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p-adic L-functions and the Rationality of Darmon Cycles

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Abstract. Darmon cycles are a higher weight analogue of Stark–Heegner points. They yield local cohomology classes in the Deligne representation associated with a cuspidal form on $\Gamma_0(N)$ of even weight $k_0 \geq 2$. They are conjectured to be the restriction of global cohomology classes in the Bloch–Kato Selmer group defined over narrow ring class fields attached to a real quadratic field. We show that suitable linear combinations of them obtained by genus characters satisfy these conjectures. We also prove p-adic Gross–Zagier type formulas, relating the derivatives of p-adic L-functions of the weight variable attached to imaginary (resp. real) quadratic fields to Heegner cycles (resp. Darmon cycles). Finally we express the second derivative of the Mazur–Kitagawa p-adic L-function of the weight variable in terms of a global cycle defined over a quadratic extension of $\mathbb{Q}$.

1 Introduction

Let $S_{k_0}(\Gamma_0(N))$ be the space of modular forms on $\Gamma_0(N)$ of even weight $k_0 \geq 2$ and suppose that $N = pM$ is a decomposition into prime factors with p a rational prime not dividing M. Let $K/\mathbb{Q}$ be a real quadratic field such that p is inert and the primes dividing M are split in K. When $k_0 = 2$, Darmon [10] offers a p-adic construction of local points in the Mordel–Weil group $A_f(K_p)$ that are conjectured to be global points and to be subject to a reciprocity law analogous to the one provided by the theory of complex multiplication. Here f is a new modular form of conductor N, and $A_f/\mathbb{Q}$ is the abelian variety attached to it by the Eichler–Shimura construction. The theory as been extended in [11], where the construction has been lifted to the p-new quotient of the Jacobian $J_0(N)$.

This paper focuses instead on the higher weight case $k_0 > 2$. In [19] a p-adic integration theory is offered that is a higher weight counterpart of Darmon's one. Section 2 presents a lift of this p-adic integration theory from the new part to the entire p-new part, in almost the same way as the theory developed in [11] offers a lift of the theory developed in [10] (where the newform is also assumed to have rational coefficients). Indeed, by means of this p-adic integration theory and following the construction of [23, Section 4.2], we are able to construct a monodromy module $\mathbf{D} \in MF_{\mathbb{Q}_p}(\phi, N)$, the category of filtered Frobenius modules over $\mathbb{Q}_p$ that should be thought of as being a realization in the category of filtered Frobenius modules of the p-new part of the motive of weight k_0 modular forms. The existence of this "modular symbol theoretic" p-adic integration theory is essentially encoded in Proposition 2.8,

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which borrows from the techniques of [23]. In [23], a different cohomological approach allows us to develop a p-integration theory that also covers the compact case of a more general Shimura curve: once the existence of both the "modular symbol theoretic" p-adic integration theory and the "cohomological theoretic" p-adic integration theory developed in [23] have been established and the latter is specialized to a modular curve, it can be shown that they are equivalent, and the monodromy modules are isomorphic (see [23, Section 6]). However, the modular symbol approach is suitable for the computations developed in the present paper and seems to be more convenient in providing computational evidences to the conjectures formulated in [23].

Let us a fix a complete field extension $F_p/\mathbb{Q}_p$. Suppose that there exists a prime $q \parallel N$ different from p. We can consider a factorization $N = pN^+N^-$, where N^- is an odd number of primes. By the Jacquet–Langlands correspondence, the Eichler Shimura relations, and the Brauer–Nesbitt principle (see for example [15, Lemma 5.9]), the Deligne representation $V_{[f]}$ attached to the modular form f can be realized inside the p-adic étale cohomology of the Shimura curve $X = X_{N^+,pN^-}$ attached to the indefinite quaternion algebra $\mathcal{B}$ of discriminant pN^- and an Eichler order of level N^+ in $\mathcal{B}$. Set $n := k_0 - 2$ and $m := n/2$. More generally, [15, Lemma 5.9] explains how to construct a Chow motive $\mathcal{M}_n$ over $\mathbb{Q}$ whose p-adic realization $V(m+1) := H_p(\mathcal{M}_{n,\overline{\mathbb{Q}}}, \mathbb{Q}_p(m+1))$ affords representations for all modular forms on $\Gamma_0(N)$ that are new at the primes dividing pN^-. One has a p-adic étale Abel–Jacobi map:

$$cl^{m+1}_{0,F_p}\colon CH^{m+1}(\mathcal{M}_{n,F_p}) \to \mathrm{Ext}^1_{G_{F_p}}\left(\mathbb{Q}_p, V(m+1)\right),$$

where CH^{m+1} is the Chow group of codimension $m+1$ cycles, $\mathcal{M}_{n,F_p}$ denotes the base change to F_p, and we write V again to denote the restriction of the global representation V to the local group G_{F_p}. Let $\mathbb{D} := \mathbb{D}_{st}(V)$ (resp. $\mathbb{D}_{[f]} := \mathbb{D}_{st}(V_{[f]})$) be the associated filtered Frobenius module. Then $\mathbb{D}_{[f]} := \mathbb{D}_{st}(V_{[f]})$ is indeed a $K_f \otimes \mathbb{Q}_p$-monodromy module (see [15, Section 7]). The above ext group is explicitly computed in [15, (49)]:

$$\begin{aligned} IS : \mathrm{Ext}^1_{G_{F_p}}\left(\mathbb{Q}_p, V(m+1)\right) &= \mathrm{Ext}^1_{MF}\left(F, \mathbb{D}(m+1)\right) = \mathbb{D}_{F_p}/F^{m+1} \\ &= M_k(X, F_p)^\vee = M_k(\Gamma', F_p)^\vee. \end{aligned}$$

Here $(-)^\vee$ denotes the F_p-dual space and $M_k(X, F_p)$ is the space of weight k-modular forms on X, while $M_k(\Gamma', F_p)$ denotes the space of weight k modular forms on the Mumford curve $\Gamma' \backslash \mathcal{H}_p$, defined over F_p, and the last equality holds assuming $F_p \supset \mathbb{Q}_{p^2}$. Here $\mathcal{H}_p$ is the p-adic upper halfplane, and Γ' is the arithmetic group defined in subsection 5.3.2: it is associated with the Eichler order of level N^+ in $\mathcal{B}$, and it is a subgroup of the norm one elements in the definite quaternion algebra ramified at the primes ∞N^-. Indeed the last of the above identifications comes from the identification $X^{an} = \Gamma' \backslash \mathcal{H}_p$ over $\mathbb{Q}_{p^2}$ provided by the Cerednik–Drinfeld Theorem. In this way the p-adic étale Abel–Jacobi can be interpreted as

$$\log \Phi^{AJ}\colon CH^{m+1}(\mathcal{M}_{n,F_p}) \to M_k(\Gamma', F_p)^\vee.$$

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We can consider the projection onto the f-isotypic component, thus getting a p-adic Abel–Jacobi map with values in $e_{[f]}M_k(\Gamma', F_p)^\vee$. Here $e_{[f]}$ is the idempotent in the Hecke algebra corresponding to the modular form f.

Let R be the Eichler $\mathbb{Z}[1/p]$-order consisting of matrices in $\mathbb{M}_2(\mathbb{Z}[1/p])$ that are upper triangular $\mod M$, set $\widetilde{\Gamma} := R^\times$ and denote by $\Gamma \subset \widetilde{\Gamma}$ the subgroup of matrices with determinant 1. Write $\mathbf{D}_{F_p}$ to denote the filtered F_p-vector space attached to the base change of $\mathbf{D}$ to F_p. First, our integration theory is a morphism

$$\Phi\colon \left(\Delta^0 \otimes \operatorname{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n\right)_\Gamma \to \mathbf{D}_{F_p}/F^{m+1}.$$

Here $\Delta^0 := \operatorname{Div}^0 \mathbb{P}^1(\mathbb{Q})$, $F_p^0 := F_p \cap \mathbb{Q}_p^{\mathrm{ur}}$, $\operatorname{Div}^0(\mathcal{H}_p^{\mathrm{ur}})$ denotes the degree zero divisor supported on $\mathbb{Q}_p^{\mathrm{ur}} - \mathbb{Q}_p$ that is fixed by the action of the Galois group $G_{\mathbb{Q}_p^{\mathrm{ur}}}/F_p^0$, $\mathbf{P}_n$ is the space of polynomials of degree $\leq n = k_0 - 2$ with coefficients in F_p, and F^{m+1} is the $m+1$-step in the filtration of our monodromy module. In order to be able to construct the right analogue of the notion of Stark–Heegner points, following the ideas of [10], we lift the above morphism to

$$\Phi^{AJ}\colon (\Delta^0 \otimes \operatorname{Div}(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n)_\Gamma \to \mathbf{D}_{F_p}/F^{m+1}.$$

The left-hand side should be regarded as being a substitute for the local Chow group. Indeed the Darmon cycles are defined as being suitable elements

$$j_\Psi \in \left(\Delta^0 \otimes \operatorname{Div}(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n\right)_\Gamma$$

attached to the optimal embeddings $\Psi\colon \mathcal{O} \hookrightarrow R$, where $\mathcal{O}$ is an order of K of conductor prime to ND_K, D_K being the discriminant of $K/\mathbb{Q}$. One of the main differences with the weight 2 setting and with the cohomological approach followed in [23, Section 6] is the lack of uniqueness of the lifting Φ^{AJ}. In any case one can show that the values $\Phi^{AJ}(j_\Psi)$ are well defined quantities, *i.e.,* they do not depend on the choice of the p-adic Abel-Jacobi map Φ^{AJ}. Furthermore, the p-adic Abel–Jacobi images $\Phi^{AJ}(j_\Psi)$ agree with the p-adic Abel–Jacobi images of the Darmon cycles considered in [23, Section 6].

Suppose that $f \in S_{k_0}(\Gamma_0(N))$ is a normalized newform and denote by K_f the field generated by the Fourier coefficients of f. Attached to the modular form f, there is a $K_f \otimes \mathbb{Q}_p$-monodromy module $\mathbf{D}_{[f]}$ that appears like a quotient of $\mathbf{D}$ in the category $MF_{\mathbb{Q}_p}(\phi, N)$ of filtered Frobenius modules over $\mathbb{Q}_p$; we can consider the p-adic Abel–Jacobi map $\Phi^{AJ}_{[f]}$ obtained by Φ^{AJ} followed by this projection, taking values in $\mathbf{D}_{[f],F_p}/F^{m+1}$. The construction of these monodromy modules, that follows [23, Section 4.2], is reviewed in Subsection 2.3. They are built from a space $\mathbf{MS}^{c,w_\infty}$, which is obtained from the cuspidal part of the space of modular symbols with values in the F_p-dual of $\mathbf{P}_n$ and depends on the choice w_∞ of a sign at infinity. In Section 3 we show how to realize our p-adic Abel–Jacobi map as taking values in $\mathbf{MS}^{c,w_\infty,\vee}$:

$$\log \Phi^{AJ}\colon \left(\Delta^0 \otimes \operatorname{Div}(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n\right)_\Gamma \to \mathbf{D}_{F_p}/F^{m+1} \stackrel{\simeq}{\to} \mathbf{MS}^{c,w_\infty,\vee}.$$

Our p-adic integration theory can be used to produce local cohomology classes in

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$Ext^1_{G_{F_p}}(\mathbb{Q}_p, V_{[f]})$ as follows. Thanks to a combination of the work of Bertolini, Darmon and Iovita with a result of Colmez (see Theorem 4.11), there is an isomorphism $\varphi : \mathbf{D}_{[f]} \simeq \mathbb{D}_{[f]}$. Let $F = H/K$ be the narrow ring class field attached to the order $\mathcal{O}$ and choose a local embedding $H \hookrightarrow F_p$ (assuming $F_p \supset \mathbb{Q}_{p^2}$). Then we find an identification of the tangent spaces

$$\varphi\colon \mathbf{D}_{[f],F_p}/F^{m+1} \simeq \mathbb{D}_{[f],F_p}/F^{m+1} \overset{\simeq}{\to} \mathrm{Ext}^1_{G_{F_p}}(\mathbb{Q}_p, V_{[f]}),$$

where the last identification, provided by the Bloch–Kato exponential map, is indeed an isomorphism in our setting. Let $\chi\colon G_{H/K} \to \mathbb{C}^\times$ be a character and set

$$j^\chi := \sum_{\sigma \in G_{H^+_{\mathcal{O}}/K}} \chi^{-1}(\sigma)\, j_{\sigma\Psi} \in \left(\Delta^0 \otimes \mathrm{Div}(\mathcal{H}^{\mathrm{ur}}_p) \otimes \mathbf{P}_n\right)_\Gamma \otimes \chi.$$

Here $(-) \otimes \chi$ denotes a suitable scalar extension. For every global field F set $MW_f(F) := \Im(e_{[f]} \circ cl^{m+1}_{0,F})$. Let H_χ/K be the extension out by the character χ. Denote by $MW(H_\chi)^\chi$ the χ-part of $MW(H_\chi)$. As in [23, Section 5] one can formulate rationality conjectures asserting that

$$\varphi\left(\Phi^{AJ}(j^\chi)\right) \in \mathrm{res}_p\left(MW(H_\chi)^\chi\right).$$

Our local cohomology classes are the same as the ones defined in [23, Section 6] in the more general setting of a Shimura curve, when the theory is specialized to a modular curve. One of the main motivations of this paper is indeed to provide instances where the conjectures formulated there, or rather some of their consequences, can be proved.

Fix once and for all an identification $\mathbb{C} \simeq \mathbb{C}_p$. Denote by $K_{[f]}$ the field generated by the Fourier coefficients of f and all its companion cusp forms. Assuming $F_p \supset K_{[f]}$ the tangent space $\mathbf{D}_{[f],F_p}/F^{m+1} = \mathbf{MS}^{c,w_\infty,\vee}$ (resp. $\mathbb{D}_{[f],F_p}/F^{m+1} = e_{[f]}M_k(\Gamma, F_p)^\vee$) splits into $\sigma(f)$-components corresponding to the companion forms $\sigma(f)$ of f. Write Φ^{AJ}_f (resp. $\log \Phi^{AJ}_f$) to denote the f-component of the above p-adic Abel–Jacobi maps, so that

$$\Phi^{AJ}_{[f]} = \bigoplus_\sigma \Phi^{AJ}_{\sigma(f)} \quad \left(\text{resp. } \log \Phi^{AJ}_{[f]} = \bigoplus_\sigma \log \Phi^{AJ}_{\sigma(f)}\right).$$

Attached to the modular form f there is a modular symbol $I_f \in \mathbf{MS}^{c,w_\infty}$ (resp. a rigid analytic modular form f^{rig}) generating the f-component of $\mathbf{MS}^{c,w_\infty}$ (resp. $M_k(\Gamma, F_p)$).

Let $\chi\colon G_{H/K} \to \mathbb{C}^\times$ be a genus character attached to the pair (χ_1, χ_2) of Dirichlet character. Note that the values $\chi_i(-M)$ do not depend on $i = 1, 2$. The identification $\mathbb{C} \simeq \mathbb{C}_p$ determines a prime $\mathfrak{p}$ of K_f above p, and we can decompose $V_{[f]}$, $\mathbf{D}_{[f]}$, $\mathbb{D}_{[f]}$, and $MW_f(H_\chi)^\chi$ according to the decomposition $K_f \otimes \mathbb{Q}_p = \bigoplus_{\mathfrak{p}'|p} K_{f,\mathfrak{p}'}$, where $K_{f,\mathfrak{p}'}$ denotes the $\mathfrak{p}'$-adic completion of K_f at $\mathfrak{p}'$. We will write $V_{[f],\mathfrak{p}}$, $\mathbf{D}_{[f],\mathfrak{p}}$, $\mathbb{D}_{[f],\mathfrak{p}}$, and $MW_{f,\mathfrak{p}}(H_\chi)^\chi$ to denote the $\mathfrak{p}$-component, so that $MW_{f,\mathfrak{p}}(H_\chi)^\chi$ is naturally a $K_{f,\mathfrak{p}}$-vector space, and the f-component of $\mathbf{D}_{[f],F_p}/F^{m+1}$ (resp. $\mathbb{D}_{[f],F_p}/F^{m+1}$) appears in $\mathbf{D}_{[f],\mathfrak{p},F_p}/F^{m+1}$ (resp. $\mathbb{D}_{[f],\mathfrak{p},F_p}/F^{m+1}$).

One of the main results is the following theorem, which is implied by the conjectures formulated in [23, Section 5].

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Theorem 1.1 *Suppose that $N = pM$, that there exists a prime $q \parallel M$, and that*

$$\chi_i(-M) = (-1)^{\frac{k_0}{2}} w_M$$

Then

(i) *there is $y_f^\chi \in MW_f(H_\chi)^\chi$ such that $\varphi(\Phi_{[f]}^{AJ}(j^\chi)) = \mathrm{res}_p(y_f^\chi)$;*
(ii) *if $y_{f,\mathfrak{p}}^\chi \neq 0$, we have $MW_{f,\mathfrak{p}}(H_\chi)^\chi = K_{f,\mathfrak{p}} y_{f,\mathfrak{p}}^\chi$.*

The proof of this theorem follows the strategy developed in [2, 3] in the weight 2 setting. Indeed we also obtain p-adic Gross-Zagier formulas that are of independent interest and an higher weight analogue of the main results of [2, 3].

Let $\mathcal{W} := \mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{G}_m)$ be the weight space, viewed as a rigid analytic space over $\mathbb{Q}_p$. The integers $\mathbb{Z}$ are embedded in $\mathcal{W}$ by sending the integer k to the function $t \mapsto t^{k-2}$. Let $U \subset \mathcal{W}$ be a small enough open affinoid disk centered at k_0. We will define p-adic L-functions

$$L_p(f/K, \chi, -)\colon U \to \mathbb{C}_p, \quad L_p(f/K', \chi, -)\colon U \to \mathbb{C}_p.$$

of the weight variable attached to the real quadratic field K or an imaginary quadratic field K' such that we can write $N = pN^+N^-$, where the primes dividing N^+ are split in K and the primes dividing pN^- are inert, squarefree, and even in number.

When $K/\mathbb{Q}$ is a real quadratic field satisfying the above assumptions, we obtain the following formula, relating the second derivative of the above p-adic L-function to the p-adic Abel-Jacobi image of the Darmon cycles.

Theorem 1.2 *Let $\chi\colon G_{H^+/K} \to \mathbb{C}^\times$ be a genus character (here H^+ is the narrow Hilbert ring class field). Then*

$$\frac{d^2}{d\kappa^2}[L_p(f/K, \chi, \kappa)]_{\kappa=k_0} = \begin{cases} 2D_K^{\frac{k_0-2}{2}} \log \Phi_f^{AJ}(j^\chi)(I_f)^2 & \textit{if } \chi_i(-M) = (-1)^{\frac{k_0}{2}} w_M, \\ 0 & \textit{if } \chi_i(-M) = (-1)^{\frac{k_0-2}{2}} w_M. \end{cases}$$

Now let $K'/\mathbb{Q}$ be an imaginary quadratic field and consider a factorization $N = pN^+N^-$ as above. We now focus on a genus character χ of the imaginary quadratic field K'. Denote by H'_χ/K' the extension cut out by the character χ and by $y^\chi \in CH^{m+1}(\mathcal{M}_{n,H'_\chi})^\chi$ the corresponding Heegner cycle. There is a decomposition

$$MW_f(H'_\chi)^\chi = MW_f(\mathbb{Q}_{\chi_1})^{\chi_1} \oplus MW_f(\mathbb{Q}_{\chi_2})^{\chi_2},$$

where $\mathbb{Q}_{\chi_i}/\mathbb{Q}$ denotes the quadratic extension cut out by the Dirichlet character χ_i. Furthermore, $cl_{0,f}^{m+1}(y^\chi)$ belongs precisely to one between $MW_f(\mathbb{Q}_{\chi_1})^{\chi_1}$ and $MW_f(\mathbb{Q}_{\chi_2})^{\chi_2}$.

We obtain the following formula, this time relating the second derivative of the above p-adic L-function to the p-adic Abel–Jacobi image of an Heegner cycle.

Theorem 1.3 *Let $\chi\colon G_{H'/K'} \to \mathbb{C}^\times$ be a genus character (here H' is the Hilbert ring class field). If $cl_{0,f}^{m+1}(y^\chi) \in MW_f(\mathbb{Q}_{\chi_i})^{\chi_i}$, we have*

$$\frac{d^2}{d\kappa^2}\big[L_p(f/K', \chi, \kappa)\big]_{\kappa=k_0} = \begin{cases} 2 \log \Phi_f^{AJ}(y^\chi)(f^{\mathrm{rig}})^2 & \textit{if } \chi_i(p) = -w_p, \\ 0 & \textit{if } \chi_i(p) = w_p. \end{cases}$$

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Hence the second derivative of $L_p(f/K', \chi, \kappa)$ at k_0 encodes information about the restriction at p of $cl_{0,f}^{m+1}(y^\chi)$: when $\chi_i(p) = -w_p$, it is zero precisely when (the f-component of) the restriction of $cl_{0,f}^{m+1}(y^\chi)$ at p is zero. Information on the exact position of $cl_{0,f}^{m+1}(y^\chi)$, *i.e.,* which one is the character χ_i in the above statement, is given in Lemma 5.28.

We will also consider the restriction of the Mazur–Kitagawa p-adic L-function $L_p(f, \omega, \kappa, s)$ to the critical line $L_p(f, \omega, \kappa, \kappa/2)$.

