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ON DISCRETE GENERALISED TRIANGLE GROUPS

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A generalised triangle group has a presentation of the form

$$\langle x, y \mid x^k = y^l = R^m(x, y) = 1 \rangle$$

where R is a cyclically reduced word involving both x and y. When $R = xy$, these classical triangle groups have representations as discrete groups of isometries of $S^2, \mathbf{R}^2, \mathbf{H}^2$ depending on

$$\chi = \frac{1}{k} + \frac{1}{l} + \frac{1}{m} - 1.$$

In this paper, for other words R, faithful discrete representations of these groups in $\mathrm{Isom}^+\mathbf{H}^3 = PSL(2, \mathbf{C})$ are considered with particular emphasis on the case $R = [x, y]$ and also on the relationship between the Euler characteristic χ and finite covolume representations.

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1. Introduction

In this article, we consider generalised triangle groups, i.e. groups Γ with a presentation of the form

$$\Gamma = \Gamma(k, l; m) = \langle x, y \mid x^k = y^l = R^m(x, y) = 1 \rangle \quad k, l, m \geqq 2$$

where $R(x, y)$ is a cyclically reduced word in the free product on x, y which involves both x and y. These groups have been studied for their group theoretical interest [**8**, **7**, **13**], for topological reasons [**2**], and more recently for their connections with hyperbolic 3-manifolds and orbifolds [**12**, **10**].

Here we will be concerned with faithful discrete representations $\rho: \Gamma \to PSL(2, \mathbf{C})$ with particular emphasis on the cases where the Kleinian group $\rho(\Gamma)$ has finite covolume. In Theorem 3.2, we give necessary conditions on the group Γ so that it should admit such a faithful discrete representation of finite covolume.

For certain generalised triangle groups where the word $R(x, y)$ has a specified form, faithful discrete representations as above have been constructed by Helling–Mennicke–Vinberg [**12**] and by the first author [**11**]. In this paper, the cases where $R(x, y)$ is the commutator of x, y are considered in detail. Starting with Coxeter [**3**] the study of these groups has a long history. We determine completely when these groups have faithful discrete representations, which have finite covolume (Theorem 2.1 and Corollary 3.3) and when they are arithmetic (Theorem 4.2).

2. Commutator case

In this section let

$$\Gamma = \Gamma(k,l;m) = \langle x, y \,|\, x^k = y^l = [x,y]^m = 1 \rangle \tag{1}$$

where $[x,y] = xyx^{-1}y^{-1}$. Furthermore, we assume that $2 \leqq k \leqq l \leqq \infty$ and $2 \leqq m \leqq \infty$. In the cases where k, l, m is ∞ then there is no corresponding relation. In the Euler characteristic formulae that appear later, we assume that if $i = \infty$ then $1/i = 0$.

Theorem 2.1. *Let Γ be as defined at* (1) *above.*

(i) *Γ has a faithful discrete representation $\rho: \Gamma \to PSL(2,\mathbf{C})$ if and only if $(k,l;m) \neq (2,3;2), (3,3;2), (2,3;3), (2,4;2)$.*

(ii) *If $(k,l;m) = (3,3;3), (4,4;2), (3,4;2)$ then Γ has a faithful discrete representation of finite covolume and in all other cases given in* (i), *Γ has a faithful discrete representation of infinite covolume.*

Notation. Throughout, we will use $\tau(X)$ to denote the trace of $X \in SL(2,\mathbf{C})$.

Proof. First let us recall some results on traces. If $X, Y \in SL(2,\mathbf{C})$ and $\alpha = \tau(X)$, $\beta = \tau(Y)$ and $\gamma = \tau(XY)$ then

$$\tau[X,Y] = \alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma - 2. \tag{2}$$

Also, any two-generator non-elementary subgroup $\langle X, Y \rangle$ of $SL(2,\mathbf{C})$ is determined up to conjugacy by α, β, γ.

We now deal with the four exceptional cases excluded in (i). These groups have been investigated in [15] (see also [3,5]) and the groups $\Gamma(2,3;2)$, $\Gamma(3,3;2)$ are finite of orders 24, 288 respectively while the groups $\Gamma(2,3;3)$, $\Gamma(2,4;2)$ are infinite and solvable. We see immediately that the second group cannot have a faithful discrete representation in $PSL(2,\mathbf{C})$. Now let $\rho: \Gamma(2,3;2) \to PSL(2,\mathbf{C})$ be a homomorphism with $\rho(x) = PX$, of order 2 and $\rho(y) = PY$ of order 3. (Here P denotes the natural projection $P: SL(2,\mathbf{C}) \to PSL(2,\mathbf{C})$). Then from (2), $0 = \tau[X,Y] = \gamma^2 - 1$. Thus $\rho(xy) = PXY$ has order 3. But then $\rho(\Gamma)$ is A_4 and cannot have order 24, so that ρ cannot be faithful. A similar argument rules out the remaining two groups.

Note that the group $\Gamma(2,2;m)$ is dihedral of order $4m$ and has a faithful discrete representation in $PSL(2,\mathbf{C})$. This remains true if $m = \infty$.

In all other cases where k, l, m are finite, equation (2) admits a solution with $\alpha = 2\cos \pi/k$, $\beta = 2\cos \pi/l$, $\tau[X,Y] = -2\cos \pi/m$ and $\gamma \in \mathbf{C} \backslash \mathbf{R}$ and a corresponding homomorphism $\rho: \Gamma(k,l;m) \to PSL(2,\mathbf{C})$ can be constructed (see [11]). This remains true if any of k, l, m become infinite if we choose the corresponding matrices to be parabolic.

