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ON THE CENTRAL SERIES OF A RING

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The study of group types was completed by Meldrum [1]. The concept of ring type described here is based on analogous definitions.

The series $R=R_0 \supset R_1 \supset \cdots \supset R_\alpha=R_{\alpha+1}$ is the *lower central series* for the ring R if $R_{\gamma+1}=RR_\gamma+R_\gamma R$ for ordinal number γ and $R_\gamma=\bigcap_{\delta<\gamma} R_\delta$ if γ is a limit ordinal. The *upper central series* for R is the series $0=J_0 \subset J_1 \subset \cdots \subset J_\beta=J_{\beta+1}$ where $J_{\gamma+1}=\{x \in R: xR+Rx \subseteq J_\gamma\}$ for every ordinal number γ and $J_\gamma=\bigcup_{\delta<\gamma} J_\delta$ if γ is a limit ordinal. The *length* of the upper central series is the smallest ordinal number β for which $J_\beta=J_{\beta+1}$. The length of the lower central series is defined similarily. We shall say the ring has *type* (β, α) if the length of the upper central series is β and the length of the lower central series is α.

In this paper, ring types are determined for three important classes of rings: nilpotent, power nilpotent, and commutative. Although ring types for non-commutative rings are not completely determined, a partial result is given in Theorem 5.

The following lemma is very useful since it shows how to construct rings of many types from the few examples given in this paper.

LEMMA 1. *If $R=A\oplus B$, A has type (k_1, k_2) and B has type (l_1, l_2), then R has type* $(\max\{k_1, l_1\}, \max\{k_2, l_2\})$.

The proof is based on the fact that the ideals R_γ and J_γ in the lower and upper central series for R are the direct sums of the corresponding terms in the lower and upper central series for A and B.

The following theorem is easy to prove:

THEOREM 1. *If R is a nilpotent ring such that $R^n\neq 0$ while $R^{n+1}=0$, then R has type (n, n).*

COROLLARY. *There are nilpotent rings of type (n, n) for every natural number n.*

The ring R is *power nilpotent* if the last term in its lower central series is 0; R is *weakly nilpotent* if the last term in its upper central series is R.

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The following example shows that for every ordinal number $\beta \geq \omega$ (ω is the first nonfinite ordinal) there are rings of type (β, ω) which are both weakly nilpotent and power nilpotent.

EXAMPLE 1. Let $\beta=\gamma+n$ where γ is a limit ordinal number and n is a nonnegative integer. Let R be the commutative ring generated by the set $S=\{x_\alpha : \alpha$ is an ordinal number, but not a limit ordinal and $\alpha \leq \beta\}$ with the relations:

(1) $x_\alpha^2=0$ for all $x_\alpha \in S$ where $\alpha<\beta$; if β is not a limit ordinal, then $x_\beta^n \neq 0$ while $x_\beta^{n+1}=0$.

(2) Suppose that $x_{\delta_i} \in S$ for all i in $[1, m]$. Then the product $x_{\delta_1} \cdot \cdots \cdot x_{\delta_m}=0$ if α, the smallest of the ordinal numbers $\{\delta_1, \ldots, \delta_m\}$, has the form $\alpha=\eta+k$ where η is zero or a limit ordinal number and k is a natural number less than m.

If $x_{\alpha_{i,j}}$ stands for an arbitrary element in S, if z is a nonzero element in R, and if $z=\sum_{i=1}^{g} L_i(\prod_{j=1}^{h_i} x_{\alpha_{i,j}})$, where $L_i \in Z$, then every factorization of z in R has fewer than $h+1$ factors where $h=\max\{h_1, \ldots, h_g\}$. It follows that $z \notin R^{h+1}$ and hence $z \notin R_\omega$, the ω-th term of the lower central series for R.