Theorem 1.4 *Suppose that there exists $q \parallel M$ and let ω be a quadratic Dirichlet character such that*

$$\omega(-N) = (-1)^{\frac{k_0-2}{2}} w_N \text{ and } \omega(p) = a_p p^{-\frac{k_0-2}{2}} = -w_p.$$

Then:

(i) *the p-adic L-function $L_p(f, \omega, \kappa, \kappa/2)$ vanishes to order*

$$\mathrm{ord}_{\kappa=k_0} L_p(f, \omega, \kappa, \kappa/2) \geq 2;$$

(ii) *there exists $y^\omega \in CH^{m+1}(\mathcal{M}_{n,\mathbb{Q}_\omega})^\omega$ and $t \in K_f^\times$ such that*

$$\frac{d^2}{d\kappa^2}\big[L_p(f, \omega, \kappa, \kappa/2)\big]_{\kappa=k_0} = t \cdot \log \Phi_f^{AJ}(y^\omega)(f^{\mathrm{rig}})^2;$$

(iii) *if $cl_{0,f}^{m+1}(y_\mathfrak{p}^\omega) \neq 0$, then $MW_{f,\mathfrak{p}}(\mathbb{Q}_\omega)^\omega = K_{f,\mathfrak{p}} cl_{0,f}^{m+1}(y_\mathfrak{p}^\omega)$.*

Again, $\mathbb{Q}_\omega$ is the extension cut out by the character ω, while $(-)^\omega$ denotes the ω-component. Hence again the second derivative of the Mazur–Kitagawa p-adic L-function $L_p(f, \omega, \kappa, \kappa/2)$ at k_0 encodes information on (the f-component of) the restriction at p of $cl_{0,f}^{m+1}(y^\omega)$, whose $\mathfrak{p}$-component generates $MW_{f,\mathfrak{p}}(\mathbb{Q}_\omega)^\omega$ when nonzero. In particular,

$$\frac{d^2}{d\kappa^2}\big[L_p(f, \omega, \kappa, \kappa/2)\big]_{\kappa=k_0} \neq 0 \Rightarrow MW_{f,\mathfrak{p}}(\mathbb{Q}_\omega)^\omega = K_{f,\mathfrak{p}} cl_{0,f}^{m+1}(y_\mathfrak{p}^\omega).$$

2 p-adic Integration Theory, $\mathcal{L}$-invariants, and the Monodromy Module of Weight k_0 Modular Forms

Let S_k be the space of modular forms of even weight $k > 2$, endowed with the $\mathbb{GL}_2^+(\mathbb{Q})$-action

$$f \mid \gamma := \det \gamma^{k-1}(cz+d)^{-k} f(\gamma z),$$

with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

For every integer N denote as usual by $S_k(\Gamma_0(N)) := S_k^{\Gamma_0(N)}$ the $\Gamma_0(N)$-invariants, *i.e.,* the space of weight k modular forms on $\Gamma_0(N)$. Let $\mathbb{T}_N$ be the Hecke $\mathbb{Q}$-algebra generated by the operators T_l for $l \nmid N$ and U_l for $l \mid N$ acting on $S_k(\Gamma_0(N))$. Then $\dim_\mathbb{Q} \mathbb{T}_N = \dim_\mathbb{C} S_k(\Gamma_0(N))$ (see [25, Theorem 3.51]). The number field generated

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by the Fourier coefficients of a normalized modular form f is denoted by K_f. The spaces $S_k(\Gamma_0(N))$ are endowed with the Petersson inner product $\langle -, -\rangle_k$.

Let $\mathbf{P}_{k-2}$ be the space of polynomials of degree $\leq k-2$, endowed with the following right $\mathbb{GL}_2$-action:

$$P(X)\mathbf{M} := (cX+d)^{k-2} P\Big(\frac{aX+b}{cX+d}\Big) \text{ for } P \in \mathbf{P}_{k-2}(K_p), \tag{2.1}$$

where $\mathbf{M} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{GL}_2$.

Usually we do not specify any field in the notation, and we write $\mathbf{P}_{k-2} = \mathbf{P}_{k-2}(F)$ when such a choice has been made. The dual space $\mathbf{V}_{k-2}(F) := \mathrm{Hom}_F(\mathbf{P}_{k-2}(F), F)$ is then endowed with a natural $\mathbb{GL}_2$-left action by the rule$(\mathbf{M}\Lambda)(P) := \Lambda(P\mathbf{M})$. The same notation $\mathbf{V}_{k-2}$ be used to indicate that the choice of a field has been made. Indeed, whenever $\mathbf{V}$ and $\mathbf{W}$ are vector spaces over some field F, we set $\mathbf{V}^\vee := \mathrm{Hom}_F(\mathbf{V}, F)$ and $\mathbf{V} \otimes \mathbf{W}$ without reference to the field.

We recall that $\mathbf{P}_{k-2}$ and $\mathbf{V}_{k-2}$ carry a non-degenerate $\mathbb{GL}_2$-invariant bilinear form (see for example [5, Sec. 1.2] or [15, (33)]):

$$\langle -, -\rangle_{\mathbf{P}_{k-2}} \colon \mathbf{P}_{k-2} \otimes \mathbf{P}_{k-2} \to F, \quad \langle -, -\rangle_{\mathbf{V}_{k-2}} \colon \mathbf{V}_{k-2} \otimes \mathbf{V}_{k-2} \to F.$$

Let $\Delta := \mathrm{Div}\,\mathbb{P}^1(\mathbb{Q})$ and $\Delta^0 := \mathrm{Div}^0\,\mathbb{P}^1(\mathbb{Q})$ be respectively the space of divisors and degree zero divisors supported at the cusps with coefficients in some field F, endowed with their natural action by fractional linear transformation by $\mathbb{GL}_2(\mathbb{Q})$. For any space $\mathbf{V}$ endowed with an action by $G \subset \mathbb{GL}_2(\mathbb{Q})$ (a congruence group of $\mathbb{SL}_2(\mathbb{Z})$ in the applications), set $\mathcal{BS}(\mathbf{V}) := \mathrm{Hom}(\Delta, \mathbf{V})$ and $\mathcal{MS}(\mathbf{V}) := \mathrm{Hom}(\Delta^0, \mathbf{V})$, equipped with the natural induced actions. There is a canonical exact sequence

$$0 \to \mathbf{V} \to \mathcal{BS}(\mathbf{V}) \to \mathcal{MS}(\mathbf{V}) \to 0. \tag{2.2}$$

We also write $\mathcal{BS}_G(\mathbf{V}) := \mathcal{BS}(\mathbf{V})^G$ and $\mathcal{MS}_G(\mathbf{V}) := \mathcal{MS}(\mathbf{V})^G$ to denote the G-invariants. Finally when $\mathbf{V} = \mathbf{V}_{k-2} = \mathbf{V}_{k-2}(F)$ we will occasionally write $\mathcal{MS}^k = \mathcal{MS}^k(F)$ (and $\mathcal{MS}^k_G = \mathcal{MS}^k_G(F)$ for the invariants).

Recall the Bruhat–Tits tree $\mathcal{T}$ at p, whose vertices $\mathcal{V} = \mathcal{V}(\mathcal{T})$ are the homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$. Let $L_* := \mathbb{Z}_p^2$ be the standard $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$ and set $L_\infty := \mathbb{Z}_p \oplus p\mathbb{Z}_p$. Write $\mathcal{E} = \mathcal{E}(\mathcal{T})$ to denote the set of ordered edges and choose the following orientation $\mathcal{E} = \mathcal{E}^+ \sqcup \mathcal{E}^-$: write $\mathcal{V}^+$ (resp. $\mathcal{V}^-$) to denote the set of those vertices v such that the distance $d(v, v_*)$ is even (resp. odd), where $v_* := [L_*]$; define $\mathcal{E}^+$ (resp. $\mathcal{E}^-$) to be the set of those edges e with source $s(e) \in \mathcal{V}^+$ (resp. $s(e) \in \mathcal{V}^-$).

We denote by $\mathcal{C}_0(\mathcal{E}, \mathbf{V})$ (resp. $\mathcal{C}(\mathcal{V}, \mathbf{V})$) the space of maps $c : \mathcal{E} \to \mathbf{V}$ such that $c(\bar{e}) = -c(e)$ (resp. $\mathcal{C}(\mathcal{V}, \mathbf{V})$ the set of all maps $c \colon \mathcal{V} \to \mathbf{V}$). The set of harmonic cocycles $\mathcal{C}_{\mathrm{har}}(\mathcal{E}, \mathbf{V})$ is defined by the following exact sequence (see [13, Lemma 24] for the right exactness):

$$0 \to \mathcal{C}_{\mathrm{har}}(\mathcal{E}, \mathbf{V}) \to \mathcal{C}_0(\mathcal{E}, \mathbf{V}) \xrightarrow{\varphi_s} \mathcal{C}(\mathcal{V}, \mathbf{V}) \to 0 \tag{2.3}$$

$$\varphi_s(c)(v) := \sum_{s(e)=v} c(e).$$

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It will be also useful to consider the following exact sequence:

$$0 \to \mathbf{V} \to \mathcal{C}(\mathcal{V}, \mathbf{V}) \stackrel{\partial^*}{\to} \mathcal{C}_0(\mathcal{E}, \mathbf{V}) \to 0 \qquad (\partial^* c)(e) := c(s(e)) - c(t(e)). \tag{2.4}$$

Let $F_p/\mathbb{Q}_p$ be any complete local field. Let $\mathcal{A}_{k-2}(\mathbb{P}^1(\mathbb{Q}_p)) := \mathcal{A}_{k-2}(\mathbb{P}^1(\mathbb{Q}_p), F_p)$ denote the space of F_p-valued locally analytic functions on $\mathbb{Q}_p$ with a pole of order at most $k-2$ at ∞. The same formula (2.1) endows it with a $\mathbb{GL}_2(\mathbb{Q}_p)$-module structure. This space sits in the following exact sequence

$$0 \to \mathbf{P}_{k-2} \to \mathcal{A}_{k-2}(\mathbb{P}^1(\mathbb{Q}_p)) \to \mathcal{A}_{k-2}(\mathbb{P}^1(\mathbb{Q}_p))/\mathbf{P}_{k-2} \to 0.$$

Define the spaces $\mathcal{D}^0_{k-2}(\mathbb{P}^1(\mathbb{Q}_p))$, $\mathcal{D}_{k-2}(\mathbb{P}^1(\mathbb{Q}_p))$ and $\mathbf{V}_{k-2}$ by taking the (continuous) F_p-dual exact sequence

$$0 \to \mathcal{D}^0_{k-2}(\mathbb{P}^1(\mathbb{Q}_p)) \to \mathcal{D}_{k-2}(\mathbb{P}^1(\mathbb{Q}_p)) \to \mathbf{V}_{k-2} \to 0.$$

It will also be convenient to consider the subspace $\mathcal{D}^{0,b}_{k-2}(\mathbb{P}^1(\mathbb{Q}_p)) \subset \mathcal{D}_{k-2}(\mathbb{P}^1(\mathbb{Q}_p))$ of bounded distributions, *i.e.,* those μ for which there is a constant A such that, for all $i \geq 0$, $j \geq 0$, and all $a \in \mathbb{Z}_p$,

$$|\mu((x-a)^i | a + p^j\mathbb{Z}_p)| \leq p^{A - j(i-1-k/2)}.$$

We recall that there is a standard basis for the topology on $\mathbb{P}^1(\mathbb{Q}_p)$ obtained from the open compact subsets $U_e \subset \mathbb{P}^1(\mathbb{Q}_p)$ corresponding to the ends of $\mathcal{T}$ originating from e.

Note that with the only possible exception of S_k, the above spaces are endowed with an action by the full group $\mathbb{GL}_2(\mathbb{Q})$. Hence the matrix $W_\infty = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ acts on these spaces; furthermore, since it normalizes the groups of the form $\Gamma_0(N)$, the cohomology groups $H^i(\Gamma_0(N), -)$ are endowed with a natural W_∞-action. Suppose that $\mathbf{V}$ is a characteristic $\neq 2$ vector space endowed with a W_∞-action (the characteristic will be 0 in our applications): we denote by $\mathbf{V}^{w_\infty}$ the direct summand of $\mathbf{V} = \mathbf{V}^+ \oplus \mathbf{V}^-$ on which $W_\infty = w_\infty \in \{\pm 1\}$.

We recall that there is a $\mathbb{GL}_2^+(\mathbb{Q})$-equivariant map

$$\widetilde{I}_- : S_k \otimes_{\mathbb{R}} \mathbb{C} \to \mathcal{MS}^k(\mathbb{C}) \tag{2.5}$$

$$\widetilde{I}_f\{x - y\}(P) := 2\pi i \int_x^y f(z)P(z,1)dz \in \mathbb{C}.$$

The composition of this morphism with the boundary map δ arising from the exact sequence (2.2) by taking the $\Gamma_0(N)$-invariants identifies $S_k(\Gamma_0(N)) \otimes_{\mathbb{R}} \mathbb{C}$ with the image of $H^1_c(\Gamma_0(N), \mathbf{V}_{k-2}(\mathbb{C}))$ in the group $H^1(\Gamma_0(N), \mathbf{V}_{k-2}(\mathbb{C}))$, usually called the parabolic cohomology subgroup $H^1_{\mathrm{par}}(\Gamma_0(N), \mathbf{V}_{k-2}(\mathbb{C}))$. The identification

$$\delta \circ \widetilde{I}_- : S_k(\Gamma_0(N)) \otimes_{\mathbb{R}} \mathbb{C} \stackrel{\simeq}{\to} H^1_{\mathrm{par}}(\Gamma_0(N), \mathbf{V}_{k-2}(\mathbb{C})) \tag{2.6}$$

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is called the Eichler–Shimura isomorphism. Since $\mathbf{V}_{k-2}$ is an irreducible $\Gamma_0(N)$-module (in light of the assumption $k > 2$, see for example [14, 6.1 Lemma 2]), the following sequence is exact by definition of the parabolic cohomology and Hecke equivariant:

$$0 \to \mathcal{BS}^k_{\Gamma_0(N)}(\mathbb{C}) \to \mathcal{MS}^k_{\Gamma_0(N)}(\mathbb{C}) \to H^1_{\mathrm{par}}(\Gamma_0(N), \mathbf{V}_{k-2}(\mathbb{C})) \to 0. \tag{2.7}$$

More generally we define $H^1_{\mathrm{par}}(G, \mathbf{V}) := \delta(\mathcal{MS}_G(\mathbf{V}))$.

We recall the following theorem of Shimura.

Proposition 2.1 *If f is new of level N, there exist complex periods $\Omega_f^{\pm} \in \mathbb{C}$ such that*

$$I_f^{\pm} := (\Omega_f^{\pm})^{-1}\widetilde{I}_f^{\pm} \in \mathcal{MS}^{k,\pm}_{\Gamma_0(N)}(K_f)$$

The periods $\Omega_f^{\pm}$ can be chosen such that $\Omega_f^{+}\Omega_f^{-} = \langle f, f\rangle_k$.

Once we make the choice of a sign $w_\infty \in \{\pm 1\}$, we set $\Omega_f := \Omega_f^{w_\infty}$ and $I_f := I_f^{w_\infty}$.

As in the introduction we let $k_0 > 2$ be a fixed even weight and set for shortness $n := k_0 - 2$, $m := n/2 = (k_0 - 2)/2$.

2.1 Decomposition into Eisenstein and Cuspidal Parts

Whenever M is a $\mathbb{T}_N$-module, we say that it admits an Eisenstein/cuspidal decomposition if there exists a Hecke operator T_l for some $l \nmid N$ such that:

(a) we can write $M = M^e \oplus M^c$;
(b) the operator $t_l := T_l - l^{k-1} - 1$ is zero on M^e and is invertible on M^c.

The following lemmas are easily established.

Lemma 2.2 *Whenever $M = M^e \oplus M^c$ admits an Eisenstein/cuspidal decomposition, $M^* \subset M$ with $* = e, c$ is a $\mathbb{T}_N$-submodule, and furthermore the decomposition is unique.*

*Let M_1 (resp. M_2) be a $\mathbb{T}_N$-module (resp. $\mathbb{T}_M$-module). If $f\colon M_1 \to M_2$ is a Hecke equivariant morphism (*i.e.*, a morphism such that $T_l f = f T_l$ for every $l \nmid MN$) and there exists T_l with $l \nmid MN$ such that the properties (*a*) and (*b*) are satisfied by M_1 and M_2,*

$$f = f^e \oplus f^c \colon M_1^e \oplus M_1^c \to M_2^e \oplus M_2^c \text{ with } f^*\colon M_1^* \to M_2^* \text{ for } * = e, c.$$

In particular

$$\ker(f) = \ker(f^e) \oplus \ker(f^c) \text{ and } \operatorname{coker}(f) = \operatorname{coker}(f^e) \oplus \operatorname{coker}(f^c)$$

admit an Eisenstein/cuspidal decomposition.

Lemma 2.3 *Suppose that we are given an exact sequence*

$$0 \to E \to M \to C \to 0$$

of Hecke modules such that $t_l = 0$ on E and is invertible on C. Then there exists a unique Hecke equivariant section $C \hookrightarrow M$, $M = M^e \oplus M^c$ admitting an Eisenstein/cuspidal decomposition, $M^e = E$ and $M^c = C$.

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We are now going to describe the Eisenstein/cuspidal decompositions of some spaces that will be of interest to us. Recall the groups $\Gamma_0(M)$, $\Gamma_0(pM)$, and Γ from the introduction.

Eisenstein/Cuspidal Decomposition of $\mathcal{MS}_{\Gamma_0(M)}(\mathbf{V}_n)$, $\mathcal{MS}_{\Gamma_0(pM)}(\mathbf{V}_n)$, and $\mathcal{MS}_{\Gamma}(\mathcal{C}_{\mathrm{har}}(\mathcal{E},\mathbf{V}_n))$

The exact sequence (2.7) endows $\mathcal{MS}_{\Gamma_0(M)}(\mathbf{V}_n)$ and $\mathcal{MS}_{\Gamma_0(pM)}(\mathbf{V}_n)$ with Eisenstein/cuspidal decompositions in light of Lemma 2.3: indeed in [19, Section 7.2] a careful study of the action of the Hecke operators on $\mathcal{BS}_{\Gamma_0(M)}(\mathbf{V}_n)$ shows the existence of l such that $t_l = 0$ on $\mathcal{BS}_{\Gamma_0(M)}(\mathbf{V}_n)$ and $\mathcal{BS}_{\Gamma_0(pM)}(\mathbf{V}_n)$; on the other hand, by the Ramanujan–Petersson conjecture proved by Deligne, this Hecke operator is invertible on $H^1_{\mathrm{par}}(\Gamma_0(N),\mathbf{V}_n)$.

Taking the Γ-invariants from the exact sequence (2.3) with $\mathbf{V} = \mathcal{MS}^n$ (and using Shapiro's Lemma) gives the commutative diagram

(2.8)
$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{C}_{\mathrm{har}}(\mathcal{E},\mathcal{MS}(\mathbf{V}_n))^{\Gamma} & \longrightarrow & \mathcal{C}_0(\mathcal{E},\mathcal{MS}(\mathbf{V}_n))^{\Gamma} & \longrightarrow & \mathcal{C}(\mathcal{V},\mathcal{MS}(\mathbf{V}_n))^{\Gamma} \\ & & \| & & \| & & \| \\ 0 & \longrightarrow & \mathcal{MS}^{\mathrm{har}}_{\Gamma_0(pM)}(\mathbf{V}_n) & \longrightarrow & \mathcal{MS}_{\Gamma_0(pM)}(\mathbf{V}_n) & \longrightarrow & \mathcal{MS}_{\Gamma_0(M)}(\mathbf{V}_n)^2, \end{array}$$

where $\mathcal{MS}^{\mathrm{har}}_{\Gamma_0(pM)}(\mathbf{V}_n)$ is by definition the image of $\mathcal{C}_{\mathrm{har}}(\mathcal{E},\mathcal{MS}(\mathbf{V}_n))^{\Gamma}$ under Shapiro's isomorphism. The lower right arrow can be described explicitly in terms of corestriction as in [13, Section 3.2]. Thanks to Lemma 2.2 we can endow $\mathcal{MS}_{\Gamma}(\mathcal{C}_{\mathrm{har}}(\mathcal{E},\mathbf{V}_n)) = \mathcal{C}_{\mathrm{har}}(\mathcal{E},\mathcal{MS}(\mathbf{V}_n))^{\Gamma}$ with a natural Eisenstein/cupidal decomposition.

Remark 2.4 Let $\mathbb{T} := \mathbb{T}^{p-new}_{pM}$ be the p-new quotient of the Hecke algebra $\mathbb{T}_{pM}$. It follows from Lemma 2.2 and the Eichler-Shimura isomorphism (2.6) that $\mathcal{MS}_{\Gamma}(\mathcal{C}_{\mathrm{har}}(\mathcal{E},\mathbf{V}_n(F)))^c$ is a free rank two module over $\mathbb{T}_F := \mathbb{T}\otimes_{\mathbb{Q}} F$.

Eisenstein/Cuspidal Decomposition of $\mathcal{MS}_{\Gamma}(\mathbf{V}_n)$, $H^1(\Gamma,\mathcal{MS}(\mathbf{V}_n))$ and $H_1(\Gamma,\Delta^0\otimes\mathbf{P}_n)$

Sequence (2.4) (together with Shapiro's Lemma) produces the long exact sequence

$$\begin{aligned} (2.9)\quad 0 \to \mathcal{MS}_{\Gamma}(\mathbf{V}_n) \to \mathcal{MS}_{\Gamma_0(M)}(\mathbf{V}_n)^2 \to \mathcal{MS}_{\Gamma_0(pM)}(\mathbf{V}_n) \\ \xrightarrow{\delta} H^1(\Gamma,\mathcal{MS}(\mathbf{V}_n)) \to 0, \end{aligned}$$

where the zero on the right is a consequence of $H^1(\Gamma_0(M),\mathcal{MS}(\mathbf{V}_n)) = 0$ (see [19, Section 7.1]). Thanks to Lemma 2.2 we can endow $\mathcal{MS}_{\Gamma}(\mathbf{V}_n)$ and $H^1(\Gamma,\mathcal{MS}(\mathbf{V}_n))$ with an Eisenstein/cuspidal decomposition. It follows that

$$H_1(\Gamma,\Delta^0\otimes\mathbf{P}_n) = H^1(\Gamma,\mathcal{MS}(\mathbf{V}_n))^{\vee}$$

is naturally endowed with a cuspidal decomposition too.

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Eisenstein/Cuspidal Decomposition of $H^1(\Gamma, \mathrm{Hom}(\mathbf{P}_n, \mathbf{V}))$

The groups $H^1(G, \mathbf{V}_n)$ with $G = \Gamma_0(N), \Gamma_0(pN)$ have an Eisenstein/cuspidal decomposition. The long exact sequence obtained from (2.4) and Shapiro's Lemma gives $H^1(\Gamma, \mathbf{V}_n)$ an Eisenstein/cuspidal decomposition too. Let $\mathbf{V}$ be a finite dimensional vector space endowed with the trivial Γ-action. By the universal coefficient theorem

$$H^1(\Gamma, \mathrm{Hom}(\mathbf{P}_n, \mathbf{V})) = H^1(\Gamma, \mathbf{V}_n) \otimes \mathbf{V},$$

and the Eisenstein/cuspidal decomposition on $H^1(\Gamma, \mathbf{V}_n)$ induces an Eisenstein/cuspidal decomposition on $H^1(\Gamma, \mathrm{Hom}(\mathbf{P}_n, \mathbf{V}))$.