Now suppose

$$\frac{1}{k}+\frac{1}{k}+\frac{1}{m}\geqq 1 \quad \text{and} \quad \frac{1}{l}+\frac{1}{l}+\frac{1}{m}\geqq 1$$

so that, in addition to the five cases discussed above, we have $(3,3;2),(4,4;2),(3,4;2)$. The triangle subgroups $\Gamma_x=\langle x,yx^{-1}y^{-1}\rangle$, $y^{-1}\Gamma_xy=\langle y^{-1}xy,x^{-1}\rangle$, $\Gamma_y=\langle y,xy^{-1}x^{-1}\rangle$, $x^{-1}\Gamma_yx=\langle x^{-1}yx,y^{-1}\rangle$ are spherical or Euclidean. Under the representation ρ, these triangle groups have fixed points, denoted $O_x,\rho(y^{-1})O_x,O_y,\rho(x^{-1})O_y$, either in $\mathbf{H}^3$ or on $\partial\mathbf{H}^3$, which form the vertices of a tetrahedron **[10]** (see also **[12]** and Section 4). This tetrahedron has finite volume and checking face identifications and the angle sum at edge cycles establishes that it is a fundamental region for $\rho(\Gamma)$, so that, by Poincaré's Theorem, we have a faithful representation of Γ. Thus for these groups (i) and the first part of (ii) are established.

Now suppose that the triangle subgroups are not all spherical or Euclidean. Suppose that Γ_x is hyperbolic, which, we note, occurs precisely when $y^{-1}\Gamma_xy$ is hyperbolic. In that case, one can regard O_x as being an "ideal" vertex lying outside $\mathbf{H}^3$. More precisely, there exists a unique geodesic plane intersecting the axes of the rotations $\rho(xyx^{-1}y^{-1})$, $\rho(x^{-1}yxy^{-1})$, $\rho(x)\in\rho(\Gamma_x)$ orthogonally in points A,B,C respectively. Similarly, there will be a plane intersecting the axes of $\rho(x)$, $\rho(y^{-1}x^{-1}yx)$, $\rho(xy^{-1}x^{-1}y)\in\rho(y^{-1}\Gamma_xy)$ in the points D,E,F. (See Figure 1). Furthermore, it can be shown that these two planes do not intersect, nor do they intersect the opposite faces of the tetrahedron **[11]**. We are assuming in this case, that Γ_y is spherical or Euclidean. (The abandoning of the order $k\leqq l$ is unimportant here). Let $\tilde{\Gamma}$ be the group generated by $\rho(x)=\xi,\rho(y)=\eta,\rho_1,\rho_2$ where ρ_1,ρ_2 are the reflections in the planes ABC,DEF, so that ξ is a rotation of order k around the axis CD and η a rotation about the axis $O_y\xi^{-1}O_y$. Using the same procedure as in the finite volume case above, we obtain that the "tetrahedron" truncated at the two "ideal" vertices, by the planes ABC and DEF is a fundamental region for $\tilde{\Gamma}$ Finally, let $\tilde{\Gamma}^+$ denote the orientation-preserving subgroup of $\tilde{\Gamma}$. Now $\tilde{\Gamma}^+$ is of index 2 in $\tilde{\Gamma}$ and is generated by $\xi,\eta,\rho_1\eta\rho_1,\rho_1\rho_2$. Thus two copies of the truncated tetrahedron obtained by reflecting one across the face DEF as shown in Figure 1 form a fundamental region for $\tilde{\Gamma}^+$.

The face identifications are given by the following

$$\begin{array}{ll} ABC & \xrightarrow{\rho_2\rho_1} \rho_2A\rho_2B\rho_2C \\ \rho_2CDCB\xi^{-1}O_yE\rho_2\xi^{-1}O_y\rho_2B & \xrightarrow{\xi} \rho_2CDCAO_yF\rho_2O_y\rho_2A \\ \xi^{-1}O_yO_yEF & \xrightarrow{\eta} \xi^{-1}O_yO_yBA \\ \rho_2O_y\rho_2\xi^{-1}O_yEF & \xrightarrow{\rho_2\eta\rho_2} \rho_2O_y\rho_2\xi^{-1}O_y\rho_2B\rho_2A \end{array}$$

and consequently we have the following cycles of edges

(a) $C\rho_2C \xrightarrow{\xi} C\rho_2C$

(b) $O_y\xi^{-1}O_y \xrightarrow{\eta} O_y\xi^{-1}O_y$

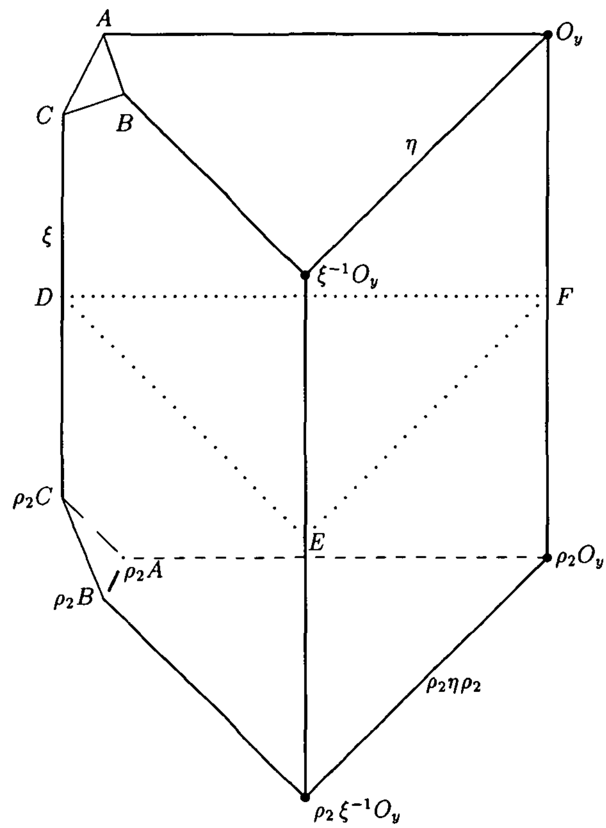

FIGURE 1

(b′) $\rho_2 O_y \rho_2 \xi^{-1} O_y \xrightarrow{\rho_2\eta\rho_2} \rho_2 O_y \rho_2 \xi^{-1} O_y$

(c) $E\xi^{-1}O_y \xrightarrow{\xi} FO_y \xrightarrow{\eta} AO_y \xrightarrow{\xi^{-1}} B\xi^{-1}O_y \xrightarrow{\xi^{-1}} E\xi^{-1}O_y$

