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ON BLOCK-SCHEMATIC STEINER SYSTEMS $S(t, t+1, v)$

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1. Introduction. A Steiner system $S(t, k, v)$ is a collection of k-subsets, called blocks, of a v-set of points with the property that any t-subset of points is contained in a unique block. We assume $1 < t < k < v$. A Steiner system is called *block-schematic* if the blocks form an association scheme with the relations determined by size of intersection. Ito and Patton [**3**] proved that if $S(4, 5, v)$ is block-schematic, then $v = 11$. The purpose of this paper is to extend this result, and we prove the following theorem.

THEOREM. *A Steiner system $S(t, t+1, v)$ is block-schematic if and only if one of the following holds:* (i) $t = 2$, (ii) $t = 3, v = 8$, (iii) $t = 4, v = 11$, (iv) $t = 5, v = 12$.

It is well known that $S(3, 4, 8)$, $S(4, 5, 11)$ and $S(5, 6, 12)$ are unique (cf. [**6**]), and $S(2, k, v)$, $S(3, 4, 8)$ and $S(4, 5, 11)$ are block-schematic (cf. [**1**], [**2**]). Now, since the automorphism groups of $S(5, 6, 12)$, $S(4, 5, 11)$, and $S(3, 4, 8)$ are the Mathieu group M_{12}, the Mathieu group M_{11}, and the three transitive group of order 1344 respectively, it is not difficult to check that they have the following property: If B_1, B_2, B_3 and B_4 are blocks with $|B_1 \cap B_2| = |B_3 \cap B_4|$, then there exists a automorphism σ such that $\sigma(B_1) = B_3$ and $\sigma(B_2) = B_4$. Hence, $S(5, 6, 12)$ is block-schematic, also. Thus, in order to prove the theorem, it is sufficient to show that if $S(t, t+1, v)$ is block-schematic ($t \geqq 3$), then $t = 3, v = 8$, or $t = 4, v = 11$, or $t = 5, v = 12$.

2. Notation and preliminaries. For a Steiner system $S = S(t, k, v)$ we use λ_i $(0 \leqq i \leqq t)$ to represent the number of blocks which contain the given i points of S. Then we have

$$\lambda_i = \frac{(v-i)(v-i-1)\dots(v-t+1)}{(k-i)(k-i-1)\dots(k-t+1)} \quad (0 \leqq i \leqq t).$$

For a block B of S we use x_i $(0 \leqq i \leqq k)$ to denote the number of blocks each of which has exactly i points in common with B. By a theorem of [**4**], the number x_i depends on S, but not on the choice of a block B, and the

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following equality holds for $i = 0, 1, \ldots, t-1$:

$$x_i + \binom{i+1}{i} x_{i+1} + \ldots + \binom{t-1}{i} x_{t-1} = (\lambda_i - 1)\binom{k}{i}.$$

We remark that $x_t = \ldots = x_{k-1} = 0$ and $x_k = 1$.

Let $B_1, \ldots, B_{\lambda_0}$ be the blocks of S. Let A_h $(0 \leqq h \leqq k)$ be the h-adjacency matrix of S of degree λ_0 defined by

$$A_h(i,j) = \begin{cases} 1 & \text{if } |B_i \cap B_j| = h, \\ 0 & \text{otherwise.} \end{cases}$$

We remark that $A_t = \ldots = A_{k-1} = 0$ (the zero matrix) and $A_k = I$ (the identity matrix). If S is block-schematic, then

$$A_i A_j = \sum_{h=0}^{k} \mu(i,j,h) A_h \quad (0 \leqq i, j \leqq k)$$

where $\mu(i, j, h)$ is a non-negative integer defined by the following: When there exist blocks B_p and B_q with $|B_p \cap B_q| = h$,

$$\mu(i,j,h) = |\{B_r \,|\, |B_p \cap B_r| = i, \quad |B_q \cap B_r| = j, 1 \leqq r \leqq \lambda_0\}|,$$

and when there exist no blocks B_p and B_q with $|B_p \cap B_q| = h$, $\mu(i, j, h) = 0$. Now, the following equalities are easily verified:

$$\mu(i,j,h) = \mu(j,i,h), \quad \mu(i,j,k) = \delta_{ij} x_i$$
$$\mu(i,j,h) x_h = \mu(h,j,i) x_i = \mu(h,i,j) x_j.$$

3. Proof of the theorem. Let S be a Steiner system $S(t, t+1, v)$ with $t \geqq 3$.

LEMMA 1. *$v - t$ is not divisible by any prime p with $p \leqq t+1$.*

Proof. Let p be any prime with $p \leqq t+1$. Now,

$$\lambda_{t+1-p} = \frac{(v-t-1+p)\ldots(v-t+1)}{p\ldots 2}.$$

If $v - t$ is divisible by p, then λ_{t+1-p} is not an integer, a contradiction.