Lemma 2.5 *We have* $H^1(\Gamma, \mathrm{Hom}(\mathbf{P}_n, \mathbf{V}))^c = H_{\mathrm{par}}(\Gamma, \mathrm{Hom}(\mathbf{P}_n, \mathbf{V})) = 0$.

Proof The claim is reduced $\mathbf{V} = K$, and we may apply [23, Lemma 3.10]. ■

Taking the cuspidal parts from the exact sequence (2.9) and applying Lemma 2.2, we get the exact sequence

$$0 \to \mathcal{MS}_\Gamma(\mathbf{V}_n)^c \to \mathcal{MS}_{\Gamma_0(M)}(\mathbf{V}_n)^{2,c} \to \mathcal{MS}_{\Gamma_0(pM)}(\mathbf{V}_n)^c \xrightarrow{\delta^c} H^1(\Gamma, \mathcal{MS}(\mathbf{V}_n))^c \to 0.$$

Lemma 2.6 *The boundary map δ^c restricts to give an isomorphism:*

$$\delta^c \colon \mathcal{MS}^{p-new}_{\Gamma_0(pM)}(\mathbf{V}_n)^c \xrightarrow{\simeq} H^1(\Gamma, \mathcal{MS}(\mathbf{V}_n))^c.$$

Proof The proof is analogous to [23, Lemma 2.9] ■

2.2 p-adic Integration Theory

Until the end of this section we fix a complete field extension $F_p/\mathbb{Q}_p$, and we will work over this field. Consider the natural map

$$R \colon \mathcal{D}^0_n(\mathbb{P}^1(\mathbb{Q}_p)) \to \mathcal{C}_{\mathrm{har}}(\mathcal{E}, \mathbf{V}_n)$$
$$R(\mu)_e(P) := \mu(P\chi_{U_e}).$$

It induces a map

$$R \colon \mathcal{MS}\Big(\mathcal{D}^0_n\big(\mathbb{P}^1(\mathbb{Q}_p)\big)\Big) \to \mathcal{MS}\big(\mathcal{C}_{\mathrm{har}}(\mathcal{E}, \mathbf{V}_n)\big) = \mathcal{C}_{\mathrm{har}}\big(\mathcal{E}, \mathcal{MS}(\mathbf{V}_n)\big).$$

Write $\Gamma_{e_*}\backslash\Gamma = \bigsqcup_{e\in\mathcal{E}} \Gamma_{e_*}\gamma_e$, where $\gamma_e e = e_*$ and e_* is the unique positive oriented edge whose stabilizer in Γ is $\Gamma_{e_*} = \Gamma_0(pM)$. Whenever $\mathbf{V}$ is a Γ_{e_*}-module endowed with a (possibly infinite) norm $|\cdot|$, define the following norm on $\mathcal{C}_0(\mathcal{E}, \mathbf{V})$:

$$\|c\| := \sup_{e\in\mathcal{E}^+} |\gamma_e c(e)| \in \mathbb{R} \cup \infty.$$

The above definition does not depend on the choice of coset representatives.

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Lemma 2.7 *Taking the invariants, the Shapiro isomorphism*

$$\mathcal{C}_0(\mathcal{E},\mathbf{V})^{\Gamma} \simeq \mathbf{V}^{\Gamma_{e_*}}$$
$$c \mapsto c(e_*)$$

respects the norms $\|\cdot\|$ *and* $|\cdot|$.

Proof The Γ-module identification $\mathcal{C}_0(\mathcal{E},\mathbf{V}) = \mathcal{C}(\mathcal{E}^+,\mathbf{V})$ respects the norms defined on the right-hand side by the same formula. Then $\mathcal{C}(\mathcal{E}^+,\mathbf{V})$ is identified with the induced module $\mathrm{Ind}_{\Gamma_{e_*}}^{\Gamma}\mathbf{V}$. Mapping v to $c^v(e) := \gamma_e^{-1}e$ gives an explicit inverse to the Shapiro isomorphism. The claim easily follows from the definition of the norms. ■

Proposition 2.8 *Taking* Γ*-invariants yields an isomorphism*

$$\mathcal{MS}_\Gamma(\mathcal{D}_n^{0,b}(\mathbb{P}^1(\mathbb{Q}_p))) \overset{\simeq}{\to} \mathcal{MS}_\Gamma(\mathcal{C}_{\mathrm{har}}(\mathcal{E},\mathbf{V}_n)) = \mathcal{C}_{\mathrm{har}}(\mathcal{E},\mathcal{MS}(\mathbf{V}_n))^{\Gamma}.$$

Proof Let $|\cdot|$ be a (finite) Γ_{e_*}-invariant norm on $\mathbf{V}_n$ that must exist, since $\mathbf{V}_n$ is finite dimensional and $\Gamma_{e_*} \subset \mathbb{GL}_2(L_*)$ is contained in a compact subgroup of $\mathbb{GL}_2(\mathbb{Q}_p)$. Endow $\mathcal{C}_0(\mathcal{E},\mathbf{V}_n)$ with the same norm $\|\cdot\|$ considered in Lemma 2.7. Let $\mathcal{C}_0^b(\mathcal{E},\mathbf{V}_n)$ (resp. $\mathcal{C}_{\mathrm{har}}^b(\mathcal{E},\mathbf{V}_n)$) be the subspace of those elements of $\mathcal{C}_0(\mathcal{E},\mathbf{V}_n)$ (resp. $\mathcal{C}_{\mathrm{har}}(\mathcal{E},\mathbf{V}_n)$) having a finite norm.

Consider the Γ-modules

$$\mathrm{Hom}(\Delta^0,\mathcal{C}_*(\mathcal{E},\mathbf{V}_n)) = \mathcal{C}_*(\mathcal{E},\mathrm{Hom}(\Delta^0,\mathbf{V}_n)) \text{ with } * = 0,\mathrm{har},$$

Define on $\mathrm{Hom}(\Delta^0,\mathbf{V}_n)$ a norm by the formula

$$|m|' := \sup_{x,y\in\mathbb{P}^1(\mathbb{Q})} |m(x-y)|.$$

Note that the above formula defines a Γ_{e_*}-invariant norm on $\mathrm{Hom}(D,\mathbf{V}_n)$, since the norm on $\mathbf{V}_n$ was Γ_{e_*}-invariant. Furthermore, taking the Γ_{e_*}-invariants we see that the above norm is finite on $\mathrm{Hom}_{\Gamma_{e_*}}(\Delta^0,\mathbf{V}_n)$. Indeed for every $\gamma\in\Gamma_{e_*}$, thanks to the Γ_{e_*}-invariance of the norm on $\mathbf{V}_n$, we have

$$|m(\gamma^{-1}x-\gamma^{-1}y)| = |\gamma m(\gamma^{-1}x-\gamma^{-1}y)| = |(\gamma m)(x-y)|;$$

hence, whenever $m \in \mathrm{Hom}_{\Gamma_{e_*}}(\Delta^0,\mathbf{V}_n)$, the sup can be taken over all a set of representatives for the set of Γ_{e_*}-equivalence classes of $\mathbb{P}^1(\mathbb{Q})$. Thanks to Lemma 2.7 we also know that setting

$$\|m\|' := \sup_{e\in\mathcal{E}^+} |\gamma_e m(e)|'$$

defines a finite norm on $\mathcal{C}_0(\mathcal{E},\mathrm{Hom}(\Delta^0,\mathbf{V}_n))^{\Gamma}$, and hence also on the subset

$$\mathrm{Hom}_\Gamma\big(\Delta^0,\mathcal{C}_{\mathrm{har}}(\mathcal{E},\mathbf{V}_n)\big) \subset \mathrm{Hom}_\Gamma\big(\Delta^0,\mathcal{C}_0(\mathcal{E},\mathbf{V}_n)\big).$$

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Making explicit the definition of the above norms, we see that

$$\begin{aligned}\|m\|' &:= \sup_{e\in\mathcal{E}^+} |\gamma_e m(e)|' = \sup_{\substack{e\in\mathcal{E}^+\\ x,y\in\mathbb{P}^1(\mathbb{Q}_p)}} |(\gamma_e m(e))(x-y)| \\ &= \sup_{\substack{e\in\mathcal{E}^+\\ x,y\in\mathbb{P}^1(\mathbb{Q}_p)}} |\gamma_e m(e)(\gamma_e^{-1}x-\gamma_e^{-1}y)| = \sup_{\substack{e\in\mathcal{E}^+\\ x,y\in\mathbb{P}^1(\mathbb{Q}_p)}} |\gamma_e m(e)(x-y)|\end{aligned}$$

must be finite on $\mathcal{C}_{\mathrm{har}}(\mathcal{E},\mathcal{MS}(\mathbf{V}_n))^\Gamma$. In particular, for every $x,y\in\mathbb{P}^1(\mathbb{Q})$ and $m\in\mathrm{Hom}_\Gamma(\Delta^0,\mathcal{C}_{\mathrm{har}}(\mathcal{E},\mathbf{V}_n))$, we find

$$\|m(x-y)\| = \sup_{e\in\mathcal{E}^+} |\gamma_e m(e)(x-y)| \le \|m\|' < \infty.$$

In other words, for every $x,y\in\mathbb{P}^1(\mathbb{Q}_p)$, we have $m(x-y)\in\mathcal{C}^b_{\mathrm{har}}(\mathcal{E},\mathbf{V}_n)$, so that the natural inclusion of $\mathrm{Hom}_\Gamma(\Delta^0,\mathcal{C}^b_{\mathrm{har}}(\mathcal{E},\mathbf{V}_n))$ in $\mathrm{Hom}_\Gamma(\Delta^0,\mathcal{C}_{\mathrm{har}}(\mathcal{E},\mathbf{V}_n))$ is really an identity

$$\mathrm{Hom}_\Gamma\big(\Delta^0,\mathcal{C}^b_{\mathrm{har}}(\mathcal{E},\mathbf{V}_n)\big) = \mathrm{Hom}_\Gamma\big(\Delta^0,\mathcal{C}_{\mathrm{har}}(\mathcal{E},\mathbf{V}_n)\big). \tag{2.10}$$

By a theorem of Amice–Velu–Teitelbaum (see [12] for the apropriate formulation) the morphism R restricts to give an isomorphism $\mathcal{D}_n^{0,b}(\mathbb{P}^1(\mathbb{Q}_p)) \simeq \mathcal{C}^b_{\mathrm{har}}(\mathcal{E},\mathbf{V}_n)$, and the claim follows from (2.10). ■

Recall our fixed working field F_p and let $F_p^0 := F_p\cap\mathbb{Q}_p^{\mathrm{ur}}$ be the maximal absolutely unramified subfield of F_p. Write $\mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}})$ (resp. $\mathrm{Div}(\mathcal{H}_p^{\mathrm{ur}})$) to denote the degree zero divisors (resp. the divisors) supported on $\mathbb{Q}_p^{\mathrm{ur}}-\mathbb{Q}_p$ that are fixed by the action of the Galois group $G_{\mathbb{Q}_p^{\mathrm{ur}}/F_p^0}$.

Definition 2.9 Define pairings

$$\begin{aligned}&\big(\Delta^0\otimes\mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}})\otimes\mathbf{P}_n\big)\otimes\mathcal{MS}\Big(\mathcal{D}_n^0\big(\mathbb{P}^1(\mathbb{Q}_p)\big)\Big)\to F_p\\ &\qquad\qquad (s-r)\otimes(\tau_2-\tau_1)\otimes P\otimes\mu\mapsto \int_{\tau_1}^{\tau_2} P\omega_\mu^{\log}\{r\to s\}\\ &\big(\Delta^0\otimes Div^0(\mathcal{H}_p)\otimes\mathbf{P}_n\big)\otimes\mathcal{MS}\Big(\mathcal{D}_n^0\big(\mathbb{P}^1(\mathbb{Q}_p)\big)\Big)\to F_p\\ &\qquad\qquad (s-r)\otimes(\tau_2-\tau_1)\otimes P\otimes\mu\mapsto \int_{\tau_1}^{\tau_2} P\omega_\mu^{\mathrm{ord}}\{r\to s\},\end{aligned}$$

where

$$\int_r^s\int_{\tau_1}^{\tau_2} P\omega_\mu^{\log} := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_p\Big(\frac{t-\tau_2}{t-\tau_1}\Big)P(t)d\mu\{r\to s\}(t)$$

and

$$\int_r^s\int_{\tau_1}^{\tau_2} P\omega_\mu^{\mathrm{ord}} := \sum_{e:\ \mathrm{red}(\tau_1)\to\mathrm{red}(\tau_2)} \int_{U(e)} P(t)d\mu\{r\to s\}(t).$$

Since the pairings are Γ-invariant, they give pairings

$$\Psi^{\log}, \Psi^{\mathrm{ord}} : \left(\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n\right)_\Gamma \otimes \mathcal{MS}_\Gamma\left(\mathcal{D}_n^0\left(\mathbb{P}^1(\mathbb{Q}_p)\right)\right) \to F_p.$$

Hence there are two morphisms

$$\Psi^{\log}, \Psi^{\mathrm{ord}} : \left(\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n\right)_\Gamma \to \mathcal{MS}_\Gamma\left(\mathcal{D}_n^0\left(\mathbb{P}^1(\mathbb{Q}_p)\right)\right)^\vee.$$

From now on we will identify, via Proposition 2.8,

$$\mathbf{MS} := \mathcal{MS}_\Gamma\left(\mathcal{D}_n^{0,b}\left(\mathbb{P}^1(\mathbb{Q}_p)\right)\right) = \mathcal{MS}_\Gamma\left(\mathcal{C}_{\mathrm{har}}(\mathcal{E}, \mathbf{V}_n)\right).$$

Consider the exact sequence

$$0 \to \Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n \to \Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n \to \Delta^0 \otimes \mathbf{P}_n \to 0,$$

yielding the boundary map

$$H_1(\Gamma, \Delta^0 \otimes \mathbf{P}_n) \xrightarrow{\partial} (\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n)_\Gamma.$$

Recall that we have introduced Eisenstein/cuspidal decompositions on both $H_1(\Gamma, \Delta^0 \otimes \mathbf{P}_n)$ and $\mathbf{MS}$. Let p^c be the projection onto the cuspidal part of $\mathbf{MS}^\vee$.

Theorem 2.10 *The morphism*

$$p^c \circ \Psi_\partial^{\mathrm{ord}} := p^c \circ \Psi^{\mathrm{ord}} \circ \partial \colon H_1(\Gamma, \Delta^0 \otimes \mathbf{P}_n) \to \mathbf{MS}^{c,\vee}$$

is surjective and induces an isomorphism

$$H_1(\Gamma, \Delta^0 \otimes \mathbf{P}_n)^c \xrightarrow{\simeq} \mathbf{MS}^{c,\vee}.$$

Proof The proof is the same as [23, Theorem 3.11] with obvious modifications and Lemma 2.6 in place of [23, Lemma 2.9]. ∎

Definition 2.11 The morphisms

$$\Phi^{\log}, \Phi^{\mathrm{ord}} : (\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n)_\Gamma \to \mathbf{MS}^{c,\vee}$$

are by definition $\Phi^* := p_c \circ \Psi^*$ with $* = \log, \mathrm{ord}$.

The above theorem allows us to define the Orton $\mathcal{L}$-invariant.

Corollary 2.12 *There exists a unique* $\mathcal{L} \in \mathrm{End}_{\mathbb{T}_{\mathbb{Q}_p}}(\mathbf{MS}^{c,\vee})$ *such that*

$$\Phi^{\log} \circ \partial = \mathcal{L} \circ \Phi^{\mathrm{ord}} \circ \partial \colon H_1(\Gamma, \Delta^0 \otimes \mathbf{P}_n) \to \mathbf{MS}^{c,\vee}.$$

Proof The corollary can be deduced from Theorem 2.10 exactly as [23, Corollary 3.13] is deduced from [23, Theorem 3.11]. ∎

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2.3 The Monodromy Module of Weight k_0 Modular Forms

Choose a sign w_∞ and set

$$\mathbf{D} = \mathbf{D}^{w_\infty} := \mathbf{MS}^{c,\vee,w_\infty} \oplus \mathbf{MS}^{c,\vee,w_\infty}.$$

Note that $\mathbf{D}$ is a free rank two module over $\mathbb{T}_{F_p}$ by Remark 2.4.

Define

$$\Phi := -\Phi^{\log} \oplus \Phi^{\mathrm{ord}} \colon \left(\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n\right)_\Gamma \to \mathbf{D}.$$

According to Corollary 2.12 and Theorem 2.10,

$$F^{m+1} := \left\{ (-\mathcal{L}x, x) : x \in \mathbf{MS}^{c,\vee,w_\infty} \right\} = \Im(\Phi \circ \partial) \tag{2.11}$$

is a free rank one $\mathbb{T}_{F_p}$-submodule.

Let σ be the absolute Frobenius automorphism of F_p^0. Write

$$\mathbf{D}(F_p^0) := \mathbf{MS}^{c,\vee,w_\infty}(F_p^0)^2.$$

Then we can consider the σ-linear automorphism $\sigma_{\mathbf{D}} := 1 \otimes \sigma$ on

$$\mathbf{D}(F_p^0) = \mathbf{D}(\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} F_p^0.$$

We define a structure of filtered Frobenius module over F_p on $\mathbf{D}$ as follows.

(a) The filtration is

$$\mathbf{D} = F^0 \supsetneq F^1 = \cdots = F^{k-1} \supsetneq F^k = 0,$$
$$F^{m+1} = \left\{ (-\mathcal{L}x, x) : x \in \mathbf{MS}^{c,\vee,w_\infty} \right\}, \text{ for } m := (k-2)/2.$$

(b) The Frobenius operator φ is defined on $\mathbf{D}(F_p^0)$ by the equation $\varphi = U_p \otimes \sigma_{\mathbf{D}} \oplus pU_p \otimes \sigma_{\mathbf{D}}$, *i.e.*,

$$\varphi(x, y) := \left(U_p \sigma_{\mathbf{D}}(x), pU_p \sigma_{\mathbf{D}}(y)\right).$$

(c) The monodromy operator N is defined on $\mathbf{D}(F_p^0)$ by the rule $N(x, y) = (y, 0)$.

It is easily checked that the above conditions do indeed define a filtered Frobenius module structure on $\mathbf{D}$, defined over $\mathbb{Q}_p$ if we have taken $F_p = \mathbb{Q}_p$. The filtered Frobenius module $\mathbf{D}$ over F_p is indeed obtained from the one over $\mathbb{Q}_p$ by base change from $MF_{\mathbb{Q}_p}(\phi, N)$ to $MF_{F_p}(\phi, N)$. Since the Hecke algebra $\mathbb{T}$ is commutative, every element of this ring commutes with φ and N. Furthermore, $F^{m+1} \subset \mathbf{D}$ is a (rank one) $\mathbb{T}_{F_p}$-submodule. Indeed, $\mathbf{D}$ is a rank two $\mathbb{T}_{\mathbb{Q}_p}$-monodromy module over F_p.

Let $f \in S_k(\Gamma_0(pM))$ be a normalized p-new weight k eigenform. Denote by $I_f^{w_\infty} \in \mathcal{MS}^k_{\Gamma_0(M)}(K_f)$ the modular symbol attached to the choice of the sign w_∞ that was chosen to define $\mathbf{D}$, appropriately normalized by means of Proposition 2.1. Let $K_{[f]}$ be the composition of the fields K_{f^σ}, where f^σ is the modular form obtained from f by applying the automorphism $\sigma \in G_{\mathbb{Q}}$ to the Fourier coefficients of f. Up to extending F_p we can fix an embedding $K_{[f]} \subset F_p$.

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Let

$$\mathbf{MS}_f^{c,w_\infty} := F_p I_f^{w_\infty} \hookrightarrow \mathbf{MS}^{c,w_\infty}$$

be the f-eigencomponent of $\mathbf{MS}^{c,w_\infty}$ on which the Hecke algebra acts through

$$\mathbb{T} \to K_f \subset F_p. \tag{2.12}$$

Write

$$\mathbf{MS}_{[f]}^{c,w_\infty} := \bigoplus_\sigma \mathbf{MS}_{f^\sigma}^{c,w_\infty}.$$

Note that the above sum can be indexed by the $[K_f : \mathbb{Q}]$ embeddings of K_f in $\overline{\mathbb{Q}}$. The inclusion $\mathbf{MS}_{[f]}^{c,w_\infty} \subset \mathbf{MS}^{c,w_\infty}$ gives rise to a morphism

$$e_{[f]} \colon \mathbf{D} \twoheadrightarrow \mathbf{D}_{[f]},$$

where we define

$$\mathbf{D}_{[f]} := \mathbf{MS}_{[f]}^{c,\vee,w_\infty} \oplus \mathbf{MS}_{[f]}^{c,\vee,w_\infty}.$$

We also note that $\mathbf{D}_{[f]} = \bigoplus_\sigma \mathbf{D}_{f^\sigma}$, where $\mathbf{D}_f$ is similarly defined.

Hence we can consider

$$\Phi_{[f]} \colon \left(\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n\right)_\Gamma \xrightarrow{\Phi} \mathbf{D} \overset{e_{[f]}}{\twoheadrightarrow} \mathbf{D}_{[f]}.$$

Since $\mathbf{MS}_{[f]}^{c,w_\infty} \subset \mathbf{MS}^{c,w_\infty}$ is a Hecke submodule, setting $F_{[f]}^{m+1} := e_{[f]}(F^{m+1})$, it is easily checked that $\mathbf{D}_{[f]}$ gets a structure of filtered F_p-vector space with multiplication by $K_f \otimes \mathbb{Q}_p$. The same remark applies to $\mathbf{D}_f$, the Hecke algebra acting though (2.12). In this way

$$\mathbf{D}_{[f]} = \bigoplus_\sigma \mathbf{D}_{f^\sigma} \tag{2.13}$$

is a decomposition of filtered F_p-vector spaces endowed with multiplication by the Hecke algebra.

