functions and applied classical results of complex analysis (see, e.g. [26] and the references therein). In recent years this classical approach has been renewed in combination with other methods (see, e.g. [18] and [23]). However, its application is limited to skip-free processes because of technical reasons (see Remark 3).

In this paper we are primarily interested in answering the question: how do the tail asymptotics change when upward jumps are unbounded? For this, we consider the asymptotics of the stationary probability of the tail set, i.e.

$$\{(i_1, i_2) \in \mathbb{Z}_+^2; c_1 i_1 + c_2 i_2 \geq x\}$$

as x goes to ∞ for each $c_1, c_2 \geq 0$, where $\mathbb{Z}_+$ is the set of all nonnegative integers, and $\boldsymbol{c} = (c_1, c_2)$ is called a directional vector if $c_1^2 + c_2^2 = 1$. We are interested in its decay rate, where α is said to be the decay rate of function $p(x)$ if

$$\alpha = -\lim_{n \to \infty} \frac{1}{x} \log p(x).$$

We aim to derive the tail decay rates of the marginal distributions in all directions. These decay rates will be geometrically obtained from the curves which are produced by the modeling primitives, that is, one-step transitions of the reflecting random walk. In this way, we answer our question, which simultaneously resolves issues (a) and (b).

Obviously, if the tail decay rates are positive then the one-step transitions of the random walk must have light tails, that is, they decay exponentially fast. Thus, we assume that this light tail condition holds. Then the mean drifts of the reflecting random walk in the interior and on the boundary faces are finite. Using these means, Fayolle *et al.* [12] characterized stability, that is, the existence of the stationary distribution. However, their result was incomplete as they did not consider all cases, which we redress in Lemma 1. If the mean drifts vanish in all directions then it is not hard to see that the stationary distribution does not have light tails in all directions. We also formally verify this fact. Thus, we assume that not all the mean drifts vanish in addition to the light tail condition for the one-step transitions.

Under these assumptions and provided the stability conditions hold, we solve the decay rate problem for the marginal stationary distribution in each direction. The decay rate may be considered a rough asymptotic, but we also study some finer asymptotics, called exact asymptotics. We say that the tail probability has exact asymptotics for some function f if the ratio of the tail probability at level x to $f(x)$ converges to a positive constant as x goes to ∞. In particular, if f is exponential (or geometric), it is said to be exactly exponential (or geometric). We derive some sufficient conditions for the tail asymptotic to be exactly exponential.

The difficulty of the tail asymptotic problem mainly arises from reflections at the boundary of the quadrant. We have two major boundary faces corresponding to the coordinate axes.

The reflections on these faces may or may not influence the tail asymptotics. Foley and McDonald [16] classified them as the jitter, branch, and cascade cases. When modeling the tail asymptotics, these considerations need to be taken into account. However, the problem is further complicated by the unbounded jumps.

To overcome these difficulties, we employ a new approach. Based on the stability conditions, we first derive the convergence domain of the moment generating function of the stationary distribution. For this, we use a stationary inequality, which was recently introduced by the second author [26] (see also [6]), and a lower bound for the large deviations of the stationary distribution given in [5]. Once the domain is obtained, it is expected that the decay rate would be obtained through the boundary of the domain. However, this is not immediate. We need sharper lower bounds for the large deviations in coordinate directions. For this, we use a method based on Markov additive processes (see, e.g. [28]).

We apply one of our main results, Theorems 2 and 3, to a batch arrival network with two nodes to see how the modeling primitives influence the tail asymptotics. We use the linear convex order for this. We also show that the stochastic upper bound for the stationary distribution obtained by Miyazawa and Taylor [27] is not tight unless one of the nodes has no batch arrival.

This paper is organized as follows. In Section 2 we formally introduce the reflecting random walk, and discuss its basic properties, including stability and the stationary equations. In Section 3 we present the main results on the domain and tail asymptotics, Theorems 1–3, and discuss the linear convex order. We prove Theorems 1–3 in Section 4, and apply them to a two-node network with exogenous batch arrivals in Section 5. Finally, in Section 6 we give some remarks on extensions and further work.

2. Double M/G/1-type process

Denote the state space by $S = \mathbb{Z}_+^2$. Recall that $\mathbb{Z}_+$ is the set of all nonnegative integers. Similarly, $\mathbb{Z}$ denotes the set of all integers. Define the boundary faces of S as

$$S_0 = \{(0,0)\}, \qquad S_1 = \{(i,0) \in \mathbb{Z}_+^2; \, i \geq 1\}, \qquad S_2 = \{(0,i) \in \mathbb{Z}_+^2; \, i \geq 1\}.$$

Let $\partial S = \bigcup_{i=0}^2 S_i$ and $S_+ = S \setminus \partial S$. We refer to ∂S and S_+ as the boundary and interior of S, respectively.

Let $\{\boldsymbol{Y}(\ell); \, \ell = 0, 1, \ldots\}$ be the random walk on $\mathbb{Z}^2$. That is, its one-step increments $\boldsymbol{Y}(\ell+1) - \boldsymbol{Y}(\ell)$ are independent and identically distributed. We denote a random vector subject to the distribution of these increments by $\boldsymbol{X}^{(+)}$. We generate a reflecting process $\{\boldsymbol{L}(\ell)\} \equiv \{(L_1(\ell), L_2(\ell))\}$ from this random walk $\{\boldsymbol{Y}(\ell)\}$ in such a way that it is a discrete Markov chain with state space S and transition probabilities $p(\boldsymbol{i}, \boldsymbol{j})$ given by

$$\mathbb{P}(\boldsymbol{L}(\ell+1) = \boldsymbol{j} | \mid \boldsymbol{L}(\ell) = \boldsymbol{i}) = \begin{cases} \mathbb{P}(\boldsymbol{X}^{(+)} = \boldsymbol{j} - \boldsymbol{i}), & \boldsymbol{j} \in S, \, \boldsymbol{i} \in S_+, \\ \mathbb{P}(\boldsymbol{X}^{(k)} = \boldsymbol{j} - \boldsymbol{i}), & \boldsymbol{j} \in S, \, \boldsymbol{i} \in S_k, \, k = 0, 1, 2, \end{cases}$$

where $\boldsymbol{X}^{(k)}$ is a random vector taking values in $\mathbb{Z}^2$. For this process to be nondefective, it is assumed that $\boldsymbol{X}^{(+)} \geq (-1,-1)$, $\boldsymbol{X}^{(1)} \geq (-1, 0)$, $\boldsymbol{X}^{(2)} \geq (0, -1)$, and $\boldsymbol{X}^{(0)} \geq (0,0)$. Here, inequalities of vectors are defined componentwise.

The vector $\boldsymbol{L}(\ell)$ is reflected at the boundary ∂S and skip free in the direction to ∂S. In particular, its entries $L_1(\ell)$ and $L_2(\ell)$ have similar transitions to the queue length process of the M/G/1 queue. We therefore refer to $\{\boldsymbol{L}(\ell)\}$ as a double M/G/1-type process.

Let $\mathbb{R}$ be the set of all real numbers. Similar to $\mathbb{Z}_+$, let $\mathbb{R}_+$ be the all nonnegative real numbers. We denote the moment generating functions of $\boldsymbol{X}^{(+)}$ and $\boldsymbol{X}^{(k)}$ by γ and γ_k, that is, for $\boldsymbol{\theta} \equiv (\theta_1, \theta_2) \in \mathbb{R}^2$,

$$\gamma(\boldsymbol{\theta}) = \mathbb{E}(\mathrm{e}^{\langle \boldsymbol{\theta}, \boldsymbol{X}^{(+)} \rangle}), \qquad \gamma_k(\boldsymbol{\theta}) = \mathbb{E}(\mathrm{e}^{\langle \boldsymbol{\theta}, \boldsymbol{X}^{(k)} \rangle}), \quad k = 0, 1, 2,$$

where $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ denotes the inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. As usual, $\mathbb{R}^2$ is considered to be a metric space with Euclidean norm $\|\boldsymbol{a}\| \equiv \sqrt{\langle \boldsymbol{a}, \boldsymbol{a} \rangle}$. In this paper we assume the following conditions.

(i) The random walk $\{\boldsymbol{Y}(\ell)\}$ is irreducible and aperiodic.

(ii) The reflecting process $\{\boldsymbol{L}(\ell)\}$ is irreducible and aperiodic.

(iii) For each $\boldsymbol{\theta} \in \mathbb{R}^2$ satisfying $\theta_1 > 0$ or $\theta_2 > 0$, there exist $t > 0$ and $t_k > 0$ such that $1 < \gamma(t\boldsymbol{\theta}) < \infty$ and $1 < \gamma_k(t_k\boldsymbol{\theta}) < \infty$ for $k = 0, 1, 2$.

(iv) Either $\mathbb{E}(X_1^{(+)}) \neq 0$ or $\mathbb{E}(X_2^{(+)}) \neq 0$ for $\boldsymbol{X}^{(+)} = (X_1^{(+)}, X_2^{(+)})$.

Conditions (i) and (iii) are stronger than those actually required, but we use them to simplify arguments. For example, except for the exact asymptotics, (iii) can be weakened to the following condition (see Remark 2).

(iii′) $\gamma(\boldsymbol{\theta})$ and $\gamma_k(\boldsymbol{\theta})$ for $k = 0, 1, 2$ are finite for some $\boldsymbol{\theta} > \boldsymbol{0}$.

In the rest of this section we discuss three basic topics. First, we consider necessary and sufficient conditions for stability of the reflecting random walk $\{\boldsymbol{L}(\ell)\}$, that is, the existence of its stationary distribution, and explain why (iv) is assumed. We then formally define rough and exact asymptotics.

2.1. Stability condition and tail asymptotics

Fayolle *et al.* [12] claimed to obtain necessary and sufficient conditions for stability (see Theorem 3.3.1 therein). However, the proof of their Theorem 3.3.1 is incomplete because important steps are omitted. A better proof can be found in [10]. Furthermore, some exceptional cases are missing as we will see. Nevertheless, their necessary and sufficient conditions can be made to be valid under a minor amendment.

In [12], the necessary and sufficient conditions are separately considered according to whether all the mean drifts are null, that is, $\mathbb{E}(X_1^{(+)}) = \mathbb{E}(X_2^{(+)}) = 0$ or not. One may easily guess that the null drift implies that the stationary distribution has a heavy tail in all directions, that is, the tail decay rates in all directions vanish. We formally prove this fact in Remark 7. Thus, we can assume that (iv) holds when studying the light tail asymptotics.

We now present the stability conditions of [12] under condition (iv). We will consider their geometric interpretations. For this, we introduce some notation. For $k = 1, 2$, let

$$m_k = \mathbb{E}(X_k^{(+)}), \qquad m_k^{(1)} = \mathbb{E}(X_k^{(1)}), \qquad m_k^{(2)} = \mathbb{E}(X_k^{(2)}).$$

Define the vectors

$$\boldsymbol{m} = (m_1, m_2), \qquad \boldsymbol{m}^{(1)} = (m_1^{(1)}, m_2^{(1)}), \qquad \boldsymbol{m}^{(2)} = (m_1^{(2)}, m_2^{(2)}),$$
$$\boldsymbol{m}_\perp^{(1)} = (m_2^{(1)}, -m_1^{(1)}), \qquad \boldsymbol{m}_\perp^{(2)} = (-m_2^{(2)}, m_1^{(2)}).$$

Note that $\boldsymbol{m} \neq \boldsymbol{0}$ by condition (iv). Obviously, $\boldsymbol{m}_\perp^{(k)}$ is orthogonal to $\boldsymbol{m}^{(k)}$ for each $k = 1, 2$. Below we present the stability conditions of [12] using these vectors, in which we make some minor corrections for the missing cases.

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Lemma 1. (Corrected Theorem 3.3.1 of [12].) *If $\boldsymbol{m} \neq \boldsymbol{0}$, the reflecting random walk $\{\boldsymbol{Z}(\ell)\}$ has the stationary distribution, that is, it is stable, if and only if one of the following assertions holds.*

(I) $m_1 < 0, m_2 < 0, \langle \boldsymbol{m}, \boldsymbol{m}_\perp^{(1)} \rangle < 0$, *and* $\langle \boldsymbol{m}, \boldsymbol{m}_\perp^{(2)} \rangle < 0$.

(II) $m_1 \geq 0, m_2 < 0, \langle \boldsymbol{m}, \boldsymbol{m}_\perp^{(1)} \rangle < 0$. *In addition to these conditions, $m_2^{(2)} < 0$ is required if $m_1^{(2)} = 0$.*

(III) $m_1 < 0, m_2 \geq 0, \langle \boldsymbol{m}, \boldsymbol{m}_\perp^{(2)} \rangle < 0$. *In addition to these conditions, $m_1^{(1)} < 0$ is required if $m_2^{(1)} = 0$.*

Remark 1. The additional conditions $m_2^{(2)} < 0$ for $m_1^{(2)} = 0$ in (II) and $m_1^{(1)} < 0$ for $m_2^{(1)} = 0$ in (III) are missing in Theorem 3.3.1 of [12]. To see their necessity, let us assume that $m_1^{(2)} = 0$ in (II). This implies that the reflecting random walk cannot escape from the second coordinate axis except for the origin once it hits the axis. On the other hand, $m_2^{(2)}$ may take any value because there is no constraint on $m_2^{(2)}$ in the first three conditions of (II). Hence, $m_2^{(2)} < 0$ is necessary for stability. By symmetry, the extra condition is similarly required for (III). It is not hard to see the sufficiency of these conditions if the proof in [10] and [12] is traced, and so we omit its verification.

We will see that the stability conditions are closely related to the curves $\gamma(\boldsymbol{\theta}) = 1$ and $\gamma_k(\boldsymbol{\theta}) = 1$. We therefore introduce the following notation:

$$\Gamma = \{\boldsymbol{\theta} \in \mathbb{R}^2; \gamma(\boldsymbol{\theta}) < 1\}, \qquad \partial\Gamma = \{\boldsymbol{\theta} \in \mathbb{R}^2; \gamma(\boldsymbol{\theta}) = 1\},$$
$$\Gamma_k = \{\boldsymbol{\theta} \in \mathbb{R}^2; \gamma_k(\boldsymbol{\theta}) < 1\}, \qquad \partial\Gamma_k = \{\boldsymbol{\theta} \in \mathbb{R}^2; \gamma_k(\boldsymbol{\theta}) = 1\}, \qquad k = 1, 2.$$

Remark 2. (a) Γ and Γ_k are convex sets because γ and γ_k are convex functions. Furthermore, Γ is a bounded set by condition (i).

(b) By condition (iii), $\partial\Gamma$ and $\partial\Gamma_k$ are continuous curves. If we replace (iii) by the weaker assumption (iii′) then $\partial\Gamma$ and $\partial\Gamma_k$ may be empty sets. In these cases, we redefine them as the boundaries of Γ and Γ_k, respectively. This does not change our arguments as long as the solution of $\gamma(\boldsymbol{\theta}) = 1$ or $\gamma_k(\boldsymbol{\theta}) = 1$ is not analytically used. Thus, we will see that the rough asymptotics are still valid under (iii′), but this is not the case for the exact asymptotics.

Note that $\boldsymbol{m}$ and $\boldsymbol{m}^{(k)}$ are normal to the tangents of $\partial\Gamma$ and $\partial\Gamma_k$ at the origin and toward the outside of Γ and Γ_k, respectively (see Figure 1). From this observation, we obtain the following lemma, which gives a geometric interpretation of the stability condition.

Lemma 2. *Under conditions (iii) and (iv), the reflecting random walk $\{\boldsymbol{L}(\ell)\}$ has the stationary distribution if and only if $\Gamma \cap \Gamma_k$ contains a vector $\boldsymbol{\theta}$ such that $\theta_k > 0$ for each $k = 1, 2$. Furthermore, if this is the case, at least for either $k = 1, 2$, there exists a $\boldsymbol{\theta} \in \Gamma \cap \Gamma_k$ such that $\theta_k > 0$ and $\theta_{3-k} \leq 0$.*

Proof. See Appendix A.

Throughout the paper, we assume stability, that is, we assume that either stability condition (I), (II), or (III) holds, in addition to conditions (i)–(iv). We are now ready to formally introduce tail asymptotics for the stationary distribution of the double M/G/1-type process. We denote this stationary distribution by ν. Let $\boldsymbol{L} \equiv (L_1, L_2)$ be the random vector subject to the distribution ν, that is,

$$\nu(\boldsymbol{i}) = \mathbb{P}(\boldsymbol{L} = \boldsymbol{i}), \qquad \boldsymbol{i} \in S.$$

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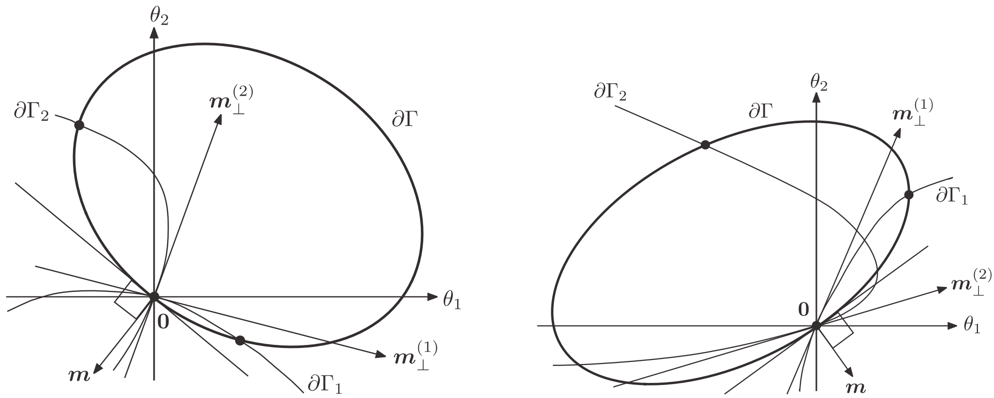

FIGURE 1: Vectors for conditions (I) and (II).

We are interested in the tail asymptotic behavior of this distribution. For this, we define the rough and exact asymptotics. We refer to vector $\boldsymbol{c} \in \mathbb{R}^2$ as a direction vector if $\|\boldsymbol{c}\| = 1$. For an arbitrary direction vector $\boldsymbol{c} \geq \mathbf{0}$, we define $\alpha_{\boldsymbol{c}}$ as

$$\alpha_{\boldsymbol{c}} = -\lim_{x\to\infty} \frac{1}{x} \log \mathbb{P}(\langle \boldsymbol{c}, \boldsymbol{L}\rangle \geq x),$$

as long as it exists. This $\alpha_{\boldsymbol{c}}$ is referred to as a decay rate in the direction $\boldsymbol{c}$. Thus, if the decay rate $\alpha_{\boldsymbol{c}}$ exists, $\mathbb{P}(\langle \boldsymbol{c}, \boldsymbol{L}\rangle \geq x)$ is lower and upper bounded by $\mathrm{e}^{-(\alpha_{\boldsymbol{c}}+\varepsilon)x}$ and $\mathrm{e}^{-(\alpha_{\boldsymbol{c}}-\varepsilon)x}$, respectively, for any $\varepsilon > 0$ and sufficiently large $x \in \mathbb{R}$. If there exists a function f and a positive constant b such that $\mathbb{P}(\langle \boldsymbol{c}, \boldsymbol{L}\rangle \geq x) \sim bf(x)$, that is,

$$\lim_{x\to\infty} \frac{\mathbb{P}(\langle \boldsymbol{c}, \boldsymbol{L}\rangle \geq x)}{f(x)} = b, \tag{1}$$

then $\mathbb{P}(\langle \boldsymbol{c}, \boldsymbol{L}\rangle \geq x)$ is said to have exact asymptotic $f(x)$. In particular, if f is exponential, that is, $f(x) = \mathrm{e}^{-\alpha x}$ for some $\alpha > 0$, then it is called an exactly exponential asymptotic. It is worth noting that the random variable $\langle \boldsymbol{c}, \boldsymbol{L}\rangle$ only takes a countable number of real values. Hence, we must be careful about their periodicity.

