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Transformations and Colorings of Groups

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Abstract. Let G be a compact topological group and let $f\colon G \to G$ be a continuous transformation of G. Define $f^*\colon G \to G$ by $f^*(x) = f(x^{-1})x$ and let $\mu = \mu_G$ be Haar measure on G. Assume that $H = \operatorname{Im} f^*$ is a subgroup of G and for every measurable $C \subseteq H$, $\mu_G((f^*)^{-1}(C)) = \mu_H(C)$. Then for every measurable $C \subseteq G$, there exist $S \subseteq C$ and $g \in G$ such that $f(Sg^{-1}) \subseteq Cg^{-1}$ and $\mu(S) \geq (\mu(C))^2$.

A subset S of a group G is called *symmetric* if there exists $g \in G$ such that $gS^{-1}g = S$. This notion was introduced in [6] and turned out to be fruitful enough, especially from the point of view of Ramsey theory (see [1, 2, 5]).

Let G be a compact topological group, let μ be Haar measure on G, and let $r \in \mathbb{N}$. Denote by $s_r(G)$ the least upper bound of real $\varepsilon > 0$ such that for every measurable r-coloring of G, there exists a monochrome symmetric subset $S \subseteq G$ with $\mu(S) \geq \varepsilon$. In [2] it was proved that if G is Abelian, then $s_r(G) \geq 1/r^2$. (For finite Abelian groups this inequality was proved earlier in [4].) Actually, it has been shown that for every measurable $C \subseteq G$, there exists a measurable symmetric $S \subseteq C$ with $\mu(S) \geq (\mu(C))^2$. The estimate $s_r(G) \geq 1/r^2$ is optimal. For example, for the circle group $\mathbb{T}$, $s_r(\mathbb{T}) = 1/r^2$. In the non-Abelian case the estimate fails: for the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, we have $s_2(Q) = 1/8 = 1/(2 \cdot 2^2)$. But this was the only known counter-example.

The aim of this note is to prove the following three theorems.

Given a group G and $f\colon G \to G$, define the dual mapping $f^*\colon G \to G$ by $f^*(x) = f(x^{-1})x$. Notice that $(f^*)^* = f$ and that if G is Abelian and f is an endomorphism, then f^* is also an endomorphism.

Theorem 1 *Let G be a compact topological group and let $f\colon G \to G$ be a continuous transformation of G such that $H = \operatorname{Im}\ f^*$ is a subgroup of G and for every measurable $C \subseteq H$, $\mu_G((f^*)^{-1}(C)) = \mu_H(C)$. Then for every measurable $C \subseteq G$, there exist $S \subseteq C$ and $g \in G$ such that $f(Sg^{-1}) \subseteq Cg^{-1}$ and $\mu(S) \geq (\mu(C))^2$.*

The class of continuous transformations $f\colon G \to G$ satisfying the required condition in Theorem 1 is big enough. In the finite case, it consists of mappings dual to the mappings of the form $h\colon G \to G$ where $H = \operatorname{Im} h$ is a subgroup of G and $|h^{-1}(x)| = |G{:}H|$ for all $x \in H$. In the Abelian case, it contains all continuous endomorphisms of G, in particular, the inversion. In the last case, Theorem 1

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gives the result from [2] cited above. Indeed, $g(S \cup gS^{-1}g)^{-1}g = gS^{-1}g \cup S$ and $gS^{-1}g = (Sg^{-1})^{-1}g \subseteq C$.

Notice also that the inclusion $f(Sg^{-1}) \subseteq Cg^{-1}$ is equivalent to $f(Sg^{-1})g \subseteq C$, and if $f\colon G \to G$ is a homomorphism, $f(Sg^{-1})g = f(S)f^*(g)$.

The second theorem gives a general enough construction of compact topological groups G with $s_r(G) < 1/r^2$.