We write $\mathcal{L}_{[f]} \in \mathrm{End}_{\mathbb{T}_{\mathbb{Q}_p}}(\mathbf{MS}_{[f]}^{c,w_\infty})$ (resp. $\mathcal{L}_f \in \mathrm{End}_{\mathbb{T}_{\mathbb{Q}_p}}(\mathbf{MS}_f^{c,w_\infty})$) to denote the $\mathcal{L}$-invariant corresponding to the modular form f (of course depending a priori on the choice of w_∞), *i.e.,* the image of $\mathcal{L}$ acting on $\mathbf{MS}_{[f]}^{c,w_\infty}$ (resp. $\mathbf{MS}_f^{c,w_\infty}$). It is also characterized by exploiting a property similar to the one in Corollary 2.12 (see [23, Section 4.3] for details). We have $\mathcal{L}_{[f]} \in K_f \otimes \mathbb{Q}_p$, and $\mathcal{L}_f \in F_p$ is its image under (2.12). Then we have

$$F_{[f]}^{m+1} = \{(-\mathcal{L}_{[f]}x, x) : x \in \mathbf{MS}_{[f]}^{c,\vee,w_\infty}\} \subset \mathbf{D}_{[f]},$$
$$F_f^{m+1} = \{(-\mathcal{L}_f x, x) : x \in \mathbf{MS}_f^{c,\vee,w_\infty}\} \subset \mathbf{D}_f.$$

Remark 2.13 Indeed, $\mathbf{D}_{[f]}$ has a natural $\mathbb{Q}_p$-structure compatible that can be used to define on $\mathbf{D}_{[f]}$ the structure of a $K_f \otimes \mathbb{Q}_p$-monodromy module over $\mathbb{Q}_p$, *i.e.,* we could have taken $F_p = \mathbb{Q}_p$. In this way $e_{[f]}$ becomes an epimorphism in $MF_{F_p}(\phi, N)$

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On the other hand $\mathbf{D}_f$ is only defined when $K_f \subset F_p$. Assuming $K_{[f]} \subset F_p$, the decomposition (2.13) of filtered F_p-vector spaces endowed with multiplication by the Hecke algebra produces a decomposition

$$\mathbf{D}_{[f]}/F_{[f]}^{m+1} = \bigoplus_{\sigma} \mathbf{D}_{f^\sigma}/F_{f^\sigma}^{m+1} \tag{2.14}$$

of the tangent space of $\mathbf{D}_{[f]} \in MF_{F_p}(\phi, N)$. The Hecke algebra acts through (2.12) on the f-component.

We can also consider

$$\Phi_{[f]} \colon (\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n)_\Gamma \xrightarrow{\Phi} \mathbf{D}/F^{m+1} \overset{e_{[f]}}{\twoheadrightarrow} \mathbf{D}_{[f]}/F^{m+1}.$$

The same construction holds for the inclusion $\mathbf{MS}_f^{c,w_\infty} \subset \mathbf{MS}^{c,w_\infty}$ and produces the analogous morphisms Φ_f. We will write e_f to denote the projection onto the f-component.

2.4 The p-adic Abel–Jacobi Maps in the Darmon Setting

Consider the exact sequence

$$\cdots \longrightarrow H_i(\Gamma, \Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n) \longrightarrow H_i(\Gamma, \Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n) \longrightarrow H_i(\Delta^0 \otimes \mathbf{P}_n) \longrightarrow \cdots$$

obtained from the short exact sequence

$$0 \to \Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n \to \Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n \to \Delta^0 \otimes \mathbf{P}_n \to 0.$$

Let $\mathbf{V}$ be any F_p-vector space, regarded as a trivial Γ-module. The application of $\mathrm{Hom}(-, \mathbf{V})$ produces the exact sequence

$$\begin{aligned}\mathrm{Hom}\big((\Delta^0 \otimes \mathbf{P}_n)_\Gamma, \mathbf{V}\big) &\to \mathrm{Hom}\big((\Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n)_\Gamma, \mathbf{V}\big) \to \\ &\mathrm{Hom}\big((\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n)_\Gamma, \mathbf{V}\big) \to \mathrm{Hom}\big(H_1(\Gamma, \Delta^0 \otimes \mathbf{P}_n), \mathbf{V}\big)\end{aligned} \tag{2.15}$$

It is convenient to give the following definition.

Definition 2.14 A $\mathbf{V}$-valued definite integration theory is an element

$$\Phi \in \mathrm{Hom}\Big(\big(\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n\big)_\Gamma, \mathbf{V}\Big).$$

A $\mathbf{V}$-valued semidefinite integration theory lifting Φ is an element

$$\Phi^{AJ} \in \mathrm{Hom}\Big(\big(\Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n\big)_\Gamma, \mathbf{V}\Big)$$

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such that the image via the middle arrow of (2.15) is Φ. One can also define the notion of $\mathbf{V}$-valued, positive oriented, definite integration theory and the notion of $\mathbf{V}$-valued, positive oriented, semidefinite integration theory by means of the exact sequence

$$(2.16)\quad 0 \to \Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}\,+}) \otimes \mathbf{P}_n \to \Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p^{\mathrm{ur}\,+}) \otimes \mathbf{P}_n \to \Delta^0 \otimes \mathbf{P}_n \to 0,$$

where $\mathcal{H}_p^{\mathrm{ur}\,+}$ denotes the subset of those $\tau \in \mathcal{H}_p^{\mathrm{ur}}$ whose reduction $\mathrm{red}(\tau) \in \mathcal{V}^+$.

In particular we can consider the $\mathbf{D}/F^{m+1}$-valued integration theory obtained by Φ followed by the projection onto the quotient $\mathbf{D}/F^{m+1}$, that we will denote again by the same symbol by abuse of notation:

$$\Phi\colon (\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n)_\Gamma \to \mathbf{D}/F^{m+1}.$$

Definition 2.15 A *p-adic Abel–Jacobi map* (in the Darmon setting) is any $\mathbf{D}/F^{m+1}$-valued semidefinite integration theory lifting the above integration theory Φ (eventually positive oriented).

Proposition 2.16 *There exists a $\mathbf{D}/F^{m+1}$-valued semidefinite integration theory Φ^{AJ} lifting the $\mathbf{D}/F^{m+1}$-valued integration theory Φ. In particular the restriction of Φ^{AJ} to $(\Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p^{\mathrm{ur}\,+}) \otimes \mathbf{P}_n)_\Gamma$ provides a $\mathbf{D}/F^{m+1}$-valued positive oriented semidefinite integration theory lifting the restriction of Φ to the group $(\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}\,+}) \otimes \mathbf{P}_n)_\Gamma$.*

Proof The claim follows from (2.15) specialized to $\mathbf{V} = \mathbf{D}/F^{m+1}$, in light of (2.11). ∎

Remark 2.17 One of the main differences with the weight 2 setting, as well as with the cohomological approach followed in [23], is in the lack of uniqueness of a semidefinite integration theory. In fact, note that two different liftings differ by an element of

$$\mathrm{Hom}\big((\Delta^0 \otimes \mathbf{P}_n)_\Gamma, \mathbf{V}\big),$$

as follows from the exactness of (2.15). In any case we will be able to define the p-adic Abel–Jacobi image of the Darmon cycles $j_\Psi \in (\Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n)_\Gamma$ by showing that $\Phi^{AJ}(j_\Psi)$ does not depend on the choice of the p-adic Abel–Jacobi map Φ^{AJ} (see Proposition 2.22).

2.5 Darmon Cycles

Let $K/\mathbb{Q}$ be a real quadratic field of discriminant D_K and recall our factorization $N = pM$. We make the following assumption.

Assumption 2.18 (Darmon hypothesis) The prime p is inert in K, while the primes dividing M are split.

Choose embeddings $\sigma\colon K \to \mathbb{R}$ and $\sigma_p\colon K \to K_p$ that we will use to regard K like a subfield of both $\mathbb{R}$ and the quadratic unramified extension $\mathbb{Q}_{p^2}/\mathbb{Q}_p$. In particular

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the inequalities involving $<$ make sense between elements of K, and we can consider $\mathcal{H}_p(K) := \mathcal{H}_p \cap K$. We may also view $\sqrt{D_K} \in K_p$ via σ_p. We denote by $\mathcal{E}mb = \mathcal{E}mb(K, \mathbb{M}_2(\mathbb{Q}))$ the set of all the $\mathbb{Q}$-algebra embeddings of K into $\mathbb{M}_2(\mathbb{Q})$. Whenever $\mathcal{O}$ is a $\mathbb{Z}[1/p]$-order of conductor c prime to $D_K N$ we also denote by $\mathcal{E}mb(\mathcal{O}, \mathcal{R})$ the set of $\mathbb{Z}[1/p]$-embeddings of $\mathcal{O}$ into our fixed Eichler $\mathbb{Z}[1/p]$-order $\mathcal{R}$. Define the $\mathbb{Z}[1/p]$-order associated with $\Psi \in \mathcal{E}mb$ as being $\mathcal{O}_\Psi := \Psi^{-1}(\mathcal{R})$, so that for every fixed $\mathbb{Z}[1/p]$-order $\mathcal{O}$ as above $\mathcal{E}mb(\mathcal{O}, \mathcal{R}) \subset \mathcal{E}mb$ is the subset of those $\Psi \in \mathcal{E}mb$ such that $\mathcal{O}_\Psi = \mathcal{O}$. Attached to the embedding $\Psi \in \mathcal{E}mb$ there are the following data:

- the two fixed points $\tau_\Psi, \overline{\tau}_\Psi \in \mathcal{H}_p$ for the action of $\Psi(K^\times)$ on $\mathcal{H}_p(K)$, ordered in such a way that the action of $K^\times$ on the tangent space at τ_Ψ is through the character $z \mapsto z/\overline{z}$;
- the unique fixed vertex $v_\Psi \in \mathcal{V}$ for the action of $\Psi(K^\times)$ on $\mathcal{V}$, which is nothing but the reduction $\mathrm{red}(\tau_\Psi) = \mathrm{red}(\overline{\tau}_\Psi)$;
- the unique polynomial up to sign P_Ψ in $\mathbf{P}_2$, which is fixed by the action of $\Psi(K^\times)$ on $\mathbf{P}_2 \otimes \det^{-1}$ and satisfies $\langle P_\Psi, P_\Psi \rangle_{\mathbf{P}_2} = -D_K/4$ (the pairing being defined as in [5]), which we fix by the choice
$$P_\Psi := \mathrm{Tr}\Big(\Psi(\sqrt{D_K}/2) \cdot \Big(\begin{smallmatrix} X & -X^2 \\ 1 & -X \end{smallmatrix}\Big)\Big) \in \mathbf{P}_2.$$
The other one is obtained replacing $\sqrt{D_K}/2$ with $-\sqrt{D_K}/2$;
- the stabilizer Γ_Ψ of Ψ in Γ, which is nothing but
$$\Gamma_\Psi = \Psi(K^\times) \cap \Gamma = \Psi(\mathcal{O}_1^\times),$$
where $\mathcal{O}_1^\times$ stands for the subgroup of $\mathcal{O}^\times$ of norm 1 and $\mathcal{O} = \mathcal{O}_\Psi$ is the associated order;
- the generator $\gamma_\Psi \in \Gamma_\Psi/\{\pm 1\} \simeq \mathbb{Z}$, which is the image $\gamma_\Psi := \Psi(u)$ of the unique generator of $u \in \mathcal{O}_1^\times$ such that $\sigma(u) > 1$.

For each $\tau \in \mathcal{H}_p(K) := \mathcal{H}_p \cap K$ (use σ_p to view K as a subfield of K_p), we say that τ has positive orientation at p if $\mathrm{red}(\tau) \in \mathcal{V}^+$. We write $\mathcal{H}_p^+(K)$ to denote the set of positive oriented elements in $\mathcal{H}_p(K)$. We say that $\Psi \in \mathcal{E}mb^+ \subset \mathcal{E}mb$ has positive orientation whenever $v_\Psi \in \mathcal{V}^+$, *i.e.*, $\tau_\Psi, \overline{\tau}_\Psi \in \mathcal{H}_p^+(K)$. It is possible to introduce the notion of negative oriented embeddings and then we have $\mathcal{E}mb = \mathcal{E}mb^+ \sqcup \mathcal{E}mb^-$. We also denote by $\mathcal{E}mb^+(\mathcal{O}, \mathcal{R})$ the subset of positive oriented embeddings of conductor c. The group Γ naturally acts on $\mathcal{E}mb$ by conjugation, preserving all the subsets we introduced.

We note that the association $\Psi \mapsto (\tau_\Psi, P_\Psi, \gamma_\Psi)$ satisfies the following property under the conjugation action by $\gamma \in \Gamma$:

$$(\tau_{\gamma\Psi\gamma^{-1}}, P_{\gamma\Psi\gamma^{-1}}, \gamma_{\gamma\Psi\gamma^{-1}}) = (\gamma\tau_\Psi, \gamma P_\Psi := P_\Psi\gamma^{-1}, \gamma\gamma_\Psi\gamma^{-1}). \tag{2.17}$$

Once we fix $x \in \mathbb{P}^1(\mathbb{Q})$, we can consider

$$j\colon \mathcal{E}mb(\mathcal{O}, \mathcal{R}) \to \Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p) \otimes \mathbf{P}_n,$$
$$j_\Psi := (\gamma_\Psi x - x) \otimes \tau_\Psi \otimes D_K^{-\frac{k_0-2}{4}} P_\Psi^m.$$

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Lemma 2.19 *The image* $[j_\Psi]$ *of* j_Ψ *in* $(\Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n)_\Gamma$ *does not depend on the choice of* $y \in \Gamma x$ *that was made to define it. Furthermore it does not depend on the choice of a representative of the class* $[\Psi]$ *of* Ψ *in* $\mathcal{E}mb$*, so that the above association gives a well defined map*

$$j\colon \Gamma\backslash\mathcal{E}mb(\mathcal{O},\mathcal{R}) \to (\Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n)_\Gamma.$$

Proof The proof is easy. ∎

The following remark on the property of the data attached to $\Psi \in \mathcal{E}mb$ will be useful later.

Remark 2.20 We have

$$(\tau_{\overline{\Psi}}, P_{\overline{\Psi}}, \gamma_{\overline{\Psi}}) = (\overline{\tau}_\Psi, -P_\Psi, \gamma_\Psi^{-1}).$$

Indeed the equality $(\tau_{\overline{\Psi}}, P_{\overline{\Psi}}) = (\overline{\tau}_\Psi, -P_\Psi)$ is clear. To see that $\gamma_{\overline{\Psi}} = \gamma_\Psi^{-1}$, simply note that, since the norm of u is one, $u^{-1} = \overline{u}$. Thus,

$$\gamma_{\overline{\Psi}} := \overline{\Psi}(u) = \Psi(\overline{u}) = \Psi(u^{-1}) =: \gamma_\Psi^{-1}.$$

Definition 2.21 The Darmon cycle attached to the embedding Ψ is the element

$$[j_\Psi] \in (\Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n)_\Gamma,$$

also denoted by j_Ψ by abuse of notation.

The following proposition allows us to define the p-adic Abel–Jacobi image of the Darmon cycles.

Proposition 2.22 *For every* $\Phi \in \mathrm{Hom}((\Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n)_\Gamma, \mathbf{V})$*, let* Φ_i^{AJ} *with* $i = 1, 2$ *be two* $\mathbf{V}$*-valued indefinite integration theories lifting the integration theory* Φ*. Then we have*

$$\Phi_1^{AJ}\big(\big[(\gamma_\Psi x - x) \otimes \tau \otimes P\big]\big) = \Phi_2^{AJ}\big(\big[(\gamma_\Psi x - x) \otimes \tau \otimes P\big]\big),$$

for any $\tau \in \mathcal{H}_p$ *and any* $P \in K_p P_\Psi^m$*. In particular,*

$$\Phi_1^{AJ}\big([j_\Psi]\big) = \Phi_2^{AJ}\big([j_\Psi]\big).$$

The same result holds for positive oriented $\mathbf{V}$*-valued integration theories.*

Proof By Remark 2.17 $\Phi_1^{AJ} - \Phi_2^{AJ}$ belongs to $\mathrm{Hom}((\Delta^0 \otimes \mathbf{P}_n)_\Gamma, \mathbf{V})$. More explicitly, this simply means that we may write $(\Phi_1^{AJ} - \Phi_2^{AJ}) = \Delta \circ \pi$ for some $\Delta \in \mathrm{Hom}((\Delta^0 \otimes \mathbf{P}_n)_\Gamma, \mathbf{V})$, where π is the quotient map with source $(\Delta^0 \otimes \mathrm{Div}(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n)_\Gamma$ and target $(\Delta^0 \otimes \mathbf{P}_n)_\Gamma$. In other words, for every $x, y \in \mathbb{P}^1(\mathbb{Q})$, $\tau \in \mathcal{H}_p$ and every $P \in \mathrm{Hom}(\mathbf{P}_n, \mathbf{V})$,

$$\Phi_1^{AJ}\big((x-y) \otimes \tau \otimes P\big) - \Phi_2^{AJ}\big((x-y) \otimes \tau \otimes P\big) = \Delta(x - y \otimes P).$$

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We will show that every element $\Delta \in \operatorname{Hom}((\Delta^0 \otimes \mathbf{P}_n)_\Gamma, \mathbf{V})$ satisfies

$$\Delta((\gamma_\Psi x - x) \otimes P) = 0 \text{ for } P \in K_p P_\Psi^m,$$

from which the claim will follow in light of the above equality.

Consider the function

$$c_x \colon \gamma \to \Delta((\gamma x - x) \otimes -) \in \operatorname{Hom}(\mathbf{P}_n, \mathbf{V}).$$

It is a crossed homomorphism from Γ to $\operatorname{Hom}(\mathbf{P}_n, \mathbf{V})$, because $\mathbf{V}$ is endowed with the trivial Γ-action. Let $\mathbf{c}_x$ be the class of c_x in $H^1(\Gamma, \operatorname{Hom}(\mathbf{P}_n, \mathbf{V}))$.

Consider the exact sequence (2.2)

$$0 \to \operatorname{Hom}(\mathbf{P}_n, \mathbf{V}) \to \mathcal{BS}\big(\operatorname{Hom}(\mathbf{P}_n, \mathbf{V})\big) \to \mathcal{MS}\big(\operatorname{Hom}(\mathbf{P}_n, \mathbf{V})\big) \to 0.$$

We claim that $\mathbf{c}_x = -\delta\boldsymbol{\Delta}$, where we regard Δ as an element of

$$\operatorname{Hom}\big((\Delta^0 \otimes \mathbf{P}_n)_\Gamma, \mathbf{V}\big) = \operatorname{Hom}_\Gamma\big((\Delta^0 \otimes \mathbf{P}_n), \mathbf{V}\big) = \mathcal{MS}_\Gamma\big(\operatorname{Hom}(\mathbf{P}_n, \mathbf{V})\big),$$

and δ is the boundary map arising from the above exact sequence. Once we will have established this fact, the claim will follow from Lemma 2.5, since then we will know that $\mathbf{c}_x = -\delta\boldsymbol{\Delta} = 0$. But this means that there exists $\Lambda \in \operatorname{Hom}(\mathbf{P}_n, \mathbf{V})$ such that $\partial\Lambda = c$, *i.e.*, for every $\gamma \in \Gamma$ and every $P \in \mathbf{P}_n$,

$$\Delta\big((\gamma x - x) \otimes P\big) = c_x(\gamma)(P) = \Lambda(\gamma^{-1}P) - \Lambda(P).$$

But (2.17) implies that $K_p P_\Psi^m \subset \mathbf{P}_n^{\Gamma_\Psi}$; evaluating at $\gamma_\Psi x - x \otimes P$ with $P \in K_p P_\Psi^m$ gives $\Delta(\gamma_\Psi x - x \otimes P) = 0$.

Hence it remains to prove the equality $\mathbf{c}_x = -\delta\boldsymbol{\Delta}$. By definition, $\delta\Delta$ is obtained choosing an element $\widetilde{\Delta} \in \mathcal{BS}(\operatorname{Hom}(\mathbf{P}_n, \mathbf{V}))$ such that $\widetilde{\Delta}(x - y) = \Delta(x - y)$ for every degree zero divisor $x - y$ and then noticing that

$$\begin{aligned} \gamma \mapsto \gamma\widetilde{\Delta} - \widetilde{\Delta} &= (\gamma\widetilde{\Delta})(y) - \widetilde{\Delta}(y) \\ &= \widetilde{\Delta}(\gamma^{-1}y)(\gamma^{-1}-) - \widetilde{\Delta}(y)(-) \in \operatorname{Hom}(\mathbf{P}_n, \mathbf{V}) \end{aligned}$$

is a constant function, independent of the choice of the divisor y at which to evaluate it. Taking $y = \gamma x$ for any given γ, we find that the above cocycle is

$$\gamma \mapsto \widetilde{\Delta}(x)(\gamma^{-1}-) - \widetilde{\Delta}(\gamma x)(-) \in \operatorname{Hom}(\mathbf{P}_n, \mathbf{V}).$$

On the other hand, up to the identification

$$\begin{aligned} &\operatorname{Hom}(\Delta^0 \otimes \mathbf{P}_n, \mathbf{V}) = \mathcal{MS}(\operatorname{Hom}(\mathbf{P}_n, \mathbf{V})), \\ &\qquad c_x(\gamma)(P) = \Delta(\gamma x - x)(P) = \widetilde{\Delta}(\gamma x)(P) - \widetilde{\Delta}(x)(P). \end{aligned}$$

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Hence the sum $c_x + \delta\Delta$ to be considered is

$$\widetilde{\Delta}(\gamma x)(P) - \widetilde{\Delta}(x)(P) + \widetilde{\Delta}(x)(\gamma^{-1}P) - \widetilde{\Delta}(\gamma x)(P) = \widetilde{\Delta}(x)(\gamma^{-1}P) - \widetilde{\Delta}(x)(P),$$

and we have to show that this is a coboundary.

But now a coboundary in $H^1(\Gamma, \mathrm{Hom}(\mathbf{P}_n, \mathbf{V}))$ is of the form $\gamma \mapsto \gamma\Lambda - \Lambda$ with $\Lambda \in \mathrm{Hom}(\mathbf{P}_n, \mathbf{V})$, *i.e.*, $(\partial\Lambda)(\gamma)(P) = \Lambda(\gamma^{-1}P) - \Lambda(P)$. We can now take $\Lambda = \widetilde{\Delta}(x)(-) \in \mathrm{Hom}(\mathbf{P}_n, \mathbf{V})$ so that

$$(\partial(\widetilde{\Delta}(x)(-)))(\gamma)(P) = \widetilde{\Delta}(x)(\gamma^{-1}P) - \widetilde{\Delta}(x)(P).$$

The same proof applies for positive oriented $\mathbf{V}$-valued integration theories, exploiting the long exact sequence obtained from (2.16) in place of (2.15). ■

Now we are in the position to define the p-adic Abel–Jacobi image of the Darmon cycles.

Definition 2.23 The p-adic Abel–Jacobi image of the Darmon cycle attached to the embedding Ψ is the element

$$\Phi^{AJ}(j_\Psi) = \Phi^{AJ}([j_\Psi]) \in \mathbf{D}/F^{m+1},$$

where Φ^{AJ} is any p-adic Abel–Jacobi map.