Let λ be any ordinal number smaller than β and adopt the convention that $x_\delta=0$ if $\delta=0$ or a limit ordinal. Then it can be shown by induction on λ that J_λ, the λ-th term of R's upper central series, is spanned as an additive group by the set of all monomials $x_{\delta_1} \cdot \cdots \cdot x_{\delta_m}$, $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_m$ where either $\delta_1 \leq \lambda$ or $\delta_1=\lambda+k$, $k \in N$ and $m>k$. It follows from this that x_α first occurs in R's upper central series at J_α for all ordinal numbers $\alpha \leq \beta$. Hence R's upper central series does end at $J_\beta=R$.

The next example shows that there exist power nilpotent rings of type $(n, \gamma+n)$ where n is any nonnegative integer and γ is any limit ordinal number.

EXAMPLE 2. Let R be the ring of all $(\gamma+n)$ by $(\gamma+n)$ matrices with only a finite number of nonzero integer entries and with only zeros on the main diagonal and to the left of the main diagonal. Addition and multiplication in R are the usual matrix addition and multiplication. Let R have the lower central series $R \supset R_1 \supset \cdots \supset R_{\gamma+n}$. Computation shows that R_α is the ring of all matrices in R in which all the entries are zeros on the α diagonals parallel to the main diagonal and just to the right of the main diagonal. Since there are exactly $\gamma+n$ such diagonals, $R_{\gamma+n}=0$, while $R_{\gamma+n-1} \neq 0$. Whenever $m \leq n$, J_m, the m-th term of R's upper central series, is the subring of R consisting of all matrices in R in which all the entries are zeros on the last m diagonals parallel to the main diagonal and to the right of it. The subring J_n of R is the last term of the upper central series since in the matrices in R there are no "last" $n+1$ diagonals to the right of the main diagonal.

THEOREM 2. *A power nilpotent ring must have one of the following ring types: (n, n) where n is a nonnegative integer or $(m, \gamma+n)$ where n is a nonnegative integer, γ is a limit ordinal number, and m is an ordinal number $\geq n$.*

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Proof. Let R be a power nilpotent ring. If R has a finite lower central series, then R is nilpotent and therefore has type (n, n) for some nonnegative integer n. If the lower central series for R is not finite, suppose that R's lower central series has length $\gamma+n$ where n is a nonnegative integer and γ is a limit ordinal number. Then it is easy to see that $R_\gamma \subseteq J_n$, where J_n is the n-th term of R's upper central series and R_γ is the γ-th term of R's lower central series. However, if $R_\gamma \subseteq J_{n-1}$, then $R_{\gamma+n-1}=0$, which is a contradiction. Hence $J_n \neq J_{n-1}$ and R must have type $(m, \gamma+n)$ where m is an ordinal number $\geq n$.

From the fact that a direct sum of power nilpotent rings is a power nilpotent ring and the results given above it follows that there exist power nilpotent rings of each type mentioned in the statement of Theorem 2.

Example 3 will show that there are rings of type $(0, \xi)$ for every ordinal number ξ. We will fix the value of ξ for the rest of this paper. The example is based on certain properties of central series and the ordinal numbers which are established in several lemmas below.

Let A_0 be the class of all ordinal numbers. Let $\lambda_0=1$; define $\lambda_{\alpha+1}=\lambda_\alpha\omega$, and if α is a limit ordinal number, let $\lambda_\alpha=\inf\{\delta \in A_0 : \delta>\lambda_\gamma \text{ for all } \gamma<\alpha\}$. Let A_α be the smallest subclass of A_0 with the properties:

(1) $\lambda_\alpha \in A_\alpha$,

(2) if δ, $\eta \in A_\alpha$, then $\delta+\eta \in A_\alpha$, and

(3) if $B \subset A_\alpha$ and B is a set, then the $\inf\{\delta \in A_0\colon \delta>\eta \text{ for all } \eta \in B\} \in A_\alpha$.

Note that $A_0 \supset A_1 \supset \cdots \supset A_\alpha \supset \cdots$, and that if α is a limit ordinal number, then $A_\alpha = \bigcap_{\gamma<\alpha} A_\gamma$ since $\lambda_\alpha=\inf\{\delta \in A_0\colon \lambda_\gamma<\delta \text{ for all } \gamma<\alpha\}$ is an element in A_γ for all $\gamma<\alpha$.