Definition 1. (a) A countable set A of real numbers is said to be δ-arithmetic at ∞ for some $\delta > 0$ if δ is the largest number such that, for some $x_0 > 0$, $\{x \in A; x \geq x_0\}$ is a subset of $\{\delta n; n \in \mathbb{Z}_+\}$. On the other hand, A is said to be asymptotically dense at ∞ if there is a positive number a for each $\varepsilon > 0$ such that, for all $x \geq a$, $|x - y| < \varepsilon$ for some $y \in A$.

(b) A random variable X taking countably many real values at most is said to be δ-arithmetic for some integer $\delta > 0$ if δ is the largest positive number such that $\{x \in \mathbb{R}; \mathbb{P}(X = x) > 0\} \subset \{\delta n; n \in \mathbb{Z}\}$.

The following fact is an easy consequence of Lemma 2 and the corollary in [13, Section V.4a], but we prove it in Appendix B for completeness.

Lemma 3. *For a directional vector $\boldsymbol{c} \geq \mathbf{0}$, let $K_{\boldsymbol{c}} = \{\langle \boldsymbol{c}, \boldsymbol{n}\rangle; \boldsymbol{n} \in \mathbb{Z}_+^2\}$. Then $K_{\boldsymbol{c}}$ is asymptotically dense at ∞ if and only if neither c_1 nor c_2 vanishes and c_1/c_2 is irrational. Otherwise, $K_{\boldsymbol{c}}$ is arithmetic at ∞.*

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Owing to this lemma, the x in (1) runs over either arithmetic numbers δn for some $\delta > 0$ or real numbers. In particular, if K_c is 1-arithmetic at ∞ then we replace x by integer n. For example, this n is used for the asymptotics $\mathbb{P}(L_k = n,\ L_{3-k} = i) \sim f(n, i)$ for each fixed $i \in \mathbb{Z}_+$.

2.2. Moment generating function and stationary equation

There are two typical approaches to represent the stationary distribution of the double M/G/1-type process. One is a traditional expression using either generating or moment generating functions. Another is a matrix-analytic expression viewing one of the coordinates as a background state. In this paper we will use both approaches because they have their own merits. We first consider the stationary equation using moment generating functions. Since the states are vectors of nonnegative integers, it is natural to ask why moment generating functions are not used. This question will be answered at the end of this section.

We denote the moment generating function of the stationary random vector $\boldsymbol{L}$ by

$$\varphi(\boldsymbol{\theta}) \equiv \mathbb{E}(e^{\langle \boldsymbol{\theta}, \boldsymbol{L} \rangle}), \qquad \boldsymbol{\theta} \in \mathbb{R}^2.$$

We define a light tail for the stationary distribution ν as in [26].

Definition 2. The stationary distribution ν is said to have a light tail in all directions if there is a positive $\boldsymbol{\theta} \in \mathbb{R}^2$ such that $\varphi(\boldsymbol{\theta}) < \infty$. Otherwise, it is said to have a heavy tail in some direction.

Define the convergence domain $\mathcal{D}$ of the moment generating function φ as

$$\mathcal{D} = \text{the interior of } \{\boldsymbol{\theta} \in \mathbb{R};\ \varphi(\boldsymbol{\theta}) < \infty\}.$$

Then, we can expect that the tail asymptotic of the stationary distribution is obtained through the boundary of the domain $\mathcal{D}$. Obviously, $\mathcal{D}$ is a convex set because φ is a convex function on $\mathbb{R}^2$. Let us derive the stationary equation for this $\mathcal{D}$. Let

$$\varphi_+(\boldsymbol{\theta}) = \mathbb{E}(\mathrm{e}^{\langle \boldsymbol{\theta}, \boldsymbol{L} \rangle} 1(\boldsymbol{L} > \boldsymbol{0})),$$
$$\varphi_k(\theta_k) = \mathbb{E}(\mathrm{e}^{\theta_k L_k} 1(L_k \geq 1,\ L_{3-k} = 0)), \qquad k = 1, 2,$$

where $1(\cdot)$ is the indicator function. Then

$$\varphi(\boldsymbol{\theta}) = \varphi_+(\boldsymbol{\theta}) + \varphi_1(\theta_1) + \varphi_2(\theta_2) + \varphi_0(0),$$

where $\varphi_0(0) = \mathbb{P}(\boldsymbol{L} = \boldsymbol{0})$. From this relation and the stationary equation,

$$\boldsymbol{L} \stackrel{\mathrm{D}}{=} \boldsymbol{L} + \boldsymbol{X}^{(+)} 1(\boldsymbol{L} \in S_+) + \sum_{k \in \{0,1,2\}} \boldsymbol{X}^{(k)} 1(\boldsymbol{L} \in S_k), \tag{2}$$

where '$\stackrel{\mathrm{D}}{=}$' denotes equality in distribution and the random vectors on the right-hand side are assumed to be independent, we have

$$(1 - \gamma(\boldsymbol{\theta}))\varphi_+(\boldsymbol{\theta}) = \sum_{k \in \{1,2\}} (\gamma_k(\boldsymbol{\theta}) - 1)\varphi_k(\theta_k) + (\gamma_0(\boldsymbol{\theta}) - 1)\varphi_0(0), \tag{3}$$

as long as $\varphi(\boldsymbol{\theta}) < \infty$. This equation holds at least for $\boldsymbol{\theta} \leq \boldsymbol{0}$.

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Equation (3) is equivalent to the stationary equation of the Markov chain $\{\boldsymbol{L}(\ell)\}$ and, therefore, characterizes the stationary distribution. Thus, the stationary distribution can be obtained if we can solve (3) for the unknown function φ, or, equivalently, φ_+, φ_1, φ_2, and φ_0. However, this is known to be a notoriously difficult problem. The problem is somewhat simplified when the jumps are skip free (see, e.g. [11], [20], and [26]). This point is detailed below.

Remark 3. (*The kernel method and generating function.*) To obtain useful information from (3), it is key to consider it on the surface obtained from $1-\gamma(\boldsymbol{\theta}) = 0$. This enables us to express $\varphi_i(\theta_i)$ in terms of the other $\varphi_j(\theta_j)$ under the constraint that $1-\gamma(\boldsymbol{\theta}) = 0$. We can then apply analytic extensions for $\varphi_i(\theta_i)$ using complex variables for $\boldsymbol{\theta}$. This analytic approach is called the kernel method (see, e.g. [18], [23], and [26]). In the kernel method, the generating function is more convenient to use than the moment generating function. Let $\tilde{\gamma}(z_1, z_2) = \gamma(\log z_1, \log z_2)$. Then $\tilde{\gamma}(z_1, z_2)$ is the generating function corresponding to $\gamma(\boldsymbol{\theta})$. Note that $z_1 z_2 \tilde{\gamma}(z_1, z_2)$ is a polynomial in z_1 and z_2. In particular, for the skip-free, two-dimensional reflecting random walk, $z_1 z_2(1-\tilde{\gamma}(z_1, z_2)) = 0$, which corresponds to $1-\gamma(\boldsymbol{\theta}) = 0$, is a quadratic equation in z_i for each fixed z_{3-i}. Hence, it can be solved algebraically; see [11]. However, the problem becomes more difficult if the random walk is not skip free. If the jumps are unbounded, the equation $1-\tilde{\gamma}(z_1, z_2) = 0$ has infinitely many solutions in the complex number field for each fixed z_2 (or z_1), and there may be no hope to solve the equation. Even if these roots are found, it would be difficult to analytically extend $\varphi_i(z_i)$.

As the kernel method based on complex analysis is difficult to use for the M/G/1-type process, we look at the problem differently, and consider the equation $1-\gamma(\boldsymbol{\theta}) = 0$ in the real number field. In this case, it has at most two solutions of θ_i for each fixed θ_{3-i} because $\gamma(\boldsymbol{\theta})$ is a two variable convex function. However, the stationary equation (3) is only valid for $\boldsymbol{\theta} \le \boldsymbol{0}$. We therefore introduce a new tool, called the stationary inequality, and work on Γ and Γ_k for $k = 1, 2$. For this, moment generating functions are more convenient because in this case Γ and Γ_k will be convex. This is not necessarily true for generating functions because two variable generating functions may not be convex.

Although we mainly use moment generating functions, we do not exclude the use of generating functions. In fact, they are convenient when considering the tail asymptotics in coordinate directions. For other directions, we again need moment generating functions because $c_1 L_1 + c_2 L_2$ may not be periodic.

3. Convergence domain and main results

The aim of this section is to present the main results on the domain $\mathcal{D}$ and the tail asymptotics. The proofs are given in Section 4. We first give a key tool for finding the domain $\mathcal{D}$, which allows us to extend the valid region of (3) from $\{\boldsymbol{\theta} \in \mathbb{R}^2; \boldsymbol{\theta} \le \boldsymbol{0}\}$.

Lemma 4. *For $\boldsymbol{\theta} \in \mathbb{R}^2$, $\varphi(\boldsymbol{\theta}) < \infty$ and (3) holds if either of the following conditions is satisfied:*

(a) $\boldsymbol{\theta} \in \Gamma$ *and* $\varphi_k(\boldsymbol{\theta}) < \infty$ *for* $k = 1, 2$,

(b) $\boldsymbol{\theta} \in \Gamma \cap \Gamma_1$ *and* $\varphi_2(\theta_2) < \infty$,

(c) $\boldsymbol{\theta} \in \Gamma \cap \Gamma_2$ *and* $\varphi_1(\theta_1) < \infty$.

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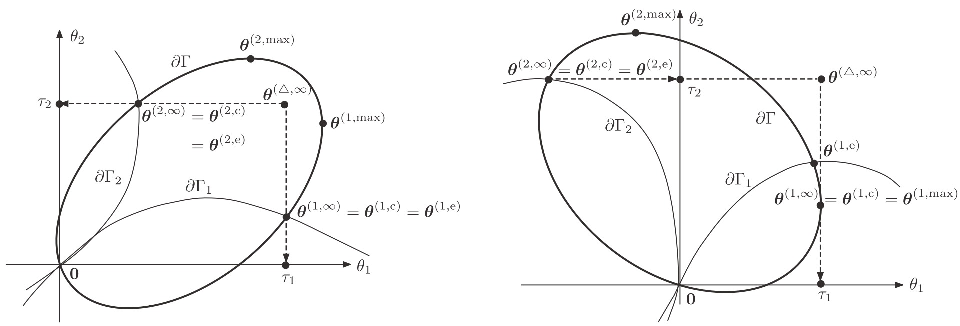

FIGURE 2: Typical figures for (D1).

This lemma is related to Lemma 6.1 of [26] and proved in Appendix C. It suggests that finding Γ, $\Gamma \cap \Gamma_1$, and $\Gamma \cap \Gamma_2$ is important for ensuring that $\varphi(\boldsymbol{\theta})$ is finite. These sets are not empty by Lemma 2, and are obviously convex. Let us denote their extreme points by

$$\boldsymbol{\theta}^{(k,\max)} = \arg \max_{(\theta_1,\theta_2)} \{\theta_k; \gamma(\theta_1, \theta_2) = 1\}, \qquad \boldsymbol{\theta}^{(k,\min)} = \arg \min_{(\theta_1,\theta_2)} \{\theta_k; \gamma(\theta_1, \theta_2) = 1\},$$

$$\boldsymbol{\theta}^{(k,\mathrm{c})} = \arg \sup_{(\theta_1,\theta_2)} \{\theta_k; \boldsymbol{\theta} \in \Gamma \cap \Gamma_k\}, \qquad \boldsymbol{\theta}^{(k,\mathrm{e})} = \arg \max_{(\theta_1,\theta_2)} \{\theta_k; \boldsymbol{\theta} \in \partial\Gamma \cap \partial\Gamma_k\},$$

where the superscripts c and e stand for the convergence parameter and edge, respectively (see Figure 2). Their meanings will be clarified in the context of the Markov additive process $\{\boldsymbol{Z}^{(1)}(\ell)\}$.

It is easy to see that, for $k = 1, 2$,

$$\boldsymbol{\theta}^{(k,\mathrm{c})} = \begin{cases} \boldsymbol{\theta}^{(k,\mathrm{e})}, & \gamma_k(\boldsymbol{\theta}^{(k,\max)}) > 1, \\ \boldsymbol{\theta}^{(k,\max)}, & \gamma_k(\boldsymbol{\theta}^{(k,\max)}) \le 1. \end{cases}$$

Using these points, we classify their configurations into three categories:

(D1) $\theta_1^{(2,\mathrm{c})} < \theta_1^{(1,\mathrm{c})}$ and $\theta_2^{(1,\mathrm{c})} < \theta_2^{(2,\mathrm{c})}$,

(D2) $\boldsymbol{\theta}^{(2,\mathrm{c})} \le \boldsymbol{\theta}^{(1,\mathrm{c})}$,

(D3) $\boldsymbol{\theta}^{(1,\mathrm{c})} \le \boldsymbol{\theta}^{(2,\mathrm{c})}$.

We have excluded the case in which $\theta_1^{(2,\mathrm{c})} \ge \theta_1^{(1,\mathrm{c})}$ and $\theta_2^{(1,\mathrm{c})} \ge \theta_2^{(2,\mathrm{c})}$ because it is impossible (see Figures 2 and 3). These categories were first introduced in [25] for double quasi-birth-and-death processes, and shown to be useful for tail asymptotic problems.

Using this classification, we define the vector $\boldsymbol{\tau} \equiv (\tau_1, \tau_2)$ as

$$\boldsymbol{\tau} = \begin{cases} (\theta_1^{(1,\mathrm{c})}, \theta_2^{(2,\mathrm{c})}) & \text{if (D1) holds}, \\ (\overline{\xi}_1(\theta_2^{(2,\mathrm{c})}), \theta_2^{(2,\mathrm{c})}) & \text{if (D2) holds}, \\ (\theta_1^{(1,\mathrm{c})}, \overline{\xi}_2(\theta_1^{(1,\mathrm{c})})) & \text{if (D3) holds}, \end{cases} \tag{4}$$

where $\overline{\xi}_1$ and $\overline{\xi}_2$ are defined as

$$\overline{\xi}_1(\theta_2) = \max\{\theta; \gamma(\theta, \theta_2) = 1\}, \qquad \overline{\xi}_2(\theta_1) = \max\{\theta; \gamma(\theta_1, \theta) = 1\}.$$

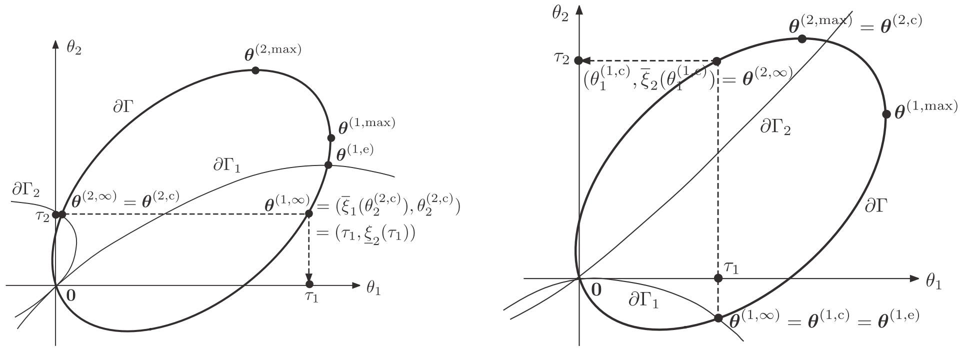

FIGURE 3: Typical figures for (D2) and (D3).

We also define

$$\underline{\xi}_1(\theta_2) = \min\{\theta_1; \gamma(\theta_1, \theta_2) = 1\}, \qquad \underline{\xi}_2(\theta_1) = \min\{\theta_2; \gamma(\theta_1, \theta_2) = 1\}.$$

The domain $\mathcal{D}$ is then obtained as follows.

Theorem 1. *Under conditions (i)–(iv) and the stability condition, we have*

$$\mathcal{D} = \{\boldsymbol{\theta} \in \Gamma_{\max}; \boldsymbol{\theta} < \boldsymbol{\tau}\}, \tag{5}$$

where $\Gamma_{\max} = \{\boldsymbol{\theta} \in \mathbb{R}^2;$ *there exists* $\boldsymbol{\theta}' \in \Gamma, \boldsymbol{\theta} < \boldsymbol{\theta}'\}$. *Furthermore, for* $k = 1, 2$,

$$\tau_k = \sup\{\theta_k \geq 0; \varphi_k(\theta_k) < \infty\}. \tag{6}$$

Remark 4. Theorem 1 is more general than Theorem 3.1 of [25] because it fully identifies the domain and relaxes the skip-free condition given in [25].

We now consider the decay rate of $\mathbb{P}(\langle \boldsymbol{c}, \boldsymbol{L}\rangle \geq x)$ as x goes to ∞. In some cases, we also derive its exact asymptotic. From the domain $\mathcal{D}$ obtained in Theorem 1, we can expect that this decay rate is given by

$$\alpha_{\boldsymbol{c}} = \sup\{x \geq 0; x\boldsymbol{c} \in \mathcal{D}\}. \tag{7}$$

One may consider this to be immediate, claiming that the tail decay rate of a distribution on $[0, \infty)$ is obtained by the convergence parameter of its moment generating function. This claim has been used in the literature, but it is not true (see Appendix D for a counterexample). Thus, we do need to prove (7).

We simplify the notation $\alpha_{\boldsymbol{e}_k}$ to α_k for $\boldsymbol{e}_1 = (1, 0)$ and $\boldsymbol{e}_2 = (0, 1)$. Note that

$$\alpha_1 = \begin{cases} \tau_1, & \bar{\xi}_2(\tau_1) \geq 0, \\ \beta_1, & \bar{\xi}_2(\tau_1) < 0, \end{cases} \qquad \alpha_2 = \begin{cases} \tau_2, & \bar{\xi}_1(\tau_2) \geq 0, \\ \beta_2, & \bar{\xi}_1(\tau_2) < 0, \end{cases} \tag{8}$$

where β_k is the positive solution x of $\gamma(x\boldsymbol{e}_k) = 1$ (see Figure 4).

In the next two theorems we present asymptotic results, which we prove in the next section.

Theorem 2. *Under conditions (i)–(iv) and the stability condition, we have,*

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(L_k \geq n, \ L_{3-k} = i) = -\tau_k, \qquad i \in \mathbb{Z}_+, k = 1, 2. \tag{9}$$

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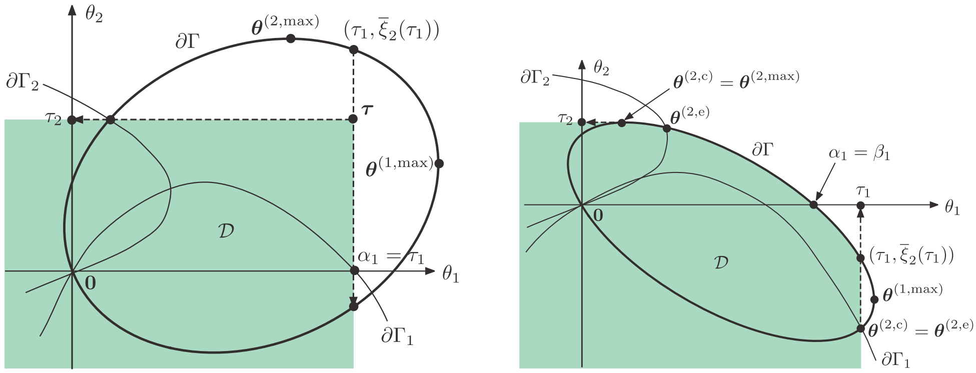

FIGURE 4: Typical figures for $\alpha_1 = \tau_1$ and $\alpha_1 = \beta_1$ (the shaded regions represent the domains).