(c′) $E\rho_2\xi^{-1}O_y \xrightarrow{\xi} F\rho_2 O_y \xrightarrow{\rho_2\eta\rho_2} \rho_2 A\rho_2 O_y \xrightarrow{\xi^{-1}} \rho_2 B\rho_2\xi^{-1}O_y \xrightarrow{\rho_2\eta^{-1}\rho_2} E\rho_2\xi^{-1}O_y$

(d) $AB \xrightarrow{\eta^{-1}} FE \xrightarrow{\rho_2\eta\rho_2} \rho_2 A\rho_2 B \xrightarrow{\rho_1\rho_2} AB$

(e) $AC \xrightarrow{\xi^{-1}} BC \xrightarrow{\rho_2\rho_1} \rho_2 B\rho_2 C \xrightarrow{\xi} \rho_2 A\rho_2 C \xrightarrow{\rho_1\rho_2} AC$

The conditions of Poincaré's theorem (see **[16]**) can be shown to hold and the group $\tilde{\Gamma}^+$ has the presentation

$$\tilde{\Gamma}^+ = \langle x, y, r_2yr_2, r_2r_1 \mid x^k = y^l = [x,y]^m = 1,$$
$$(r_2yr_2)^l = [x, r_2yr_2]^m = 1, [x^{-1}, r_2r_1] = 1, y^{-1}r_2yr_2(r_2r_1)^{-1} = 1\rangle.$$

This abstract group then has a faithful discrete representation in $PSL(2,\mathbf{C})$ with finite covolume.

Using now the last relation $r_2r_1=y^{-1}r_2yr_2$ to eliminate this generator and setting $\theta=r_2yr_2$ we get

$$\tilde{\Gamma}^+=\langle x,y,\theta \mid x^k=y^l=[x,y]^m=1,\\ \theta^l=[x,\theta]^m=1,[x^{-1},y^{-1}\theta]=1\rangle. \tag{3}$$

Now the group $\Gamma(k,l;m)$ can be embedded in $\tilde{\Gamma}^+$. Setting $\theta=y$ we obtain $\Gamma(k,l;m)$ as a factor group. Therefore the subgroup generated by x and y in $\tilde{\Gamma}^+$ is isomorphic to $\Gamma(k,l;m)$. This group is of infinite index, since setting $x=1$ yields the factor group $\mathbf{Z}_l*\mathbf{Z}_l$. So finally we obtain a faithful representation of $\Gamma(k,l;m)$ as a discrete subgroup of $PSL(2,\mathbf{C})$ with infinite covolume.

In the cases where

$$\frac{1}{k}+\frac{1}{k}+\frac{1}{m}<1 \quad \text{and} \quad \frac{1}{l}+\frac{1}{l}+\frac{1}{m}<1,$$

all four vertices of the tetrahedron are "ideal" and so are truncated by planes. Using similar notation and methods to those given above, the orientation preserving subgroup $\hat{\Gamma}^+$ of the group $\hat{\Gamma}$ is again generated by the same ξ,η,ρ_1 and ρ_2, with additionally, in this case, the two reflections ρ_3 and ρ_4 in the planes GHK and LMN, respectively. In this way, a fundamental domain of finite volume (see Figure 2) was constructed in [**11**].

For this polyhedron the faces are identified in the following way

$$\begin{array}{ll}
ABC & \xrightarrow{\rho_2\rho_1} \rho_2A\rho_2B\rho_2C\\
GHK & \xrightarrow{\rho_2\rho_3} \rho_2G\rho_2H\rho_2K\\
LMN & \xrightarrow{\rho_2\rho_4} \rho_2L\rho_2M\rho_2N\\
\rho_2CDCBNME\rho_2M\rho_2N\rho_2B & \xrightarrow{\xi} \rho_2CDCAKGF\rho_2G\rho_2K\rho_2A\\
HLMEFG & \xrightarrow{\eta} HLNBAK\\
\rho_2H\rho_2L\rho_2MEF\rho_2G & \xrightarrow{\rho_2\eta\rho_2} \rho_2H\rho_2L\rho_2N\rho_2B\rho_2A\rho_2K.
\end{array}$$

From these identifications we deduce the following cycles of edges

(a) $C\rho_2C \xrightarrow{\xi} C\rho_2C$

(b) $HL \xrightarrow{\eta} HL$

(b′) $\rho_2H\rho_2L \xrightarrow{\rho_2\eta\rho_2} \rho_2H\rho_2L$

(c) $EM \xrightarrow{\xi} FG \xrightarrow{\eta} AK \xrightarrow{\xi^{-1}} BN \xrightarrow{\eta^{-1}} EM$

(c′) $E\rho_2M \xrightarrow{\xi} F\rho_2G \xrightarrow{\rho_2\eta\rho_2} \rho_2A\rho_2K \xrightarrow{\xi^{-1}} \rho_2B\rho_2N \xrightarrow{\rho_2\eta^{-1}\rho_2} E\rho_2M$

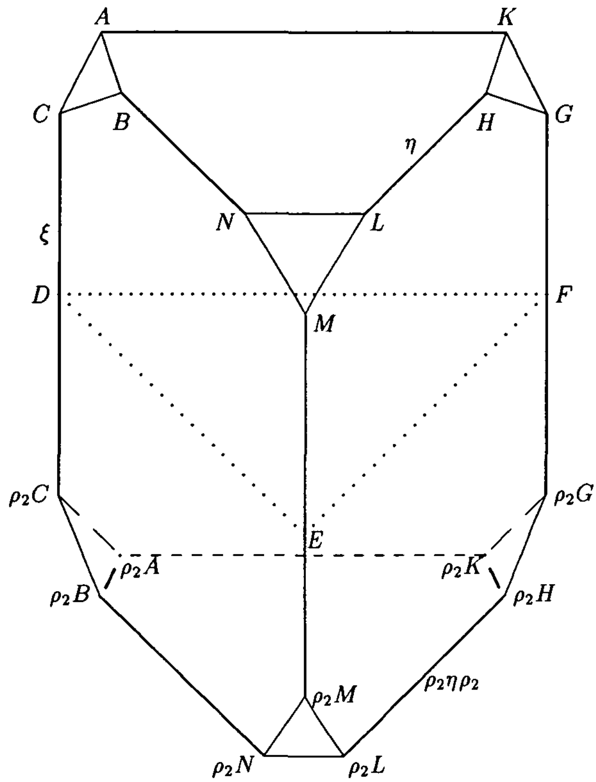

FIGURE 2

(d) $AB \xrightarrow{\eta^{-1}} FE \xrightarrow{\rho_2\eta\rho_2} \rho_2A\rho_2B \xrightarrow{\rho_1\rho_2} AB$