Lemma 2. *Let Q be any commutative ring with lower central series $Q \supset Q_1 \supset \cdots \supset Q_\beta=Q_{\beta+1}$. Then:*

(1) *$Q_\alpha Q_\gamma \subseteq Q_{\alpha+\gamma}$ for all ordinal pairs (α, γ) for which α is a limit ordinal, and*

(2) *$\bigcap_{n\varepsilon N} (Q_{\lambda_\alpha})^n \subseteq Q_{\lambda_{(\alpha+1)}}$, where N is the set of natural numbers.*

It is easy to prove (1) by transfinite induction and (2) follows easily from (1).

Now let δ be any nonzero ordinal number. Let α_1 be the largest ordinal number such that $\lambda_{\alpha_1}\leq\delta$, and let δ_1 be the largest ordinal number in A_{α_1} such that $\delta_1\leq\delta$. For every natural number $j\geq 2$ let α_j be defined recursively as the largest ordinal number such that $\delta_1+\cdots+\delta_{j-1}+\lambda_{\alpha_j}\leq\delta$, and let δ_j be the largest ordinal number in A_{α_j} such that $\delta_1+\cdots+\delta_j\leq\delta$. Eventually, for some natural number n, $\delta_1+\cdots+\delta_n=\delta$ since $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a strictly decreasing set of ordinal numbers.

The representation $\delta=\delta_1+\cdots+\delta_n$ of an ordinal number given above will be called its *limit form*.

Lemma 3. *The limit form of every nonzero ordinal number is unique.*

Proof. Given an ordinal number δ, the ordinal number δ_1, the first term in δ's limit form, is uniquely determined. The ordinal number δ_j is uniquely determined once the ordinals $\delta_1, \ldots, \delta_{j-1}$ have been determined. Hence the sum, $\delta=\delta_1+\cdots+\delta_n$, is composed of uniquely determined terms.

There exists an ordinal number ρ such that $\lambda_\rho>\xi$. Let G be the set of all ordinal numbers less than λ_ρ. Let R be the ring generated by the set $\{x_\delta:\delta\in G\}$ with the defining relations:

(1) R is commutative.

(2) If δ has the limit form $\delta=\delta_1+\cdots+\delta_m$ where $\delta_m\in A_{\alpha_m}\sim A_{\alpha_m+1}$, then δ_m must be the n-th ordinal in the usual ordering of the ordinals in A_{α_m} for some natural number n. We impose the condition on R that every generator x_δ satisfies the relationship: $(x_\delta)^{n+1}=x_{\delta_1+\cdots+\delta_{m-1}}$, for every $\delta\in G$.

Lemma 4. *Let R have the lower central series $R\supset R_1\supset\cdots\supset R_\beta=R_{\beta+1}$. Let α be an ordinal number. Then $x_\delta\in R_{\lambda_\alpha}$ if $\delta\in G\cap A_\alpha$.*

Proof. If $\alpha=1$, then $\delta\in G\cap A_1$ implies that δ is a limit ordinal number. Since $x_\delta=(x_{\delta+n})^{n+1}$ for every natural number n, it follows that $x_\delta\in R_\omega=R_{\lambda_1}$. Suppose that $x_\delta\in R_\alpha$ for all $\delta\in G\cap A_\alpha$. Also suppose that $\theta\in G\cap A_{\alpha+1}$. Then θ must be the γ-th ordinal number in A_α, where γ is a limit ordinal.

Let μ_n be the $(\gamma+n)$-th ordinal in A_α, namely, $\theta+(\lambda_\alpha)n$. Then $(x_{\mu_n})^{n+1}=x_\theta$ and hence $x_\theta\in(R_{\lambda_\alpha})^{n+1}$ for all natural numbers n. By Lemma 2, $x_\theta\in R_{\lambda_{\alpha+1}}$. If α is a limit ordinal number, then $A_\alpha=\bigcap_{\gamma<\alpha}A_\gamma$. Hence $\delta\in G\cap A_\alpha$ implies that $\delta\in G\cap A_\gamma$ for all $\gamma<\alpha$, and therefore $x_\delta\in\bigcap_{\gamma<\alpha}R_{\lambda_\gamma}=R_{\lambda_\alpha}$. By transfinite induction on α the lemma follows.