Furthermore, if $\tau_k \neq \theta_k^{(k,\max)}$ *and if the Markov additive kernel* $\{A_n^{(k)}; n \geq -1\}$ *is* 1*-arithmetic, then we have the following asymptotics for some constant* $b_{ki} > 0$*:*

$$\liminf_{n\to\infty} \mathrm{e}^{\tau_k n}\mathbb{P}(L_k \geq n,\ L_{3-k} = i) \geq b_{ki}, \qquad i \in \mathbb{Z}_+,\ k = 1, 2.$$

In particular, for (D1) with $k = 1, 2$*, (D2) with* $k = 2$*, and (D3) with* $k = 1$*, this is refined as*

$$\lim_{n\to\infty} \mathrm{e}^{\tau_k n}\mathbb{P}(L_k \geq n,\ L_{3-k} = i) = c_{ki}, \qquad i \in \mathbb{Z}_+ \text{ for some } c_{ki} > 0.$$

Remark 5. (a) In the case (D1), if $\tau_k = \theta_k^{(k,\max)}$ and $\boldsymbol{\theta}^{(k,\max)} \neq \boldsymbol{\theta}^{(k,\mathrm{e})}$, then we can show that

$$\lim_{n\to\infty} \mathrm{e}^{\tau_k n}\mathbb{P}(L_k \geq n,\ L_{3-k} = i) = 0. \tag{10}$$

This is immediate from Remark 6(b) below. We conjecture that (10) also holds for $\tau_k = \theta_k^{(k,\max)}$ and $\boldsymbol{\theta}^{(k,\max)} = \boldsymbol{\theta}^{(k,\mathrm{e})}$.

(b) When we apply the kernel method to the skip-free case, $\varphi_k(z)$ with complex variable z (or the corresponding generating function) has a branch point at $z = \tau_k$ for $\tau_k = \theta_k^{(k,\max)}$ under the case (D1), which is a dominant singular point of $\varphi_k(z)$ (see [20]). Thus, the rightmost point $\boldsymbol{\theta}^{(k,\max)}$ corresponds with a branch point of the complex analysis. This is also the reason why (10) holds because $\mathbb{P}(L_k = n,\ L_{3-k} = i)$ has exact asymptotics of the form $n^c\mathrm{e}^{-\tau_k n}$ with $c = -\frac{1}{2}, -\frac{3}{2}$ as $n \to \infty$.

Theorem 3. *Under the conditions of Theorem 2, we have, for any directional vector* $\boldsymbol{c} \geq \boldsymbol{0}$*,*

$$\lim_{x\to\infty} \frac{1}{x}\log \mathbb{P}(\langle \boldsymbol{c}, \boldsymbol{L}\rangle \geq x) = -\alpha_{\boldsymbol{c}}, \tag{11}$$

where we recall that $\alpha_{\boldsymbol{c}} = \sup\{x \geq 0; x\boldsymbol{c} \in \mathcal{D}\}$*. Furthermore, assume that* $\gamma(\alpha_{\boldsymbol{c}}\boldsymbol{c}) = 1$*,* $\gamma_k(\alpha_{\boldsymbol{c}}\boldsymbol{c}) \neq 1$*, and* $\alpha_{\boldsymbol{c}}c_k \neq \tau_k$ *for* $k = 1, 2$*. Then we have, for some positive constant* $b_{\boldsymbol{c}}$*,*

$$\lim_{x\to\infty} \mathrm{e}^{\alpha_{\boldsymbol{c}} x}\mathbb{P}(\langle \boldsymbol{c}, \boldsymbol{L}\rangle \geq x) = b_{\boldsymbol{c}}, \tag{12}$$

where x *runs over* $\{\delta n; n \in \mathbb{Z}_+\}$ *if* $\langle \boldsymbol{c}, \boldsymbol{X}^{(+)}\rangle$ *is* δ*-arithmetic for some* $\delta > 0$*, while it runs over* $\mathbb{R}_+$ *otherwise, which is equivalent to* $c_2 \neq 0$ *and* c_1/c_2 *not rational.*

Remark 6. (a) Since α_k may be less than τ_k, the decay rate of $\mathbb{P}(L_k \geq n)$ may be different from that of $\mathbb{P}(L_k \geq n, L_{3-k} = 0)$.

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(b) Similar to Remark 5, if $\alpha_k = \theta_k^{(k,\max)}$ and $\boldsymbol{\theta}^{(k,\max)} \neq \boldsymbol{\theta}^{(k,\mathrm{e})}$, then we can show that $\varphi(\boldsymbol{\theta}^{(k,\max)}) < \infty$ using expression (33) below. This implies that

$$\lim_{n\to\infty} \mathrm{e}^{\alpha_k n}\mathbb{P}(L_k \geq n) = 0.$$

This obviously implies (10).

(c) For the jumps that are skip free, finer exact asymptotics were obtained for coordinate directions in [20], where the kernel method discussed in Section 2.2 was partially used. One may think to apply the same approach as in [20]. Unfortunately, this approach does not work because of the same reason discussed in Remark 3.

We now consider how the decay rates τ_k and $\alpha_{\boldsymbol{c}}$ are influenced by the modeling primitives. Obviously, if $\gamma(\boldsymbol{\theta})$, $\gamma_1(\boldsymbol{\theta})$, and $\gamma_2(\boldsymbol{\theta})$ are increased by changing the modeling primitives, then the open sets Γ, Γ_1, and Γ_2 are diminished.

Lemma 5. *Under the assumptions of Theorem 2, if the distributions of $\boldsymbol{X}^{(+)}$, $\boldsymbol{X}^{(1)}$, and $\boldsymbol{X}^{(2)}$ are changed to increase $\gamma(\boldsymbol{\theta})$, $\gamma_1(\boldsymbol{\theta})$, and $\gamma_2(\boldsymbol{\theta})$ for each fixed $\boldsymbol{\theta} \in \mathbb{R}^2$, then the decay rates τ_k and $\alpha_{\boldsymbol{c}}$ are decreased.*

Here decreasing and increasing are used in the weaker sense. This convention will be used throughout the paper. To materialize the monotone property in Lemma 5, we use the following stochastic order for random vectors.

Definition 3. (*[30].*) (a) For random variables X and Y, the distribution of X is said to be less than the distribution of Y in the convex order, denoted by $X \leq_{\mathrm{cx}} Y$, if, for any convex function f from $\mathbb{R}$ to $\mathbb{R}$,

$$\mathbb{E}(f(X)) \leq \mathbb{E}(f(Y)),$$

as long as the expectations exist.

(b) For two-dimensional random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, if $\langle\boldsymbol{\theta}, \boldsymbol{X}\rangle \leq_{\mathrm{cv}} \langle\boldsymbol{\theta}, \boldsymbol{Y}\rangle$ for all $\boldsymbol{\theta} \in \mathbb{R}^2$ then the distribution of $\boldsymbol{X}$ is said to be less than the distribution of $\boldsymbol{Y}$ in the linear convex order, denoted by $\boldsymbol{X} \leq_{\mathrm{lcx}} \boldsymbol{Y}$.

For the linear convex order, Koshevoy and Mosler [22] gave several equivalent conditions (see also [34]), such as Strassen's characterization for the convex order which characterizes the variability of this order (see Theorem 2.6.6 of [30]). That is, $\boldsymbol{X} \leq_{\mathrm{lcx}} \boldsymbol{Y}$ if and only if there is a random variable $U_{\boldsymbol{\theta}}$ for each $\boldsymbol{\theta} \in \mathbb{R}^2$ such that $\mathbb{E}(U_{\boldsymbol{\theta}} \mid \langle\boldsymbol{\theta}, \boldsymbol{X}\rangle) = 0$ and

$$\langle\boldsymbol{\theta}, \boldsymbol{Y}\rangle \stackrel{\mathrm{D}}{=} \langle\boldsymbol{\theta}, \boldsymbol{X}\rangle + U_{\boldsymbol{\theta}}. \tag{13}$$

Since e^x is a convex function, the following fact is immediate from Theorems 2 and 3 and Lemma 5.

Corollary 1. *If the distributions of $\boldsymbol{X}^{(+)}$, $\boldsymbol{X}^{(1)}$, and $\boldsymbol{X}^{(2)}$ are increased in the linear convex order, then the decay rates τ_k and $\alpha_{\boldsymbol{c}}$ are decreased for $k = 1, 2$ and any direction $\boldsymbol{c} \geq \boldsymbol{0}$, where the stability condition is unchanged due to (13).*

This corollary shows how the decay rates are degraded by increasing the variability of the transition jumps $\boldsymbol{X}^{(+)}$, $\boldsymbol{X}^{(1)}$, and $\boldsymbol{X}^{(2)}$.

4. Proofs of the theorems

The aim of this section is to prove Theorems 1, 2, and 3. Before we present these proofs, we need two auxiliary results: another representation of the stationary distribution using a Markov additive process, and an iteration algorithm for deriving the convergence domain $\mathcal{D}$.

4.1. Occupation measure and Markov additive process

We first represent the stationary distribution using a so-called censoring. That is, the stationary probability of each state is computed as the expected number of visiting times to that state between the returning times to the boundary or its face. The set of these expected numbers is called an occupation measure.

Let U be a subset of S, and let $\sigma^U = \inf\{\ell \ge 1; \boldsymbol{L}(\ell) \in U\}$. For this U, define the distribution g^U of the first return state and the occupation measure h^U as

$$g^U(\boldsymbol{i}, \boldsymbol{j}) = \mathbb{P}(\boldsymbol{L}(\sigma^U) = \boldsymbol{j},\, \sigma^U < \infty \mid \boldsymbol{L}(0) = \boldsymbol{i}), \qquad \boldsymbol{i}, \boldsymbol{j} \in U,$$

$$h^U(\boldsymbol{i}, \boldsymbol{j}) = \mathbb{E}\biggl(1(\sigma^U < \infty) \sum_{\ell=0}^{\sigma^U-1} 1(\boldsymbol{L}(\ell) = \boldsymbol{j}) \biggm| \boldsymbol{L}(0) = \boldsymbol{i}\biggr), \qquad \boldsymbol{i}, \boldsymbol{j} \in S \setminus U.$$

Let G^U and H^U be the matrices whose $(\boldsymbol{i}, \boldsymbol{j})$th entries are $g^U(\boldsymbol{i}, \boldsymbol{j})$ and $h^U(\boldsymbol{i}, \boldsymbol{j})$, respectively. The matrix H^U may also be considered a potential kernel (see, e.g. [32, Section 2.1] for this kernel). Note that G^U is stochastic since $\{\boldsymbol{L}(\ell)\}$ has the stationary distribution ν. Let ν^U be the stationary distribution of G^U, which is uniquely determined up to a constant multiplier by $\nu^U G^U = \nu^U$. By $\mathbb{E}_{\nu^U}(\sigma^U)$ we denote the expectation of σ^U with respect to ν^U. By the existence of the stationary distribution, $\mathbb{E}_{\nu^U}(\sigma^U)$ is finite. Then, as discussed in Section 2 of [29], it follows from censoring on the set U that

$$\nu(\boldsymbol{j}) = \frac{1}{\mathbb{E}_{\nu^U}(\sigma^U)} \sum_{\boldsymbol{i} \in U} \nu^U(\boldsymbol{i}) \sum_{\boldsymbol{k} \in S \setminus U} p(\boldsymbol{i}, \boldsymbol{k}) h^U(\boldsymbol{k}, \boldsymbol{j}), \qquad \boldsymbol{j} \in S \setminus U, \tag{14}$$

where $p(\boldsymbol{i}, \boldsymbol{k}) = \mathbb{P}(\boldsymbol{L}(1) = \boldsymbol{k} \mid \boldsymbol{L}(0) = \boldsymbol{i})$.

We here need the distribution ν^U because U may not be a singleton. The censored process is particularly useful when the occupation measure is simple. For example, if we choose ∂S for U then $h^U(\boldsymbol{i}, \boldsymbol{j})$ is obtained from the random walk $\{\boldsymbol{Y}(\ell)\}$, which is simpler than $\{\boldsymbol{L}(\ell)\}$.

We next choose $U = S_0 \cup S_2$ for $H^U \equiv \{h^U(\boldsymbol{i}, \boldsymbol{j}); \boldsymbol{i}, \boldsymbol{j} \in S \setminus U\}$. This occupation measure has been used to find the tail asymptotics of the marginal stationary distribution (see, e.g. [21], [25], and [28]). For each $m, n \ge \ell \ge 1$, we denote the $\mathbb{Z}_+ \times \mathbb{Z}_+$ matrix whose (i, j)th entry is $h^{U_\ell}((m, i), (n, j))$ by $H^{(1,\ell)}_{mn}$, where $U_\ell = \{0, 1, \dots, \ell - 1\} \times \mathbb{Z}_+$. Since this $H^{(1,m)}_{m(m+n)}$ does not depend on $m \ge \ell$, we simply denote it by $H^{(1)}_n$ for $n \ge 0$, where $H^{(1)}_0$ is the identity matrix.

For nonnegative integers $m, n \ge 0$, let

$$\begin{aligned} &r^{(1)}((m, i), (m + n, j)) \\ &\quad = \sum_{\ell=1}^{\infty} \mathbb{P}(\boldsymbol{L}(\ell) = (m + n, j), m < L_1(\ell) \le \min(L_1(1), \dots, L_1(\ell - 1)) \mid \boldsymbol{L}(0) = (m, i)). \end{aligned} \tag{15}$$

For $m \ge 1$, $r^{(1)}((m, i), (m + n, j))$ does not depend on m, so, for each $n \ge 1$, we denote the matrix whose (i, j)th entry is $r^{(1)}((m, i), (m + n, j))$ by $R^{(1)}_n$ for $n \ge 1$. On the other hand, for $m = 0$, we denote the corresponding matrix by $R^{(1)}_{0n}$ for $n \ge 1$.

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For each $n \geq 0$, let $\boldsymbol{\nu}_n^{(1)}$ be the vector whose ith entry is $\nu(n, i)$. Similar to (14), it follows from censoring with respect to $U = U_n$ that

$$\boldsymbol{\nu}_n^{(1)} = \boldsymbol{\nu}_0^{(1)} R_{0n}^{(1)} + \sum_{\ell=1}^{n-1} \boldsymbol{\nu}_\ell^{(1)} R_{n-\ell}^{(1)}, \qquad n \geq 1. \tag{16}$$

Using the well-known identity

$$\sum_{n=0}^{\infty} s^n H_n^{(1)} = \left(I - \sum_{n=1}^{\infty} s^n R_n^{(1)}\right)^{-1},$$

(16) can be written as

$$\boldsymbol{\nu}_n^{(1)} = \boldsymbol{\nu}_0^{(1)} \sum_{\ell=1}^{n} R_{0\ell}^{(1)} H_{n-\ell}^{(1)} \geq \boldsymbol{\nu}_1^{(1)} H_{n-1}^{(1)}, \qquad n \geq 1. \tag{17}$$

These formulae can also be found in [28] and [29]. Thus, $\boldsymbol{\nu}_n^{(1)}$ is given in terms of $\boldsymbol{\nu}_0^{(1)}$, $\{R_{0n}^{(1)}\}$, and $\{H_n^{(1)}\}$, expressions which will now also be useful.

We consider a Markov additive process generated from the reflecting process $\{\boldsymbol{L}(\ell)\}$ by removing the boundary transitions on $S_0 \cup S_2$, and present a useful identity called the Wiener–Hopf factorization.

Denote the Markov additive process by $\{\boldsymbol{Z}^{(1)}(\ell)\}$. Specifically, let $\{A_n^{(1)}; n \geq -1\}$ be its Markov additive kernel, that is, the (i, j)th entry of matrix $A_n^{(1)}$ is defined as

$$[A_n^{(1)}]_{ij} = \mathbb{P}(\boldsymbol{Z}^{(1)}(\ell+1) = (m+n, j) \mid \boldsymbol{Z}^{(1)}(\ell) = (m, i)), \qquad n, m \in \mathbb{Z},\ i, j \in \mathbb{Z}_+.$$

Obviously, the right-hand side is independent of m, and

$$[A_n^{(1)}]_{ij} = \begin{cases} \mathbb{P}(\boldsymbol{X}^{(+)} = (n, j - i)), & i \geq 1,\ j \geq i - 1, \\ \mathbb{P}(\boldsymbol{X}^{(1)} = (n, j - i)), & i = 0,\ j \geq 0, \end{cases}$$

for $n \geq -1$, where n and i, j are referred to as the level and background states, respectively. Define matrix $A_*^{(1)}(t)$ for $t > 0$ by

$$[A_*^{(1)}(t)]_{ij} = \sum_{n=-1}^{\infty} t^n [A_n^{(1)}]_{ij}, \qquad t > 0, i,\ j \in \mathbb{Z}_+,$$

as long as they exist. This matrix is called a matrix generating function of the transition kernel $A_n^{(1)}$. Similarly, we denote the matrix generating function $R_n^{(1)}$ by $R_*^{(1)}(t)$. Note that (15) is also valid for $\boldsymbol{Z}^{(1)}(\ell)$ instead of $\boldsymbol{L}(\ell)$. Hence, $R_*^{(1)}(t)$ is well defined for the Markov additive process $\{\boldsymbol{Z}^{(1)}(\ell)\}$. Then, we have the Wiener–Hopf factorization

$$I - A_*^{(1)}(t) = (I - R_*^{(1)}(t))(I - G_*^{(1)}(t)), \tag{18}$$

where $G_*^{(1)}(t)$ is the $\mathbb{Z} \times \mathbb{Z}$ matrix whose (i, j)th entry is given by

$$g_*^{(1)}(t)(i, j) = \mathbb{E}(t^{Z_1^{(1)}(\sigma_1^{-0})} 1(Z_2^{(1)}(\sigma_1^{-0}) = j) \mid Z_2^{(1)}(0) = i),$$

where $\sigma_1^{-0} = \inf\{\ell \geq 1; Z_1^{(1)}(\ell) \leq 0\}$. Factorization (18) goes back to [1], but t is limited to a complex number satisfying $|t| \leq 1$ for simplicity. The present version is valid as long as $A_*^{(1)}(t)$ exists. This fact was formally proved in [29], but has often been ignored in the literature (see, e.g. [28]), even though it is crucial to the tail asymptotic problem.

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4.2. Iteration algorithm and bounds

To prove Theorem 1, we prepare a series of lemmas. The main body of the proof will be given in the next section. We first construct a sequence in the closure of $\mathcal{D}$, denoted by $\overline{\mathcal{D}}$, that converges to $\boldsymbol{\tau}$ defined in (4). For this, we rewrite (3) as

$$\begin{aligned}(1-\gamma(\boldsymbol{\theta}))\varphi_+(\boldsymbol{\theta}) + (1-\gamma_k(\boldsymbol{\theta}))\varphi_k(\theta_k) \\ = (\gamma_{3-k}(\boldsymbol{\theta})-1)\varphi_{3-k}(\theta_{3-k}) + (\gamma_0(\boldsymbol{\theta})-1)\varphi_0(0).\end{aligned} \tag{19}$$

We expand the confirmed region of $\varphi(\boldsymbol{\theta}) < \infty$ using (3) and (19) with the help of Lemma 4.