Theorem 2 *Let A be a compact topological Abelian group, let f be the inversion of A, and let $G = A \rtimes \mathbb{C}_4$ be the semidirect product with respect to the homomorphism $\mathbb{C}_4 \ni j \mapsto f^j \in \operatorname{Aut}(A)$. Then for every $r \geq 2$, $1/2r^2 \leq s_r(G) \leq 1/2s_r(A)$. In particular, if $s_r(A) = 1/r^2$, then $s_r(G) = 1/2r^2$.*

We do not know whether there is a compact topological group G with $s_r(G) < 1/2r^2$ for some r.

The third theorem is concerned with arbitrary infinite Abelian groups and their endomorphisms.

Theorem 3 *Under the generalized continuum hypothesis, for every infinite Abelian group G, an endomorphism $f\colon G \to G$ and a finite coloring of G, there exist $S \subseteq G$ of arbitrarily large cardinality $< |G|$ and $g \in G$ such that $S \cup (f(S) + f^*(g))$ is monochrome.*

The proof of Theorem 1 is based on the following lemma.

Lemma 4 *Let G be a compact topological group and let $f\colon G \to G$ be a measurable transformation of G. Then for every measurable $C \subseteq G$ there exist $S \subseteq C$ and $g \in G$ such that $f(Sg^{-1}) \subseteq Cg^{-1}$ and*

$$\mu(S) \geq \int_G \chi_C(x) \int_G \chi_C(f^*(y)x)\, dy dx,$$

where $\chi_C(x)$ is the characteristic function of $C \subseteq G$.

Proof For every $y \in G$, denote $S(y) = C \cap f^{-1}(Cy^{-1})y$. Then

$$S(y) \subseteq C, \quad f(S(y)y^{-1}) \subseteq Cy^{-1}, \quad \mu(S(y)) = \int_G \chi_{S(y)}(x)\, dx.$$

It is easy to check that

$$\chi_{C\cap D}(x) = \chi_C(x)\chi_D(x), \quad \chi_{Cy}(x) = \chi_C(xy^{-1}), \quad \chi_{h^{-1}(C)}(x) = \chi_C(h(x)).$$

Consequently, $\chi_{S(y)}(x) = \chi_C(x)\chi_C(f(xy^{-1})y)$ and

$$\mu(S(y)) = \int_G \chi_C(x)\chi_C(f(xy^{-1})y)\, dx.$$

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Integrating this equation, we obtain

$$\begin{aligned}\int_G \mu(S(y))\,dy &= \int_G\int_G \chi_C(x)\chi_C(f(xy^{-1})y)\,dxdy\\ &= \int_G \chi_C(x)\int_G \chi_C(f(xy^{-1})y)\,dydx\\ &= \int_G \chi_C(x)\int_G \chi_C(f(y^{-1})yx)\,dydx\\ &= \int_G \chi_C(x)\int_G \chi_C(f^*(y)x)\,dydx.\end{aligned}$$

By the theorem of the mean, there exists $g \in G$ such that

$$\mu(S(g)) \geq \int_G \chi_C(x)\int_G \chi_C(f^*(y)x)\,dydx.$$

Put $S = S(g)$. ■

Proof of Theorem 1 By Lemma 4, it suffices to prove that

$$\int_G \chi_C(x)\int_G \chi_C(x+y-f(y))\,dydx \geq (\mu(C))^2.$$

Denote $H = \operatorname{Im}(1-f)$ and $F = G/H$. Then

$$\begin{aligned}\int_G \chi_C(x)\int_G \chi_C(x+y-f(y))\,dydx &= \int_G \chi_C(x)\int_H \chi_C(x+z)\,dzdx\\ &= \int_F\int_H \chi_C(x+y)\int_H \chi_C(x+y+z)\,dzdyd\dot{x}\\ &= \int_F\int_H \chi_C(x+y)\int_H \chi_C(x+z)\,dzdyd\dot{x}\\ &= \int_F\Big(\int_H \chi_C(x+y)\,dy\Big)^2 d\dot{x}\\ &\geq \Big(\int_F\int_H \chi_C(x+y)dyd\dot{x}\Big)^2\\ &= \Big(\int_G \chi_C(x)\,dx\Big)^2\\ &= (\mu(C))^2.\end{aligned}$$