(e) $AC \xrightarrow{\xi^{-1}} BC \xrightarrow{\rho_2\rho_1} \rho_2B\rho_2C \xrightarrow{\xi} \rho_2A\rho_2C \xrightarrow{\rho_1\rho_2} AC$

(f) $NL \xrightarrow{\eta^{-1}} ML \xrightarrow{\rho_2\rho_4} \rho_2M\rho_2L \xrightarrow{\rho_2\eta\rho_2} \rho_2N\rho_2L \xrightarrow{\rho_4\rho_2} NL$

(g) $KH \xrightarrow{\eta^{-1}} GH \xrightarrow{\rho_2\rho_3} \rho_2G\rho_2H \xrightarrow{\rho_2\eta\rho_2} \rho_2K\rho_2H \xrightarrow{\rho_3\rho_2} KH$

(h) $KG \xrightarrow{\xi^{-1}} NM \xrightarrow{\rho_2\rho_4} \rho_2N\rho_2M \xrightarrow{\xi} \rho_2K\rho_2G \xrightarrow{\rho_1\rho_2} KG$

Again by the theorem of Poincaré the group $\hat{\Gamma}^+$ has the presentation

$$\begin{aligned}\hat{\Gamma}^+ = \langle x, y, r_2yr_2, r_2r_1, r_2r_3, r_2r_4 \,|\, x^k = y^l = [x,y]^m = 1,\\ (r_2yr_2)^l = [x, r_2yr_2]^m = 1, [x^{-1}, r_2r_1] = 1, r_2r_4x(r_2r_1)^{-1} = x,\\ r_2r_3r_2yr_2(r_2r_3)^{-1} = y, r_2r_4r_2yr_2(r_2r_4)^{-1} = y, y^{-1}r_2yr_2(r_2r_1)^{-1} = 1\rangle.\end{aligned}$$

and so has a faithful discrete representation in $PSL(2, \mathbf{C})$ with finite covolume.

We are now in a position to eliminate the generators r_2yr_2 and r_2r_1 using the last two relations. Setting additionally $\theta_3=r_2r_3$ and $\theta_4=r_2r_4$ we get

$$\hat{\Gamma}^+=\langle x,y,\theta_3,\theta_4 \mid x^k=y^l=[x,y]^m=1,[x,\theta_4^{-1}y\theta_4]^m=1,$$
$$[x^{-1},y^{-1}\theta_4^{-1}y\theta_4]=1,\theta_4x(y^{-1}\theta_4^{-1}y\theta_4)^{-1}=x,\theta_3\theta_4^{-1}y\theta_4\theta_3^{-1}=y\rangle.$$

Note that $r_2yr_2=\theta_4^{-1}y\theta_4$ and $r_2r_1=y^{-1}\theta_4^{-1}y\theta_4$.

Setting now $\theta_3=\theta_4=1$ we get as before $\Gamma(k,l;m)$ as a factor group of $\hat{\Gamma}^+$ and so the group $\Gamma(k,l;m)$ can be embedded in $\hat{\Gamma}^+$. Therefore again the subgroup generated by x and y in $\hat{\Gamma}^+$ is isomorphic to $\Gamma(k,l;m)$. This group is again of infinite index, since setting $y=1$ and $\theta_4=1$ we obtain the factor group $Z_k*\mathbf{Z}$. So finally we obtain a faithful representation of $\Gamma(k,l;m)$ as a discrete subgroup of $PSL(2,\mathbf{C})$ with infinite covolume. This deals with the cases where k,l,m are all finite.

If m is infinite, or both k,l are infinite, then Γ has a discrete faithful representation as a Fuschian group. Now fix the finite values of k and m. Let G be the abstract group obtained from the presentations at (3) or (4), depending on the values of k and m, by omitting the relations $y^l=1$ and $\theta^l=1$. Then for l large enough, we have a sequence of representations $\rho_l:G\to PSL(2,\mathbf{C})$ whose images are the groups $\tilde{\Gamma}^+$ or $\hat{\Gamma}^+$ and so have finite covolume. In each case, the representation depends on the parameter γ defined at (2) for fixed k,m. As $l\to\infty$, the representations ρ_l converge algebraically to ρ, whose parameter value is 2. Since the representations are discrete of finite covolume, algebraic convergence implies geometric convergence [**24**], and so $\rho(G)$ is discrete. Now suppose that for some $W\in G$, $\rho(W)=1$. Then $\rho_l(W)$ converges to 1. But by a compactness argument in the geometric topology, there is a lower bound on $d(\rho_l(W)x,x)$ for $x\in\mathbf{H}^3$. Thus for large enough l we must have $\rho_l(W)=1$, which will be a consequence of the defining relations in $\tilde{\Gamma}^+$ or $\hat{\Gamma}^+$. But since this is true for all large l, we must have $W=1$, so that ρ is faithful. As before $\Gamma(k,\infty;m)$ can be embedded as a subgroup of infinite index in $\rho(G)$ and the proof is complete.

Remarks 1. Note that Theorem 2.1(ii) does not quite give necessary and sufficient conditions for the existence of faithful discrete representations of finite covolume. This will be pursued farther in Corollary 3.3.