Let $\Gamma_k^{(0)} = \{\boldsymbol{\theta} \in \Gamma_k \cap \Gamma; \theta_{3-k} \le 0\}$ for $k = 1, 2$. Note that $\Gamma_k^{(0)}$ is not empty by Lemma 2. Obviously, $\varphi_{3-k}(\theta_{3-k})$ is finite for $\boldsymbol{\theta} \in \Gamma_k^{(0)}$. Hence, by Lemma 4, $\varphi_+(\boldsymbol{\theta})$ and $\varphi_k(\theta_k)$ are finite for $\boldsymbol{\theta} \in \Gamma_k^{(0)}$. Thus, $\varphi(\boldsymbol{\theta})$, $\varphi_1(\theta_1)$, and $\varphi_2(\theta_2)$ are finite for $\boldsymbol{\theta} \in \Gamma_1^{(0)} \cup \Gamma_2^{(0)}$. We define $\boldsymbol{\theta}^{(\Delta,0)} \equiv (\theta_1^{(\Delta,0)}, \theta_2^{(\Delta,0)})$ by

$$\theta_k^{(\Delta,0)} = \sup\{\theta_k; (\theta_1,\theta_2) \in \Gamma_k^{(0)}\}, \qquad k = 1, 2.$$

By Lemma 2, at least one of $\theta_1^{(\Delta,0)}$ and $\theta_2^{(\Delta,0)}$ is positive under condition (iv).

We now define $\boldsymbol{\theta}^{(\Delta,n)} = (\theta_1^{(\Delta,n)}, \theta_2^{(\Delta,n)})$ for $n \ge 1$ by

$$\theta_k^{(\Delta,n)} = \sup\{\theta_k; \boldsymbol{\theta} \in \Gamma_k \cap \Gamma,\ \theta_{3-k} \le \theta_{3-k}^{(\Delta,n-1)}\}.$$

Then $\boldsymbol{\theta}^{(\Delta,n)}$ is nondecreasing in n, and $\boldsymbol{\theta}^{(\Delta,n)} \le \boldsymbol{\theta}^{\max}$ from our definition. Thus, the sequence $\boldsymbol{\theta}^{(\Delta,n)}$ converges to a finite positive vector because $\boldsymbol{\theta}^{(\Delta,1)} > \mathbf{0}$. Denote this limit by $\boldsymbol{\theta}^{(\Delta,\infty)} \equiv (\theta_1^{(\Delta,\infty)}, \theta_2^{(\Delta,\infty)})$. Since $\Gamma_k \cap \Gamma$ is a bounded convex set, we can see that

$$\theta_k^{(\Delta,\infty)} = \sup\{\theta_k; \boldsymbol{\theta} \in \Gamma_k \cap \Gamma,\ \theta_{3-k} \le \theta_{3-k}^{(\Delta,\infty)}\}, \qquad k = 1, 2. \tag{20}$$

This can be considered as a fixed-point equation, and we have the following solution.

Lemma 6. $\boldsymbol{\theta}^{(\Delta,\infty)} = \boldsymbol{\tau}$, *and* $\varphi(\boldsymbol{\theta})$ *is finite for all* $\boldsymbol{\theta} < \boldsymbol{\theta}^{(k,\infty)}$ *for* $k = 1, 2$, *where*

$$\boldsymbol{\theta}^{(1,\infty)} = (\theta_1^{(\Delta,\infty)}, \underline{\xi}_2(\theta_1^{(\Delta,\infty)})), \qquad \boldsymbol{\theta}^{(2,\infty)} = (\underline{\xi}_1(\theta_2^{(\Delta,\infty)}), \theta_2^{(\Delta,\infty)}).$$

Proof. See Appendix E.

We need two more lemmas. Recall that $\boldsymbol{c} \in \mathbb{R}^2$ is called a direction vector if $\|\boldsymbol{c}\| = 1$. Let $\mathbf{1}$ be the vector of 1s. The dimension of $\mathbf{1}$ is either 2 or ∞; the distinction will be clear from the context.

Lemma 7. *Let* $\Delta(\boldsymbol{a}) = \{\boldsymbol{x} \in \mathbb{R}^2; \mathbf{0} \le \boldsymbol{x} < \boldsymbol{a}\}$ *for* $\boldsymbol{a} \ge \mathbf{1}$. *Then, under conditions (i), (ii), and (iii), we have, for any direction vector* $\boldsymbol{c} > \mathbf{0}$ *and any* $\boldsymbol{a} \ge \mathbf{1}$,

$$\liminf_{x\to\infty} \frac{1}{x} \log \mathbb{P}(\boldsymbol{L} \in x\boldsymbol{c} + \Delta(\boldsymbol{a})) \ge -\sup\{\langle \boldsymbol{\theta}, \boldsymbol{c}\rangle; \gamma(\boldsymbol{\theta}) \le 1\}; \tag{21}$$

therefore, $\varphi(\boldsymbol{\theta})$ *is infinite for* $\boldsymbol{\theta} \notin \overline{\Gamma}_{\max}$, *where* $\overline{\Gamma}_{\max}$ *is the closure of* $\Gamma_{\max}$.

Remark 7. Lemma 7 is valid without condition (iv). Hence, it can be used for $\mathbb{E}(X_1^{(+)}) = \mathbb{E}(X_2^{(+)}) = 0$. In this case, the right-hand side of (21) is 0, so the stationary distribution ν cannot have a light tail.

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The first part of Lemma 7 is obtained in Theorem 3.1 of [5] (see also Theorem 1.6 therein). However, these theorems use Theorem 1.2 therein, whose proof is largely omitted for the lower bound. We therefore give a self-contained proof of Lemma 7 in Appendix F. We need one more lemma.

Lemma 8. *For each $k = 1, 2$,*

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(L_1 \geq n,\ L_2 = i) \geq -\theta_1^{(1,\mathrm{c})}, \qquad i \in \mathbb{Z}_+, \tag{22}$$

and, therefore, $\theta_k > \theta_k^{(k,\mathrm{c})}$ implies that $\varphi_k(\theta_k) = \infty$, so $\varphi(\boldsymbol{\theta}) = \infty$.

Proof. By symmetry, it is sufficient to prove the lemma for $k = 1$. From (17), we have

$$\mathbb{P}(L_1 \geq n,\ L_2 = i) = \sum_{\ell=n}^{\infty} [\boldsymbol{\nu}_n^{(1)}]_i \geq \left[\boldsymbol{\nu}_0^{(1)} R_{01}^{(1)} \sum_{\ell=n-1}^{\infty} H_\ell^{(1)}\right]_i, \qquad i \in \mathbb{Z}_+.$$

Hence, (22) follows from Theorem 4.1 of [21].

We are now ready to prove Theorem 1.

4.3. Proof of Theorem 1

We first prove that $\{\boldsymbol{\theta} \in \Gamma_{\max}; \boldsymbol{\theta} < \boldsymbol{\tau}\} \subset \mathcal{D}$. Since $\boldsymbol{\theta} \in \Gamma_{\max}$ implies the existence of $\boldsymbol{\theta}' \in \Gamma$ such that $\boldsymbol{\theta}' > \boldsymbol{\theta}$, it is sufficient to prove that

$$\Gamma_{\boldsymbol{\tau}} \equiv \{\boldsymbol{\theta} \in \Gamma; \boldsymbol{\theta} < \boldsymbol{\tau}\} \subset \mathcal{D}.$$

For cases (D2) and (D3), this $\Gamma_{\boldsymbol{\tau}}$ is a subset of $\{\boldsymbol{\theta} \in \mathbb{R}^2; \boldsymbol{\theta} < \boldsymbol{\theta}^{(2,\infty)}\}$ and $\{\boldsymbol{\theta} \in \mathbb{R}^2; \boldsymbol{\theta} < \boldsymbol{\theta}^{(1,\infty)}\}$, respectively, on which $\varphi(\boldsymbol{\theta})$ is finite by Lemma 6. Hence, we only need to consider the case (D1). In this case, $\varphi_k(\theta) < \infty$ for $\theta < \theta_k^{(k,\infty)} = \tau_k$, $k = 1, 2$, and, therefore, $\varphi(\boldsymbol{\theta}) < \infty$ for $\boldsymbol{\theta} \in \Gamma$ by Lemma 4. Thus, we have $\Gamma_{\boldsymbol{\tau}} \subset \mathcal{D}$. Obviously, this also implies that

$$\tau_k = \theta_k^{(k,\infty)} \leq \sup\{\theta_k \geq 0; \varphi_k(\theta_k) < \infty\}. \tag{23}$$

We now prove that $\mathcal{D} \subset \{\boldsymbol{\theta} \in \Gamma_{\max}; \boldsymbol{\theta} < \boldsymbol{\tau}\}$. Because of the symmetric roles of θ_1 and θ_2, it is sufficient to prove that either $\theta_1 > \tau_1$ or $\gamma(\boldsymbol{\theta}) > 1$ with $\boldsymbol{\theta} \notin \Gamma_{\max}$ implies that $\varphi(\boldsymbol{\theta}) = \infty$. The latter is immediate from Lemma 7, and so we only need to prove that

$$\theta_1 > \tau_1 \quad \Longrightarrow \quad \varphi_1(\theta_1) = \infty, \tag{24}$$

which together with (23) verifies (6). We prove (24) for the cases (D1), (D2), and (D3) separately.

We first consider the case in which (D1) or (D3) holds. In this case, $\theta_1^{(1,\infty)} = \theta_1^{(1,\mathrm{c})}$, and, therefore, Lemma 8 verifies the claim.

We now consider the case in which (D2) holds. In this case, $\theta_1^{(1,\infty)} = \overline{\xi}_1(\theta_2^{(2,\mathrm{c})})$ and $\theta_2^{(2,\mathrm{c})} = \theta_2^{(2,\infty)}$. Then, applying the same argument for θ_1, we have $\varphi(\boldsymbol{\theta}) = \infty$ and $\varphi_2(\theta_2) = \infty$ for $\theta_2 > \theta_2^{(2,\infty)}$. If $\theta_1^{(1,\infty)} = \theta_1^{(1,\mathrm{c})}$ then Lemma 8 again verifies (24). Hence, we assume that $\theta_1^{(1,\infty)} < \theta_1^{(1,\mathrm{c})}$.

In what follows, we consider the stationary equation (3) for $\theta_1 \in (0, \theta_1^{(1,\infty)})$. Let $\theta_2 = \underline{\xi}_2(\theta_1)$. Since $\varphi(\boldsymbol{\theta}) < \infty$ for this $\boldsymbol{\theta}$, (3) is valid, and yields

$$\begin{aligned}&(1 - \gamma_1(\theta_1, \underline{\xi}_2(\theta_1)))\varphi_1(\theta_1)\\ &\quad = (\gamma_2(\theta_1, \underline{\xi}_2(\theta_1)) - 1)\varphi_2(\underline{\xi}_2(\theta_1)) + (\gamma_0(\theta_1, \underline{\xi}_2(\theta_1)) - 1)\varphi_0(0).\end{aligned} \tag{25}$$

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We increase θ_1 to $\theta_1^{(1,\infty)}$ in this equation. Note that we can find $\varepsilon_0 > 0$ such that $\gamma_1(\theta_1, \underline{\xi}_2(\theta_1)) \neq 1$ and $\gamma_2(\theta_1, \underline{\xi}_2(\theta_1)) \neq 1$ for $\theta_1 \in [\theta_1^{(1,\infty)} - \varepsilon_0, \theta_1^{(1,c)}]$. Since $\underline{\xi}_2(\theta_1^{(1,\infty)}) = \theta_2^{(2,\infty)}$ and $\underline{\xi}_2(\theta)$ is increasing for $\theta \in (\theta_1^{(1,\infty)} - \varepsilon_0, \theta_1^{(1,c)})$, we have, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\lim_{\theta_1 \uparrow \theta_1^{(1,\infty)} + \varepsilon} \varphi_2(\underline{\xi}_2(\theta_1)) = \infty.$$

This and (25) verify (24). This completes the proof.

4.4. Proof of Theorem 2

The proof of Theorem 2 is immediate from the following two lemmas that provide suitable bounds for the tail probabilities.

Lemma 9. *Under the assumptions of Theorem 1,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(L_k \geq n, L_{3-k} = i) \leq -\tau_k, \qquad i \in \mathbb{Z}_+,\ k = 1, 2, \tag{26}$$

and, for any directional vector $\boldsymbol{c} \geq \boldsymbol{0}$,

$$\limsup_{x\to\infty} \frac{1}{x} \log \mathbb{P}(\langle \boldsymbol{c}, \boldsymbol{L}\rangle \geq x) \leq -\sup\{u \geq 0; u\boldsymbol{c} \in \mathcal{D}\}. \tag{27}$$

Proof. Equation (26) is immediate from (5) and (6). To see (27), we use the Markov inequality

$$\mathrm{e}^{ux}\mathbb{P}(\langle \boldsymbol{c}, \boldsymbol{L}\rangle \geq x) \leq \mathbb{E}(\mathrm{e}^{\langle u\boldsymbol{c}, \boldsymbol{L}\rangle}), \qquad u \geq 0,\ n = 0, 1, \ldots.$$

Taking the logarithm of both sides, dividing by $x > 0$, and letting $x \to \infty$ yields (27).

Lemma 10. *Under the assumptions of Theorem 1,*

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(L_1 \geq n,\ L_2 = i) \geq -\tau_1, \qquad i \in \mathbb{Z}_+. \tag{28}$$

Furthermore, assume that the Markov additive kernel $\{A_n^{(k)}; n \geq -1\}$ *is* 1*-arithmetic concerning the additive component. If either (D1) with* $\tau_1 = \theta_1^{(1,c)} < \theta_1^{(1,\max)}$ *or (D3) holds then, for some positive vector* $\boldsymbol{b}$,

$$\lim_{n\to\infty} \mathrm{e}^{\tau_1 n} \boldsymbol{\nu}_n^{(1)} = \boldsymbol{b}, \tag{29}$$

while, if (D2) with $\tau_1 < \theta_1^{(1,c)}$ *holds then, for some constants* $b' > 0$,

$$\liminf_{n\to\infty} \mathrm{e}^{\tau_1 n} \boldsymbol{\nu}_n^{(1)} \geq b'\boldsymbol{x}, \tag{30}$$

where the limit is taken componentwise and $\boldsymbol{x}$ *is the left-invariant vector of* $A_*^{(1)}(\tau_1)$.

Proof. If $\tau_1 = \theta_1^{(1,\max)}$ then we obviously have (28) by Lemma 7. Otherwise, (28) follows from (29) and (30) or their δ-arithmetic versions for each positive integer δ. Thus, it remains to prove (29) and (30).

We first consider the case where either (D1) with $\theta_1^{(1,c)} < \theta_1^{(1,\max)}$ or (D3) holds. In this case, we have $\tau_1 = \theta_1^{(1,e)} = \theta_1^{(1,c)}$ and, therefore, $\tau_1 < \theta_1^{(1,\max)}$ by the definition of τ_1 given in (4). By Lemma 4.2 of [21], we already know that $A_*^{(1)}(\mathrm{e}^{\theta_1^{(1,e)}})$ is positive recurrent. Furthermore, $\varphi_2(\theta_2) < \infty$ for $\theta_2 < \tau_2$ and $\gamma_1(\boldsymbol{\theta}^{(1,e)}) = 1$. Hence, using the notation $\boldsymbol{\nu}_n^{(1)}$ of Section 2, we

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can verify all the conditions of Theorem 4.1 of [28] since $\{A_n^{(k)}\}$ is 1-arithmetic. Thus, we obtain (29).

We now consider the case where (D2) with $\tau_1 < \theta_1^{(1,\mathrm{c})}$ holds. We use the same idea applied to the double quasi-birth-and-death process, a special case of the double M/G/1-type process, in [25] (see Proposition 3.1 therein and Lemma 2.2.1 of [24]). However, the skip-free condition for the reflecting process is crucial to the arguments in [24] and [25], and, therefore we cannot directly use these arguments since the skip-free condition is not satisfied for the present model. Thus, we need some more ideas.

Similar to the case (D3) for $k = 1$, (D2) implies that $\tau_2 = \theta_2^{(2,\mathrm{e})}$ and, for a positive constant $b_\ell^{(2)}$ for each $\ell \in \mathbb{Z}_+$,

$$\lim_{i\to\infty} \mathrm{e}^{\tau_2 i} \nu(\ell, i) = b_\ell^{(2)}.$$

Furthermore, let x_i be the ith entry of the left-invariant vector $\boldsymbol{x}$ of $A_*^{(1)}(\mathrm{e}^{\tau_1})$, where $\tau_2 = \underline{\xi}_1(\tau_1)$. Then it follows from Theorem A.1 and (A.11) of [21] that the complex variable generating function $x_*(z) \equiv \sum_{\ell=0}^{\infty} z^\ell x_\ell$ has a simple pole at $z = \tau_2$ and no other pole on $|z| = \tau_2$ because of the 1-arithmetic condition; therefore, for some positive constant c_0,

$$\lim_{i\to\infty} \mathrm{e}^{\tau_2 i} x_i = c_0. \tag{31}$$

Hence, we have

$$\lim_{i\to\infty} \boldsymbol{\nu}_\ell^{(1)}(i) x_i^{-1} = c_0 \lim_{i\to\infty} \mathrm{e}^{\tau_2 i} \nu(\ell, i) = c_0 b_\ell^{(2)}, \qquad \ell \in \mathbb{Z}_+.$$

Recall the Markov additive process $\{\boldsymbol{Z}^{(1)}(\ell)\}$ introduced in Section 4.1. We now change the measure of this process using τ_1 and $\boldsymbol{x}$ in such a way that the new kernel is defined as

$$\tilde{A}_n^{(1)} = \Delta_{\boldsymbol{x}}^{-1} (\mathrm{e}^{\tau_1 n} A_n^{(1)})^\top \Delta_{\boldsymbol{x}}, \qquad n \geq -1,$$

where $\Delta_{\boldsymbol{x}}$ is the diagonal matrix whose ith diagonal entry is x_i. Obviously, $\tilde{A}_n^{(1)}(i, j)$ is a nondefective Markov additive kernel. We denote the Markov additive process with this kernel by $\{\tilde{\boldsymbol{Z}}^{(1)}(\ell)\} \equiv \{(\tilde{Z}_1^{(1)}(\ell), \tilde{Z}_2^{(1)}(\ell))\}$. Let

$$\tilde{A}^{(1)} \equiv \sum_{n=-1}^{\infty} \tilde{A}_n^{(1)},$$

which is the transition probability matrix of the background process $\{\tilde{Z}_2^{(1)}(n)\}$. By Lemma A.2 of [21], $\tilde{A}^{(1)}$ must be transient because $\tau_1 < \theta_1^{(1,\mathrm{c})} = \theta_1^{(1,\mathrm{e})}$. Under this change of measure, $R_n^{(1)}$ and $H_n^{(1)}$ are similarly changed to $\tilde{R}_n^{(1)}$ and $\tilde{H}_n^{(1)}$. Namely,

$$\tilde{R}_n^{(1)} = \Delta_{\boldsymbol{x}}^{-1} (\mathrm{e}^{\tau_1 n} R_n^{(1)})^\top \Delta_{\boldsymbol{x}}, \quad n \geq 1, \qquad \tilde{H}_n^{(1)} = \Delta_{\boldsymbol{x}}^{-1} (\mathrm{e}^{\tau_1 n} H_n^{(1)})^\top \Delta_{\boldsymbol{x}}, \quad n \geq 0.$$

It is worth noting that $\tilde{R}^{(1)} \equiv \sum_{n=1}^{\infty} \tilde{R}_n^{(1)}$ is stochastic because $\boldsymbol{x}$ is also the left-invariant vector of $R_*^{(1)}(\mathrm{e}^{\tau_1})$ by the Wiener–Hopf factorization (18). For $n = 1, 2, \ldots$, let $\tilde{\sigma}_1^{(1)}(0) = 0$ and

$$\tilde{\sigma}_1^{(1)}(n) = \inf\{\ell \geq \tilde{\sigma}_1^{(1)}(n-1);\ \tilde{Z}_1^{(1)}(\ell) - \tilde{Z}_1^{(1)}(\tilde{\sigma}_1^{(1)}(n-1)) \geq 1\}.$$