As in [10] the set $\Gamma\backslash\mathcal{E}mb^{+}(\mathcal{O}, \mathcal{R})$ is naturally endowed with an action by the (narrow) Picard group $\mathrm{Pic}^{+}(\mathcal{O})$ attached to the order $\mathcal{O}$. The class field theory identifies canonically $\mathrm{Pic}^{+}(\mathcal{O})$ with the Galois group over K of the narrow ring class field $H^{+}_{\mathcal{O}}$.

$$\mathrm{rec}\colon\ \mathrm{Pic}^{+}(\mathcal{O}) \xrightarrow{\simeq} G_{H^{+}_{\mathcal{O}}/K}.$$

In this way $G_{H^{+}_{\mathcal{O}}/K}$ acts on $\Gamma\backslash\mathcal{E}mb^{+}(\mathcal{O}, \mathcal{R})$.

Remark 2.24 As in [10, after Lemma 5.7] it is possible to introduce the notion of oriented embeddings $\mathcal{E}mb^{+\mathfrak{d}}(\mathcal{O}, \mathcal{R})$ by fixing a homomorphism

$$\mathfrak{d}\colon \mathcal{O} \to \mathbb{Z}/M\mathbb{Z}.$$

Then Γ preserves $\mathcal{E}mb^{+\mathfrak{d}}(\mathcal{O}, \mathcal{R})$ so that it makes sense to consider the quotient $\Gamma\backslash\mathcal{E}mb^{+\mathfrak{d}}(\mathcal{O}, \mathcal{R})$, and this set becomes a torsor under the action of $\mathrm{Pic}^{+}(\mathcal{O})$. Furthermore, the Atkin–Lehner involution W_{l^e} at the primes dividing $l^e \parallel M$ transitively permutes the possible orientations, while the Atkin–Lehner involution W_p reverses the orientation at p.

Let $\chi\colon G_{H^{+}_{\mathcal{O}}/K} \to \mathbb{C}^{\times}$ be a character. It is convenient to introduce the following linear combination:

$$j^{\chi} := \sum_{\sigma \in G_{H^{+}_{\mathcal{O}}/K}} \chi^{-1}(\sigma) j_{\sigma\Psi} \in (\Delta^0 \otimes \mathrm{Div}(\mathcal{H}^{\mathrm{ur}}_p) \otimes \mathbf{P}_n)^{\chi}_{\Gamma}. \tag{2.18}$$

3 Review of the p-adic Abel-Jacobi Map in the Darmon Setting

Consider the following commutative diagram:

$$\begin{tikzcd}
(\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n)_\Gamma \arrow[r, "\Phi"] \arrow[d, equal] & \mathbf{D}/F^{m+1} \arrow[d, "f"] \\
(\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n)_\Gamma \arrow[r, "\log\Phi"] & \mathbf{MS}^{c,\vee,w_\infty}
\end{tikzcd} \tag{3.1}$$

where:

- $f(x, y) = -x - \mathcal{L}y$, which is easily checked to be well defined, *i.e.*, $f(F^{m+1}) = 0$, and is an isomorphism;
- $\log\Phi := \Phi^{\log} - \mathcal{L}\Phi^{\mathrm{ord}}$.

Note also that, by Corollary 2.12,

$$\log\Phi \circ \partial = (\Phi^{\log} - \mathcal{L}\Phi^{\mathrm{ord}}) \circ \partial = 0,$$

a fact that can also be deduced by the commutativity of the diagram and the equality $\Phi \circ \partial = 0$. Since f is an isomorphism we can identify Φ and $\log\Phi$. It is clear that the above discussion applies to $\mathbf{D}_{[f]}$ or $\mathbf{D}_f$ when f is a modular form. Hence we will write $\log\Phi_f = e_f \circ \log\Phi$.

We will use the following notation for the branches of p-adic logarithm. We let $\log_0$ be the branch of the p-adic logarithm such that $\log_0(p) = 0$, and for every $\lambda \in F_p$ we let

$$\log_\lambda := \log_0 - \lambda\,\mathrm{ord}_p \colon F_p^\times \to F_p$$

be the branch of the p-adic logarithm such that $\log_\lambda(p) = -\lambda$.

Note that the definition of the monodromy module $\mathbf{D}$, as well as Φ, depends in a crucial way on the choice of a branch of the p-adic logarithm, since $\Phi^{\log}$ depends on this choice. Write $\Phi^{\log_\lambda}$,$\mathcal{L}^\lambda$, Φ^λ, and $\log\Phi^\lambda$ to emphasize the dependence on this choice. The dependence on λ appears in $\mathbf{D}$ in the definition of the filtration, so that we write F_λ^{m+1}.

Proposition 3.1 *For every $\lambda \in F_p$*

$$\Phi^{\log_\lambda} = \Phi^{\log_0} - \lambda\Phi^{\mathrm{ord}} \in \mathrm{Hom}\Big(\big(\Delta^0 \otimes \mathrm{Div}^0(\mathcal{H}_p^{\mathrm{ur}}) \otimes \mathbf{P}_n\big)_\Gamma, \mathbf{MS}^{c,\vee,w_\infty}\Big).$$

Proof We need to evaluate $\Phi^{\log_\lambda}(x - y \otimes \tau_2 - \tau_1 \otimes P)$ at $m \in \mathbf{MS}^{c,\vee,w_\infty}$ in order to prove the proposition. By definition:

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_\lambda\Big(\frac{t-\tau_2}{t-\tau_1}\Big) P(t)dm\{x \to y\}(t) =$$
$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_0\Big(\frac{t-\tau_2}{t-\tau_1}\Big)P(t)dm\{x \to y\}(t)$$
$$+ -\lambda \int_{\mathbb{P}^1(\mathbb{Q}_p)} \mathrm{ord}_p\Big(\frac{t-\tau_2}{t-\tau_1}\Big)P(t)dm\{x \to y\}(t).$$

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Thus we need to check the formula

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} \operatorname{ord}_p\left(\frac{t-\tau_2}{t-\tau_1}\right)P(t)dm\{x\to y\}(t) = \sum_{e:\ v_1\to v_2} \int_{U_e} P(t)dm\{x\to y\}(t).$$

The proof of [6, Lemma 2.5] gives the claim. ■

Lemma 3.2 *For every $\lambda\in F_p$, $\mathcal{L}^\lambda = \mathcal{L}^0 - \lambda$.*

Proof Proposition 3.1 implies, in light of Corollary 2.12

$$\begin{aligned}\mathcal{L}^\lambda\circ\Phi^{\mathrm{ord}}\circ\partial &= \Phi^{\log_\lambda}\circ\partial = \Phi^{\log_0}\circ\partial - \lambda\Phi^{\mathrm{ord}}\circ\partial\\ &= \mathcal{L}^0\circ\Phi^{\mathrm{ord}}\circ\partial - \lambda\Phi^{\mathrm{ord}}\circ\partial\\ &= (\mathcal{L}^0-\lambda)\circ\Phi^{\mathrm{ord}}\circ\partial,\end{aligned}$$

the equality taking place in $\operatorname{Hom}(H_1(\Gamma,\Delta^0\otimes\mathbf{P}_n),\mathbf{MS}^{c,\vee,w_\infty})$. Now the claim follows from Theorem 2.10, arguing as in Corollary 2.12. ■

Suppose that in diagram (3.1) we have chosen the standard branch $\log_0$ of the p-adic logarithm. Choosing a different branch $\log_\lambda$ of the p-adic logarithm we find, thanks to Proposition 3.1 and Lemma 3.2,

- $f^\lambda(x,y) = -x-(\mathcal{L}^0-\lambda)y$;
- $\log\Phi^\lambda = \Phi^{\log_\lambda} - \mathcal{L}^\lambda\Phi^{\mathrm{ord}} = \log\Phi^0$.

In particular we see that $\log\Phi^\lambda$ does not depend on the choice of a branch of the p-adic logarithm.

Assume now that f is a new modular form. We have $\mathcal{L}_f$, and

$$\mathcal{L}_{[f]}\in\operatorname{End}_{\mathbb{T}_{\mathbb{Q}_p}}\left(\mathbf{MS}_{[f]}^{c,\vee,w_\infty,}\right)$$

acts diagonally via the matrix $\operatorname{diag}(\mathcal{L}_{f^\sigma}:\sigma)$ on $\mathbf{MS}_{[f]}^{c,w_\infty}$ with respect to the decomposition (2.14). Choosing the branch of the p-adic logarithm $\lambda=\mathcal{L}_f^0$ so that $\mathcal{L}_f^\lambda=0$, the above expressions simplify and become

- $f^\lambda(x,y)=-x$;
- $\log\Phi_f^0 = \log\Phi_f^\lambda = \Phi_f^{\log_\lambda}$.

Also recall that f^λ is an isomorphism.

Proposition 3.3 *Let $f\in S_{k_0}(\Gamma_0(N))$ be a new modular form. Then the $\mathbf{D}_f/F_0^{m+1}$-valued integration theory Φ_f^0 is equivalent via f^0 to the $\mathbf{MS}_f^{c,w_\infty}$-valued integration theory*

$$\log\Phi_f^0 = \Phi_f^{\log_{\mathcal{L}_f^0}}.$$

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4 Families of Modular Forms and Families of Modular Symbols

Let $\mathcal{W} := \mathrm{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{G}_m)$ be the weight space, viewed as a rigid analytic space over $\mathbb{Q}_p$, and suppose for simplicity that $p \neq 2$. The integers $\mathbb{Z}$ are embedded in $\mathcal{W}$ by sending the integer k to the function $t \mapsto t^{k-2}$. Note that this normalization follows [2] but not [4], where the integer k is sent to the function $t \mapsto t^k$. If $U \subset \mathcal{W}$ is an open affinoid defined over the local field K_p, every $\kappa \in U(K_p)$ can be uniquely written as a product $\kappa(t) = \varepsilon(t)\chi(t)\langle t\rangle^s$, where $\varepsilon\colon \mathbb{Z}_p^\times \to K_p^\times$ is a character of order $p-1$, $\chi\colon \mathbb{Z}_p^\times \to K_p^\times$ is a character of order p, and $s \in \mathcal{O}_{K_p}$. We can uniquely write every element of $\mathbb{Z}_p^\times$ as a product $t = [t]\langle t\rangle$, where $[t] \in \mu_{p-1}$, the group of $p-1$-roots of unity, and $\langle t\rangle \in 1 + p\mathbb{Z}_p$. With our normalization an integer $k \in U$ corresponds to the character $k(t) = [t]^{k-2}\langle t\rangle^{k-2}$, *i.e.,* $\varepsilon(t) = [t]^{k-2}$, $\chi = 1$, and $s = k-2$. In general, up to shrinking U in a neighbourhood of $k_0 \in \mathbb{Z}$, we can assume $\varepsilon(t) = [t]^{k_0-2}$ and $\chi = 1$ for every $\kappa \in U(K_p)$, so that $\kappa(t) = [t]^{k_0-2}\langle t\rangle^s$. In this case we also set $(\kappa/2)(t) := [t]^{\frac{k_0-2}{2}}\langle t\rangle^{\frac{s}{2}}$. Then we define, for every $\alpha \in \mathbb{Q}_p^{nr,\times}$,

$$\begin{aligned}
\langle\alpha\rangle^{\kappa-k_0} &:= \langle\alpha\rangle^{s-k_0+2} = \exp\big((s-k_0+2)\log_0(\alpha)\big),\\
\langle\alpha\rangle^{\frac{\kappa-k_0}{2}} &:= \langle\alpha\rangle^{\frac{s}{2}-\frac{k_0-2}{2}},\\
\langle\alpha\rangle^{\kappa-\kappa/2-1} &:= \langle\alpha\rangle^{\kappa-k_0}\big(\langle\alpha\rangle^{\frac{\kappa-k_0}{2}}\big)^{-1}\langle\alpha\rangle^{\frac{k_0-2}{2}},\\
\langle\alpha\rangle^{\kappa/2-1} &:= \langle\alpha\rangle^{\frac{\kappa-k_0}{2}}\langle\alpha\rangle^{\frac{k_0-2}{2}}.
\end{aligned}$$

Note that the first two expressions make sense for every $\kappa \in U$, since $\langle\alpha\rangle \in 1 + p\mathcal{O}_{\mathbb{Q}_p^{nr}}$ and $\log_0(\alpha) \in p\mathcal{O}_{\mathbb{Q}_p^{nr}}$ (since $p \neq 2$ the exponential converges in $p\mathcal{O}_{\mathbb{C}_p}$); the subsequent two expressions are defined using the other two, *i.e.,* $(-)^{-1}$ and $\alpha^{\frac{k_0-2}{2}}$ have the obvious meaning.

We fix the following notation to be in force for the rest of this paper. We let $W := \mathbb{Q}_p^2 - \{0\}$ be the set of non-zero vectors in $\mathbb{Q}_p^2$ and consider the natural continuous (for the p-adic topologies) projection

$$\pi\colon W \to \mathbb{P}^1(\mathbb{Q}_p)$$
$$\pi((x,y)) := x/y.$$

For any $\mathbb{Z}_p$-lattice L in $\mathbb{Q}_p^2$, we denote by $L' := L - pL$ the set of primitive vectors of L and we write $|L| := p^{\mathrm{ord}_p(\det B)}$ for B any $\mathbb{Z}_p$-basis of L. Recall that we let $L_* := \mathbb{Z}_p^2$ be the standard $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^2$ and we set $L_\infty := \mathbb{Z}_p \oplus p\mathbb{Z}_p$. Recall the Bruhat–Tits tree $\mathcal{T}$ whose set of oriented edges we denoted by $\mathcal{E} = \mathcal{E}(\mathcal{T})$. If $e \in \mathcal{E}$, let $L_{s(e)}$ and $L_{t(e)}$ be lattices whose homothety classes represent the source and the target of e, chosen in such a way that $L_{s(e)} \supset L_{t(e)}$ with index p. To the edge e we associate the open compact subsets $W_e \subset W$ and $U_e \subset \mathbb{P}^1(\mathbb{Q}_p)$, defined by the rules

$$W_e := L'_{s(e)} \cap L'_{t(e)} \quad\text{and}\quad U_e := \pi(W_e).$$

We remark that W_e depends on the choices of $L_{s(e)}$ and $L_{t(e)}$, so that W_e is well defined (as a function of e) up to multiplication by elements of $\mathbb{Q}_p^\times$. On the other hand

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U_e is well defined and in fact it is the set of ends originating from e, when making the canonical identification $\mathcal{E}^\infty(\mathcal{T}) = \mathbb{P}^1(\mathbb{Q}_p)$ between ends of $\mathcal{T}$ and $\mathbb{P}^1(\mathbb{Q}_p)$. In particular we recall that these subsets U_e form a basis for the p-adic topology of $\mathbb{P}^1(\mathbb{Q}_p)$. We write $W_\infty = L'_* \cap L'_\infty$ to denote the set W_e obtained from the edge $e_\infty = (v_*, v_\infty)$, where $v_* = [L_*]$ and $v_\infty = [L_\infty]$.

For every open compact subset $X \subset \mathbb{Q}_p^2$ or $X \subset \mathbb{P}^1(\mathbb{Q}_p)$, write $\mathcal{A}(X)$ for the $\mathbb{Q}_p$-space of locally analytic functions on X, as defined in [4, Sec. 2]. Denote by $\mathcal{D}(X) := \mathrm{Hom}_{cont}(\mathcal{A}(X), \mathbb{Q}_p)$ the continuous $\mathbb{Q}_p$-dual space, called the space of locally analytic distributions on X. As usual, for any $\mu \in \mathcal{D}(X)$ and $F \in \mathcal{A}(X)$, we write $\int_X F d\mu$ to denote the value of μ at F; then it is clear what we mean by $\int_Y F d\mu$ for any open compact subset $Y \subset X$.

We let $\mathbb{GL}_2(\mathbb{Q}_p)$ act on the left on $\mathbb{Q}_p^2$ by viewing elements of $\mathbb{Q}_p^2$ as column vectors. There is an induced action on W and $\mathcal{T}$ as well as an induced action of the subgroup $\mathbb{GL}_2(\mathbb{Z}_p)$ on L'_*; the action of the scalar matrices $\mathbb{Z}_p^\times$ on W preserves the set L' for any lattice L and will be denoted as $t(x, y) := (tx, ty)$.

It follows that $\mathcal{A}(L'_*)$ is endowed with a right $\mathbb{GL}_2(\mathbb{Z}_p)$-action, and its continuous dual $\mathbb{D} := \mathcal{D}(L'_*)$ is endowed with a natural left $\mathbb{GL}_2(\mathbb{Z}_p)$-action. Denote by $R := \mathcal{D}(\mathbb{Z}_p^\times)$ the space of locally analytic distributions on $\mathbb{Z}_p^\times$.

There is a natural R-module structure on $\mathbb{D}$,

$$R \times \mathbb{D} \to \mathbb{D} \; (r, \mu) \mapsto r\mu,$$

defined by the formula

$$\int_{L'_*} F(x, y) d(r\mu)(x, y) := \int_{\mathbb{Z}_p^\times} \Big(\int_{L'_*} F(tx, ty) d\mu(x, y) \Big) dr(t).$$

Fix an integer $k \geq 0$ and let $U \subset \mathcal{W}$ be an affinoid disk such that $k \in U$, defined over a finite extension K_p of $\mathbb{Q}_p$. We can define a structure of R-algebra on the K_p-affinoid algebra $A(U)$ of U by means of the formula

$$r \mapsto \Big[\kappa \mapsto \int_{\mathbb{Z}_p^\times} \kappa(t) dr(t) \Big].$$

We denote by $\mathbb{D}_U := A(U) \widehat{\otimes}_R \mathbb{D}$ the completed tensor product over R. Now fix any $\kappa \in U$ and define, for any $\mathbb{Z}_p^\times$-stable open compact $X \subset \mathbb{Q}_p^2$:

$$\mathcal{A}^{(\kappa)}(X) := \{ F \in \mathcal{A}(X) : F(tx, ty) = \kappa(t) F(x, y) \text{ for all } t \in \mathbb{Z}_p^\times \}.$$

In [4, Section 3] it is explained how to define a continuous R-bilinear map

$$\int_X : \mathcal{A}^{(\kappa)}(X) \times \mathbb{D}_U \to K_p$$

that we denote $\int_X F(x, y) d\mu_U$.

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For every integer $n \in \mathbb{Z}$ we will also be interested in the subspace $\mathcal{A}_n^{(\kappa)}(W) \subset \mathcal{A}^{(\kappa)}(W)$ consisting of those functions $F \in \mathcal{A}^{(\kappa)}(W)$ such that $F(px, py) = p^n F(x, y)$.

Finally note that for every homogeneous function $F \in \mathcal{A}_n^{(n)}(W)$ of degree n, we can consider the locally analytic function on $\mathbb{P}^1(\mathbb{Q}_p)$ with a pole of order at most n at ∞ defined by the rule $F(t) := F(t, 1)$. Conversely, given a locally analytic function on $\mathbb{P}^1(\mathbb{Q}_p)$ with a pole of order at most n at ∞, we can consider the homogeneous function of degree n defined by $F(x, y) := y^n F(x/y)$. In this way we establish a $\mathbb{GL}_2$-equivariant bijection between these spaces. The space $\mathbf{P}_n$, with the action previously considered, corresponds to the space of homogeneous polynomials of degree n, and we will denote again by $P = P(x, y)$ the polynomial attached to $P = P(t)$.

Lemma 4.1 *For all $\alpha \in \mathbb{Q}_p^{nr,\times}$, $\kappa \in U$, and $t \in \mathbb{Z}_p^{\times}$,*

- $\langle t\alpha\rangle^{\kappa-k_0} = \kappa(t)t^{-(k-2)}\langle\alpha\rangle^{\kappa-k_0}$;
- $\langle t\alpha\rangle^{\frac{\kappa-k_0}{2}} = (\kappa/2)(t)t^{-\frac{k_0-2}{2}}\langle\alpha\rangle^{\frac{\kappa-k_0}{2}}$;
- $\langle t\alpha\rangle^{\kappa-\kappa/2-1} = \kappa(t)(\kappa/2)^{-1}(t)\langle\alpha\rangle^{\kappa-\kappa/2-1}$;
- $\langle t\alpha\rangle^{\kappa/2-1} = (\kappa/2)(t)\langle\alpha\rangle^{\kappa/2-1}$.

Furthermore, for every $k \in \mathbb{Z} \cap U$ and for every $\alpha \in \mathbb{Z}_p^{\times}$, we have

- $\langle\alpha\rangle^{k-k_0} = \alpha^{k-k_0}$;
- $\langle\alpha\rangle^{\frac{k-k_0}{2}} = \alpha^{\frac{k-k_0}{2}}$;
- $\langle\alpha\rangle^{k-k/2-1} = \alpha^{k-k/2-1}$;
- $\langle\alpha\rangle^{k/2-1} = \alpha^{k/2-1}$.

Suppose that $X \subset L'_*$ is an open compact subset preserved by the action of $\mathbb{Z}_p^{\times}$. Whenever $\mu \in \mathbb{D}_U$, $P \in \mathbf{P}_n$ and $\alpha, \beta \in \mathcal{A}(X)$ satisfy $\alpha(tx) = t\alpha(x)$ and $\beta(tx) = t\beta(x)$, it makes sense to consider the following functions on U:

$$\kappa \mapsto \mu\big(P\langle\alpha\rangle^{\kappa-k_0}\chi_X\big),$$

$$\kappa \mapsto \mu\big(P\langle\alpha\rangle^{\frac{\kappa-k_0}{2}}\langle\beta\rangle^{\frac{\kappa-k_0}{2}}\chi_X\big),$$

$$\kappa \mapsto \mu\big(\langle\alpha\rangle^{\kappa/2-1}\langle\beta\rangle^{\kappa-\kappa/2-1}\chi_X\big).$$

These functions are analytic.