LEMMA 2.

$$x_{t-1} = (v-t-1)(t+1)t/4,$$
$$x_{t-2} = (v-t-1)(t+1)t(t-1)(v-t-5)/36,$$
$$x_{t-3} = (v-t-1)(t+1)t(t-1)(t-2) \times \{v^2 - (2t+9)v + t^2 + 9t + 26\}/576.$$

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Proof. By a theorem of [**4**], we have the following:

$$x_{t-1} = (\lambda_{t-1} - 1)\binom{t+1}{t-1},$$

$$x_{t-2} + \binom{t-1}{t-2}x_{t-1} = (\lambda_{t-2} - 1)\binom{t+1}{t-2},$$

$$x_{t-3} + \binom{t-2}{t-3}x_{t-2} + \binom{t-1}{t-3}x_{t-1} = (\lambda_{t-3} - 1)\binom{t+1}{t-3}.$$

By the above three equalities, we obtain Lemma 2.

From now on, let us suppose that S is a block-schematic Steiner system $S(t, t+1, v)$ with $t \geqq 3$.

LEMMA 3.

$$x_{t-1}^2 = \mu_{t-3}x_{t-3} + \mu_{t-2}x_{t-2} + \mu_{t-1}x_{t-1} + x_{t-1},$$

where

$$\mu_j = \mu(t-1, t-1, j) \ (j = t-3, t-2, t-1).$$

Proof. Since S is block-schematic, we have

$$A_{t-1}^2 = \sum_{h=0}^{t+1} \mu(t-1, t-1, h)A_h.$$

Let $\mathscr{A}$ be the all -1 column vector of degree λ_0. Then,

$$A_{t-1}^2\mathscr{A} = \sum_{h=0}^{t+1} \mu(t-1, t-1, h)A_h\mathscr{A}.$$

Therefore,

$$x_{t-1}^2 = \sum_{h=0}^{t+1} \mu(t-1, t-1, h)x_h.$$

Since $(t-4) + 3 + 3 > t + 1$, we have $\mu(t-1, t-1, h) = 0$ for $h \leqq t - 4$.

From now on, let us assume $\mu_j = \mu(t-1, t-1, j)$ $(j = t-3, t-2, t-1)$.

LEMMA 4. $1 \leqq \mu_{t-3} \leqq 12$.

Proof. First we show that $\mu_{t-3} \leqq 12$. By Lemma 2, we have $x_{t-3} > 0$. Let B_1 and B_2 be blocks with $|B_1 \cap B_2| = t - 3$. If B is a block with $|B_1 \cap B| = t - 1$ and $|B_2 \cap B| = t - 1$, then we have $B \supset B_1 \cap B_2$. And, if B' is a block $(\neq B)$ with $B_1 \cap B' = B_1 \cap B$ and $|B_2 \cap B'| = t - 1$, then we have

$$B' \supset B_1 \cap B_2 \quad \text{and} \quad (B_2 \cap B') \cap (B_2 \cap B) = B_1 \cap B_2.$$

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So,

$$|\{B|\ B \text{ a block}, |B_1 \cap B| = t-1, |B_2 \cap B| = t-1\}| \leqq \binom{4}{2} \times 2 = 12.$$

Next, we show that $1 \leqq \mu_{t-3}$. Let $\alpha_1, \ldots, \alpha_{t-1}$ be $t-1$ points of S, and $B_1, \ldots, B_{\lambda_{t-1}}$ be λ_{t-1} blocks with

$$B_i \supset \{\alpha_1, \ldots, \alpha_{t-1}\}\ (i = 1, \ldots, \lambda_{t-1}).$$

Set

$$B_1 - \{\alpha_1, \ldots, \alpha_{t-1}\} = \{\alpha_t, \alpha_{t+1}\}.$$

Let α_{t+2} be a point with $B_1 \not\ni \alpha_{t+2}$, and B_0 be the block which contains $\{\alpha_1, \ldots, \alpha_{t-3}, \alpha_t, \alpha_{t+1}, \alpha_{t+2}\}$. If $B_0 \cap B_i = \{\alpha_1, \ldots, \alpha_{t-3}\}$ for some i $(2 \leqq i \leqq \lambda_{t-1})$, then we have

$$|B_0 \cap B_i| = t-3, |B_0 \cap B_1| = t-1 \text{ and } |B_i \cap B_1| = t-1.$$