Then we can see, from a version of (18) for $\{\tilde{A}_n\}$, that $\tilde{R}_n^{(1)}$ is the transition probability matrix at the first ascending ladder epoch, that is, its (i, j)th entry is given by

$$\tilde{r}_n^{(1)}(i, j) = \mathbb{P}(\tilde{Z}_1^{(1)}(\tilde{\sigma}_1^{(1)}(1)) - \tilde{Z}_1^{(1)}(0) = n,\ \tilde{Z}_2^{(1)}(\tilde{\sigma}_1^{(1)}(1)) = j \mid \tilde{Z}_2^{(1)}(0) = i), \qquad n \geq 1.$$

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Since $\tilde{R}^{(1)}$ is stochastic, this implies that $\tilde{Z}_1(\tilde{\sigma}_1^{(1)}(n))$ drifts to ∞ as $n \to \infty$ with probability 1. Let $\tilde{h}_n^{(1)}(i, j)$ be the (i, j)th entry of $\tilde{H}_n^{(1)}$. Since

$$\tilde{h}_n^{(1)}(i, j) = 1(n = 0)I + \sum_{\ell=1}^{n} \sum_{k=0}^{\infty} \tilde{r}_\ell^{(1)}(i, k)\tilde{h}_{n-\ell}^{(1)}(k, j),$$

we have, for $n \geq 1$,

$$\begin{aligned}\tilde{h}_n^{(1)}(i, j) &= \sum_{\ell=1}^{n} \mathbb{P}(\tilde{Z}_1^{(1)}(\tilde{\sigma}_1^{(1)}(\ell)) - \tilde{Z}_1^{(1)}(0) = n,\ \tilde{Z}_2^{(1)}(\tilde{\sigma}_1^{(1)}(\ell)) = j \mid \tilde{Z}_2^{(1)}(0) = i) \\ &= \mathbb{P}\bigg(\bigcup_{\ell=1}^{n} \{\tilde{Z}_1^{(1)}(\tilde{\sigma}_1^{(1)}(\ell)) - \tilde{Z}_1^{(1)}(0) = n\} \cap \{\tilde{Z}_2^{(1)}(\tilde{\sigma}_1^{(1)}(\ell)) = j\} \,\bigg|\, \tilde{Z}_2^{(1)}(0) = i\bigg),\end{aligned}$$

where the second equality follows because $\tilde{Z}_1^{(1)}(\tilde{\sigma}_1^{(1)}(\ell))$ is increasing in ℓ. Thus, $\tilde{h}_n^{(1)}(i, j)$ is obtained as the solution of the Markov renewal equation, which is uniformly bounded by 1. However, we cannot apply the standard Markov renewal theorem because its background kernel $\tilde{A}^{(1)}$ is transient. Nevertheless, we can show that, for some constant $a > 0$,

$$\lim_{n\to\infty} \tilde{H}_n^{(1)}\mathbf{1} = \frac{1}{a}\mathbf{1}. \tag{32}$$

Intuitively, this may be obvious because both entries of $\tilde{\boldsymbol{Z}}^{(1)}(n)$ go to ∞ as $n \to \infty$ and they behave like a random walk asymptotically when they become large. However, we need to prove (32). Since this proof is quite technical, we defer it to Appendix G.

It follows from (17) using the vector row $\boldsymbol{y} = \{\mathrm{e}^{-\tau_2\ell}; \ell \geq 0\}$ that

$$\mathrm{e}^{\tau_1 n}\boldsymbol{\nu}_n^{(1)} \geq \boldsymbol{\nu}_1^{(1)}\Delta_{\boldsymbol{x}}^{-1}\Delta_{\boldsymbol{x}}(\mathrm{e}^{\tau_1 n}H_{n-1}^{(1)})\Delta_{\boldsymbol{x}}^{-1}\Delta_{\boldsymbol{x}} = \mathrm{e}^{\tau_1}(\Delta_{\boldsymbol{x}}\tilde{H}_{n-1}^{(1)}\Delta_{\boldsymbol{x}}^{-1}(\boldsymbol{\nu}_1^{(1)})^\top)^\top.$$

Hence, applying the bounded convergence theorem, (31), and (32), we obtain (30). This completes the proof.

Proof of Theorem 2. Because of symmetry, we need only prove the $k = 1$ case. The rough asymptotic (9) is immediate from Lemmas 9 and 10. The remaining part is also immediate from the second part of Lemma 10.

4.5. Proof of Theorem 3

We first consider where the ray $x\boldsymbol{c}$ with $x > 0$ intersects $\mathcal{D}$ in the (θ_1, θ_2)-plane. By Theorem 1, there are three possibilities.

(a) It intersects the vertical line $\theta_1 = \tau_1$.

(b) It intersects the horizontal line $\theta_2 = \tau_2$.

(c) It intersects $\partial\Gamma_{\max} \equiv$ the boundary of $\Gamma_{\max}$ (see Figure 5).

Possibilities (a) and (b) are symmetric, so we only need to consider possibilities (a) and (c).

We first consider possibility (a), for which

$$\alpha_{\boldsymbol{c}} = \sqrt{(\tau_1)^2 + \left(\frac{c_2}{c_1}\tau_1\right)^2} = \frac{1}{c_1}\tau_1,$$

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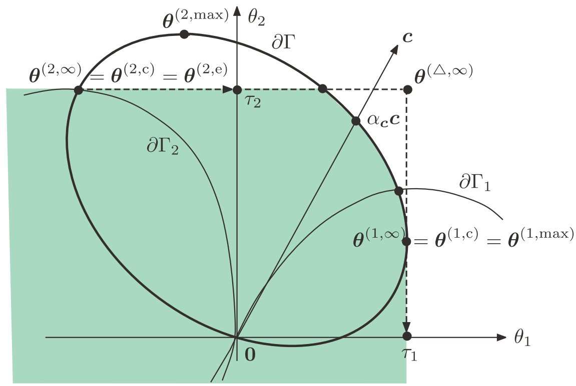

FIGURE 5: Typical figures for possibility (c).

because $\langle \boldsymbol{c}, \boldsymbol{c}\rangle = 1$. On the other hand, it follows from (28) that

$$\liminf_{x\to\infty} \frac{1}{x} \log \mathbb{P}(\langle \boldsymbol{c}, \boldsymbol{L}\rangle > x) \geq \frac{1}{c_1} \liminf_{x\to\infty} \frac{c_1}{x} \log \mathbb{P}\bigg(L_1 > \frac{1}{c_1}x,\ L_2 = 0\bigg)$$
$$\geq -\frac{1}{c_1}\tau_1.$$

Hence, combining this with the upper bound (27) of Lemma 9, we have (11).

We now consider possibility (c), for which (11) is immediate from Lemmas 7 and 9.

It remains to prove (12). For this, we consider a singular point of the analytic function $\varphi(z\boldsymbol{c})$ obtained from the moment generating function of $\langle \boldsymbol{c}, \boldsymbol{L}\rangle$. From (3),

$$(1-\gamma(z\boldsymbol{c}))\varphi(z\boldsymbol{c}) = \sum_{k=1}^{2}(\gamma_k(z\boldsymbol{c}) - \gamma(z\boldsymbol{c}))\varphi_k(c_k z) + (\gamma_0(z\boldsymbol{c}) - \gamma(z\boldsymbol{c}))\varphi_0(0) \qquad (33)$$

for $(\operatorname{Re} z)\boldsymbol{c} \in \mathcal{D}$. From the assumptions on $\alpha_{\boldsymbol{c}}$, we first observe that the right-hand side of (33) is analytic for $\operatorname{Re} z <\equiv \min(\tau_1/c_1, \tau_2/c_2)$, which is greater than $\alpha_{\boldsymbol{c}}$ (see Figure 5).

Let us consider the root of equation $1-\gamma(z\boldsymbol{c}) = 0$ for $\operatorname{Re} z = \alpha_{\boldsymbol{c}}$ in the complex number field. We first note that the root $z = \alpha_{\boldsymbol{c}}$ is simple because $\gamma(t\boldsymbol{c})$ is a convex function of $t \in \mathbb{R}$ and $\gamma(\boldsymbol{0}) = 1$. If $\langle \boldsymbol{c}, \boldsymbol{X}^{(+)}\rangle$ is δ-arithmetic for some $\delta > 0$ then $\langle \boldsymbol{c}, \boldsymbol{X}^{(+)}\rangle/\delta$ is integer valued and 1-arithmetic, and, therefore, the roots are of the form that $\alpha_{\boldsymbol{c}} + 2\pi \mathrm{i} k/\delta$ for $k \in \mathbb{Z}$, where $\mathrm{i} = \sqrt{-1}$. If there is no $\delta > 0$ for $\langle \boldsymbol{c}, \boldsymbol{X}^{(+)}\rangle$ to be δ-arithmetic then we can see that there is no root other than $z = \alpha_{\boldsymbol{c}}$ because $\gamma(z)$ is the function of e^z.

We now consider the two possibilities separately. First assume that $\langle \boldsymbol{c}, \boldsymbol{X}^{(+)}\rangle$ is δ-arithmetic for some integer $\delta > 0$. Since $\langle \boldsymbol{c}, \boldsymbol{X}^{(+)}\rangle/\delta$ is integer valued, we consider generating functions instead of moment generating functions. That is, we change the complex variable z to $w = \mathrm{e}^{\delta z}$ in $\varphi(z\boldsymbol{c})$, $\varphi_k(c_k z)$, $\gamma(z\boldsymbol{c})$, and $\gamma_k(z\boldsymbol{c})$, respectively denoted by $f(w)$, $f_k(w)$, $g(w)$, and $g_k(w)$, which are generating functions. Since the analytic properties of the original functions are transformed to these generating functions, we can see that $f(w)$ is analytic for $|w| < \mathrm{e}^{\delta\alpha_{\boldsymbol{c}}}$, and there is no singular point on the circle $|w| = \mathrm{e}^{\delta\alpha_{\boldsymbol{c}}}$ except for $w = \mathrm{e}^{\delta\alpha_{\boldsymbol{c}}}$. Since the circle in the complex plane is compact and $g(w) = 1$ has a simple root at $w = \mathrm{e}^{\delta\alpha_{\boldsymbol{c}}}$, we can analytically expand $f(w)$ given by

$$f(w) = \frac{1}{1-g(w)}\left(\sum_{k=1}^{2}(g_k(w)-g(w))f_k(w) + (g_0(w\boldsymbol{c})-g(w))\varphi_0(0)\right)$$

to the region $\{w \in \mathbb{C}; |w| < e^{\delta\alpha_c} + \varepsilon\}$ for some $\varepsilon > 0$ except for $w = e^{\delta\alpha_c}$. Since h must be singular at $w = e^{\delta\alpha_c}$ and $1 - g(w) = 0$ has a single root there, $f(w)$ has a simple pole at $w = e^{\delta\alpha_c}$. Hence, we can apply the asymptotic inversion technique for a generating function of a 1-arithmetic distribution (see, e.g. Theorem VI.5 of [14]). Thus, we have (12).

We now assume that $\langle \boldsymbol{c}, \boldsymbol{X}^{(+)}\rangle$ is not δ-arithmetic for any $\delta > 0$. In this case, we return to the moment generating function. We already observed that $\gamma(z\boldsymbol{c}) = 1$ has no other root for $\operatorname{Re} z = \alpha_{\boldsymbol{c}}$ than $z = \alpha_{\boldsymbol{c}}$ in the complex number field. Moreover, the circle $\{e^z \in \mathbb{C}; \operatorname{Re} z = \alpha_{\boldsymbol{c}}\}$ is a compact set and $1 = |\gamma(z\boldsymbol{c})| \le \gamma(\operatorname{Re} z\boldsymbol{c})$. Hence, for any sequence of complex numbers z_n for $n = 1, 2, \ldots$ such that $\gamma(z_n\boldsymbol{c}) = 1$, we have $\operatorname{Re} z_n > \alpha_{\boldsymbol{c}}$, and $\operatorname{Re} z_n$ cannot converge to $\alpha_{\boldsymbol{c}}$ as $n \to \infty$. If it was a converging sequence then $\{e^{z_n} \in \mathbb{C}; n = 1, 2, \ldots\}$ has a converging subsequence, and, therefore, $\gamma(z\boldsymbol{c}) = 1$ for all $z \in \mathbb{C}$ by the uniqueness of the analytic extension, which is a contradiction. This proves that $1 - \gamma(z\boldsymbol{c}) = 0$ has a single root at $z = \alpha_{\boldsymbol{c}}$ in the complex region $\{z \in \mathbb{C}; 0 < \operatorname{Re} z < \alpha_{\boldsymbol{c}} + \varepsilon\}$ for some $\varepsilon > 0$. Hence, $\varphi(z\boldsymbol{c})$ is analytically extendable for $\operatorname{Re} z < \alpha_{\boldsymbol{c}} + \varepsilon$ except at the simple pole at $z = \alpha_{\boldsymbol{c}}$. To this analytic function, we apply Lemma 6.2 of [7], which is an adaptation of the asymptotic inversion due to Doetsch [8]. This yields (12).

5. Application to queueing networks

In this section we consider a two-node Markovian network with batch arrivals. This network generalizes the Jackson network so that each node may have batch arrivals at once. By applying Theorem 3, it is easy to compute the decay rates at least numerically not only for this modification but also for the further modification in which the service rates are changed when either one of the nodes is empty because they can be formulated as the double M/G/1-type process. Note that special cases of these models have been studied in the literature (see, e.g. [15], [17], [18], [19], and [20]), where the exact asymptotics have been studied. In Theorem 3 we only examine the rough asymptotics.

The aim of this section is twofold. First, we consider the influence of the variability of the batch size distributions on the decay rates. Second, we examine whether the upper bound in [27] for the stationary distribution is tight for the decay rate, assuming that there is no simultaneous arrival as in [27]. This tightness has never been considered.

5.1. Two-node Markovian network with batch arrivals

Consider a continuous-time Markovian network with two nodes, numbered 1 and 2. We assume the following arrivals and service. Batches of customers arrive either at one of the two nodes or simultaneously at both nodes from outside, which we describe by a compound Poisson process with rate $\lambda > 0$ and joint batch size distribution F. The service times at node k ($k = 1, 2$) are independent and have common exponential distribution with mean μ_k^{-1}, which are also independent of the batch arrivals. Because of this exponential assumption, the service discipline is irrelevant as long as servers are busy. A customer who completes service at node 1 moves to node 2 with probability p_{12} or leaves the network with probability $1 - p_{12}$. Similarly, a departing customer from node 2 goes to node 1 with probability p_{21} or outside with

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probability $1 - p_{21}$. To exclude trivial cases, we assume that

(A1) $0 < p_{12} < 1$ and $0 < p_{21} < 1$.

We assume without loss of generality that $\lambda + \mu_1 + \mu_2 = 1$.

Let $\boldsymbol{B} \equiv (B_1, B_2)$ be a random vector subject to the joint batch size distribution F, and denote its moment generating function by $\hat{F}$. We assume that

(A2) for each nonzero $\boldsymbol{\theta} \geq \mathbf{0}$, $\sup\{t > 0; \hat{F}(t\boldsymbol{\theta}) < \infty\} = \infty$.

Thus, F has light tails. The simplest model of this type is two parallel queues with two simultaneous arrivals (no batch arrival at each node), whose exact tail asymptotics were studied in [15]. This result was recently generalized to a network without batch arrival in [20]. The present batch arrival network is more general than these models, but we can only derive the decay rates in some cases.

Let L_{tk} be the queue length of node k at time t. Clearly, (L_{t1}, L_{t2}) is a continuous-time Markov chain. We assume the intuitive stability condition,

$$\rho_1 \equiv \frac{\lambda(b_1 + b_2 p_{21})}{(1 - p_{12}p_{21})\mu_1} < 1, \qquad \rho_2 \equiv \frac{\lambda(b_2 + b_1 p_{12})}{(1 - p_{12}p_{21})\mu_2} < 1, \tag{34}$$

where $b_k = \mathbb{E}(B_k)$. One can verify that this condition is identical to the stability condition given in Lemma 1 using fact that

$$\langle \boldsymbol{m}, \boldsymbol{m}_{\perp}^{(1)} \rangle = \mu_2(\lambda(b_1 + b_2 p_{21}) - \mu_1(1 - p_{12}p_{21})),$$
$$\langle \boldsymbol{m}, \boldsymbol{m}_{\perp}^{(2)} \rangle = \mu_1(\lambda(b_2 + b_1 p_{12}) - \mu_2(1 - p_{12}p_{21})).$$

Thus, (34) is indeed the stability condition for (L_{t1}, L_{t2}) to have the stationary distribution, which is denoted by ν.

By the well-known uniformization, we reformulate the continuous-time Markov chain $\{(L_{t1}, L_{t2}); t \geq 0\}$ as a discrete-time Markov chain with the same stationary distribution ν. This discrete-time Markov chain is a double M/G/1-type process, denoted by $\{(L_1(\ell), L_2(\ell)); \ell = 0, 1, \ldots\}$.

This queueing network is more general than the model studied in [27] in the sense that simultaneous arrivals at both nodes may occur. If there is no simultaneous arrival at both nodes then the model becomes a special case of the network in [27] because batch departures are not allowed. We will consider this special case in Section 5.3 when examining the quality of the upper bound in [27].

It is known that batch arrivals and/or simultaneous arrivals make it very difficult to obtain the stationary distribution, while it is straightforward if there is no such arrival. The latter network is the Jackson network, which has the stationary distribution of a product form as is well known.

5.2. Influence of batch size distributions

For the double M/G/1-type process for the batch arrival network, we compute $\gamma(\boldsymbol{\theta})$ and $\gamma_k(\boldsymbol{\theta})$ for $k = 1, 2$:

$$\gamma(\boldsymbol{\theta}) = \lambda\hat{F}(\boldsymbol{\theta}) + \mu_1 \mathrm{e}^{-\theta_1}(1 - p_{12} + p_{12}\mathrm{e}^{\theta_2}) + \mu_2 \mathrm{e}^{-\theta_2}(1 - p_{21} + p_{21}\mathrm{e}^{\theta_1}), \tag{35}$$
$$\gamma_1(\boldsymbol{\theta}) = \lambda\hat{F}(\boldsymbol{\theta}) + \mu_1 \mathrm{e}^{-\theta_1}(1 - p_{12} + p_{12}\mathrm{e}^{\theta_2}) + \mu_2, \tag{36}$$
$$\gamma_2(\boldsymbol{\theta}) = \lambda\hat{F}(\boldsymbol{\theta}) + \mu_1 + \mu_2 \mathrm{e}^{-\theta_2}(1 - p_{21} + p_{21}\mathrm{e}^{\theta_1}). \tag{37}$$

Hence, if the distribution of $\boldsymbol{B}$ is increased in the linear convex order then $\gamma(\boldsymbol{\theta})$, $\gamma_1(\boldsymbol{\theta})$, and $\gamma_2(\boldsymbol{\theta})$ are increased for each fixed $\boldsymbol{\theta} \in \mathbb{R}^2$ as long as they exist. Corollary 1 then yields the following result.