2. In [**14**] it is shown that, if a discrete group $G=G(k,l;m)$ is generated by two primitive elliptic elements of orders k,l whose axes do not intersect and such that the two planes spanned by the common perpendicular to the two axes and each axis in turn, intersect orthogonally, and whose commutator is elliptic of order m, then the triples $(k,l;m)$ satisfy precisely the conditions given in Theorem 2.1(i) and furthermore, that such groups exist for all these triples. Thus these groups $G(k,l;m)$ satisfy the relations at (1). For the triples given in Theorem 2.1(ii), which give rise to lattices, the method of proof in [**14**] using Poincaré's theorem, indicates further that (1) is a set of defining relations for $G(k,l;m)$. Thus by rigidity, these groups will be conjugate in $PSL(2,\mathbf{C})$ to the groups $\Gamma(k,l;m)$. Thus in Theorem 2.1, in these cases, the groups can be generated by elements whose axes satisfy the orthogonality condition of [**14**] (see

Section 4). It is not clear, that for other finite covolume generalised triangle groups, this geometric condition need hold.

3. Euler characteristic conditions

In the examples considered in Theorem 2.1, the groups $\Gamma(k,l;m)$ which have finite covolume representations satisfy

$$\frac{1}{k}+\frac{1}{l}+\frac{1}{m}\geqq 1.$$

Also, in the cases considered in [12] where $R(x,y)=xyx^{-1}yxy^{-1}$, the groups which have finite covolume representations satisfy

$$\frac{1}{k}+\frac{1}{l}+\frac{1}{m}\geqq 1.$$

In this section, we consider the necessity of this condition for generalised triangle groups. Thus let

$$\Gamma=\Gamma(k,l;m)=\langle x,y\,|\,x^k=y^l=R^m(x,y)=1\rangle \tag{5}$$

with $2\leqq k\leqq l\leqq\infty$, $2\leqq m\leqq\infty$, $R(x,y)$ is a cyclically reduced word which is not a proper power in the free product on x and y and involves both x and y.

In the cases where $R(x,y)$ is a product of conjugates of elements of finite order, like those discussed in Section 2, a representation may give rise to elliptic elements with intersecting axes. In these cases we make repeated use of the following result (see [17]).

Theorem 3.1. *Let x,y be elliptic elements of orders n,m, where $n\leqq m$, which generate a discrete subgroup of $PSL(2,\mathbf{C})$. Suppose that the axes of x and y intersect in a point in $\mathbf{H}^3$. Then one of the following must occur:* (i) $n=2, m\geqq 2$, (ii) $(n,m)=(3,3)$, (iii) $(n,m)=(3,4)$, (iv) $(n,m)=(3,5)$, (v) $(n,m)=(4,4)$, (vi) $(n,m)=(5,5)$.

Theorem 3.2. *Let Γ be as at* (5) *above. Suppose, further, that one of the following conditions holds.*

(i) $m\geqq 4$.

(ii) $m=3$ *and the word $R(x,y)$ does not involve an element of order* 2.

If Γ has a faithful representation $\rho:\Gamma\to PSL(2,\mathbf{C})$ such that $\rho(\Gamma)$ is a Kleinian group of finite covolume, then

$$\frac{1}{k}+\frac{1}{l}+\frac{1}{m}\geqq 1. \tag{6}$$

Proof. Note, if $k=l=2$ then (6) holds. Thus we assume $l\geqq 3$. Now assume that Γ is a Kleinian group of finite covolume.

In this argument we have to distinguish generalised triangle groups which have the following property which we refer to as

Property E: $R(x,y)$ is conjugate in the free product on x,y to a word uv for some elements u,v of orders $p,q(\geqq 2)$ respectively with

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{m}>1.$$

Suppose (ii) holds or $m\geqq 4$ and Γ does not have *Property E*. In these cases, it is shown in [**9**] that Γ has a rational Euler characteristic $\chi(\Gamma)$ and

$$\chi(\Gamma)=-1+\frac{1}{k}+\frac{1}{l}+\frac{1}{m}.$$

But if Γ has finite covolume, then $\chi(\Gamma)\geqq 0$. (See e.g. [**20**]) and so (6) holds.

It remains to consider the cases where $m\geqq 4$ and Γ does have *Property E*. Without loss of generality, we may assume that, for some word t in the free product on x,y we have either

(a) $u=x^\alpha, v=ty^\beta t^{-1}$ k,l finite, $1\leqq\alpha<k$, $\alpha\mid k$, $1\leqq\beta<l$, $\beta\mid l$ or

(b) $u=x^\alpha, v=tx^\beta t^{-1}$ k finite, $1\leqq\alpha,\beta<k$, $\alpha\mid k$ or

(c) $u=y^\alpha, v=ty^\beta t^{-1}$ l finite, $1\leqq\alpha,\beta<l$, $\alpha\mid l$.

(a). $R(x,y)=uv=x^\alpha ty^\beta t^{-1}$. Since the subgroup $\langle x^\alpha, ty^\beta t^{-1}\rangle$ is finite, it has a fixed point in $\mathbf{H}^3$. Consequently, the axes of x^α and $ty^\beta t^{-1}$ intersect and hence so do the axes of x, tyt^{-1}. Since Γ has *Property E* and $m\geqq 4$, it follows from Theorem 3.1 that $(k,l)=(2,l),(3,4),(4,4)$. Furthermore, the above argument shows that $(xtyt^{-1})^n=1$ for some finite integer n and this relation must be a consequence of the relations of the group as given at (5).

Case 1. $(k,l)=(2,l)$. Now $l\geqq 3$ and $\alpha=1$. From

$$\frac{1}{2}+\frac{1}{q}+\frac{1}{m}>1,$$

we must have $q=2$ or $q=3$ and $m=4$ or 5. Now if $z=R(x,y)=xty^\beta t^{-1}$ then

$$H=\langle x, ty^\beta t^{-1}\rangle=\langle z, ty^\beta t^{-1}\rangle.$$

Now $l\geqq 3$, $m\geqq 4$, $q\mid l$ so the same argument as above using Theorem 3.1 implies that $(l,m)=(3,4),(3,5),(4,4)$. In all cases (6) holds.