Proof One first has to check the homogeneity properties of $\langle\alpha\rangle^{\kappa-k}$ and $\langle\alpha\rangle^{\frac{\kappa-k_0}{2}}$, and then use their properties to check those of $\langle\alpha\rangle^{\kappa-\kappa/2-1}$ and $\langle\alpha\rangle^{\kappa/2-1}$. The second statement follows from the fact that, whenever $k(t) = [t]^{k-2}\langle t\rangle^{k-2}$ is an integer in U, we can assume $[t]^{k-2} = [t]^{k_0-2}$, so that $k \equiv k_0 \mod (p-1)$. It follows that $[\alpha]^{k-k_0} = 1$ whenever $\alpha \in \mathbb{Z}_p^{\times}$, and then

$$\langle\alpha\rangle^{k-k_0} := \langle\alpha\rangle^{k-2-k_0+2} = \langle\alpha\rangle^{k-k_0} = \langle\alpha\rangle^{k-k_0}[\alpha]^{k-k_0} = \alpha^{k-k_0}.$$

The claim $\langle\alpha\rangle^{\frac{k-k_0}{2}} = \alpha^{\frac{k-k_0}{2}}$ follows in a similar way, and the other two equations follow from the definition of $\langle\alpha\rangle^{\kappa-\kappa/2-1}$ and $\langle\alpha\rangle^{\kappa/2-1}$.

The fact that the above functions are well defined follows because $P\alpha^{\kappa-k}$, $P\alpha^{\frac{\kappa-k_0}{2}}\beta^{\frac{\kappa-k_0}{2}}$, and $\alpha^{\kappa/2-1}\beta^{\kappa-\kappa/2-1}$ belong to $\mathcal{A}^{(\kappa)}$, so that we can apply μ. Finally, to show that they are indeed analytic, one can follow [4, Lemma 4.5]. ■

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The following proposition will be useful for the computation of the derivatives of p-adic L-functions.

Proposition 4.2 *Let $k_0 \in \mathbb{Z}^{\geq 2}$ and $P \in \mathbf{P}_n$ with $n = k_0 - 2$. For every lattice L and every $\tau_1, \tau_2 \in \mathcal{H}_p$,*

$$\begin{aligned}\frac{d}{d\kappa}\Big(\int_{L'} P(x,y)\langle x-\tau_1 y\rangle^{\frac{\kappa-k}{2}}\langle x-\tau_2 y\rangle^{\frac{\kappa-k}{2}}dI\{r\to s\}(x,y)\Big)_{\kappa=k_0}\\ =\frac{1}{2}\frac{d}{d\kappa}\Big(\int_{L'} P(x,y)\langle x-\tau_1 y\rangle^{\kappa-k}dI\{r\to s\}(x,y)\Big)_{\kappa=k_0}\\ +\frac{1}{2}\frac{d}{d\kappa}\Big(\int_{L'} P(x,y)\langle x-\tau_2 y\rangle^{\kappa-k}dI\{r\to s\}(x,y)\Big)_{\kappa=k_0}.\end{aligned}$$

Proof Use the explicit formula of [4, Remark 4.7] for the derivatives appearing on the right-hand side and compare it with an analogous formula for the left-hand side. Note also that it makes sense to consider the derivatives in light of Lemma 4.1. ■

4.1 Families of Modular Symbols

We let f be a weight k_0 newform on the modular curve $X = X_0(N)$, where $N = pM$ is a factorization into prime factors and p is a prime. The Hecke operator at p acts on f with eigenvalues

$$f \mid U_p = \pm p^{\frac{k_0-2}{2}} f.$$

A p-adic family of modular forms deforming f is the data of an affinoid disk $U \subset \mathcal{W}$ in the weight space, such that $k_0 \in U$ and a formal q-expansion

$$f_\infty = \sum_n a_n(\kappa)q^n,\ a_n(\kappa) \in A(U)$$

such that:

- for every $k \in U \cap \mathbb{Z}^{\geq k_0}$, the specialization f_k is a weight k modular eigenform;
- $f_{k_0} = f$.

Since the slope of the U_p operator acting on f is strictly less than $k_0 - 1$, there exists such a family, which we assume to be an eigenfamily of modular forms of slope $(k_0-2)/2$, up to shrinking U (by [8, Corollary B5.7.1]). Note that whenever $k \neq k_0$, the modular form f_k is old at p. There is a unique normalized new eigenform $f_k^\# \in S_k(\Gamma_0(M))$ such that

$$f_k(z) = f_k^\#(z) - p^{k-1}a_p(k)^{-1}f_k^\#(pz). \tag{4.1}$$

We set $f_{k_0}^\# = f_{k_0}$.

Let $\widetilde{I}_k^\# \in \mathcal{MS}^k_{\Gamma_0(M)}(\mathbb{C})$ (resp. $\widetilde{I}_k \in \mathcal{MS}^k_{\Gamma_0(pM)}(\mathbb{C})$) be the modular symbol attached to $f_k^\#$ (resp. f_k) by rule (2.5). Recall the periods $\Omega_k^{\#\pm} \in \mathbb{C}$ attached to $f_k^\#$ by means of Proposition 2.1, allowing us to define the modular symbols

$$I_k^{\#\pm} := (\Omega_k^{\#\pm})^{-1}\widetilde{I}_k^\# \in \mathcal{MS}^{k,\pm}_{\Gamma_0(M)}(K_k).$$

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Here K_k is a short notation for the field generated by the Fourier coefficients of $f_k^{\#}$, which is equal to the field generated by the Fourier coefficients of f_k.

From now on we will choose a sign $w_\infty \in \{\pm 1\}$, which is compatible with the same choice that was made to construct the filtered Frobenius module **D**. We set

$$\Omega_k^{\#} := \Omega_k^{\# w_\infty} \text{ and } I_k^{\#} := I_k^{\# w_\infty}.$$

The right-hand side of formula (4.1) translates into the right-hand side of the subsequent formula, so that we may use to define the K_k-valued modular symbol symbol I_k, having the same eigenvalues as f_k:

$$I_k\{r \to s\}(P) = \widetilde{I}_k^{\#}\{r \to s\}(P) - p^{k-1} a_p(k)^{-1} \widetilde{I}_k^{\#}\{r/p \to s/p\}(P(x, y/p)). \tag{4.2}$$

Recall the space $\mathbb{D}_U := \mathbb{D}\widehat{\otimes}_R A(U)$ previously introduced. For each $k \in \mathbb{Z}^{\geq 2} \cap U$ define a weight k specialization map

$$\rho_k \colon \mathcal{MS}_{\Gamma_0(M)}(\mathbb{D}_U) \to \mathcal{MS}^k_{\Gamma_0(pM)}(\mathbb{C}_p)$$
$$\rho_k(I)\{r \to s\}(P) := \int_{W_\infty} P(x, y) dI\{r \to s\}(x, y).$$

The following result is due to G. Stevens.

Theorem 4.3 *There exists $I_\infty \in \mathcal{MS}_{\Gamma_0(M)}(\mathbb{D}_U)$ such that:*
- *for every $k \in \mathbb{Z}^{\geq 2} \cap U$, $\rho_k(I_\infty) = \lambda(k) I_k$ for some $\lambda(k) \in \mathbb{C}_p^\times$;*
- *$\rho_{k_0}(I_\infty) = I_{k_0}$.*

By Shapiro's Lemma the modular symbol $I_\infty \in \mathcal{MS}_{\Gamma_0(M)}(\mathbb{D}_U)$ gives rise to a family of distributions $\{I_L\}_{L \subset \mathbb{Q}_p^2}$ indexed by the lattices in $\mathbb{Q}_p^2$, which is $\widetilde{\Gamma}$-invariant for the natural action of $\widetilde{\Gamma}$ on the induced module$\mathcal{C}(\mathcal{L}, \mathcal{MS}_{\Gamma_0(M)}(\mathcal{D}_U(*)))$ of maps I_* from the set $\mathcal{L}$ of lattices in $\mathbb{Q}_p^2$ to the disjoint union of the spaces $\mathcal{MS}_{\Gamma_0(M)}(\mathcal{D}_U(L'))$ with $L \in \mathcal{L}$ such that $I_L \in \mathcal{D}_U(L')$.

Definition 4.4 The family $\{I_L\}_{L \in \mathcal{L}}$ is defined by the rule

$$I_L\{r \to s\}(F) := I_{L_*}\{\gamma r \to \gamma s\}(F \mid \gamma^{-1}) = \int_{L'_*} (F \mid \gamma^{-1}) I_{L_*}\{\gamma r \to \gamma s\},$$

for any locally analytic function $F \in \mathcal{A}^U(L')$, where $\gamma L = L_*$ and $\gamma \in \widetilde{\Gamma}$.

Lemma 4.5 *Let $\kappa \in U$ and let $L_2 \subset L_1$ be an index p sublattice of L_1 and let $e = ([L_1], [L_2])$ be the corresponding edge. Then*

$$I_{L_2}\{r \to s\}(F) = a_p I_{L_1}\{r \to s\}(F)$$

for every locally analytic function $F \in \mathcal{A}^\kappa(W_e)$.

Proof See [4, Lemma 6.3]. ∎

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The specialization property of $I_\infty \in \mathcal{MS}_{\Gamma_0(pM)}(\mathbb{D}_U)$ can be explicitly written

$$\rho_k(I_\infty)\{r \to s\}(P\chi_{W_\infty}) = \lambda(k) I_k\{r \to s\}(P) \text{ for every } P \in \mathbf{P}_{k-2}. \tag{4.3}$$

The following corollary describes the specialization in terms of the modular symbol $I_k^\#$.

Corollary 4.6 *For all $k \in \mathbb{Z} \cap U$ and all $P \in \mathbf{P}_{k-2}$*

$$\rho_k(I_\infty)\{r \to s\}(P) = \lambda(k)(1 - p^{k-2} a_p(k)^{-2}) I_k^\#\{r \to s\}(P).$$

Proof This is proved in [3, Proposition 2.4] using Lemma 4.5, (4.2), and (4.1). ■

For every lattice L, define a modular symbol $\pi_*(I_L) \in \mathcal{MS}(\mathcal{D}_n(\mathbb{P}^1(\mathbb{Q}_p)))$ by the rule

$$\pi_*(I_L)\{r \to s\}(F) := |L|^{-\frac{k_0-2}{2}} I_L\{r \to s\}(F(x, y)), \tag{4.4}$$

where F is a locally analytic function on $\mathbb{P}^1(\mathbb{Q}_p)$ with a pole of order at most $k_0 - 2$ at ∞ and $F(x, y) := y^{k_0-2} F(x/y)$. Recall the exact sequence (2.8). Thanks to the new assumption on f, it can be used to attach to the modular symbol $I_{k_0} \in \mathcal{MS}^k_{\Gamma_0(pM)}(K_{k_0})$ an harmonic modular symbol $I_{k_0}^{\text{har}}$ belonging to

$$\mathbf{MS} := \mathcal{MS}_\Gamma\left(\mathcal{D}_n^{0,b}\left(\mathbb{P}^1(\mathbb{Q}_p)\right)\right) = \mathcal{MS}_\Gamma\left(\mathcal{C}_{\text{har}}(\mathcal{E}, \mathbf{V}_n)\right),$$

where the identification is provided by Proposition 2.8.

Corollary 4.7 *For all lattices L such that $[L]$ is even, $\pi_*(I_L) = I_{k_0}^{\text{har}}$, the modular symbol in $\mathcal{MS}_\Gamma(\mathcal{D}_0^{k_0-2}(\mathbb{P}^1(\mathbb{Q}_p)))$ attached to f.*

Proof This is a consequence of Lemma 4.5 together with the specialization property (4.3); see [4, Proposition 6.4]. Our restriction to even lattices, that does not appear in [4], is a consequence of the fact that we are not assuming f to be a split modular form as in [4] (compare with [2, Proposition 2.12], where the analogous result is proved in the definite weight 2 setting). ■

The following definition is justified by Lemma 4.1.

Definition 4.8 The semidefinite integral attached to $r, s \in \mathbb{P}^1(\mathbb{Q})$, $\tau \in \mathcal{H}_p^+(\mathbb{Q}_p^{\text{ur}})$ and $P \in \mathbf{P}_n$ is defined by the formula

$$\int_r^s \int^\tau P\omega_f := |L_\tau|^{-\frac{k_0-2}{2}} \frac{d}{d\kappa}\left(\int_{L_\tau'} P(x, y)\langle x - \tau y\rangle^{\kappa - k_0} d\mu_{L_\tau}\{r \to s\}(x, y)\right)_{\kappa = k_0},$$

where $[L_\tau] = \text{red}(\tau)$.

We remark that the above formula does not depend on the choice of the representative L_τ, since

$$\log_0(px - p\tau y)P(px, py) = p^{k_0-2}\log_0(x - \tau y)P(x, y).$$

Proposition 4.9 *For every $\gamma \in \Gamma$ and every $\tau \in \mathcal{H}_p^+(\mathbb{Q}_p^{\mathrm{ur}})$,*

$$\int_{\gamma r}^{\gamma s}\int^{\gamma\tau} P\omega_f = \int_{\gamma r}^{\gamma s}\int^{\gamma\tau}(P \mid \gamma)\omega_f.$$

Proof See [4, Proposition 6.6]. ■

Recall the harmonic cocycle I_f.

Proposition 4.10 *For every $\tau_1, \tau_2 \in \mathcal{H}_p^+(\mathbb{Q}_p^{\mathrm{ur}})$*

$$\int_r^s\int^{\tau_2} P\omega_f - \int_r^s\int^{\tau_1} P\omega_f = \int_r^s\int_{\tau_1}^{\tau_2} P\omega_f^{\log_0} + 2a_p^{-1}(k_0)a_p'(k_0)\sum_{e:\ \mathrm{red}(\tau_1)\to\mathrm{red}(\tau_2)} I_{k_0}^{\mathrm{har}}(e)\{r \to s\}(P).$$

Proof This formula is proved in the split case in [4, Proposition 6.7]. The methods of the proof adapt to the non-split setting as explained in [2, Proposition 2.19]. ■

Combining Propositions 4.9 and 4.10 with the main result of [9] yields the following theorem.

Theorem 4.11 (Exceptional zero conjecture) *Let f be a new modular form. Then $\mathbf{D}_{[f]} \simeq \mathbb{D}_{st}(V_{[f]})$, the filtered Frobenius module attached to the modular form f.*

Proof Propositions 4.9 and 4.10 imply that the f-component of the $\mathcal{L}$-invariant of $\mathbf{D}_{[f]}$ is $-2a_p^{-1}(k_0)a_p'(k_0)$. This is also the f-component of the $\mathcal{L}$-invariant of $\mathbb{D}_{st}(V_{[f]})$, by the main result of [9]. As explained in Subsection 2.3, the $\mathcal{L}$-invariant of $\mathbf{D}_{[f]}$ is the direct sum of the $\mathcal{L}$-invariants attached to the companion forms of f, and a similar fact is true for $\mathbb{D}_{st}(V_{[f]})$. Working with the companion forms of f, we see that $\mathbf{D}_{[f]}$ and $\mathbb{D}_{st}(V_{[f]})$ have the same $\mathcal{L}$-invariant. As explained in [23, §4], that suffices to find the isomorphism. ■

Corollary 4.12 *Choose the branch of the p-adic logarithm corresponding to $\lambda = \mathcal{L}_f^0$. Then the symbol $\int_r^s\int^\tau P\omega_f$ satisfies*

$$\int_r^s\int^{\tau_2} P\omega_f - \int_r^s\int^{\tau_1} P\omega_f = \int_r^s\int_{\tau_1}^{\tau_2} P\omega_f^{\log_\lambda}.$$

Proof The corollary follows by combining Propositions 4.10 and 3.1. ■

Corollary 4.13 *We have*

$$\int_x^{\gamma_\Psi x}\int^{\tau_\Psi} P_\Psi^m\omega_f = D_K^{\frac{k_0-2}{4}}\log\Phi^{AJ}(j_\Psi)(I_f).$$

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Proof By Corollary 4.12 the $\mathbf{MS}_f^{c,\vee,w_\infty}$-valued semidefinite integration theory

$$\int_r^s \int^{\tau_2} P\omega_f$$

lifts the integration theory

$$\int_r^s \int_{\tau_1}^{\tau_2} P\omega_f^{\log_{\mathcal{L}_f^0}}.$$

Hence the claim follows from Propositions 3.3 and 2.22, which allows us to compute the p-adic Abel-Jacobi image of the Darmon cycles using any p-adic Abel–Jacobi map. ∎

4.2 Families of Modular Forms on Definite Quaternion Algebras

Let N^- be a squarefree positive integer divisible by an odd number of primes and let B be the rational definite quaternion algebra ramified at $N^-\infty$. Let $\mathcal{O}_B$ be any maximal order in B. Write $\widehat{\mathbb{Z}}$ to denote the profinite completion of $\mathbb{Z}$ and set $\widehat{B} := B \otimes \widehat{\mathbb{Z}}$. Let $\Sigma = \prod_l \Sigma_l$ be any decomposable open compact subgroup of $\widehat{B}^\times$ and let $\mathbf{V}$ be any K_p-vector space, equipped with a left action of Σ_p.

For every prime l let $\mathbb{H}_l$ be a the unique (up to isomorphism) quaternion division algebra over $\mathbb{Q}_l$. We can choose $\mathbb{Q}_l$-algebra isomorphisms $\iota_l \colon B \otimes \mathbb{Q}_l \xrightarrow{\simeq} \mathbb{M}_2(\mathbb{Q}_l)$ sending $\mathcal{O}_B \otimes \mathbb{Z}_l$ isomorphically onto $\mathbb{M}_2(\mathbb{Z}_l)$ for every $l \nmid N^-\infty$, as well as $\mathbb{Q}_l$-algebra isomorphisms $\iota_l \colon B \otimes \mathbb{Q}_l \xrightarrow{\simeq} \mathbb{H}_l$ for every $l \mid N^-\infty$, so that $\iota_l(\mathcal{O}_B \otimes \mathbb{Z}_l)$ is the unique maximal order $\mathcal{O}_{\mathbb{H}_l}$ of $\mathbb{H}_l$. Setting

$$\widetilde{\Gamma}_\Sigma := \iota_p\Big(\mathcal{O}_B[1/p] \cap \prod_{l \neq p} \Sigma_l\Big)$$

and letting Γ_Σ be the subgroup of $\widetilde{\Gamma}_\Sigma$ of elements of determinant 1, we can give the following definition of a $\mathbf{V}$-valued p-adic automorphic form on B of level Σ (see [4, Sec. 1]).

Definition 4.14 A $\mathbf{V}$-valued p-adic automorphic form on B of level Σ is a function $\varphi \colon \mathbb{GL}_2(\mathbb{Q}_p) \to \mathbf{V}$ such that $\varphi(\gamma g u) = u^{-1}\varphi(g)$ for all $\gamma \in \widetilde{\Gamma}_\Sigma, g \in \mathbb{GL}_2(\mathbb{Q}_p)$ and $u \in \iota_p(\Sigma_p)$.

The K_p-vector space of $\mathbf{V}$-valued p-adic automorphic forms of level Σ will be denoted $S(\Sigma, V)$.

We will always assume that $\iota_p(\Sigma_p) = \Gamma_0(p\mathbb{Z}_p)$, and we write Σ_∞ to denote the open compact obtained from Σ by replacing the local condition at p with the local condition $\iota_p(\Sigma_{\infty,p}) = \mathbb{GL}_2(\mathbb{Z}_p)$. When $\mathbf{V} = \mathbf{V}_{k-2}$ we will simply write $S_k(\Sigma)$. Specializing to the case $\mathbf{V} = \mathbb{D}_U$ we obtain the notion of a p-adic family of automorphic forms (here again U is an affinoid disk in the weight space).

Definition 4.15 The space of p-adic families of automorphic forms on B of level Σ parametrized by weights in U is by definition

$$\mathbb{S}_U(\Sigma) := S(\Sigma_\infty, \mathbb{D}_U).$$

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The space $\mathbb{S}_U(\Sigma)$ of p-adic families of automorphic forms on B of level Σ comes equipped with specializations maps for every $k \in U \cap \mathbb{Z}^{\geq 2}$:

$$\rho_k \colon \mathbb{S}_U(\Sigma) \to S_k(\Sigma),$$
$$\big(\rho_k(\Phi)(g)\big)(P) := \int_{W_\infty} P(x,y)\, d\Phi(g)(x,y),$$

where $g \in \mathbb{GL}_2(\mathbb{Q}_p)$, $P \in \mathbf{P}_{k-2}$ and $P(x,y) := y^{k-2}P(x/y)$ is the corresponding degree $k-2$ homogeneous polynomial, an element of $\mathcal{A}^{(n)}(W_\infty)$. Note that the definition of ρ_k depends on the choice of W_∞, which was only defined up to multiplication by $\mathbb{Q}_p^\times$; we choose $W_\infty := L'_* \cap L'_\infty = \mathbb{Z}_p^\times \times p\mathbb{Z}_p$.

We now choose the level structure as follows. Let $N = pN^+N^-$ be a factorization into prime factors of our given integer N, where N^- corresponds to the finite primes of ramification of B. Define the open compact group Σ by the following local conditions:

$$\Sigma_l = \begin{cases} (\mathcal{O}_B \otimes \mathbb{Z}_l)^\times & l \nmid N^+ p, \\ \iota_l^{-1}(\Gamma_0(N\mathbb{Z}_l)) & l \mid N^+ p. \end{cases} \tag{4.5}$$

Write $\widetilde{\Gamma}' = \widetilde{\Gamma}_\Sigma$ for the corresponding group as well as $\Gamma' = \Gamma_\Sigma$.

By the Jacquet–Langlands correspondence, the modular form $f = f_{k_0}$ that was fixed in the previous section corresponds to a modular form $\varphi = \varphi_{k_0}$ in the above sense (*i.e.,* it has the same eigenvalues as f) for the above choice of the level. The functions f_k (resp. $f_k^\#$) similarly correspond to functions φ_k (resp. $\varphi_k^\#$) for the suitable choice of the level structure (4.5) (resp. Σ_∞). Since the stabilizer of the standard edge e_* is $\Gamma_0(p\mathbb{Z}_p)$, it is possible to attach to the modular form φ_{k_0} a cocycle $c_{k_0} \in \mathcal{C}(\mathcal{E}, \mathbf{V}_{k_0-2})^{\widetilde{\Gamma}'}$ by the rule ($n := k_0 - 2$):

$$c_{k_0}(e) := \widetilde{c}_\varphi(e) := p^{-n/2\,\mathrm{ord}_p(\det(g))} g\varphi(g), \text{ if } e = ge_*.$$

We note that, since φ is new at p, the above cocyle satisfies the rules

$$\sum_{s(e)=v} c(e) = 0, \qquad \sum_{t(e)=v} c(e) = 0, \qquad \text{and} \qquad c(\bar{e}) = w_p c(e),$$

where $w_p \in \{\pm 1\}$ is the sign of the Atkin–Lehner involution at p, which is equal to -1 if c is of split multiplicative type and is equal to 1 if c is of non-split multiplicative type (see [3, Prop. 1.4] or [5, p. 32]). Let $\Gamma' \subset \widetilde{\Gamma}'$ be the subgroup of those elements having norm 1. One can attach an harmonic cocycle in $\mathcal{C}_{\mathrm{har}}(\mathcal{E}, \mathbf{V}_{k_0-2})^{\Gamma'}$ to the cocycle c_{k_0} by the rule

$$c^{\mathrm{har}}(e) := \begin{cases} c(e) & \text{when } e \in \mathcal{E}^+, \\ -c(\bar{e}) & \text{when } e \in \mathcal{E}^-. \end{cases}$$

Let $X = X_{N^+,pN^-}$ be the Shimura curve attached to the indefinite quaternion algebra $\mathcal{B}$ ramified at the primes dividing pN^- and the choice of an Eichler order

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$\mathcal{R} = \mathcal{R}_{N^+,pN^-}$ of level N^+. By the Theorem of Cerednik–Drinfeld the above Shimura curve admits a rigid analytic uniformization at p. The modular form f corresponds to a rigid analytic modular form f^{rig} again by the Jacquet–Langlands correspondence, and the cocycle c^{har} is precisely the cocyle attached to f^{rig} by taking the residues. As a consequence of the Theorem of Amice–Velu–Teitelbaum, we may attach to the harmonic cocycle c^{har} a unique locally analytic distribution $\mu \in \mathcal{D}_n^{0,b}(\mathbb{P}^1(\mathbb{Q}_p))^{\Gamma'}$ such that $R(\mu) = c^{\mathrm{har}}$. This is the analogue of Proposition 2.8 in this definite setting.