Proposition 1. *For the batch arrival Markovian network satisfying conditions (A1), (A2), and (34), if the distribution of $\boldsymbol{B}$ is increased in the linear convex order then the decay rates τ_1, τ_2 in Theorem 2, and $\alpha_{\boldsymbol{c}}$ in Theorem 3 are decreased.*

To obtain the decay rates, we need to find $\boldsymbol{\theta}^{(k,\mathrm{e})}$ and $\boldsymbol{\theta}^{(k,\max)}$ for $k = 1, 2$, which are the roots of the equations $\gamma(\boldsymbol{\theta}) = \gamma_k(\boldsymbol{\theta}) = 1$ and $\gamma(\boldsymbol{\theta}) = 1$ satisfying $\mathrm{d}\theta_k/\mathrm{d}\theta_{3-k} = 0$, respectively. We note that $\gamma(\boldsymbol{\theta}) = \gamma_1(\boldsymbol{\theta}) = 1$ is equivalent to $\gamma_1(\boldsymbol{\theta}) = 1$ and

$$\mathrm{e}^{\theta_2} = 1 - p_{21} + p_{21}\mathrm{e}^{\theta_1},$$

which follows from $\gamma(\boldsymbol{\theta}) = \gamma_1(\boldsymbol{\theta})$. Thus, the numerical values of the decay rates are easily computed using, e.g. MATHEMATICA®, but their analytical expressions are difficult to obtain except in the skip-free case. The model studied by Flatto and Hahn [15] is the simplest skip-free case. Theorems 2 and 3 are fully compatible with their asymptotic results.

5.3. Stochastic upper bound of Miyazawa and Taylor

We consider the stochastic upper bound for the stationary distribution ν, obtained by Miyazawa and Taylor [27]. Since their model does not allow for simultaneous arrival, we assume that either B_1 or B_2 is 0. Thus, the joint batch size distribution F can be written as

$$F(x_1, x_2) = F(x_1, 0) + F(0, x_2), \qquad x_1, x_2 \geq 0.$$

Let $F_1(x) = F(x, 0)/F(\infty, 0)$ and $F_2(x) = F(0, x)/F(0, \infty)$, and let $\lambda_1 = \lambda F(\infty, 0)$ and $\lambda_2 = \lambda F(0, \infty)$. For computational convenience, we switch to generating functions from moment generating functions. Let $\tilde{F}_k$ be the generating functions of F_k. We present the upper bound of [27] using our notation.

Proposition 2. (Corollary 3.2 and Theorem 4.1 of [27].) *If (A1), (A2), and the stability condition (34) hold, then the equations*

$$\lambda_1(\tilde{F}_1(s_1) - 1) + \mu_1 s_1^{-1}(1 - s_1) = \mu_2 p_{21} s_2^{-1}(1 - s_1), \tag{38}$$

$$\lambda_2(\tilde{F}_2(s_2) - 1) + \mu_2 s_2^{-1}(1 - s_2) = \mu_1 p_{12} s_1^{-1}(1 - s_2) \tag{39}$$

have solutions $(s_1, s_2) > \boldsymbol{0}$. Let (h_1, h_2) be the maximal solution. Then

$$\mathbb{P}(\boldsymbol{L} \geq \boldsymbol{n}) \leq h_1^{-n_1} h_2^{-n_2}, \qquad \boldsymbol{n} = (n_1, n_2) \geq \boldsymbol{0},$$

where $\boldsymbol{L}$ is a random vector subject to the stationary distribution ν.

To compare h_k with the decay rate, we let

$$\eta_k = \log h_k, \qquad k = 1, 2.$$

By Theorem 3 and Proposition 2, we have $\eta_k \leq \alpha_k$, where we recall that $\alpha_k = \alpha_{\boldsymbol{e}_k}$.

Let $s_k = \mathrm{e}^{\theta_k}$ in (35), (36), and (37). Then $\gamma(\boldsymbol{\theta})$, $\gamma_1(\boldsymbol{\theta})$, and $\gamma_2(\boldsymbol{\theta})$ can be written as

$$\lambda_1\tilde{F}_1(s_1) + \mu_1 s_1^{-1}(1 - p_{12} + p_{12}s_2) + \lambda_2\tilde{F}_2(s_2) + \mu_2 s_2^{-1}(1 - p_{21} + p_{21}s_1) = 1, \tag{40}$$

$$\lambda_1\tilde{F}_1(s_1) + \mu_1 s_1^{-1}(1 - p_{12} + p_{12}s_2) + \lambda_2\tilde{F}_2(s_2) + \mu_2 = 1, \tag{41}$$

$$\lambda_1\tilde{F}_1(s_1) + \mu_1 + \lambda_2\tilde{F}_2(s_2) + \mu_2 s_2^{-1}(1 - p_{21} + p_{21}s_1) = 1.$$

Note that (38) and (39) imply (40). That is, (h_1, h_2) satisfies (40) for variable (s_1, s_2). In other words, the point (η_1, η_2) is on the curve $\partial\Gamma$.

The question to be answered is: when is (η_1, η_2) identical to (α_1, α_2)? That is, when does the upper bounds agree with the decay rates of the marginal distributions in coordinate directions? We provide the answer in the following theorem.

Theorem 4. *Under (A1), (A2), and the stability condition (34), the decay rate $\eta_k \equiv \log t_k$ of the stochastic upper bound in [27] is identical to the decay rate α_k for $k = 1, 2$ if and only if both nodes have no batch arrival.*

Proof. The sufficiency of the single arrivals is immediate from the well-known product-form solution for the Jackson network. Thus, we only need to prove the necessity. Assume that $(\eta_1, \eta_2) = (\alpha_1, \alpha_2)$. As we already observed, this implies that $(\alpha_1, \alpha_2) \in \partial\Gamma$. Hence, from Theorem 3 and (8), we can see that neither $\alpha_1 = \beta_1$ nor $\alpha_2 = \beta_2$ is possible because $\alpha_k = \beta_k$ implies that $\alpha_{2-k} = 0$, where β_k is defined after (8). Furthermore, we cannot simultaneously have $\alpha_1 = \theta_1^{(1,\max)}$ and $\alpha_2 = \theta_2^{(2,\max)}$ because $(\alpha_1, \alpha_2) \in \partial\Gamma$. Hence, we must have either $\alpha_1 = \theta_1^{(1,\mathrm{e})}$ or $\alpha_2 = \theta_2^{(2,\mathrm{e})}$.

Suppose that $\alpha_1 = \theta_1^{(1,\mathrm{e})}$, which implies that $\eta_1 = \theta_1^{(1,\mathrm{e})}$ by our assumption. For convenience, we introduce the notation

$$t_i^{(k,\mathrm{e})} = \mathrm{e}^{\theta_i^{(k,\mathrm{e})}}, \qquad i, k = 1, 2.$$

Then $\eta_1 = \theta_1^{(1,\mathrm{e})}$ is equivalent to $h_1 = t_1^{(1,\mathrm{e})}$, and $(t_1^{(1,\mathrm{e})}, t_2^{(1,\mathrm{e})})$ is the solution of (40) and (41) for variable (s_1, s_2). Thus, it follows from (40) and (41) that

$$t_2^{(1,\mathrm{e})} = 1 - p_{21} + p_{21} t_1^{(1,\mathrm{e})}. \tag{42}$$

Substituting (42) into (41) with $(s_1, s_2) = (t_1^{(1,\mathrm{e})}, t_2^{(1,\mathrm{e})})$, we have

$$\lambda_1(\tilde{F}_1(t_1^{(1,\mathrm{e})}) - 1) + \lambda_2(\tilde{F}_2(1 - p_{21} + p_{21} t_1^{(1,\mathrm{e})}) - 1) + \mu_1(1 - p_{12}p_{21})((t_1^{(1,\mathrm{e})})^{-1} - 1) = 0.$$

Thus, $t_1^{(1,\mathrm{e})}$ is obtained as the unique positive solution of this equation. Since $h_1 = t_1^{(1,\mathrm{e})}$, this implies that

$$\lambda_1(\tilde{F}_1(h_1) - 1) + \lambda_2(\tilde{F}_2(1 - p_{21} + p_{21}h_1) - 1) + \mu_1(1 - p_{12}p_{21})(h_1^{-1} - 1) = 0. \tag{43}$$

From (38) with $(s_1, s_2) = (h_1, h_2)$ and (43), we have

$$\lambda_2 \tilde{F}_2(1 - p_{21} + p_{21}h_1) = \mu_1 p_{12} p_{21} h_1^{-1}(1 - h_1) - \mu_2 p_{21} h_2^{-1}(1 - h_1).$$

This yields

$$\lambda_2 \frac{\tilde{F}_2(1 - p_{21} + p_{21}h_1) - 1}{(1 - p_{21} + p_{21}h_1) - 1} = \mu_2 h_2^{-1} - \mu_1 p_{12} h_1^{-1}.$$

On the other hand, it follows from (39) with $(s_1, s_2) = (h_1, h_2)$ that

$$\lambda_2 \frac{\tilde{F}_2(h_2) - 1}{h_2 - 1} = \mu_2 h_2^{-1} - \mu_1 p_{12} h_1^{-1}.$$

Hence, we must have

$$\frac{\tilde{F}_2(1 - p_{21} + p_{21}h_1) - 1}{(1 - p_{21} + p_{21}h_1) - 1} = \frac{\tilde{F}_2(h_2) - 1}{h_2 - 1}.$$

Since $\tilde{F}_2(s)$ is a strictly increasing convex function, this equation holds only when either $\tilde{F}_2(s) = s$, which is equivalent to no batch arrivals at node 2, or

$$h_2 = 1 - p_{21} + p_{21}h_1. \tag{44}$$

Suppose that node 2 has batch arrivals. Then (44) holds and, therefore, (42) implies that $h_2 = t_2^{(1,\mathrm{e})}$. This is equivalent to $(\eta_1, \eta_2) = \boldsymbol{\theta}^{(1,\mathrm{e})}$. Hence, the assumption that $(\eta_1, \eta_2) = (\alpha_1, \alpha_2)$ implies that we must have the case (D2), which in turn implies that $h_2 = t_2^{(1,\mathrm{e})} = t_2^{(2,\mathrm{e})}$. Hence, by the same arguments, we have either $\tilde{F}_1(s) = s$ or

$$h_1 = 1 - p_{12} + p_{12}h_2. \tag{45}$$

However, if both of (44) and (45) hold, then $h_1 = h_2 = 1$, which is impossible. Hence, we must have $\tilde{F}_1(s) = s$, that is, node 1 has no batch arrivals.

Applying $\tilde{F}_1(h_1) = h_1$ to (41) with $(s_1, s_2) = (h_1, h_2)$ and using the fact that $h_1 > 1$, we have

$$\mu_1 h_1^{-1} = \lambda_1 + \mu_2 p_{21} h_2^{-1}. \tag{46}$$

On the other hand, from (39) with $(s_1, s_2) = (h_1, h_2)$ and (44), we have

$$\begin{aligned}\lambda_2(\tilde{F}_2(h_2) - 1) &= \mu_1 p_{12} h_1^{-1}(1 - h_2) - \mu_2 h_2^{-1}(1 - h_2) \\ &= (1 - h_2)(\mu_1 h_1^{-1} p_{12} - \mu_2 h_2^{-1}) \\ &= (1 - h_2)((\lambda_1 + \mu_2 p_{21} h_2^{-1}) p_{12} - \mu_2 h_2^{-1}),\end{aligned}$$

where the last equality is obtained by substituting (46). Rearranging terms in this equation and recalling that $\rho_2 = (\lambda_1 + \lambda_2 p_{12})/((1 - p_{12}p_{21})\mu_1)$, we have

$$\begin{aligned}\lambda_2(\tilde{F}_2(h_2) - h_2) &= (1 - h_2)((\lambda_2 + \lambda_1 p_{12}) - (1 - p_{21}p_{12})\mu_2 h_2^{-1}) \\ &= (1 - h_2)(1 - p_{21}p_{12})\mu_2(\rho_2 - h_2^{-1}).\end{aligned} \tag{47}$$

Since this ρ_2 is identical to the geometric decay rate of the Jackson network with single arrivals at both nodes, we obviously have $\rho_2 \le h_2^{-1}$. Thus, the right-hand side of (47) is not positive, but its left-hand side must be positive because node 2 has batch arrivals, which implies that $\tilde{F}_2(h_2) - h_2 > 0$. This is a contradiction, and node 2 cannot have batch arrivals. We therefore conclude that both nodes do not have batch arrivals. By symmetry, we have the same conclusion for $\alpha_2 = \theta_2^{(2,\mathrm{e})}$. This completes the proof.

Theorem 4 shows that the stochastic bound in [27] cannot be tight even for the decay rates. However, this may not exclude the case where one of the upper bounds is tight. In our proof, the tightness leads to a contradiction if either node 2 has batch arrivals or (D2) holds. This suggests that, if node 2 has no batch arrivals and if (D2) does not hold, then node 1 with $\eta_1 = \alpha_1$ may have batch arrivals. The following corollary confirms this observation.

Corollary 2. *Under the same assumptions of Theorem 4, $\eta_1 = \alpha_1$ holds if there is no batch arrival at node 2 and either (D1) with $\boldsymbol{\theta}^{(1,\max)} \notin \overline{\Gamma}_1$ or (D3) holds. These conditions are also necessary for $\eta_1 = \alpha_1$ if node 1 has batch arrivals. Similarly, $\eta_2 = \alpha_2$ holds if and only if there is no batch arrival at node 1 and either (D1) with $\boldsymbol{\theta}^{(2,\max)} \notin \overline{\Gamma}_2$ or (D2) holds. These conditions are also necessary if node 2 has batch arrivals.*

Proof. By symmetry, we only need to prove the first two claims. If there is no batch arrival at node 2 then (h_1, h_2) is obtained as the solution of (38) and (39) with $\tilde{F}_2(s_2) = s_2$, that is,

$$\lambda_1(\tilde{F}_1(s_1) - 1) + \mu_1 s_1^{-1}(1 - h_1) = \mu_2 p_{21} s_2^{-1}(1 - s_1), \tag{48}$$

$$\mu_2 s_2^{-1} = \lambda_2 + \mu_1 p_{12} s_1^{-1}. \tag{49}$$

Substituting (49) into (48) implies that

$$\lambda_1(\tilde{F}_1(s_1) - 1) - \lambda_2 p_{21}(1 - s_1) + \mu_1(1 - p_{12}p_{21})(1 - s_1)s_1^{-1} = 0.$$

On the other hand, this equation also follows from (40), (41), and the single arrivals at node 2; therefore, we have $h_1 = t_1^{(1,\mathrm{e})}$, or, equivalently, $\eta_1 = \theta_1^{(1,\mathrm{e})}$. Hence, $\eta_1 = \alpha_1$ holds if either (D1) with $\boldsymbol{\theta}^{(1,\max)} \notin \overline{\Gamma}_1$ or (D3) holds because these conditions imply that $\alpha_1 = \theta_1^{(1,\mathrm{e})}$. This proves the first claim. To prove the necessity, we note that $\eta_1 = \alpha_1$ implies that $\alpha_1 = \theta_1^{(1,\mathrm{e})}$. Assume that node 1 has batch arrivals. Then at node 2 it is not possible to have batch arrivals or for (D2) to hold, as shown in the proof of Theorem 4. Hence, it is required that node 2 has no batch arrivals and (D2) does not hold. The latter together with $\alpha_1 = \theta_1^{(1,\mathrm{e})}$ implies that either (D1) with $\boldsymbol{\theta}^{(1,\max)} \notin \overline{\Gamma}_1$ or (D3) holds. Thus, the second claim is proved.

6. Concluding remarks

In this paper we studied the tail decay asymptotics of the marginal stationary distributions for an arbitrary direction under conditions (i)–(iv) and the stability condition. Among these conditions, (i) is the most restrictive for applications. For example, it excludes a priority queue with two classes of customer. However, it can be relaxed as remarked in Section 7 of [25]. Hence, (i) is not a crucial restriction. As we noted, condition (iii) can also be relaxed to (iii′) for obtaining the decay rate. Thus, the rough tail asymptotics can be obtained under the minimum requisites.

What we have not studied in this paper is other types of asymptotic behaviors of the stationary distribution. In particular, we have not fully studied exact asymptotics. We have recently studied this problem for the skip-free reflecting random walk in [20]. For the unbounded jump case, this is a challenging problem. We hope that the convergence domain obtained in this paper will be helpful for the kernel method, as it proved to be in [20]. (Additional note) There are crucial errors in references [18] and [19], and an erratum has been announced in (2014) Letter to editors, *Queueing Systems* **76,** 105–107.

Appendix A. Proof of Lemma 2

We first claim that, for each $k = 1, 2$, there is a $\boldsymbol{\theta} \in \Gamma \cap \Gamma_k$ such that $\theta_k > 0$ if and only if

$$\langle \boldsymbol{\theta}, \boldsymbol{m} \rangle < 0, \qquad \langle \boldsymbol{\theta}, \boldsymbol{m}^{(k)} \rangle < 0, \quad \text{and} \quad \theta_k > 0. \tag{50}$$

Because of symmetry, we only prove this for $k = 1$. Define the functions f and f_1 as

$$f(u) = \mathbb{E}(\mathrm{e}^{u\langle \boldsymbol{\theta}, \boldsymbol{X}^{(+)} \rangle}), \qquad f_1(u) = \mathbb{E}(\mathrm{e}^{u\langle \boldsymbol{\theta}, \boldsymbol{X}^{(1)} \rangle}), \qquad u \in \mathbb{R},$$

as long as $f(u)$ is finite for each fixed $\boldsymbol{\theta} \in \mathbb{R}$. Obviously, by condition (iii),

$$\mathbb{E}(\langle \boldsymbol{\theta}, \boldsymbol{X}^{(+)} \rangle) = \theta_1 \mathbb{E}(X_1^{(+)}) + \theta_2 \mathbb{E}(X_2^{(+)})$$

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exists and is finite for any $\boldsymbol{\theta} \in \mathbb{R}$. Choose a $\boldsymbol{\theta} \in \mathbb{R}$ such that $\theta_1 > 0$ and $f(1), f_1(1) < \infty$. This $\boldsymbol{\theta}$ exists by (iii). Since $f(0) = 1$ and $f(u)$ is convex in u,

$$\langle \boldsymbol{\theta}, \boldsymbol{m} \rangle = \mathbb{E}(\langle \boldsymbol{\theta}, \boldsymbol{X}^{(+)} \rangle) = f'(0) < 0 \tag{51}$$

is necessary and sufficient for $f(u_0) < 1$ for some $u_0 > 0$, which is equivalent to $u_0\boldsymbol{\theta} \in \Gamma$ and $u_0\theta_1 > 0$. Let $\boldsymbol{\theta}' = u_0\boldsymbol{\theta}$. Then (51) is equivalent to $\langle \boldsymbol{\theta}', \boldsymbol{m} \rangle < 0$. Using this $\boldsymbol{\theta}'$, we apply the same arguments to function f_1 and can see that $\langle \boldsymbol{\theta}', \boldsymbol{m}^{(1)} \rangle < 0$ holds if and only if there is a $u_1 > 0$ such that $u_1\boldsymbol{\theta}' \in \Gamma_1$ and $u_1\theta'_1 > 0$. This implies that $\min(1, u_1)\boldsymbol{\theta}' \in \Gamma \cap \Gamma_1$ since Γ and Γ_1 are convex sets. Thus, the claim is proved.