Case 2. $(k,l)=(3,4)$. Then $\alpha=1$ and since we must have

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{m}>1$$

then

$$\Gamma = \langle x, y \mid x^3 = y^4 = (xty^2t^{-1})^4 = 1 \rangle.$$

Now we can assume that $\tau x = 1$, $\tau y = \sqrt{2}$. Let $\tau(xtyt^{-1}) = \mu$. Then $\tau(xty^2t^{-1}) = \sqrt{2}\mu - 1 = \pm\sqrt{2}$. But this implies that $xtyt^{-1}$ has infinite order.

Case 3. $(k, l) = (4, 4)$ and we have

$$\Gamma = \langle x, y \mid x^4 = y^4 = (x^2ty^2t^{-1})^4 = 1 \rangle.$$

Now the axes of x^2 and tyt^{-1} intersect and so $(x^2tyt^{-1})^n = 1$. Let $\mu = \tau(x^2tyt^{-1})$. So $\tau(x^2ty^2t^{-1}) = \sqrt{2}\mu = \pm\sqrt{2}$. Thus $\mu = \pm 1$ and $n = 3$. Now consider the factor group of Γ obtained by setting $x^2 = y^2 = 1$ to obtain $\mathbf{Z}_2 * \mathbf{Z}_2$, from which the relation $y^3 = 1$ must follow. This is clearly a contradiction.

(b). $R(x, y) = uv = x^\alpha tx^\beta t^{-1}$. Using the subgroup $\langle x^\alpha, tx^\beta t^{-1} \rangle$ we see, as before, that a relation of the form $(xtxt^{-1})^n = 1$ must be a consequence of the others.

We first show that $k = 2$. Suppose that $k = 3$ so that $\alpha = 1$, $\beta = \pm 1$ and $p = q = k = 3$. This violates the inequality on p, q, m. Thus if $k \neq 2$, then $k \geqq 4$. Again using Theorem 3.1, we must have $(k, m) = (4, 4)$ and

$$\Gamma = \langle x, y \mid x^4 = y^l = (x^2tx^2t^{-1})^4 = 1 \rangle.$$

with $l \geqq 4$. Arguing as above we obtain $\tau(x^2txt^{-1}) = \pm 1$. Now let $\tau(xtxt^{-1}) = \mu$ so that $\sqrt{2}\mu - \sqrt{2} = \pm 1$ from which we deduce that $xtxt^{-1}$ has infinite order. Thus $k = 2$.

(c). The argument just given shows that this case cannot arise since $l \geqq 3$.

(d). Returning to the last case where $k = 2$ so that $p = q = 2$ and

$$\Gamma = \langle x, y \mid x^2 = y^l = (xtxt^{-1})^m = 1 \rangle$$

with $3 \leqq l \leqq \infty$ and $4 \leqq m \leqq \infty$. If $a = y$ and $b = xyx^{-1}$ then $N = \langle a, b \rangle$ has index 2 in Γ and so is also a finite covolume Kleinian group. By the Reidemeister–Schreier method, N has a presentation

$$N = \langle a, b \mid a^l = b^l = S^m(a, b) = 1 \rangle$$

where $S(a, b)$ is a cyclically reduced word, not a proper power in the free product on a, b which involves both a and b. But then

$$\frac{1}{l} + \frac{1}{l} + \frac{1}{m} < 1.$$

This contradiction completes the proof of the Theorem.

We can now complete the special case considered in Theorem 2.1.

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Corollary 3.3. *Let*

$$\Gamma = \Gamma(k,l;m) = \langle x, y \mid x^k = y^l = [x,y]^m = 1\rangle$$

with $2 \leqq k \leqq l \leqq \infty$ *and* $2 \leqq m \leqq \infty$. *Then* Γ *has a finite covolume faithful discrete representation* $\rho: \Gamma \to PSL(2, \mathbf{C})$ *if and only if* $(k,l;m) = (3,3;3), (3,4;2), (4,4;2)$.

Proof (a). If $m \geqq 4$, then by Theorem 3.2, $k=2$. But in this case, Γ contains a subgroup of index 2 with presentation $\langle a,b \mid a^l = b^l = (ab)^m = 1\rangle$. This group is either Fuchsian or dihedral and so does not have a finite covolume representation.

(b). Let $m=3$. If $k \geqq 3$ then $l \geqq 3$ and so $[x,y]$ does not contain a subword which is an element of order 2. Thus, by Theorem 3.2, for a finite volume representation the only solution is $(3,3;3)$. If $k=2$, then using the subgroup of index 2 as above, and Theorem 2.1, gives no finite covolume representations.

(c). $m=2$. When $\infty > l \geqq 5$, there is, by Theorem 2.1, a faithful discrete representation of Γ, which we identify with Γ, with infinite covolume, in which $\alpha = \tau x = 2\cos \pi/k$, $\beta = \tau y = 2\cos\pi/l$. The two possible values of $\tau(xy)$ dictated by the equation (2) correspond to the choices of generators x, y and x^{-1}, y so that Γ is uniquely determined. We show that this implies that, in Γ, the elements $[x^t, y^s]$ where $1 \leqq t \leqq k/2, 1 \leqq s \leqq l/2$, $(t,k)=1, (s,l)=1, t, s$ not both 1, have infinite order. For, from **[22]**

$$\tau[x^t, y^s] - 2 = \sigma_t^2(\tau x)\sigma_s^2(\tau y)(\tau[x,y]-2) \tag{7}$$

where $\sigma_r(z)$ is the Tschebycheff polynomial where

$$\sigma_0(z)=0 \quad \sigma_1(z)=1 \quad \text{and} \quad \sigma_r(z) = z\sigma_{r-1}(z) - \sigma_{r-2}(z) \quad r \geqq 2.$$

Recall that

$$\sigma_r(2\cos\pi/k) = \frac{\sin r\pi/k}{\sin \pi/k}.$$

Thus in the case above, $\sigma_t^2(\alpha)\sigma_s^2(\beta) \geqq 2$ and $\tau[x,y]=0$. Thus $\tau[x^t, y^s] \leqq -2$ and the element $[x^t, y^s]$ has infinite order.