Let $y\in R$. Then y can be expressed in the form: $y=\sum_{j=1}^p L_jy_j$ where L_j is a non-zero integer for all j in $[1,p]$ and where the y_j are distinct elements of R of the form $y_j=\prod_{m=1}^{j_m}x_{\delta_{j,m}}$ where $\delta_{j,m}\in G$ for all j in $[1,p]$ and all m in $[1,j_m]$. If $y_h\notin R_\theta$ for at least one h in $[1,p]$, then $y\notin R_\theta$. For $y\in R_\theta$ implies that a sum of distinct terms, each of which lies in $R\sim R_\theta$ and has the form $L_j\prod_{m=1}^{j_m}x_{\delta_{j,m}}$, must equal a sum of distinct terms of the same general form in R_θ. This is impossible since there are no additive relations given in the defining relations for the ring R.

Let $L_1=R$, and let $L_n=R_{n-1}$ for all natural numbers n. Let $L_\alpha=R_\alpha$ for all non-finite ordinal numbers α.

Lemma 5. *Suppose that $L_{(\lambda_\alpha)n}=(L_{\lambda_\alpha})^n$. Then $L_{(\lambda_\alpha)n}L_\theta=L_{(\lambda_\alpha)n+\theta}$ if $\theta\le\lambda_\alpha$. Hence $L_{(\lambda_\alpha)(n+1)}=(L_{\lambda_\alpha})^{n+1}$.*

Proof. If $\theta=1$, $L_{(\lambda_\alpha)n}L_1=L_{(\lambda_\alpha)n}R=L_{(\lambda_\alpha)n+1}$. Suppose that $L_{(\lambda_\alpha)n}L_\eta=L_{(\lambda_\alpha)n+\eta}$ and $\eta<\lambda_\alpha$. Then $L_{(\lambda_\alpha)n}L_{\eta+1}=L_{(\lambda_\alpha)n}L_\eta R=L_{(\lambda_\alpha)n+\eta}R=L_{(\lambda_\alpha)n+\eta+1}$. Let θ be a limit ordinal number $\le\lambda_\alpha$. Suppose that $L_{(\lambda_\alpha)n}L_\eta=L_{(\lambda_\alpha)n+\eta}$ for all ordinal numbers $\eta<\theta$. Then $\bigcap_{\eta<\theta}L_{(\lambda_\alpha)n}L_\eta=\bigcap_{\eta<\theta}L_{(\lambda_\alpha)n+\eta}=L_{(\lambda_\alpha)n+\theta}$ since $(\lambda_\alpha)n+\theta$ is a limit ordinal. If $y\notin(L_{\lambda_\alpha})^{n+1}$, $y\in\bigcap_{\eta<\theta}L_{(\lambda_\alpha)n}L_\eta$, then y expressed in terms of the generators,

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$\{x_\delta:\delta \in G\}$, equals $\sum_{j=1}^{p} L_j y_j$ where the y_j's are distinct, where for some j, $y_j = x_{\delta_1}\cdot\ldots\cdot x_{\phi_n}x_{\phi_1}\cdot\ldots\cdot x_{\phi_s}$ where $x_{\delta_i} \in L_{\lambda_\alpha}$ for all i in $[1, n]$ and $x_{\phi_1}\cdot\ldots\cdot x_{\phi_s} \notin L_{\lambda_\alpha}$. This expression for y_j is unique up to the order of the factors if all simplifications permitted by defining relations (2) are made. Hence $x_{\phi_1}\cdot\cdots\cdot x_{\phi_s} \in \bigcap_{\eta<\theta} L_\eta$, and hence $y_j \in L_{(\lambda_\alpha)n} \bigcap_{\eta<\theta} L_\eta = L_{(\lambda_\alpha)n}L_\theta$. Every y_i is either an element in $(L_{\lambda_\alpha})^{n+1} \supseteq L_{(\lambda_\alpha)n}L_\theta$, or y_i has the same properties as y_j, and therefore $y_i \in L_{(\lambda_\alpha)n}L_\theta$ for all i in $[1, p]$. Hence, $y=\sum_{i=1}^{p} y_i \in L_{(\lambda_\alpha)n}L_\theta$ and $L_{(\lambda_\alpha)n+\theta} = L_{(\lambda_\alpha)n}L_\theta$. By Lemma 2, $L_{(\lambda_\alpha)n}L_\theta \subseteq L_{(\lambda_\alpha)n+\theta}$. Hence by transfinite induction, $L_{(\lambda_\alpha)n}L_\theta = L_{(\lambda_\alpha)n+\theta}$ if θ is an ordinal number $\leq \lambda_\alpha$.