Hence, $\mu_{t-3} \geqq 1$. Let us suppose that $B_0 \cap B_i \supsetneqq \{\alpha_1, \ldots, \alpha_{t-3}\}$ for any i $(1 \leqq i \leqq \lambda_{t-1})$. Then we have $\lambda_{t-1} \leqq 3$. Since S is a nontrivial design, we have $v \geqq 2t+2$. So,

$$((2t+2) - t + 1)/2 \leqq (v - t + 1)/2 = \lambda_{t-1} \leqq 3.$$

Hence, we have $t = 3$ and $v = 8$. On the other hand, $S(3, 4, 8)$ is a block-schematic Steiner system with $\mu(2, 2, 0) = 12$.

LEMMA 5. $9 \leqq \mu_{t-2} \leqq 18$ *holds except in the case where* $t = 3$ *and* $v = 8$. *Moreover*, $\mu_2 \leqq 15$ *holds for* $t = 4$, *and* $\mu_1 \leqq 12$ *holds for* $t = 3$. *If* S *is* $S(3, 4, 8)$, *then* $\mu_1 = 0$.

Proof. If $x_{t-2} = 0$, then by Lemma 2 we have $t = 3$ and $v = 8$. Hereafter, we assume $x_{t-2} > 0$. Let B_1 and B_2 be blocks with $|B_1 \cap B_2| = t-2$. Let α_1 and α_2 be any points of $B_1 - B_2$ and $B_2 - B_1$ respectively. There exists a unique block B_0 with

$$B_0 \supset \{\alpha_1, \alpha_2\} \cup (B_1 \cap B_2).$$

Here, $B_0 \cap (B_1 - B_2) = \{\alpha_1\}$ and $B_0 \cap (B_2 - B_1) = \{\alpha_2\}$. Hence,

$$|\{B|\ B \text{ a block}, B \supset B_1 \cap B_2, |B \cap B_1| = |B \cap B_2| = t-1\}| = 3 \times 3 = 9.$$

If B' is a block such that $B' \not\supset B_1 \cap B_2$ and $|B' \cap B_1| = |B' \cap B_2| = t-1$, then we have $|B' \cap B_1 \cap B_2| = t-3$. Therefore,

$$|\{B \mid B \text{ a block}, B \not\supset B_1 \cap B_2, |B \cap B_1| = |B \cap B_2| = t-1\}| \leqq \binom{3}{2} \times \binom{3}{2} = 9.$$

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Moreover, if $t = 3$ or 4, then we see that

$$|\{B| B \text{ a block}, |B \cap B_1 \cap B_2| = t - 3, \\ |B \cap (B_1 - B_2)| = |B \cap (B_2 - B_1)| = 2\}|$$

is at most 3 for $t = 3$, 6 for $t = 4$.
Thus, we complete the proof of Lemma 5.

LEMMA 6.

$$\frac{v - t - 3}{2} + 4(t - 1) \leqq \mu_{t-1} \leqq \frac{v - t - 3}{2} + 4(t - 1) + \left[\frac{t - 1}{2}\right].$$

Proof. By Lemma 2, we have $x_{t-1} > 0$. Let B_1 and B_2 be blocks with $|B_1 \cap B_2| = t - 1$. If B is a block with $|B \cap B_1| = |B \cap B_2| = t - 1$, then one of the following three cases holds: (I) $B \supset B_1 \cap B_2$. (II) $|B \cap B_1 \cap B_2| = t - 2$, $|B \cap (B_1 - B_2)| = |B \cap (B_2 - B_3)| = 1$. (III) $|B \cap B_1 \cap B_2| = t - 3$, $B_0 \supset B_1 - B_2$, $B_0 \supset B_2 - B_1$. There exists just $(v - t - 3)/2$ blocks B satisfying (I), and there exist just $4(t-1)$ blocks B satisfying (II), and there exist at most $[(t - 1)/2]$ blocks B satisfying (III).

LEMMA 7. $3 \leqq t \leqq 43$.

Proof. By Lemmas 2, 3, 4 and 5, we have

$$x_{t-1}^2 > x_{t-3} + 9x_{t-2}.$$

Hence,

$$36(v - t - 1)^2(t + 1)^2t^2 > (v - t - 1)(t + 1)t(t - 1) \\ \times \{(t - 2)(v - t - 1)(v - t - 8) + 144(v - t - 5)\}.$$

Since $v \geqq 2t + 2$, we have

$$\frac{36(v - t - 1)(t + 1)t}{(t - 1)(t - 3)} > (v - t - 1)(v - t - 8) + 144.$$

Let us suppose $t \geqq 44$. Then we have $36(t + 1)t/(t - 1)(t - 3) < 41$, and so,

$$41(v - t - 1) > (v - t - 1)\{(v - t - 1) - 7\} + 144.$$

Hence,

$$v - t - 1 < 24 + \sqrt{432} < 45.$$

Since $v \geqq 2t + 2$, we get $t \leqq 43$, a contradiction.