Write $\mathcal{L}$ for the set of lattices in $\mathbb{Q}_p^2 - \{0\}$ and $\mathcal{L}_0$ for the set of couples (L_1, L_2) such that $L_1 \supset L_2$. Without the normalizing condition obtained by multiplying by the determinant $p^{-n/2\,\mathrm{ord}_p(\det(g))}$, it is also possible to attach to φ_{k_0} an element (that we will denote by the same symbol) $c_{k_0} \in \mathcal{C}(\mathcal{L}_0, \mathbf{V}_{k_0-2})^{\widetilde{\Gamma}'}$. The same construction works for the modular forms φ_k (resp. $\varphi_k^{\#}$) producing elements $c_k \in \mathcal{C}(\mathcal{L}_0, \mathbf{V}_{k-2})^{\widetilde{\Gamma}'}$ (resp. $c_k^{\#} \in \mathcal{C}(\mathcal{L}, \mathbf{V}_{k-2})^{\widetilde{\Gamma}'}$) defined by the rule (and the same for $c_k^{\#}$)

$$c_k(L_1, L_2) := g\varphi(g) \text{ if } (L_1, L_2) = g(L_*, L_\infty)$$
$$(\text{resp. } c_k(L) := g\varphi(g) \text{ if } L = gL_*).$$

We further normalize the cocycles $c_k^{\#}$ for $k \neq k_0$ by the requirement

$$\langle c_k^{\#}, c_k^{\#}\rangle = 1, \tag{4.6}$$

where the inner product is the one defined at the end of [2, Section 2.2]. Then the modular form $c_k^{\#}$ is uniquely determined up to sign. The relation (4.1) translates into

$$c_k(L_1, L_2) = c_k^{\#}(L_2) - p^{k-2}a_p(k)^{-1}c_k^{\#}(L_1) = c_k^{\#}(L_2) - a_p(k)^{-1}c_k^{\#}(pL_1) \tag{4.7}$$

In fact, the correspondence can be merged in families, as it follows from results of [7].

Theorem 4.16 *There exists a family $\varphi_\infty \in \mathbb{S}_U(\Sigma)$ such that:*

- *for every $k \in \mathbb{Z}^{\geq 2} \cap U$, $\rho_k(\varphi_\infty) = \lambda_B(k)\varphi_k$ for some $\lambda_B(k) \in \mathbb{C}_p^\times$;*
- *$\rho_{k_0}(\varphi_\infty) = \varphi_{k_0}$.*

Denote by $\mathcal{C}(\mathcal{L}, \mathcal{D}(*))$ the space of maps μ_* from $\mathcal{L}$ to $\sqcup_{L\in\mathcal{L}}\mathcal{D}(L')$ such that $\mu_L \in \mathcal{D}(L')$. Define $\mathcal{C}(\mathcal{L}_0, \mathcal{D}(*))$ in a similar way, this time $\mu_{L_1,L_2} \in \mathcal{D}(W_{L_1,L_2})$, where $W_{L_1,L_2} := L_1' \cap L_2'$. The function on the lattices attached to φ_∞ obtained by Shapiro's lemma will be denoted $\mu_* \in \mathcal{C}(\mathcal{L}, \mathcal{D}^U(*))^{\widetilde{\Gamma}'}$.

Lemma 4.17 *Let $\kappa \in U$, let $L_2 \subset L_1$ be an index p sublattice of L_1, and let $e = ([L_1], [L_2])$ be the corresponding edge. Then $\mu_{L_2}(F) = (a_p\mu_{L_1})(F)$ for every locally analytic function $F \in \mathcal{A}^\kappa(W_e)$.*

Proof See [4, Lemma 4.3]. ■

The specialization property of $\varphi_\infty \in \mathbb{S}_U(\Sigma)$ can be written explicitly as

$$\varphi_\infty(g)(P\chi_{W_\infty}) = \lambda(k)\varphi_k(g)(P) \text{ for every } P \in \mathbf{P}_{k-2} \text{ and } g \in \mathbb{GL}_2(\mathbb{Q}_p).$$

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In terms of μ_* and c_k this property can be restated as follows (see [2, Lemma 2.10]):

(4.8) $$\mu_{L_1}(P\chi_{L_1,L_2}) = \lambda(k)c_k(L_1,L_2)(P) \text{ for every } P \in \mathbf{P}_{k-2} \text{ and } L_1 \supset L_2.$$

The following corollary expresses the specialization in terms of $c_k^\#$.

Corollary 4.18 *For all $k \in \mathbb{Z} \cap U$ and all $P \in \mathbf{P}_{k-2}$,*

$$\mu_L(P) = \lambda_B(k)a_p(k)(1 - p^{k-2}a_p(k)^{-2})c_k^\#(L)(P).$$

Proof This is proved in [2, Proposition 2.11] using Lemma 4.17, (4.8), and (4.7). ■

For every lattice L, define a locally analytic distribution $\pi_*(\mu_L)$ that belongs to $\mathcal{D}^{k_0-2}(\mathbb{P}^1(\mathbb{Q}_p))$ by the rule

(4.9) $$\pi_*(\mu_L)(F) := |L|^{-\frac{k_0-2}{2}}\mu_L(F(x,y)),$$

where F is a locally analytic function on $\mathbb{P}^1(\mathbb{Q}_p)$ with a pole of order at most k_0-2 at ∞ and $F(x,y) := y^{k_0-2}F(x/y)$.

Corollary 4.19 *For all lattices L such that $[L]$ is even, $\pi_*(\mu_L) = \mu$, the measure in $\mathcal{D}_0^{k_0-2}(\mathbb{P}^1(\mathbb{Q}_p))^{\Gamma'}$ attached to f.*

Proof This is a consequence of Lemma 4.17 together with the specialization property $\rho_{k_0}(\varphi_\infty) = \varphi_{k_0}$; see [4, Proposition 4.4]. Our restriction to even lattices, which does not appear in [4, Proposition 6.4], is again a consequence of the fact that we are not assuming φ to be a split modular form as in [4] (compare with [2, Proposition 2.12], where the analogous result is proved in the weight 2 setting). ■

The following definition is justified by Lemma 4.1.

Definition 4.20 The indefinite integral attached to $\tau \in \mathcal{H}_p^+(\mathbb{Q}_p^{\mathrm{ur}})$ and $P \in \mathbf{P}_n$ is

$$\int^\tau P\omega_f := |L_\tau|^{-\frac{k_0-2}{2}}\frac{d}{d\kappa}\Big(\int_{L'_\tau} P(x,y)\langle x-\tau y\rangle^{\kappa-k_0}d\mu_{L_\tau}(x,y)\Big)_{\kappa=k_0},$$

where $[L_\tau] = \mathrm{red}(\tau)$.

Proposition 4.21 *For every $\gamma \in \Gamma'$ and every $\tau \in \mathcal{H}_p^+$*

$$\int^{\gamma\tau} P\omega_f = \int^\tau (P \mid \gamma)\omega_f.$$

Proof [4, Proposition 4.4] ■

Proposition 4.22 *For every $\tau_1,\tau_2 \in \mathcal{H}_p^+(\mathbb{Q}_p^{\mathrm{ur}})$*

$$\int^{\tau_2} P\omega_f - \int^{\tau_1} P\omega_f = \int_{\tau_1}^{\tau_2} P\omega_f + 2a_p^{-1}(k_0)a'_p(k_0)\sum_{e:\ \mathrm{red}(\tau_1)\to\mathrm{red}(\tau_2)} c^{\mathrm{har}}(e)(P).$$

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Proof This is [4, Proposition 4.10]. Again the restriction to even elements of $\mathcal{H}_p^+(\mathbb{Q}_p^{\mathrm{ur}})$, which does not appear in [4], is a consequence of the fact that we are not assuming that φ is a split form. As explained in [2, Proposition 2.19], in the weight 2 setting, the non-split case can be similarly treated up to restricting to $\mathcal{H}_p^+(\mathbb{Q}_p^{\mathrm{ur}})$ and the ideas of the proof readily adapt to the higher weight case in order to remove the restriction appearing in [4]. ■

5 p-adic L-functions

5.1 The Mazur–Kitagawa p-adic L-functions

Let $g \in S_k(\Gamma_0(N))$ be an eigenform and recall the modular symbol $I_g \in \mathcal{MS}_{\Gamma_0(M)}^{k,w_\infty}(K_g)$ attached to g by means of Proposition 2.1 and the choice of a sign w_∞. Define, for our fixed g and $m \in \mathbb{N}^{>0}$, the function

$$I_{g,m}[-,-] \colon \mathbf{P}_{k-2}(K_g) \times \mathbb{Z}/m\mathbb{Z} \to K_g,$$
$$I_{g,m}[P,a] := I_g\{\infty \to a/m\}(P),$$

where the fact that $I_{g,m}[P,a]$ depends only on the class of a in $\mathbb{Z}/m\mathbb{Z}$ follows from the invariance of I_g under the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and the relation

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -a \\ 0 & m \end{pmatrix} = \begin{pmatrix} 1 & -a+m \\ 0 & m \end{pmatrix}.$$

Now let χ be any primitive Dirichlet character modulo m and consider the Gauss sum

$$\tau(\chi) := \sum_{a \in \mathbb{Z}/m\mathbb{Z}} \chi(a) e^{2\pi i a/m}.$$

Proposition 5.1 *Let $1 \le j \le k-1$ be an integer and let χ be a character such that $\chi(-1) = (-1)^{k-j-1} w_\infty = (-1)^{j-1} w_\infty$ (since k is even). Then*

$$\sum_{a \in \mathbb{Z}/m\mathbb{Z}} \chi(a) I_{g,m}[P_{j,a}, a] = \frac{(j-1)!\tau(\overline{\chi})}{(-2\pi i)^{j-1}\Omega_g} L(g,\chi,j) =: L^*(g,\chi,j),$$

where

$$P_{j,a} := \left(x - \frac{a}{m} y\right)^{j-1} y^{k-j-1}.$$

Proof As explained in [2, Proposition 1.3] the above formula is a consequence of the formula of Birch and Manin expressing special values of L-series in terms of modular symbols that can be found in [21, Formula (8.6)], after taking into account that I_g belongs to the w_∞-eigenspace for the W_∞-action, and we are assuming $\chi(-1) = (-1)^{k-j-1} w_\infty$. The assumption that χ is a quadratic character appearing in [2], which is done in view of the applications, is not needed. ■

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To the modular symbol I_∞ we attach the symbol

$$I_{\infty,m}[-,-]\colon \mathcal{A}^U(L'_*) \times \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}_p,$$
$$I_{\infty,m}[F,a] := I_\infty\{\infty \to a/m\}(F).$$

The following definition attaches a p-adic L-function

$$L_p(f,\chi,\kappa,s)\colon U \times \mathbb{Z}_p \to \mathbb{C}_p,$$
$$(\kappa,s) \mapsto L_p(f,\chi,\kappa,s)$$

to the data of f and a Dirichlet character.

Definition 5.2 The Mazur–Kitagawa p-adic L-function attached to (f,χ), where $\chi\colon \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}^\times$ is a character of conductor m, is defined by the rule

$$\begin{aligned} L_p(f,\chi,\kappa,s) &:= \sum_{a\in\mathbb{Z}/m\mathbb{Z}} \chi(ap) \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times} \Big(x - \frac{pa}{m}y\Big)^{s-1} y^{\kappa-s-1} dI_\infty\Big\{\infty \to \frac{pa}{m}\Big\} \\ &= \sum_{a\in\mathbb{Z}/m\mathbb{Z}} \chi(ap) I_{\infty,m}\Big\{\infty \to \frac{pa}{m}\Big\}(F_{s,pa}\chi_{\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times}), \end{aligned}$$

where

$$F_{s,pa} := \Big(x - \frac{pa}{m}y\Big)^{s-1} y^{\kappa-s-1}.$$

Note that whenever $(x,y) \in \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ and $\chi(ap) \neq 0$, we have $(ap,m) = 1$, so that $m \in \mathbb{Z}_p^\times$ and

$$x - \frac{pa}{m}y \in \mathbb{Z}_p^\times + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times. \tag{5.1}$$

We have defined the above Mazur–Kitagawa p-adic L-function as a two variable function, and it is indeed analytic in both variables. For the applications we have in mind it is sufficient to consider the restriction of this function to the critical line $(\kappa,\kappa/2)$. In this case we are fully justified by Lemma 4.1, since the function of the κ-variable we have defined is the linear combination of functions of the form

$$\kappa \mapsto \mu(\alpha^{\kappa/2-1}\beta^{\kappa-\kappa/2-1}\chi_X) = \mu\big(\langle\alpha\rangle^{\kappa/2-1}\langle\beta\rangle^{\kappa-\kappa/2-1}\chi_X\big).$$

Here the equality follows from (5.1) and Lemma 4.1 when $\kappa = k \in \mathbb{Z} \cap U$. Hence the right-hand side can be taken as a definition, while the notation on the left-hand side for more general $\kappa \in U$ suggests the value at the integers. This is indeed needed in order to investigate the interpolation properties of the Mazur–Kitagawa p-adic L-function, as it is done in the subsequent theorem.

Theorem 5.3 *Assume χ is a primitive character, $k \in U \cap \mathbb{Z}$, and $1 \leq j \leq k-1$ satisfies $\chi(-1) = (-1)^{j-1}w_\infty$. Then*

$$L_p(f,\chi,k,j) = \lambda(k)\big(1 - \chi(p)p^{j-1}a_p(k)^{-1}\big) L^*(f_k,\chi,j).$$

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Proof In light of the preceding remarks we can appeal to the proof of [2, Theorem 1.12], which uses Proposition 5.1 after a direct calculation (again the assumption that χ is a quadratic character is not needed). Note that strictly speaking we are only allowed to move along the line $(k, k/2)$, since we have not defined $L_p(f, \chi, \kappa, s)$ out of the line $(\kappa, \kappa/2)$ but rather remarked that it could be done; in other words, following the proof of [2, Theorem 1.12] with $j = k/2$, we can prove Corollary 5.4. ■

What really matters is the following corollary, which specializes Theorem 5.3 to $j = k/2$ and expresses the interpolation property in terms of the modular form $f_k^\#$ when $k \neq k_0$ using the relation

$$L^*(f_k, \chi, j) = (1 - \chi(p)p^{k-j-1}a_p(k)^{-1})L^*(f_k^\#, \chi, j),$$

which follows from (4.1).

Corollary 5.4 *Assume $\chi(-1) = (-1)^{\frac{k-2}{2}} w_\infty$. When $k \neq k_0$ (resp. $k = k_0$),*

$$L_p(f, \chi, k, k/2) = \lambda(k)\big(1 - \chi(p)p^{\frac{k-2}{2}} a_p(k)^{-1}\big)^2 L^*(f_k^\#, \chi, k/2),$$

$$(\text{resp. } L_p(f, \chi, k_0, k_0/2) = \big(1 - \chi(p)p^{\frac{k_0-2}{2}} a_p(k_0)^{-1}\big) L^*(f_{k_0}^\#, \chi, k_0/2)\,),$$

where (recall that $f_{k_0}^\# = f_{k_0}$)

$$L^*(f_k^\#, \chi, k/2) = \frac{(k/2-1)!\tau(\chi)}{(-2\pi i)^{k/2-1}\Omega_k} L(f_k^\#, \chi, k/2).$$

5.2 p-adic L-functions Attached to Real Quadratic Fields

In this subsection we let $K/\mathbb{Q}$ be a real quadratic field such that:

- p is inert in K;
- all the prime factors of M split in K.

Let $\Psi \in \mathcal{E}mb^+(\mathcal{O}, \mathcal{R})$ be an optimal embedding of conductor c prime to D_K, the discriminant of $K/\mathbb{Q}$, and N. Denote by $G_{H_{\mathcal{O}}^+/K}$ the Galois group of the corresponding narrow ring class field. Recall the data $(\tau_\Psi, P_\Psi, \gamma_\Psi)$ attached to it and further consider a $\mathbb{Z}_p$-lattice L_Ψ such that $[L_\Psi] = v_\Psi$. The following definition attaches a p-adic L-function to the above data:

$$\mathcal{L}_p(f/K, \Psi, -) \colon U \to \mathbb{C}_p,$$
$$\kappa \mapsto \mathcal{L}_p(f/K, \Psi, \kappa).$$

It is easily checked that the definition below does not depend on the choice of L_Ψ. As usual, here and in the following, we set $n := k_0 - 2$ and $m := n/2$ for shortness.

Definition 5.5 Let $r \in \mathbb{P}^1(\mathbb{Q})$ be any base point. The partial p-adic L-function attached to $(f/K, \Psi)$ is

$$\mathcal{L}_p(f/K, \Psi, \kappa) := |L_\Psi|^{-\frac{k_0-2}{2}} \int_{L'_\Psi} \langle P_\Psi(x, y)\rangle^{\frac{\kappa-k_0}{2}} P_\Psi^m(x, y) dI_{L_\Psi}\{r \to \gamma_\Psi r\}.$$

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The partial p-adic L-function attached to $(f/K, \chi)$, where $\chi\colon G_{H^+_{\mathcal{O}}/K} \to \mathbb{C}^\times$ is a character, is

$$\mathcal{L}_p(f/K, \chi, \kappa) := \sum_{\sigma \in G_{H^+_{\mathcal{O}}/K}} \chi^{-1}(\sigma)\mathcal{L}_p(f/K, \sigma\Psi, \kappa).$$

The p-adic L-function attached to $(f/K, \chi)$, where $\chi\colon G_{H^+_{\mathcal{O}}/K} \to \mathbb{C}^\times$ is a character, is

$$L_p(f/K, \chi, \kappa) := \mathcal{L}_p(f/K, \chi, \kappa)^2.$$

In order to justify the above definition and the fact that the above p-adic L-functions are analytic, we can appeal to Lemma 4.1, as they are built from functions of the form $\kappa \mapsto \mu(P\langle\alpha\rangle^{\frac{\kappa-k_0}{2}}\langle\beta\rangle^{\frac{\kappa-k_0}{2}}\chi_X)$.

Remark 5.6 The above p-adic L-functions depend, of course, on the choice of the modular symbol I_∞ that was used to define the family $\{I_L\}_{L\subset\mathbb{Q}_p^2}$. It can be shown that the definition depends only on the class of Ψ in $\Gamma\backslash\mathcal{E}mb^+(\mathcal{O},\mathcal{R})$. It turns out that many of the properties of these p-adic L-functions actually depend only on f/K.

We note that a suitable choice of the lattice L_Ψ can be made as follows. Since the group Γ acts transitively on the positive vertices $\mathcal{V}^+$ we can choose $\gamma \in \Gamma$ such that $\gamma v_\Psi = v_*$. Hence $v_* = v_{\gamma\Psi\gamma^{-1}}$, and $L_* = \gamma L_\Psi = L_{\gamma\Psi\gamma^{-1}}$ is associated with the embedding $\gamma\Psi\gamma^{-1} \in [\Psi]$. It is clear that this choice is the natural one in investigating the relations with the Mazur–Kitagawa p-adic L-function, whose definition was given in terms of $I_\infty = I_{L_*}$.

Note the following vanishing property of the above p-adic L-functions.

Proposition 5.7 *The p-adic L-functions vanish at k_0:*

$$\mathcal{L}_p(f/K, \Psi, k_0) = \mathcal{L}_p(f/K, \chi, k_0) = L_p(f/K, \chi, k_0) = 0.$$

Furthermore,

$$\frac{d}{d\kappa}\big[L_p(f/K, \chi, \kappa)\big]_{\kappa=k_0} = 0.$$

Proof By definition and (4.4),

$$\begin{aligned}\mathcal{L}_p(f/K, \Psi, k_0) &:= |L_\Psi|^{-\frac{k_0-2}{2}} \int_{L'_\Psi} P^m_\Psi(x,y)\,dI_{L_\Psi}\{r \to \gamma_\Psi r\}\\ &= \int_{\mathbb{P}^1(\mathbb{Q}_p)} P^m_\Psi\, d\pi_*(I_{L_\Psi})\{r \to \gamma_\Psi r\}.\end{aligned}$$

The claim now follows from Corollary 4.7. By definition, the same vanishing property holds for the other p-adic L-functions and the defining relation $L_p = \mathcal{L}_p^2$ yields the last assertion. ■

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5.2.1 Interpolation Properties of the p-adic L-functions Attached to Real Quadratic Fields and Functional Equation

The following theorem encodes the main interpolation property of the p-adic L-function $L_p(f/K,\chi,\kappa)$.

Theorem 5.8 *For all $k \in \mathbb{Z}^{\geq 2, \neq k_0} \cap U$,*

$$L_p(f/K,\chi,\kappa) = \lambda(k)^2\big(1 - p^{k-2}a_p(k)^{-2}\big)^2 D_K^{\frac{k-2}{2}} L^*(f_k^{\#}/K,\chi,k/2),$$

where

$$L^*(f_k^{\#}/K,\chi,k/2) := \frac{(\frac{k-2}{2})!^2\sqrt{D_K}}{(2\pi i)^{k-2}\Omega_k^2} L(f_k^{\#}/K,\chi,k/2).$$

Proof The proof of [3, Theorem 3.5] readily adapts to our higher weight setting. As explained in [3] the proof is reduced to Popa's formula [20, Theorem 6.3.1]. ■

Recall that a genus character of G_K is a quadratic unramified character of G_K. Such a character corresponds to a biquadratic (or quadratic when $\chi = 1$) extension of $\mathbb{Q}$, which is explicitly given by

$$H_\chi = \mathbb{Q}(\sqrt{D_1},\sqrt{D_2}) \supset \mathbb{Q}(\sqrt{D}) = K,$$

where $D_K =: D = D_1 D_2$ is a factorization of the fundamental discriminant D into factors D_i prime each other.