LEMMA 6. $L_{(\lambda_\alpha)n} = [I(\{x_\delta:\delta \in A_\alpha\})]^n$ *for all ordinal numbers* α *and all natural numbers* n, *where* $I(S)$ *is the ideal generated by the subset* S *of* R.

Proof. $L_{(\lambda_1)1} = L_\omega = I(\{x_\delta:\delta \in A_1\})$. Suppose that $L_{(\lambda_\theta)1} = I(\{x_\delta:\delta \in A_\theta\})$ for all ordinal numbers $\theta \leq \alpha$. Suppose also that $L_{(\lambda_\alpha)n} = [I(\{x_\delta:\delta \in A_\alpha\})]^n = [L_{\lambda_\alpha}]^n$. Then by Lemma 5, $L_{(\lambda_\alpha)(n+1)} = [L_{\lambda_\alpha}]^{n+1} = [I(\{x_\delta:\delta \in A_\alpha\})]^{n+1}$. Hence by induction on n, $L_{(\lambda_\alpha)n} = [I(\{x_\delta:\delta \in A_\alpha\})]^n$ for all natural numbers n. Hence $L_{\lambda_{\alpha+1}} = L_{(\lambda_\alpha)\omega} = \bigcap_{n\in N} L_{(\lambda_\alpha)n} = \bigcap_{n\in N} [I(\{x_\delta:\delta \in A_\alpha\})]^n = I(\{x_\delta:\delta \in A_{\alpha+1}\})$. By transfinite induction on α, $L_{(\lambda_\alpha)1} = I(\{x_\delta:\delta \in A_\alpha\})$ for all ordinal numbers α.

Hence R's lower central series does not end until after R_{λ_ρ} and therefore $R_\xi \neq R_{\xi+1}$.

EXAMPLE 3. Let S be the ring generated by the set $\{x_\delta:\delta \in G\}$ with the defining relations (1) and (2) given above for the ring R, and (3) let R_ξ be the ξ-th term in R's lower central series, let n be a natural number and let $\delta_i \in A_{\alpha_i} \sim A_{\alpha_i+1}$ for all i in $[1, n]$ where $\alpha_1 \geq \cdots > \alpha_n$. If $x_{\delta_1}x_{\delta_2}\cdot\ldots\cdot x_{\delta_n} \in R_\xi$ and if $x_{\phi_1}x_{\phi_2}\cdot\ldots\cdot x_{\phi_t} = x_{\delta_n}$ where $\phi_t \in A_0 \sim A_1$, then the relation $x_{\delta_1}\cdot\ldots\cdot x_{\delta_n} = x_{\delta_1}\cdot\ldots\cdot x_{\delta_{n-1}}x_{\phi_1}\cdot\ldots\cdot x_{\phi_{t-1}}(x_{\gamma+4n}x_{\gamma+4n+1} + x_{\gamma+4n+2}x_{\gamma+4n+3})$ holds where $\phi_t = \gamma + n$, and $\gamma = 0$ or $\gamma \in A_1$, the set of limit ordinals.