We now show that (50) for $k = 1$ follows from either stability condition (I), (II), or (III). We first assume that (I) holds. Since $m_2^{(1)} \geq 0$, we consider the possibility that $m_2^{(1)} = 0$. In this case, $m_1^{(1)} < 0$ by the first inequality in (I) and $m_2 < 0$. Hence, $\Gamma_1 \equiv \{\boldsymbol{\theta} \in \mathbb{R}^2; \varphi_1(\boldsymbol{\theta}) < 1\}$ is the region between the two straight lines $\theta_1 = 0$ and $\theta_1 = a$ for some $a > 0$. Since Γ is not empty by (iii), we must have (50) for some $\boldsymbol{\theta}$ such that $\theta_1 > 0$ and $\theta_2 < 0$. Assume now that $m_2^{(1)} > 0$. Set $\boldsymbol{\theta} = (\theta_1, \theta_2)$ with

$$\theta_1 = m_2^{(1)}, \qquad \theta_2 = \begin{cases} -m_1^{(1)} - \varepsilon, & m_1^{(1)} \geq 0, \\ -\varepsilon, & m_1^{(1)} < 0, \end{cases}$$

where $\varepsilon > 0$ is chosen so that $\langle \boldsymbol{m}, \boldsymbol{m}_\perp^{(1)} \rangle - \varepsilon\, m_2 < 0$ and $m_2^{(1)} m_1 - \varepsilon\, m_2 < 0$, which is possible by (I) and $m_1 < 0$. Note that $\theta_1 > 0$ and $\theta_2 < 0$ in this definition. Then, we have

$$\begin{aligned}
\langle \boldsymbol{\theta}, \boldsymbol{m} \rangle &= m_2^{(1)} m_1 - ((m_1^{(1)} + \varepsilon)1(m_1^{(1)} \geq 0) + \varepsilon 1(m_1^{(1)} < 0))m_2 \\
&= (\langle \boldsymbol{m}, \boldsymbol{m}_\perp^{(1)} \rangle - \varepsilon\, m_2)1(m_1^{(1)} \geq 0) + (m_2^{(1)} m_1 - \varepsilon\, m_2)1(m_1^{(1)} < 0) \\
&< 0, \\
\langle \boldsymbol{\theta}, \boldsymbol{m}^{(1)} \rangle &= m_2^{(1)} m_1^{(1)} - ((m_1^{(1)} + \varepsilon)1(m_1^{(1)} \geq 0) + \varepsilon 1(m_1^{(1)} < 0))m_2^{(1)} \\
&= m_2^{(1)} m_1^{(1)} 1(m_1^{(1)} < 0) - \varepsilon\, m_2^{(1)} \\
&< 0.
\end{aligned}$$

Thus, we have (50) for $k = 1$.

Let us assume that (II) holds. We consider the possibility that $m_1 = 0$. In this case, we set $\boldsymbol{\theta} = (\theta_1, \theta_2)$ with

$$\theta_1 = -m_2 - \varepsilon, \qquad \theta_2 = \varepsilon,$$

where $\varepsilon > 0$ is chosen so that $-m_2 - \varepsilon > 0$ and $\langle \boldsymbol{m}, \boldsymbol{m}_\perp^{(1)} \rangle - \varepsilon(m_1^{(1)} - m_2^{(1)}) < 0$, which is possible by (II). In this case, $\theta_1, \theta_2 > 0$. Then, we have

$$\begin{aligned}
\langle \boldsymbol{\theta}, \boldsymbol{m} \rangle &= -(m_2 + \varepsilon)m_1 + \varepsilon\, m_2 = \varepsilon\, m_2 < 0, \\
\langle \boldsymbol{\theta}, \boldsymbol{m}^{(1)} \rangle &= -(m_2 + \varepsilon)m_1^{(1)} + \varepsilon\, m_2^{(1)} = \langle \boldsymbol{m}, \boldsymbol{m}_\perp^{(1)} \rangle - \varepsilon(m_1^{(1)} - m_2^{(1)}) < 0.
\end{aligned}$$

Thus, we can assume that $m_1 > 0$. We choose $\varepsilon > 0$ such that $\langle \boldsymbol{m}, \boldsymbol{m}_\perp^{(1)} \rangle + \varepsilon\, m_1 < 0$, which is possible by (II), and set

$$\theta_1 = m_2^{(1)} + \varepsilon, \qquad \theta_2 = -m_1^{(1)}.$$

Then $\theta_1 > 0$ and $\theta_2 > 0$ since $m_2^{(1)} \geq 0$ and $m_1^{(1)} < 0$ by (II). Hence, we have

$$\begin{aligned}
\langle \boldsymbol{\theta}, \boldsymbol{m} \rangle &= (m_2^{(1)} + \varepsilon)m_1 - m_1^{(1)} m_2 = \langle \boldsymbol{m}, \boldsymbol{m}_\perp^{(1)} \rangle + \varepsilon\, m_1 < 0, \\
\langle \boldsymbol{\theta}, \boldsymbol{m}^{(1)} \rangle &= (m_2^{(1)} + \varepsilon)m_1^{(1)} - m_1^{(1)} m_2^{(1)} = \varepsilon\, m_1^{(1)} < 0.
\end{aligned}$$

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Finally, assume that (III) holds. In this case, if $m_2^{(1)} = 0$ then $m_1^{(1)} < 0$, and, therefore, Γ_1 is the region between $\theta_1 = 0$ and $\theta_1 = b$ for some $b > 0$. Since $m_1 < 0$ and $m_2 \geq 0$, we can easily see that (50) holds for $\theta_1 > 0$ and $\theta_2 \leq 0$. Thus, we can assume that $m_2^{(1)} > 0$, and, therefore, we can choose $\varepsilon > 0$ such that

$$\varepsilon m_1^{(1)} + m_2^{(1)} m_1 < 0,$$

and set $\boldsymbol{\theta} = (\varepsilon, m_1)$. Then $\theta_1 > 0, \theta_2 < 0$, and

$$\langle \boldsymbol{\theta}, \boldsymbol{m} \rangle = \varepsilon m_1 + m_1 m_2 < 0, \qquad \langle \boldsymbol{\theta}, \boldsymbol{m}^{(1)} \rangle = \varepsilon m_1^{(1)} + m_1 m_2^{(1)} < 0.$$

Thus, we have shown (50) for $k = 1$. Furthermore, $\theta_2 \leq 0$ for (I) and (III). By symmetric arguments, we obtain (50) for $k = 2$, and $\theta_1 < 0$ for (I) and (II). Thus, either one of the stability conditions of Lemma 1 implies (50). The converse is immediate from (50) and the observation presented just before this lemma.

Appendix B. Proof of Lemma 3

Obviously, if either c_1 or c_2 vanishes then $K_{\boldsymbol{c}}$ is arithmetic. Hence, we assume that $c_1 \neq 0$ and $c_2 \neq 0$. If c_1/c_2 is rational, we obviously see that $K_{\boldsymbol{c}}$ is arithmetic. Thus, we only need to prove that $K_{\boldsymbol{c}}$ is asymptotically dense at ∞ if c_1/c_2 is irrational. For this, we combine the ideas which were used in the proofs of Lemma 2 and the corollary in Section V.4a of [13].

Assume that c_1/c_2 is irrational and that $c_1 < c_2$. The latter assumption holds without loss of generality because the roles of c_1 and c_2 are symmetric. For each positive integer n, let

$$A(n) = \{c_1 m_1 - c_2 m_2;\, 0 \leq c_1 m_1 - c_2 m_2 \leq c_2,\ m_2 \leq n,\ m_1, m_2 \in \mathbb{Z}_+\}.$$

Then the number of elements of $A(n)$ is strictly increased as n is increased because of the irrationality. Hence, for each $\varepsilon > 0$, we can find a positive integer n such that there exist $u, u' \in A(n)$ such that $|u - u'| < \varepsilon$ because $A(n)$ is a subset of the interval $[0, c_2]$. Since we can find m_1, m_2, m_1', m_2' such that $m_1 > m_1'$, $u = c_1 m_1 - c_2 m_2$, and $u' = c_1 m_1' - c_2 m_2'$, we have

$$|c_1(m_1 - m_1') - c_2(m_2 - m_2')| = |u - u'| < \varepsilon.$$

Since $m_1 > m_1'$, we obviously require that $m_2 > m_2'$. Hence, setting $a = c_2(m_2 - m_2')$, for each $x \geq a$, we have $|x - y| < \varepsilon$ for some $y \in K_{\boldsymbol{c}}$.

Appendix C. The proof of Lemma 4

We only prove this lemma when condition (a) is satisfied because the other cases are similarly proved. We immediately have (3) if $\varphi_+(\boldsymbol{\theta}) < \infty$. To prove this finiteness, we apply truncation arguments for (2). For each $n = 1, 2, \ldots$, let

$$f_n(x) = \min(x, n), \qquad x \in \mathbb{R}.$$

If $x \leq n$ then $f_n(x+y) \leq x+y = f_n(x)+y$. Otherwise, if $x > n$ then $f_n(x+y) \leq n = f_n(x)$. Hence, for any $x \geq 0$ and $y \in \mathbb{R}$,

$$f_n(x + y) \leq f_n(x) + \begin{cases} y, & x \leq n, \\ 0, & x > n. \end{cases}$$

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From (2) we have

$$\langle \boldsymbol{\theta}, \boldsymbol{L}\rangle \stackrel{\mathrm{D}}{=} \langle \boldsymbol{\theta}, \boldsymbol{L}\rangle + \langle \boldsymbol{\theta}, \boldsymbol{X}^{(+)}\rangle 1(\boldsymbol{L} \in S_+) + \sum_{k\in\{0,1,2\}} \langle \boldsymbol{\theta}, \boldsymbol{X}^{(k)}\rangle 1(\boldsymbol{L} \in S_k).$$

Hence, we have, using the independence of $\boldsymbol{L}$, $\boldsymbol{X}^{(0)}$, $\boldsymbol{X}^{(1)}$, and $\boldsymbol{X}^{(2)}$,

$$\begin{aligned}\mathbb{E}(\mathrm{e}^{f_n(\langle \boldsymbol{\theta}, \boldsymbol{L}\rangle)}) \le{}& \mathbb{E}(\mathrm{e}^{f_n(\langle \boldsymbol{\theta}, \boldsymbol{L}\rangle)} 1(\boldsymbol{L} \in S_+,\ \langle \boldsymbol{\theta}, \boldsymbol{L}\rangle \le n))\mathbb{E}(\mathrm{e}^{f_n(\langle \boldsymbol{\theta}, \boldsymbol{X}^{(+)}\rangle)})\\ &+ \mathbb{E}(\mathrm{e}^{f_n(\langle \boldsymbol{\theta}, \boldsymbol{L}\rangle)} 1(\boldsymbol{L} \in S_+,\ \langle \boldsymbol{\theta}, \boldsymbol{L}\rangle > n))\\ &+ \sum_{k\in\{0,1,2\}} \mathbb{E}(\mathrm{e}^{f_n(\langle \boldsymbol{\theta}, \boldsymbol{L}\rangle)} 1(\boldsymbol{L} \in S_k,\ \langle \boldsymbol{\theta}, \boldsymbol{L}\rangle \le n))\mathbb{E}(\mathrm{e}^{f_n(\langle \boldsymbol{\theta}, \boldsymbol{X}^{(k)}\rangle)})\\ &+ \sum_{k\in\{0,1,2\}} \mathbb{E}(\mathrm{e}^{f_n(\langle \boldsymbol{\theta}, \boldsymbol{L}\rangle)} 1(\boldsymbol{L} \in S_k,\ \langle \boldsymbol{\theta}, \boldsymbol{L}\rangle > n)).\end{aligned}$$

Rewriting the left-hand side as

$$\mathbb{E}(\mathrm{e}^{f_n(\langle \boldsymbol{\theta}, \boldsymbol{L}\rangle)} 1(\boldsymbol{L} \in S_+)) + \sum_{k\in\{0,1,2\}} \mathbb{E}(\mathrm{e}^{f_n(\langle \boldsymbol{\theta}, \boldsymbol{L}\rangle)} 1(\boldsymbol{L} \in S_k)),$$

we have

$$\begin{aligned}&(1 - \mathbb{E}(\mathrm{e}^{f_n(\langle \boldsymbol{\theta}, \boldsymbol{X}^{(+)}\rangle)}))\mathbb{E}(\mathrm{e}^{f_n(\langle \boldsymbol{\theta}, \boldsymbol{L}\rangle)} 1(\boldsymbol{L} \in S_+,\ \langle \boldsymbol{\theta}, \boldsymbol{L}\rangle \le n))\\ &\quad\le \sum_{k\in\{0,1,2\}} (\mathbb{E}(\mathrm{e}^{f_n(\langle \boldsymbol{\theta}, \boldsymbol{X}^{(k)}\rangle)}) - 1)\mathbb{E}(\mathrm{e}^{f_n(\langle \boldsymbol{\theta}, \boldsymbol{L}\rangle)} 1(\boldsymbol{L} \in S_k,\ \langle \boldsymbol{\theta}, \boldsymbol{L}\rangle \le n)).\end{aligned}$$

Let n go to ∞ for this inequality. Then the monotone convergence theorem and the finiteness of $\varphi_1(\boldsymbol{\theta})$ and $\varphi_2(\boldsymbol{\theta})$ yield

$$0 \le (1 - \gamma(\boldsymbol{\theta}))\varphi_+(\boldsymbol{\theta}) \le \sum_{k\in\{1,2\}} (\gamma_k(\boldsymbol{\theta}) - 1)\varphi_k(\theta_k) + (\gamma_0(\boldsymbol{\theta}) - 1)\varphi_0(0),$$

since $\boldsymbol{\theta} \in \Gamma$ and $f_n(x)$ is nondecreasing in x. Since the right-hand side of this inequality is finite and $1 - \gamma(\boldsymbol{\theta}) > 0$, we must have $\varphi_+(\boldsymbol{\theta}) < \infty$.

Appendix D. Convergence parameter and decay rate

Let us consider a nonnegative integer-valued random variable Z with a light tail. Let

$$\alpha^* = \sup\{\alpha \ge 0; \mathbb{E}\exp(\alpha Z) < \infty\}. \tag{52}$$

By the light tail condition, $\alpha^* > 0$. We give the example that

$$\lim_{x\to\infty} x^{-1} \log \mathbb{P}(Z > x) = -\alpha^*$$

is not true. One can easily see that

$$\limsup_{x\to\infty} \frac{1}{x} \log \mathbb{P}(Z > x) = -\alpha^*.$$

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Hence, the problem is the limit infimum. Define the distribution function F of a random variable Z by

$$\overline{F}(x) = \begin{cases} 1, & x \le 1, \\ e^{-\alpha^* 2^n}, & 2^{n-1} < x \le 2^n,\ n = 1, \ldots, \end{cases}$$

where $\overline{F}(x) = 1 - F(x)$. Then,

$$\lim_{n\to\infty} \frac{1}{2^n} \log \overline{F}(2^n) = \lim_{n\to\infty} \frac{1}{2^n}(-\alpha^* 2^n) = -\alpha^*,$$

$$\lim_{n\to\infty} \frac{1}{2^n+1} \log \overline{F}(2^n+1) = \lim_{n\to\infty} \frac{1}{2^n+1}(-\alpha^* 2^{n+1}) = -2\alpha^*.$$

Thus, (52) holds, but we have

$$\liminf_{x\to\infty} \frac{1}{x} \log \mathbb{P}(Z > x) = -2\alpha^* < -\alpha^* = \limsup_{x\to\infty} \frac{1}{x} \log \mathbb{P}(Z > x).$$

The distribution function $F(x)$ is a little tricky because its increasing points are sparse as x goes to ∞. However, it is not very difficult to make a small change for it to increase at all positive integers. Similar examples are obtained in the literature (see, e.g. Section 2.3 of [31]).

Appendix E. Proof of Lemma 6

We first prove $\boldsymbol{\theta}^{(\Delta,\infty)} = \boldsymbol{\tau}$ for the three cases separately. For (D1), suppose that $\theta_k^{(k,\infty)} < \theta_k^{(k,c)}$ for $k = 1, 2$. Note that $\theta_1^{(2,\infty)} < \theta_1^{(1,\infty)}$ and $\theta_2^{(1,\infty)} < \theta_2^{(2,\infty)}$ hold by (D1). Since $\boldsymbol{\theta}^{(1,\infty)} \le \boldsymbol{\theta}^{(\Delta,\infty)}$ and $\boldsymbol{\theta}^{(2,\infty)} \le \boldsymbol{\theta}^{(\Delta,\infty)}$, at least one of $\theta_1^{(1,\infty)}$ and $\theta_2^{(2,\infty)}$ must be increased by the right-hand side of (20). This contradicts the supposition. Hence, either $\theta_1^{(1,\infty)} = \theta_1^{(1,c)}$ or $\theta_2^{(2,\infty)} = \theta_2^{(2,c)}$ holds. Suppose that $\theta_1^{(1,\infty)} = \theta_1^{(1,c)}$ and $\theta_2^{(2,\infty)} < \theta_2^{(2,c)}$. Then, from condition (D1), $\theta_2^{(2,\infty)}$ must be increased again by the right-hand side of (20). This is a contradiction. Similarly, it is impossible for $\theta_1^{(1,\infty)} < \theta_1^{(1,c)}$ but $\theta_2^{(2,\infty)} = \theta_2^{(2,c)}$. Thus, we must have $\theta_k^{(k,\infty)} = \theta_k^{(k,c)}$ for $k = 1, 2$.

For (D2), we can apply similar arguments as above if we replace $\theta_1^{(1,c)}$ by $\overline{\xi}_1(\theta_2^{(2,c)})$. For (D3), we replace $\theta_2^{(2,c)}$ by $\overline{\xi}_2(\theta_1^{(1,c)})$. Thus, we obtain (4) for all the three cases since $\tau_1 = \theta_1^{(1,\infty)}$ and $\tau_2 = \theta_2^{(2,\infty)}$.

The remaining part of this lemma is immediate since $\varphi(\boldsymbol{\theta}) < \infty$ for all $\boldsymbol{\theta} < (\theta_1^{(\Delta,n)}, \underline{\xi}_2(\theta_1^{(\Delta,n)}))$ and all $\boldsymbol{\theta} < (\underline{\xi}_1(\theta_2^{(\Delta,n)}), \theta_2^{(\Delta,n)})$ is inductively obtained by Lemma 4.

Appendix F. The proof of Lemma 7

We will use the random walk $\{\boldsymbol{Y}(\ell)\}$ introduced in Section 2. We apply the permutation arguments in Lemma 5.6 of [5] twice. Then, we have, for any positive integer n and any $\boldsymbol{x} > \boldsymbol{0}$,

$$\begin{aligned} &\mathbb{P}\Big(\boldsymbol{Y}(n) \in \boldsymbol{x} + \Delta(\boldsymbol{a}),\ \min_{1\le\ell\le n} Y_{1\ell} > 0,\ \min_{1\le\ell\le n} Y_{2\ell} > 0 \,\Big|\, \boldsymbol{Y}(0) = \boldsymbol{0}\Big) \\ &\quad \ge \frac{1}{n}\mathbb{P}\Big(\boldsymbol{Y}(n) \in \boldsymbol{x} + \Delta(\boldsymbol{a}),\ \min_{1\le\ell\le n} Y_{2\ell} > 0 \,\Big|\, \boldsymbol{Y}(0) = \boldsymbol{0}\Big) \\ &\quad \ge \frac{1}{n^2}\mathbb{P}(\boldsymbol{Y}(n) \in \boldsymbol{x} + \Delta(\boldsymbol{a}) \mid \boldsymbol{Y}(0) = \boldsymbol{0}). \end{aligned} \tag{53}$$

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We note the well-known Cramér theorem (see, e.g. Theorem 2 of [4] and Section 3.5 of [9]):

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\boldsymbol{Y}(n) \in n\boldsymbol{x} + \Delta(\boldsymbol{a})) = -\Lambda(\boldsymbol{x}),$$

where $\Lambda(\boldsymbol{x}) = \sup_{\boldsymbol{\theta}\in\mathbb{R}^2}\{\langle\boldsymbol{\theta}, \boldsymbol{x}\rangle - \log\varphi(\boldsymbol{\theta})\}$.