■

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Proof of Theorem 2 Since G contains an Abelian subgroup $H = A \times \mathbb{C}_2$, the first inequality follows from Theorem 1. To prove the second one, calculate

$$(a, f^i)(a, f^j)^{-1}(a, f^i) = \begin{cases} (2a - x, f^j) & \text{if } i \equiv j \equiv 0 \bmod (2), \\ (2a - x, f^j) & \text{if } i \equiv j \equiv 1 \bmod (2), \\ (x, f^{j+2}) & \text{if } i \equiv 0 \bmod (2) \text{ and } j \equiv 1 \bmod (2), \\ (x, f^{j+2}) & \text{if } i \equiv 1 \bmod (2) \text{ and } j \equiv 0 \bmod (2) \end{cases}$$

$$= \begin{cases} (2a - x, f^j) & \text{if } i - j \equiv 0 \bmod (2), \\ (x, f^{j+2}) & \text{if } i - j \equiv 1 \bmod (2). \end{cases}$$

Given any r-coloring $\varphi\colon A \to \mathbb{Z}_r$, define the extension $\overline{\varphi}\colon G \to \mathbb{Z}_r$ by

$$\overline{\varphi}(x, f^j) = \begin{cases} \varphi(x) & \text{if } j = 0, 1, \\ \varphi(x) + 1 & \text{if } j = 2, 3. \end{cases}$$

Let $S \subseteq G$ be monochrome (with respect to $\overline{\varphi}$) and let $(a, f^i)S^{-1}(a, f^i) = S$. Then either $S \subseteq H$ or $S \subseteq G \backslash H$. If $S \subseteq G \backslash H$, then $S_H = S \cdot (0, f) \subseteq H$ is also monochrome and $(a, 0)S_H^{-1}(a, 0) = S_H$. So we may assume that $S \subseteq H$. Put $P = S \cap A$, $Q = S \cap (H \setminus A)$, $Q_A = Q \cdot (0, f^2)$. Then $P, Q_A \subseteq A$ are monochrome (with respect to φ) and symmetric with respect to $(a, 0)$ and $\mu(S) = \mu(P) + \mu(Q_A) = \frac{1}{4}(\mu_A(P) + \mu_A(Q_A))$. It follows from this that $s_r(G) \leq \frac{1}{2} s_r(A)$. ■

Proof of Theorem 3 Consider the subgroup $H = \operatorname{Im} f^*$ of G. If $|H| < |G|$, then $|\operatorname{Ker} f^*| = |G|$, and since $\operatorname{Ker} f^* = \{x \in G : f(x) = x\}$, there is nothing to prove. So we may assume that $|H| = |G|$. Enumerate H as $\{z_\alpha : \alpha < |G|\}$. Observe that $f(H) \subseteq H$. Indeed, $f(f(-x) + x) = f(-f(x)) + f(x)$. Next, fix any $\kappa < |G|$. By the Erdős–Rado theorem (see [3]), there exists a $(\kappa + 1)$-subsequence $(x_\lambda)_{\lambda < \kappa + 1}$ in $(z_\alpha)_{\alpha < |G|}$ such that $\{f(x_\lambda) + x_\mu : \lambda < \mu < \kappa + 1\}$ is monochrome. Put

$$S = \{f(x_0) + x_\lambda : 0 < \lambda < \kappa\}.$$

Then define $z \in H$ by the simultaneous equations

$$f(f(x_0) + x_\lambda) + z = f(x_\lambda) + x_\kappa, \quad 0 < \lambda < \kappa,$$

which are equivalent to the one equation $f^2(x_0) + z = x_\kappa$, and take any $g \in (f^*)^{-1}(z)$. ■

Notice that in the case $|G| \leq \omega_1$, Theorem 3 holds in ZFC.

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