Now suppose that Γ has a faithful discrete representation ρ of finite covolume, so that necessarily, either $\tau\rho(x) \neq \pm 2\cos\pi/k$ or $\tau\rho(y) \neq \pm 2\cos\pi/l$. Choose x_1, y_1 such that $\tau\rho(x_1) = 2\cos\pi/k$ and $\tau\rho(y_1) = 2\cos\pi/l$, $x = x_1^t$, $y = y_1^s$ where $1 \leqq t \leqq k/2$, $1 \leqq s \leqq l/2$ with $(t,k)=1$, $(s,l)=1$ and s,t not both 1. Again using (7), we obtain $\tau[\rho(x_1), \rho(y_1)] \in (0,2)$. But then $[\rho(x_1), \rho(y_1)]$ is elliptic, necessarily of finite order, which is a contradiction.

This argument does not cover the cases $l=\infty$. If $k=\infty$, the group will be Fuchsian and so cannot have a finite covolume representation. If $k=2$, the subgroup of index 2 is Fuchsian. For $3 \leqq k < \infty$, Γ is easily seen to have a subgroup N of index k with presentation

$$N=\langle a_1,a_2,\ldots,a_k \mid (a_{i+1}a_i^{-1})^2=1, \quad i=1,2,\ldots,k(\mathrm{mod}\,k)\rangle.$$

Choosing as generators $\{a_1,a_2a_1^{-1},a_3a_2^{-1},\ldots,a_ka_{k-1}^{-1}\}$ we see that $N\cong \mathbf{Z} * N_1$ where

$$N=\langle \alpha_1,\alpha_2,\ldots,\alpha_{k-1} \mid \alpha_i^2=1, i=1,2,\ldots,k-1 \ (\alpha_1\alpha_2\ldots\alpha_{k-1})^2=1\rangle.$$

Now $\chi(N)=\chi(N_1)-1$ and so this group cannot have a finite covolume representation.

Noting that, if $m=2$ and $l<5$, all these cases were considered in Theorem 2.1 and the proof of the Corollary is now complete.

4. Arithmeticity

We now consider the arithmeticity of the groups given by (1). By Corollary 3.3, since arithmetic groups have finite covolume, we need only consider the groups so defined in Theorem 2.1.

We first recall some general results on arithmetic Kleinian groups Γ (see e.g. [**25**, **1**]. Let k be a number field with one complex place and A a quarternion algebra over k which is ramified at all real places. Then, if $\mathcal{O}$ is an order in A and $\rho: A\to M_2(\mathbf{C})$ a representation, then $P\rho(\mathcal{O}^1)$ is a finite covolume Kleinian group and Γ must be commensurable with some such group. (Here $\mathcal{O}^1$ denotes the elements in $\mathcal{O}$ of reduced norm 1.) Indeed if Γ is arithmetic and

$$\Gamma^{(2)}=\langle \gamma^2 \mid \gamma\in\Gamma\rangle$$

then $\Gamma^{(2)}\subset P\rho(\mathcal{O}^1)$ for some order $\mathcal{O}$ in A, [**23**, **1**]. Furthermore for arithmetic groups Γ_1,Γ_2, then, up to complex conjugation, Γ_1 is commensurable with a conjugate of Γ_2 in $PSL(2,\mathbf{C})$ if and only if the related quaternion algebras A_1,A_2 are isomorphic [**18**].

With any finite covolume Kleinian group Γ let $\bar{\Gamma}=P^{-1}(\Gamma)$ where $P:SL(2,\mathbf{C})\to PSL(2,\mathbf{C})$. Define the field $k\Gamma$ by $k\Gamma=\mathbf{Q}(\tau\gamma \mid \gamma\in\bar{\Gamma})$ and the quaternion algebra $A\Gamma$ over $k\Gamma$ by

$$A\Gamma=\{\textstyle\sum a_i\gamma_i \mid a_i\in k\Gamma, \gamma_i\in\bar{\Gamma}\}$$

The field $k\Gamma^{(2)}$ and the quaternion algebra $A\Gamma^{(2)}$ are invariants of the commensurability class of the group Γ [**19**].

In the cases where Γ is generated by two elements $x=PX$, $y=PY$ with, as before, $\alpha=\tau X$, $\beta=\tau Y$, $\gamma=\tau XY$ then

$$k\Gamma^{(2)}=\mathbf{Q}(\alpha^2,\beta^2,\gamma^2,\alpha\beta\gamma).$$

Furthermore, provided x,y are not of order 2 then

$$A\Gamma^{(2)}\cong\left(\frac{\alpha^2(\alpha^2-4),\alpha^2\beta^2(\tau[X,Y]-2)}{k\Gamma^{(2)}}\right) \tag{8}$$

(e.g. [**23**, **18**]).

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Among finite covolume Kleinian groups, one can identify those which are arithmetic [**18**]:

Theorem 4.1. *Let Γ be a finite covolume Kleinian group. Then Γ is arithmetic if and only if*

(i) *$k\Gamma^{(2)}$ has one complex place and*

(ii) *$\tau(\Gamma^{(2)})$ consists of algebraic integers, and*

(iii) *the quaternion algebra $A\Gamma^{(2)}$ is ramified at all real places.*

In the arithmetic cases, $k\Gamma^{(2)}$ and $A\Gamma^{(2)}$ coincide with the defining field and quaternion algebra.

Consider again the groups defined by (1) where $(k,l;m)=(3,3;3),(4,4;2),(3,4;2)$.

A. $(k,l;m)=(3,3;3)$. Note that in this case $\Gamma^{(2)}=\Gamma$. From equation (2) we have that $\alpha=\beta=1$ and $\gamma=\exp(\pm\pi i/3)$. Thus the defining field is $\mathbf{Q}(\sqrt{-3})$, (ii) of Theorem 4.1 is satisfied and (iii) is vacuously true. By (8)

$$A\Gamma\cong\left(\frac{-3,-3}{\mathbf{Q}(\sqrt{-3})}\right)\cong\left(\frac{1,1}{\mathbf{Q}(\sqrt{-3})}\right)\cong M_2(\mathbf{Q}(\sqrt{-3})).$$

Thus Γ is commensurable with the Bianchi group $PSL(2,O_3)$ and so is non-cocompact. Furthermore since there is only one conjugacy class of maximal orders in $M_2(\mathbf{Q}(\sqrt{-3}))$, Γ is conjugate to a subgroup of $PSL(2,O_3)$.