Since the random walk $\{\boldsymbol{Y}(\ell)\}$ is identical to that of $\{\boldsymbol{L}(\ell)\}$ as long as they are inside the quadrant S, (53) can be written as, for $\boldsymbol{y} \in S_+$,

$$\begin{aligned}\mathbb{P}(\boldsymbol{L}(n) \in \boldsymbol{x} + \Delta(\boldsymbol{a}),\ \sigma_0 > n \mid \boldsymbol{L}(0) = \boldsymbol{y})\\ \geq \frac{1}{n^2}\mathbb{P}(\boldsymbol{Y}(n) \in \boldsymbol{x} - \boldsymbol{y} + \Delta(\boldsymbol{a}) \mid \boldsymbol{Y}(0) = \boldsymbol{0}),\end{aligned}$$

where $\sigma_0 = \inf\{\ell \geq 1;\ \boldsymbol{L}(\ell) \in \partial S\}$. It follows from representation (14) with $B = \partial S$ for the stationary distribution that there are some $\boldsymbol{y}_0 \in \partial S$ and $\boldsymbol{y}_1 \in S_+$ such that $p(\boldsymbol{y}_0, \boldsymbol{y}_1) > 0$ and, for any $m \geq 1$,

$$\begin{aligned}&\mathbb{P}(\boldsymbol{L} \in n\boldsymbol{c} + \Delta(\boldsymbol{a}))\\ &\quad= \frac{1}{\mathbb{E}_\nu(\sigma_0)} \sum_{\boldsymbol{y}\in\partial S}\sum_{\boldsymbol{y}'\in S_+} \nu(\boldsymbol{y})p(\boldsymbol{y}, \boldsymbol{y}') \sum_{\ell=1}^{\infty} \mathbb{P}(\boldsymbol{L}(\ell) \in n\boldsymbol{c} + \Delta(\boldsymbol{a}),\ \sigma_0 > \ell \mid \boldsymbol{L}(0) = \boldsymbol{y}')\\ &\quad\geq \frac{1}{\mathbb{E}_\nu(\sigma_0)}\mathbb{P}(\boldsymbol{L}(m) \in n\boldsymbol{c} + \Delta(\boldsymbol{a}),\ \sigma_0 > m \mid \boldsymbol{L}(0) = \boldsymbol{y}_1)\nu(\boldsymbol{y}_0)p(\boldsymbol{y}_0, \boldsymbol{y}_1)\\ &\quad\geq \frac{1}{m^2\mathbb{E}_\nu(\sigma_0)}\mathbb{P}(\boldsymbol{Y}(m) \in n\boldsymbol{c} - \boldsymbol{y}_1 + \Delta(\boldsymbol{a}) \mid \boldsymbol{Y}(0) = \boldsymbol{0})\nu(\boldsymbol{y}_0)p(\boldsymbol{y}_0, \boldsymbol{y}_1).\end{aligned}$$

Thus, for each $t > 0$, letting $m, n \to \infty$ in such a way that $n/m \to t$, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\boldsymbol{L} \in n\boldsymbol{c} + \Delta(\boldsymbol{a})) &\geq \lim_{n\to\infty} \frac{m}{n}\frac{1}{m} \log \mathbb{P}\left(\boldsymbol{Y}(m) \in m\frac{n}{m}\boldsymbol{c} - \boldsymbol{y}_1 + \Delta(\boldsymbol{a})\right)\\ &= -\frac{1}{t}\Lambda(t\boldsymbol{c}).\end{aligned}$$

Since $t > 0$ can be arbitrary, this implies that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\boldsymbol{L} \in n\boldsymbol{c} + \Delta(\boldsymbol{a})) \geq -\inf_{t>0} \frac{1}{t}\Lambda(t\boldsymbol{c}) = -\sup\{\langle\boldsymbol{\theta}, \boldsymbol{c}\rangle;\ \gamma(\boldsymbol{\theta}) \leq 1\},$$

where the last equality is obtained from Theorem 1 of [3] (see also Theorem 13.5 of [33]).

It remains to prove that $\boldsymbol{\theta} \notin \overline{\Gamma}_{\max}$ implies that $\varphi(\boldsymbol{\theta}) = \infty$. Define the cone $C_{\max}$ as

$$C_{\max} = \{\boldsymbol{x} \in \mathbb{R}^2;\ \boldsymbol{x} = s\boldsymbol{\theta}^{(1,\max)} + t\boldsymbol{\theta}^{(2,\max)},\ s, t \geq 0\}.$$

If $\boldsymbol{\theta} \notin C_{\max} \setminus \overline{\Gamma}_{\max}$ then either $\theta_1 > \theta_1^{(1,\max)}$ or $\theta_2 > \theta_2^{(2,\max)}$ holds. Hence, $\varphi(\boldsymbol{\theta}) = \infty$ in this case by Lemma 8. Thus, we only need to consider $\boldsymbol{\theta} \in C_{\max} \setminus \overline{\Gamma}_{\max}$.

Since $\overline{\Gamma}$ is a closed convex set that contains $\boldsymbol{0}$, there exists a unique $\boldsymbol{\eta} \in \overline{\Gamma}$ that maximizes $\langle\boldsymbol{\eta}, \boldsymbol{c}\rangle$ for each $\boldsymbol{c}$. Denote this $\boldsymbol{\eta}$ by $\boldsymbol{\eta}(\boldsymbol{c})$. That is,

$$\langle\boldsymbol{\eta}(\boldsymbol{c}), \boldsymbol{c}\rangle = \sup\{\langle\boldsymbol{\eta}, \boldsymbol{c}\rangle;\ \gamma(\boldsymbol{\eta}) \leq 1\}.$$

Let $\boldsymbol{c}(\omega) = (\cos\omega, \sin\omega)$. Then the $\boldsymbol{\eta}(\boldsymbol{c}(\omega))$ continuously move on $\partial\Gamma \cap C_{\max}$ from $\boldsymbol{\theta}^{(1,\max)}$ to $\boldsymbol{\theta}^{(2,\max)}$ as ω is increased on $(-\frac{1}{2}\pi, \pi)$. As can been seen from Figure 6, there is an

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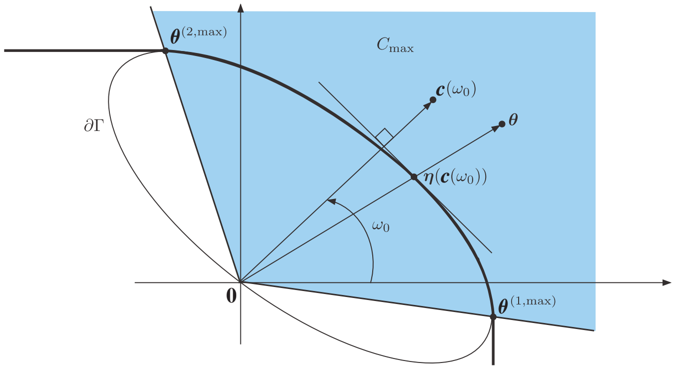

FIGURE 6: The positions of $\boldsymbol{\theta}$, $\boldsymbol{c}(\omega_0)$, $\boldsymbol{\eta}(\boldsymbol{c}(\omega_0))$, and the cone $C_{\max}$ (shaded region).

$\omega_0 \in (-\frac{1}{2}\pi, \pi)$ for $\boldsymbol{\theta} \in C_{\max} \setminus \overline{\Gamma}_{\max}$ such that $\boldsymbol{\eta}(\boldsymbol{c}(\omega_0))$ has the same direction as $\boldsymbol{\theta}$, which implies that $\boldsymbol{\theta} = a\boldsymbol{\eta}(\boldsymbol{c}(\omega_0))$ for some $a > 1$. For this ω_0, we have

$$\langle \boldsymbol{\theta}, \boldsymbol{c}(\omega_0)\rangle = a\langle \boldsymbol{\eta}(\boldsymbol{c}(\omega_0)), \boldsymbol{c}(\omega_0)\rangle > \langle \boldsymbol{\eta}(\boldsymbol{c}(\omega_0)), \boldsymbol{c}(\omega_0)\rangle.$$

Hence, Markov's inequality,

$$\varphi(\boldsymbol{\theta}) \geq \mathrm{e}^{n\langle \boldsymbol{\theta}, \boldsymbol{c}(\omega_0)\rangle}\mathbb{P}(\boldsymbol{L} \in n\boldsymbol{c}(\omega_0) + \Delta(\boldsymbol{a})),$$

and (21) yield

$$\liminf_{n\to\infty} \frac{1}{n} \log \varphi(\boldsymbol{\theta}) \geq \langle \boldsymbol{\theta}, \boldsymbol{c}(\omega_0)\rangle - \langle \boldsymbol{\eta}(\boldsymbol{c}(\omega_0)), \boldsymbol{c}(\omega_0)\rangle > 0.$$

This concludes $\varphi(\boldsymbol{\theta}) = \infty$, which completes the proof.

Appendix G. Proof of (32)

For changing measures for the random walk $\{\boldsymbol{Y}(\ell)\}$, we define $\{\hat{\boldsymbol{Y}}^{(1)}(\ell)\}$ by

$$\mathbb{P}(\hat{\boldsymbol{Y}}^{(1)}(\ell+1) = (n, j) \mid \hat{\boldsymbol{Y}}^{(1)}(\ell) = (m, i)) = \mathrm{e}^{\tau_1(n-m)-\tau_2(i-j)}\mathbb{P}(\boldsymbol{X}^{(+)} = (n-m, -(i-j))),$$

which is well defined since $\gamma(\boldsymbol{\tau}) = 1$ because of (D2). Note that, for $k = 1, 2$,

$$\mathbb{E}(\hat{Y}_k^{(1)}(\ell+1) \mid \hat{Y}_k^{(1)}(\ell) = 0) = (-1)^{k-1}\mathbb{E}(X_k^{(+)}\mathrm{e}^{\langle \boldsymbol{\tau}, \boldsymbol{X}^{(+)}\rangle}) = (-1)^{k-1}\frac{\partial}{\partial\theta_k}\gamma(\boldsymbol{\theta})\bigg|_{\boldsymbol{\theta}=\boldsymbol{\tau}}$$

is finite and positive because of (D2) and the assumption that $\tau_1 < \theta_1^{(1,\mathrm{c})}$. We denote this expectation by μ_k.

Recall that $\tilde{\sigma}_1^{(1)}(n)$ is the nth increasing instant of the Markov additive process $\{\tilde{\boldsymbol{Z}}^{(1)}(\ell)\}$. We introduce similar instants for $\{\hat{\boldsymbol{Y}}(\ell)\}$. Let

$$\hat{\zeta}_1^{(1)}(n) = \inf\{\ell \geq \hat{\zeta}_1^{(1)}(n-1); \hat{Y}_1^{(1)}(\ell) - \hat{Y}_1^{(1)}(\hat{\zeta}_1^{(1)}(n-1)) \geq 1\}.$$

For convenience, we also introduce the following events. For $1 \leq n_0 \leq n$, $m \geq 1$, and $j \geq 0$, let

$$\tilde{A}_{n_0,\ell}^{\boldsymbol{Z}}(m,k) = \{\tilde{\boldsymbol{Z}}^{(1)}(\tilde{\sigma}_1^{(1)}(\ell)) - \tilde{\boldsymbol{Z}}^{(1)}(\tilde{\sigma}_1^{(1)}(n_0)) = (m,k)\},$$
$$\hat{A}_{n_0,\ell}^{\boldsymbol{Y}}(m,k) = \{\tilde{\boldsymbol{Y}}^{(1)}(\tilde{\sigma}_1^{(1)}(\ell)) - \tilde{\boldsymbol{Y}}^{(1)}(\tilde{\sigma}_1^{(1)}(n_0)) = (m,k)\},$$

$$\tilde{B}^{Z_1}_{n_0,n}(m) = \bigcup_{\ell=n_0+1}^{n} \bigcup_{k=0}^{\infty} \tilde{A}^{\boldsymbol{Z}}_{n_0,\ell}(m,k),$$

$$\hat{B}^{Y_1}_{n_0,n}(m) = \bigcup_{\ell=n_0+1}^{n} \bigcup_{k=-\infty}^{\infty} \hat{A}^{\boldsymbol{Y}}_{n_0,\ell}(m,k),$$

$$\tilde{C}^{Z_2}_{n_0,n}(j) = \Big\{ \inf_{\tilde{\sigma}^{(1)}_1(n_0)<\ell\le\tilde{\sigma}^{(1)}_1(n)} \tilde{Z}^{(1)}_2(\ell) \ge 1,\ \tilde{Z}^{(1)}_2(\tilde{\sigma}^{(1)}_1(n_0)) = j \Big\},$$

$$\hat{C}^{Y_2}_{n_0,n}(j) = \Big\{ \inf_{\hat{\zeta}^{(1)}_1(n_0)<\ell\le\hat{\zeta}^{(1)}_1(n)} \hat{Y}^{(1)}_2(\ell) \ge 1,\ \hat{Y}^{(1)}_2(\hat{\zeta}^{(1)}_1(n_0)) = j \Big\}.$$

Since $\tilde{A}^{(1)}$ is transient, $\tilde{Z}^{(1)}_2(n)$ goes to $+\infty$ as $n \to \infty$ with probability 1. Similarly, for $k = 1, 2$, $\hat{Y}^{(1)}_k(n)$ diverges as $n \to \infty$ by $\mu_k > 0$. Hence, we can find a positive integer n_0 for any $i, j_0, k_0, \ell_0 \ge 1$ such that, for any $n \ge n_0$,

$$\sum_{j\ge j_0} \mathbb{P}(\tilde{C}^{Z_2}_{n_0,n}(j) \mid \tilde{Z}^{(1)}_2(0) = i) > 1-\varepsilon, \tag{54}$$

$$\sum_{j\ge j_0}\sum_{k\ge k_0} \mathbb{P}(\hat{C}^{Y_2}_{n_0,n}(j) \cap \{\hat{Y}^{(1)}_2(\hat{\zeta}^{(1)}_1(n)) = k\} \mid \hat{\boldsymbol{Y}}^{(1)}(0) = \boldsymbol{0}) > 1-\varepsilon, \tag{55}$$

$$\mathbb{P}(\tilde{\boldsymbol{Z}}^{(1)}(\tilde{\sigma}^{(1)}_1(n)) \ge (\ell_0, j_0) \mid \tilde{Z}^{(1)}_2(0) = i) > 1-\varepsilon. \tag{56}$$

We further choose $n_1 > n_0$ such that, for any $n \ge n_1$,

$$\mathbb{P}(\tilde{B}^{Z_1}_{0,n_0}(n) \mid \tilde{Z}^{(1)}_2(0) = i) < \varepsilon, \tag{57}$$

because $\tilde{Z}^{(1)}_1(\tilde{\sigma}^{(1)}_1(n_0))$ is finite with probability 1.

We now compute, for $n \ge n_0$ and $j \ge 1$,

$$\begin{aligned}
&\mathbb{P}(\tilde{B}^{Z_2}_{n_0,n}(n) \cap \tilde{C}^{Z_2}_{n_0,n}(j) \mid \tilde{Z}^{(1)}_2(0) = i)\\
&\quad = \sum_{\ell\ge1} \mathbb{P}(\tilde{B}^{Z_1}_{n_0,n}(n) \cap \tilde{C}^{Z_2}_{n_0,n}(j) \mid \tilde{\boldsymbol{Z}}^{(1)}(\tilde{\sigma}^{(1)}_1(n_0)) = (\ell, j))\\
&\qquad\qquad \times \mathbb{P}(\tilde{\boldsymbol{Z}}^{(1)}(\tilde{\sigma}^{(1)}_1(n_0)) = (\ell, j) \mid \tilde{Z}^{(1)}_2(0) = i)\\
&\quad = \sum_{\ell\ge1}\sum_{k\ge1}\sum_{s=n_0+1}^{n} \frac{e^{-\tau_2 k}}{x_k} \mathbb{P}(\hat{A}^{\boldsymbol{Y}}_{n_0,s}(n,k) \cap \hat{C}^{Y_2}_{n_0,n}(j)) \frac{x_j}{e^{-\tau_2 j}}\\
&\qquad\qquad \times \mathbb{P}(\tilde{\boldsymbol{Z}}^{(1)}(\tilde{\sigma}^{(1)}_1(n_0)) = (\ell, j) \mid \tilde{Z}^{(1)}_2(0) = i),
\end{aligned} \tag{58}$$

where the last equality follows from the definitions of $\{\tilde{\boldsymbol{Z}}^{(1)}(n)\}$ and $\{\hat{\boldsymbol{Y}}^{(1)}(n)\}$ and the fact that $\{\hat{\boldsymbol{Y}}^{(1)}(n)\}$ is a random walk.

We now consider application of the renewal theorem. For this, let

$$\mu^{(1)}_{\sigma} = \mathbb{E}(\hat{Y}^{(1)}_1(\hat{\zeta}^{(1)}_1(1)) \mid \hat{Y}^{(1)}_1(0) = 0).$$

Then $\mu^{(1)}_{\sigma}$ is finite because $\hat{Y}^{(1)}_1(n)$ drifts to $+\infty$ since $\mu^{(1)} > 0$, and its increments have a finite expectation (see Theorem 2.4 of [2, Chapter VIII]). Hence, it follows from the renewal theorem that

$$\lim_{n\to\infty} \sum_{\ell=1}^{n} \mathbb{P}(\hat{Y}^{(1)}_1(\hat{\zeta}^{(1)}_1(\ell)) = n) = \frac{1}{\mu^{(1)}_{\sigma}}.$$

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Obviously, this yields, for each fixed $n_0 \geq 0$,

$$\lim_{n\to\infty} \mathbb{P}(\hat{B}^{Y_1}_{n_0,n}(n)) = \frac{1}{\mu^{(1)}_\sigma}. \tag{59}$$

From (54) and (57), it follows that, for sufficiently large n,

$$\left|\mathbb{P}(\tilde{B}^{Z_1}_{n_0,n}(n) \mid \tilde{Z}^{(1)}_2(0) = i) - \sum_{j\geq j_0} \mathbb{P}(\tilde{B}^{Z_1}_{n_0,n}(n) \cap \tilde{C}^{Z_1}_{n_0,n}(j) \mid \tilde{Z}^{(1)}_2(0) = i)\right| < \varepsilon. \tag{60}$$

Similarly,

$$\left|\mathbb{P}(\hat{B}^{Y_1}_{n_0,n}(n)) - \sum_{j\geq j_0} \mathbb{P}(\hat{B}^{Y_1}_{n_0,n}(n) \cap \hat{C}^{Y_1}_{n_0,n}(j))\right| < \varepsilon. \tag{61}$$

By (31), we can choose sufficiently large j_0, k_0 such that

$$\left|1 - \frac{\mathrm{e}^{-\tau_2 k}}{x_k}\right|\left|1 - \frac{x_j}{\mathrm{e}^{-\tau_2 j}}\right| < \varepsilon, \qquad j \geq j_0,\ k \geq k_0.$$

We then sum both sides of (58) over all $j \geq j_0$. Furthermore, using (55) and (56) for sufficiently large ℓ_0, we apply (59) to the sum as $n \to \infty$, and then let $\varepsilon \to 0$ to obtain

$$\lim_{n\to\infty} \sum_{j\geq j_0} \mathbb{P}(\tilde{B}^{Z_1}_{n_0,n}(n) \cap \tilde{C}^{Z_1}_{n_0,n}(j) \mid \tilde{Z}^{(1)}_2(0) = i) = \frac{1}{\mu^{(1)}_\sigma}.$$

Hence, by (57), (60), and (61), we have

$$\limsup_{n\to\infty}\left|\mathbb{P}(\tilde{B}^{Z_1}_{n_0,n}(n) \mid \tilde{Z}^{(1)}_2(0) = i) - \frac{1}{\mu^{(1)}_\sigma}\right| < \varepsilon.$$

Equation (32) follows by letting $\varepsilon \downarrow 0$.

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