It is easy to see that elements in the ring R of the form:

$$(x_{\delta_1}\cdot\ldots\cdot x_{\delta_{n-1}}x_{\phi_1}\cdot\ldots\cdot x_{\phi_{t-1}}\cdot x_{\gamma+4n}x_{\gamma+4n+1})$$

lie in $R_{\xi+1}$ in R's lower central series. Hence if S has the lower central series $S \supset S_1 \supset \cdots \supset S_\xi$, then $R_\alpha = S_\alpha$ if $\alpha \leq \xi$. However $S_{\xi+1} = S_\xi$ due to the additive relations in S defined by the relations of type (3). Hence the ring S has type $(0, \xi)$.

The next example shows that there are commutative rings of type $(n, n-1)$ for every natural number n.

EXAMPLE 4. Let $R(k)$ be the commutative ring generated by the set $\{y_n:n \in N$, the set of natural numbers$\}$ with the relations:

(1) $y_1^{k+1} = 0$.

(2) $y_1 y_s = y_1^2$ for all natural numbers n.

(3) if $n_k > n_1, \ldots, n_{k-1}$, then $y_{n_1}y_{n_2}\cdot\cdots\cdot y_{n_k} =$

$$(y_{n_1}y_{n_2}\cdot\cdots\cdot y_{n_{k-1}}y_{4n_k}y_{4n_k+1}+y_{n_1}y_{n_2}\cdot\cdots\cdot y_{n_{k-1}}\cdot y_{4n_k+2}\cdot y_{4n_k+3}).$$

Due to relations of type (3), $R(k)^k=R(k)^{k+1}$. However $y_2y_3\cdot\cdots\cdot y_{k+1}\notin R(k)^{k-1}$ and hence $R(k)$'s lower central series has exactly $k-1$ steps. Let $R(k)$ have the upper central series $0\subset J_1\subset\cdots\subset J_k=J_{k+1}$. This series ends at J_k since

$$y_1^k\in J_1, y_1^{k-1}\in J_2, \ldots, y_1\in J_k.$$

The next theorem shows that $R(k)$'s upper central series cannot have length greater than k since its lower central series has length $k-1$. It follows that $R(k)$ has type $(k, k-1)$.

THEOREM 3. *There are no commutative rings of type (m, n) where n is a natural number and m is any ordinal number $>n+1$.*

Proof. Let R be a commutative ring. If R's lower central series ends after n steps, then $R^{n+1}=R^{n+2}$. Let R have the upper central series $0\subset J_1\subset\cdots\subset J_\alpha= J_{\alpha+1}$. If $x\in J_{n+2}$, then $xR\subseteq J_{n+1}, \ldots, xR^{n+1}\subseteq J_1$ and $xR^{n+2}=0$. Hence $xR^{n+1}=0$ which implies $x\in J_{n+1}$. Hence $J_{n+2}=J_{n+1}$ and R's upper central series has length less than $n+2$.

From the examples and theorems above, the following result can be obtained.

THEOREM 4. *The ring types realized by commutative rings are those of the form (α, β) where $\beta\geq\alpha-1$ if α is a finite ordinal number and where $\beta\geq\omega$ if α is a non-finite ordinal number.*

The next theorem shows that not all ordinal pairs (α, β) are the types of some ring R.

THEOREM 5. *There are no rings of type (m, n) where n is a natural number and m is any ordinal number $>2n+2$.*

Proof. Let R be any ring such that $R^{n+1}=R^{n+2}$. Note that $x\in J_\rho$ iff $\sum_{s=0}^{\rho} R^s x R^{\rho-s}=0$ where R^0 means that no power of R appears on that side. If $x\in J_{2n+3}$, then $\sum_{s=0}^{2n+3} R^s x R^{2n+3-s}=0$. But in each case for $s=0, 1, \ldots, 2n+3$ either $s\geq n+2$ or $2n+3-s\geq n+2$. Hence the equation above may be rewritten $\sum_{s=0}^{2n+2} R^s x R^{2n+2-s}=0$ since $R^{n+2}=R^{n+1}$. From this it follows that $x\in J_{2n+2}$ and hence R's upper central series has length $\leq 2n+2$.

It is still unknown whether there are noncommutative rings of type (m, n) where n is a natural number and m is an ordinal number $>n+1$.

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