Note that in this case, the trace equation (2) admits a solution with $\tau[A,B]=+1$. This forces the element $P(AB)$ to be of order 5 so that this representation is not faithful.

B. $(k,l;m)=(4,4;2)$. Now $\alpha=\beta=\sqrt{2}$ and $\gamma=1\pm i$. Thus $k\Gamma^{(2)}=\mathbf{Q}(i)$ and

$$A\Gamma^{(2)}\cong\left(\frac{-4,-8}{\mathbf{Q}(i)}\right)\cong M_2(\mathbf{Q}(i)).$$

Thus Γ is arithmetic and commensurable with $PSL(2,O_1)$.

C. $(k,l;m)=(3,4;2)$. In this case $\alpha=1$, $\beta=\sqrt{2}$ and $\gamma=(1\pm i)/\sqrt{2}$. Thus again $k\Gamma^{(2)}=\mathbf{Q}(i)$ and $A\Gamma^{(2)}\cong M_2(\mathbf{Q}(i))$.

Theorem 4.2. *The groups $\Gamma(k,l;m)$ defined at* (1) *with $(k,l;m)=(3,3;3),(4,4;2),(3,4;2)$ are arithmetic. $\Gamma(3,3;3)$ is conjugate to a subgroup of finite index in $PSL(2,O_3)$. The groups $\Gamma(4,4;2)$ and $\Gamma(3,4;2)$ are commensurable* (*up to conjugation*) *with $PSL(2,O_1)$ and hence with each other.*

We now investigate these group more closely using the construction in Theorem 2.1.

A. $(k,l;m)=(3,3;3)$. Under suitable normalisation we obtain

$$X=\begin{pmatrix}\exp(\pi i/3) & 0\\ 0 & \exp(-\pi i/3)\end{pmatrix}\quad Y=\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}.$$

The four triangle subgroups described in Theorem 2.1 have fixed points at $0, \infty, \exp(2\pi i/3)$, $\exp(-2\pi i/3)$. The fundamental region is then a tetrahedron with all four vertices on $\partial\mathbf{H}^3$ and dihedral angles $2\pi/3, \pi/6, \pi/6$. Thus if Li denotes the Lobachevski function, then this tetrahedron has volume

$$Li(2\pi/3)+2Li(\pi/6)=2Li(\pi/3)\approx 0.676628$$

(see e.g. [**24**]). The figure 8 knot complement has volume $6Li(\pi/3)$ and its fundamental group is of index 12 in $PSL(2, O_3)$. Note that Γ, as defined above, already has its entries in O_3 and so we have $[PSL(2,O_3):\Gamma]=4$. Now $PSL(2,O_3)=\langle a,t,l\rangle$ where

$$a=P\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad t=P\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad l=P\begin{pmatrix} \exp(\pi i/3) & 0 \\ 0 & \exp(-\pi i/3) \end{pmatrix}$$

using the notation and results in [**6**]. The subgroup Γ turns out to be the subgroup generated by l and ta.

B. $(k,l;m)=(4,4;2)$. In this case we can take

$$X=\begin{pmatrix} \exp(\pi i/4) & 0 \\ 0 & \exp(-\pi i/4) \end{pmatrix} \quad Y=\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ -1/\sqrt{2} & 0 \end{pmatrix}.$$

The vertices of the tetrahedron here are $0, \infty, -1\pm i$. The dihedral angles are thus $\pi/2$, $\pi/4, \pi/4$ and hence its volume is approximately 0.9159654. This is one of the Coxeter tetrahedra with vertices on $\partial\mathbf{H}^3$ such that the group generated by reflections in its faces is discrete. One readily calculates the generalised index to be (see e.g. [**21**])

$$[PSL(2,O_1):\Gamma(4,4;2)]=3.$$

In this case $[\Gamma:\Gamma^{(2)}]=4$ and one easily checks that $\Gamma^{(2)}\subset PSL(2,O_1)$ so that it is a subgroup of index 12.

C. $(k,l;m)=(3,4;2)$. Here

$$X=\begin{pmatrix} \exp(\pi i/3) & 0 \\ 0 & \exp(-\pi i/3) \end{pmatrix} \quad Y=\begin{pmatrix} (\sqrt{3}+1)/\sqrt{6} & \sqrt{2}/\sqrt{3} \\ -\sqrt{2}/\sqrt{3} & (\sqrt{3}-1)/\sqrt{6} \end{pmatrix}.$$

Note that in this case the fixed points of two of the triangle subgroups are in $\mathbf{H}^3$ while the other two are on $\partial\mathbf{H}^3$. These vertices are $\exp(2\pi i/3)$, $\exp(-2\pi i/3)$, $\sqrt{2}j/(\sqrt{3}+1)$, $\sqrt{2}j/(\sqrt{3}-1)$. The dihedral angles at the last vertex are $2\pi/3, \pi/4, \pi/4$ and the opposite dihedral angles are $\pi/2, \pi/4, \pi/4$. This tetrahedron admits a symmetry of order 2 in the plane through the two finite vertices and the mid-point of the opposite edge so that it is

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a union of two copies of the Coxeter tetrahedron $T(3,2,2;2,4,4)$ (e.g. [21]) and so its volume is approximately 0.3053218. The generalised index in this case is 1. We have $[\Gamma:\Gamma^{(2)}]=2$, so that some conjugate of $\Gamma^{(2)}$ is a subgroup of index 2 in $PSL(2,O_1)$. Now $PSL(2,O_1)$ is generated by the elements a,l,t,u where

$$a=P\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad l=P\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad t=P\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad u=P\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}.$$

See [6]. Now the subgroup $\langle l, ta, ua\rangle$ is isomorphic to $\Gamma^{(2)}$ and so $\Gamma^{(2)}$ will be conjugate to this subgroup.

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