LEMMA 8. $v \geqq 2t + 2$ *and* $36(t + 1)t > (t - 1)(t - 2)(v - t - 8)$.

Proof. Since S is a nontrivial design, we have $v \geqq 2t + 2$. On the other

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hand, by the proof of Lemma 7, we have

$$36(v-t-1)^2(t+1)^2t^2 > (v-t-1)(t+1)t(t-1)(t-2) \times (v-t-1)(v-t-8).$$

Hence,

$$36(t+1)t > (t-1)(t-2)(v-t-8).$$

For a Steiner system $S(t, k, v)$, generally, the number of blocks containing a point α and meeting a block B in j points $(0 \leqq j \leqq t-1)$ is jx_j/k if $\alpha \in B$, $(k-j)x_j/(v-k)$ if $\alpha \notin B$. Hence, if $\mathscr{A}$ denotes the all -1 vector of degree λ_0, and if $\mathscr{A}_\alpha$ denotes the vector with ith component 1 if $\alpha \in B_i$, 0 otherwise $(1 \leqq i \leqq \lambda_0)$, we have

$$A_j\mathscr{A}_\alpha = (jx_j/k)\mathscr{A}_\alpha + ((k-j)x_j/(v-k))(\mathscr{A} - \mathscr{A}_\alpha).$$

So, if α and β are distinct points, then

$$A_j(\mathscr{A}_\alpha - \mathscr{A}_\beta) = (j/k - (k-j)/(v-k))x_j(\mathscr{A}_\alpha - \mathscr{A}_\beta).$$

Thus for S we find

LEMMA 9. *A_j has an eigenvalue d_j $(0 \leqq j \leqq t-1)$ belonging to the eigenvector $\mathscr{A}_\alpha - \mathscr{A}_\beta$, where*

$$d_j = \left\{1 - \frac{(t+1-j)v}{(t+1)(v-t-1)}\right\}x_j.$$

LEMMA 10. $d_{t-1}^2 = \mu_{t-3}d_{t-3} + \mu_{t-2}d_{t-2} + \mu_{t-1}d_{t-1} + x_{t-1}$.

Proof. By the proof of Lemma 3, we have

$$A_{t-1}^2 = \mu_{t-3}A_{t-3} + \mu_{t-2}A_{t-2} + \mu_{t-1}A_{t-1} + x_{t-1}I.$$

Then,

$$A_{t-1}^2(\mathscr{A}_\alpha - \mathscr{A}_\beta) = (\mu_{t-3}A_{t-3} + \mu_{t-2}A_{t-2} + \mu_{t-1}A_{t-1} + x_{t-1}I) \times (\mathscr{A}_\alpha - \mathscr{A}_\beta).$$

Hence, we get Lemma 10.

By Lemmas 1–10, we get the following by computer calculations: S satisfies one of the following seven cases.

	t	v	x_{t-1}	x_{t-2}	x_{t-3}	μ_{t-1}	μ_{t-2}	μ_{t-3}	d_{t-1}	d_{t-2}	d_{t-3}
(1)	3	8	12	0	1	10	0	12	0	0	−1
(2)	3	10	18	8	3	11	9	12	3	−2	−2
(3)	3	14	30	40	20	13	9	6	9	−2	−8
(4)	4	11	30	20	15	15	15	8	8	−2	−7
(5)	4	15	50	100	100	17	11	5	20	10	−20
(6)	5	12	45	40	45	20	18	8	15	0	−15
(7)	5	16	75	200	300	22	12	5	35	40	−20

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The non-existence of Steiner system $S(4, 5, 15)$ has been proved by Mendelsohn and Hung [**5**] without any condition. So, the cases (5) and (7) do not hold. By [**5**], the number of isomorphism classes of Steiner systems $S(3, 4, 14)$ is four. Furthermore, the tables of the four classes are given in [**5**]. If S satisfies the case (3), then $\mu(2, 2, 0) = 6$. But, seeing the tables in [**5**], we get a contradiction. A similar contradiction is obtained for the well-known unique $S(3, 4, 10)$. Hence, S satisfies the case (1), (4) or (6).

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