Let χ_i (resp. ε_K) be the Dirichlet character attached to $\mathbb{Q}(\sqrt{D_i})/\mathbb{Q}$ (resp. $K/\mathbb{Q}$). Then $\varepsilon_K = \chi_1\chi_2$. We say that χ is real (resp. imaginary) whenever H_χ/K is totally real (resp. imaginary). Note that

$$1 = \varepsilon_K(-1) = \chi_1(-1)\chi_2(-1),$$

so that

$$\chi_1(-1) = \chi_2(-1), \tag{5.2}$$

depending of whether $\mathbb{Q}(\sqrt{D_i})/\mathbb{Q}$ are imaginary or real. Note that $D_K \in \mathbb{Z}_p^\times$, since by assumption p is prime to D_K. In particular $D_K^{k-2/2}$ extends on U to an analytic function $D_K^{\kappa-2/2} := \langle D_K\rangle^{\kappa-2/2}$, thanks to Lemma 4.1.

Theorem 5.9 *Let χ be a genus character such that $\chi(-1) = (-1)^{\frac{k_0-2}{2}} w_\infty$. Then*

$$L_p(f/K,\chi,\kappa) = D_K^{\frac{\kappa-2}{2}} L_p(f,\chi_1,\kappa,\kappa/2) L_p(f,\chi_2,\kappa,\kappa/2),$$

where (χ_1,χ_2) is the pair of Dirichlet characters attached to χ.

Proof The proof of [3, Theorem 3.5] adapts. ■

Remark 5.10 Let (χ_1,χ_2) be the pair attached to χ. Since the primes dividing M are split in K, it follows from (5.2) that $\chi_1(-M) = \chi_2(-M)$. Hence we shall simply write $\chi_i(-M)$.

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5.2.2 Derivatives of p-adic L-functions Attached to Real Quadratic Fields

Theorem 5.11 *Let $\Psi \in \mathcal{E}mb^{+}(\mathcal{O}, \mathcal{R})$. Then*

$$\frac{d}{d\kappa}\big[\mathcal{L}_p(f/K, \Psi, \kappa)\big]_{\kappa=k_0} = \frac{1}{2} D_K^{\frac{k_0-2}{4}} \cdot \big(\log \Phi^{AJ}(j_\Psi)(I_f) + (-1)^{m+1} \log \Phi^{AJ}(j_{\overline{\Psi}})(I_f)\big).$$

Proof Consider the factorization

$$P_\Psi(x, y) = A(x - \tau_\Psi y)(x - \overline{\tau}_\Psi y)$$

and write

$$\mathcal{L}_p(f/K, \Psi, \kappa) := |L_\Psi|^{-\frac{k_0-2}{2}} \langle A\rangle^{\frac{\kappa-k_0}{2}} \cdot \int_{L'_\Psi} P^m_\Psi(x, y)\langle x - \tau_\Psi y\rangle^{\frac{\kappa-k_0}{2}} \langle x - \overline{\tau}_\Psi y\rangle^{\frac{\kappa-k_0}{2}} dI_{L_\Psi}\{r \to \gamma_\Psi r\}.$$

In light of Proposition 5.7, the usual formula for the derivatives of the product of two functions yields

$$\frac{d}{d\kappa}\big[\mathcal{L}_p(f/K, \Psi, \kappa)\big]_{\kappa=k_0} = |L_\Psi|^{-\frac{k_0-2}{2}} \cdot \frac{d}{d\kappa}\Big[\int_{L'_\Psi} P^m_\Psi(x, y)\langle x - \tau_\Psi y\rangle^{\frac{\kappa-k_0}{2}} \langle x - \overline{\tau}_\Psi y\rangle^{\frac{\kappa-k_0}{2}} dI_{L_\Psi}\{r \to \gamma_\Psi r\}\Big]_{\kappa=k_0}.$$

By Proposition 4.2,

$$\begin{aligned} &|L_\Psi|^{-\frac{k_0-2}{2}} \frac{d}{d\kappa}\Big[\int_{L'_\Psi} P^m_\Psi(x, y)\langle x - \tau_\Psi y\rangle^{\frac{\kappa-k_0}{2}} \langle x - \overline{\tau}_\Psi y\rangle^{\frac{\kappa-k_0}{2}} dI_{L_\Psi}\{r \to \gamma_\Psi r\}\Big]_{\kappa=k_0} \\ &= \frac{1}{2}|L_\Psi|^{-\frac{k_0-2}{2}} \frac{d}{d\kappa}\Big(\int_{L'} P^m_\Psi(x, y)\langle x - \tau_\Psi y\rangle^{\kappa-k} dI_{L_\Psi}\{r \to \gamma_\Psi r\}\Big)_{\kappa=k} \\ &\quad + \frac{1}{2}|L_\Psi|^{-\frac{k_0-2}{2}} \frac{d}{d\kappa}\Big(\int_{L'} P^m_\Psi(x, y)\langle x - \overline{\tau}_\Psi y\rangle^{\kappa-k} dI_{L_\Psi}\{r \to \gamma_\Psi r\}\Big)_{\kappa=k}. \end{aligned}$$

Note now that $L_\Psi = L_{\overline{\Psi}}$ and, by Remark 2.20, $\overline{\tau}_\Psi = \tau_{\overline{\Psi}}$, $P^m_\Psi = (-1)^m P^m_{\overline{\Psi}}$ and $\gamma_\Psi = \gamma^{-1}_{\overline{\Psi}}$. It follows that the last expression is equal to

$$\frac{1}{2}\Big(\int_x^{\gamma_\Psi x}\int^{\tau_\Psi} P^m_\Psi \omega_f + (-1)^m \int_x^{\gamma^{-1}_{\overline{\Psi}} x}\int^{\tau_{\overline{\Psi}}} P^m_{\overline{\Psi}}\omega_f\Big).$$

By Lemma 2.19, replacing x by $\gamma_{\overline{\Psi}} x$ to compute the integral gives

$$\int_x^{\gamma^{-1}_{\overline{\Psi}} x}\int^{\tau_{\overline{\Psi}}} P^m_{\overline{\Psi}}\omega_f = \int_{\gamma_{\overline{\Psi}} x}^{x}\int^{\tau_{\overline{\Psi}}} P^m_{\overline{\Psi}}\omega_f = -\int_x^{\gamma_{\overline{\Psi}} x}\int^{\tau_{\overline{\Psi}}} P^m_{\overline{\Psi}}\omega_f.$$

The claim now follows from Corollary 4.13. ■

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Recall the linear combination j^χ introduced in (2.18) and set

$$\overline{j}^\chi := \sum_{\sigma \in G_{H^+_{\mathcal{O}}/K}} \chi^{-1}(\sigma) j_{\overline{\sigma\Psi}}.$$

Corollary 5.12 *Let $\chi\colon G_{H^+_{\mathcal{O}}/K} \to \mathbb{C}^\times$ be a character. Then*

$$\frac{d^2}{d\kappa^2}\big[L_p(f/K,\chi,\kappa)\big]_{\kappa=k_0} = \frac{1}{2} D_K^{\frac{k_0-2}{2}} \cdot \big(\log \Phi^{AJ}(j^\chi)(I_f) + (-1)^{m+1} \log \Phi^{AJ}(\overline{j}^\chi)(I_f)\big)^2.$$

Proof This is a consequence of Theorem 5.11, in light of Proposition 5.7. ■

Let now χ be a genus character attached to the pair (χ_1,χ_2) and let H^+ be the narrow Hilbert ring class field. Recall that by Remark 5.10 $\chi_i(-M)$ does not depend on $i = 1, 2$.

Corollary 5.13 *Let $\chi\colon G_{H^+/K} \to \mathbb{C}^\times$ be a genus character. Then*

$$\frac{d^2}{d\kappa^2}\big[L_p(f/K,\chi,\kappa)\big]_{\kappa=k_0} = \frac{1}{2} D_K^{\frac{k_0-2}{2}} \cdot \big(1 + (-1)^{m+1} w_M \chi_i(-M)\big)^2 \log \Phi^{AJ}(j^\chi)(I_f)^2.$$

Proof First of all note that, since I_f is an eigenform for the Atkin–Lehner involution W_M with eigenvalue w_M, we may write

$$w_M \log \Phi^{AJ}(j_{\overline{\sigma\Psi}})(I_f) = \log \Phi^{AJ}(j_{\overline{\sigma\Psi}})(I_f \mid W_M) = \log \Phi^{AJ}(j_{\alpha_M \overline{\sigma\Psi}\alpha_M^{-1}})(I_f).$$

Let $\sigma\Psi \in \Gamma\backslash\mathcal{E}mb^{+\mathfrak{d}}(\mathcal{O},\mathcal{R})$ be any oriented embedding as explained in Remark 2.24 and note that $\overline{\sigma\Psi}$ has the same orientation at p of Ψ, but for every prime $l^e \parallel M$ the orientation of $\overline{\sigma\Psi}$ is opposite to the orientation of $\sigma\Psi$. As explained in Remark 2.24 the Atkin–Lehner involution W_M exchanges these orientations, so that $\alpha_M\overline{\sigma\Psi}\alpha_M^{-1} \in \Gamma\backslash\mathcal{E}mb^{+\mathfrak{d}}(\mathcal{O},\mathcal{R})$. Since we noted in Remark 2.24 that $\Gamma\backslash\mathcal{E}mb^{+\mathfrak{d}}(\mathcal{O},\mathcal{R})$ is a torsor under the $G_{H^+/K}$-action, there exists a unique $\delta_{\sigma\Psi} \in G_{H^+/K}$ such that $\alpha_M\overline{\sigma\Psi}\alpha_M^{-1} = \delta_{\sigma\Psi}\sigma\Psi$. According to [3, (17)] $\delta_{\sigma\Psi} = \delta_\Psi\sigma^{-2}$, so that we find (we have $\chi^2 = 1$):

$$\begin{aligned}
&\sum_{\sigma \in G_{H^+_{\mathcal{O}}/K}} \chi(\sigma) \log \Phi^{AJ}(j_{\overline{\sigma\Psi}})(I_f)\\
&\qquad = w_M \sum_{\sigma \in G_{H^+_{\mathcal{O}}/K}} \chi(\sigma) \log \Phi^{AJ}(j_{\delta_\Psi\sigma^{-1}\Psi})(I_f)\\
&\qquad = w_M \chi(\delta_\Psi) \sum_{\sigma \in G_{H^+_{\mathcal{O}}/K}} \chi(\delta_\Psi\sigma^{-1}) \log \Phi^{AJ}(j_{\delta_\Psi\sigma^{-1}\Psi})(I_f)\\
&\qquad = w_M \chi(\delta_\Psi) \sum_{\sigma \in G_{H^+_{\mathcal{O}}/K}} \chi(\sigma) \log \Phi^{AJ}(j_{\sigma\Psi})(I_f).
\end{aligned}$$

But since χ is a genus character, [3, Proposition 1.8] tells us that $\chi(\delta_\Psi) = \chi_i(-M)$. The claim now follows from Corollary 5.12. ■

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Corollary 5.14 *Let $\chi\colon G_{H^+/K} \to \mathbb{C}^\times$ be a genus character. Then*

$$\frac{d^2}{d\kappa^2}\big[L_p(f/K,\chi,\kappa)\big]_{\kappa=k_0} = \begin{cases} 2D_K^{\frac{k_0-2}{2}} \log \Phi^{AJ}(j^\chi)(I_f)^2 & \text{if } \chi_i(-M) = (-1)^{m+1} w_M, \\ 0 & \text{if } \chi_i(-M) = (-1)^m w_M. \end{cases}$$

5.3 p-adic L-functions Attached to Imaginary Quadratic Fields

In this subsection we let $K'\mathbb{Q}$ be an imaginary quadratic field of discriminant $D_{K'}$, and we consider a factorization $N = pN^+N^-$ such that:

- p is inert or split in K';
- all the prime factors of N^+ split in K';
- N^- is the squarefree product of an odd number of primes which remain inert in K'.

Recall the definite quaternion algebra B of discriminant $N^-\infty$ and fix an identification $B_p = \mathbb{M}_2(\mathbb{Q}_p)$, so that B_p acts on the p-adic upper halfplane as well as on the Bruhat–Tits three and on the sets $\mathcal{L}$ and $\mathcal{L}_0$. As in the Darmon setting, it is possible to define the set of optimal embeddings of level N^+ and pN^+ of a $\mathbb{Z}[1/p]$-order $\mathcal{O}$ of conductor c, prime to $D_{K'}$ and N, into the corresponding Eichler order R. More precisely the definition in [2, Definition 3.2] is given in terms of optimal embeddings of a $\mathbb{Z}$-order $\mathcal{O}_\mathbb{Z}$ into an Eichler order $R_\mathbb{Z}$, this last of level N^+ or pN^+, such that $\mathcal{O} = \mathbb{Z}[1/p] \otimes \mathcal{O}_\mathbb{Z}$ and $R = \mathbb{Z}[1/p] \otimes R_\mathbb{Z}$. By [1, Lemma 2.1] the set of optimal embeddings of level pN^+ is non-empty only when p is split, so that this assumption will be implicit when considering embeddings of level pN^+. As explained in [2, Section 3.1], by the strong approximation theorem, these sets can be realized as subsets of

$$R^\times\backslash(\mathcal{E}mb(\mathcal{O},R)\times\mathcal{L}) \qquad \text{when the level is } N^+,$$

$$R^\times\backslash(\mathcal{E}mb(\mathcal{O},R)\times\mathcal{L}_0) \qquad \text{when the level is } pN^+.$$

More precisely, the elements of the first set are those represented by the couples $[\Psi, L_\Psi]$, where L_Ψ is preserved by the action of $\Psi(\mathcal{O})$, while the elements of the second set are those represented by the triples $[\Psi, L^1_\Psi, L^2_\Psi]$ such that L^1_Ψ and L^2_Ψ are both preserved under the action of $\Psi(\mathcal{O})$ (when p is split).

There are the following data attached to the optimal embeddings of level N^+, say represented by the couple $[\Psi, L_\Psi]$:

- the two fixed points $\tau_\Psi, \overline{\tau}_\Psi \in \mathcal{H}_p$ for the action of $\Psi(K'^\times)$ on $\mathcal{H}_p(K')$, ordered in such a way that the action of $K'^\times$ on the tangent space at τ_Ψ is through the character $z \mapsto z/\overline{z}$, when p is inert;
- the unique fixed vertex $v_\Psi \in \mathcal{V}$ for the action of $\Psi(K'^\times)$ on $\mathcal{V}$, which is nothing but the reduction $\mathrm{red}(\tau_\Psi) = \mathrm{red}(\overline{\tau}_\Psi)$, when p is inert;
- the lattice L_Ψ such that $[L_\Psi] = v_\Psi$, when p is inert, and the lattice L_Ψ, which is fixed by the action of the split quadratic algebra $\Psi(\mathcal{O}\otimes\mathbb{Z}_p)$ and hence admits a $\mathbb{Z}_p$-basis $\{x_\Psi, y_\Psi\}$ of eigenvectors for this action, when p is split;
- the unique polynomial up to sign P_Ψ in $\mathbf{P}_2$, which is fixed by the action of $\Psi(K'^\times)$ on $\mathbf{P}_2 \otimes \det^{-1}$ and satisfies $\langle P_\Psi, P_\Psi\rangle_{\mathbf{P}_2} = -D_{K'}$ (the pairing being defined as in

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[5]), which we fix by the choice

$$P_\Psi := \mathrm{Tr}\Big(\Psi(\sqrt{D_{K'}}) \cdot \left(\begin{smallmatrix} X & -X^2 \\ 1 & -X \end{smallmatrix}\right) \Big) \in \mathbf{P}_2,$$

the other one being obtained by replacing $\sqrt{D_{K'}}$ with $-\sqrt{D_{K'}}$; note that P_Ψ is either irreducible over $\mathbb{Q}_p$ or it splits into two linear forms corresponding to the basis $\{x_\Psi, y_\Psi\}$, according to whether p is inert or split.

Define

$$L''_\Psi := \begin{cases} L'_\Psi & \text{when } p \text{ is inert,} \\ \mathbb{Z}_p^\times x_\Psi \oplus \mathbb{Z}_p^\times y_\Psi & \text{when } p \text{ is split.} \end{cases}$$

Recall the family μ_* that was attached to φ by means of Theorem 4.16. The following definition attaches a p-adic L-function

$$\mathcal{L}_p(f/K', \Psi, -)\colon U \to \mathbb{C}_p$$
$$\kappa \mapsto \mathcal{L}_p(f/K', \Psi, \kappa)$$

to the data of the embedding $[\Psi, L_\Psi]$ of level N^+. It is easily checked that the subsequent definitions do not depend on the choice of L_Ψ such that $[L_\Psi] = v_\Psi$ when p is inert.

Definition 5.15 The partial p-adic L-function attached to $(f/K', \Psi)$ is

$$\mathcal{L}_p(f/K', \Psi, \kappa) := |L_\Psi|^{-\frac{k_0-2}{2}} \int_{L''_\Psi} \langle P_\Psi(x,y)\rangle^{\frac{\kappa-k_0}{2}} P_\Psi^m d\mu_{L_\Psi}.$$

The partial p-adic L-function attached to $(f/K', \chi)$, where $\chi\colon G_{H_{\mathcal{O}}/K'} \to \mathbb{C}^\times$ is a character, is

$$\mathcal{L}_p(f/K', \chi, \kappa) := \sum_{\sigma \in G_{H_{\mathcal{O}}/K'}} \chi^{-1}(\sigma)\mathcal{L}_p(f/K', \sigma\Psi, \kappa).$$

The p-adic L-function attached to $(f/K', \chi)$, where $\chi\colon G_{H_{\mathcal{O}}/K'} \to \mathbb{C}^\times$ is a character, is

$$L_p(f/K', \chi, \kappa) := \mathcal{L}_p(f/K', \chi, \kappa)\mathcal{L}_p(f/K', \chi^{-1}, \kappa).$$

As before, in order to justify the above definition and the fact that the above p-adic L-functions are analytic we can appeal to Lemma 4.1 after noticing that they are built from functions of the form

$$\kappa \mapsto \mu\big(P\langle\alpha\rangle^{\frac{\kappa-k_0}{2}} \langle\beta\rangle^{\frac{\kappa-k_0}{2}} \chi_X \big).$$

Of course Remark 5.6 also holds in this setting. Note the following vanishing property of the p-adic L-functions, which is proved exactly as in Proposition 5.7 using (4.9) and Corollary 4.19.

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Proposition 5.16 *Assume p is inert. The p-adic L-functions vanishes at k_0:*

$$\mathcal{L}_p(f/K', \Psi, k_0) = \mathcal{L}_p(f/K', \chi, k_0) = L_p(f/K', \chi, k_0) = 0.$$

Furthermore,

$$\frac{d}{d\kappa}\left[L_p(f/K', \chi, \kappa)\right]_{\kappa=k_0} = 0$$

5.3.1 Interpolation Properties of the p-adic L-functions Attached to Imaginary Quadratic Fields and Functional Equation

The next two theorems collect the interpolation properties of these p-adic L-functions.

Theorem 5.17 *Assume p is inert. Then, for all $k \in \mathbb{Z}^{\geq 2, \neq k_0} \cap U$,*

$$L_p(f/K', \chi, k) = \lambda_B(k)^2 a_p(k)^2 \left(1 - p^{k-2} a_p(k)^{-2}\right)^2 L^*(f_k^\#/K', \chi, k/2),$$

where

$$L^*(f_k^\#/K', \chi, k/2) := \frac{(\frac{k-2}{2})!^2 D_K^{\frac{k-1}{2}}}{(2\pi)^{k-2}\langle f_k^\#, f_k^\#\rangle} L(f_k^\#/K', \chi, k/2).$$

Proof The proof proceeds precisely along the same lines as [2, Theorem 3.8], and we repeat the ideas for the convenience of the reader. Using [2, Lemma 3.7] and noticing that $[P_\Psi(x,y)]^{\frac{k-k_0}{2}} = 1$ by Lemma 4.1, one can show that

$$P_\Psi(x,y)^{\frac{k-k_0}{2}} = |L_\Psi|^{\frac{k-k_0}{2}} \left\langle P_\Psi(x,y)\right\rangle^{\frac{k-k_0}{2}}$$

on $L''_\Psi = L'_\Psi$. Hence,

$$\mathcal{L}_p(f/K', \Psi, k) = |L_\Psi|^{-\frac{k-2}{2}} \int_{L''_\Psi} P_\Psi^{\frac{k-2}{2}}\, d\mu_{L_\Psi}.$$

Now we simply need to replace the use of [2, Proposition 2.11] with the more general, but formally identical, Corollary 4.18, which gives

$$|L_\Psi|^{-\frac{k-2}{2}} \int_{L''_\Psi} P_\Psi^{\frac{k-2}{2}}\, d\mu_{L_\Psi} = \lambda_B(k) a_p(k)\left(1 - p^{k-2}a_p(k)^{-2}\right) |L_\Psi|^{-\frac{k-2}{2}} c_k^\#(L_\Psi)\left(P_\Psi^{\frac{k-2}{2}}\right).$$

Summing together, multiplying $\mathcal{L}_p(f/K', \chi, \kappa)$ with $\mathcal{L}_p(f/K', \chi^{-1}, \kappa)$ and applying Hatcher–Hui Xue's formula

$$\begin{aligned}(5.3)\qquad &\Big(\sum_{\sigma \in G_{H_{\mathcal{O}}/K'}} \chi^{-1}(\sigma) |L_{\sigma\Psi}|^{-\frac{k-2}{2}} c_k^\#(L_{\sigma\Psi})(P_{\sigma\Psi}^{\frac{k-2}{2}})\Big) \\ &\cdot \Big(\sum_{\sigma \in G_{H_{\mathcal{O}}/K'}} \chi(\sigma) |L_{\sigma\Psi}|^{-\frac{k-2}{2}} c_k^\#(L_{\sigma\Psi})(P_{\sigma\Psi}^{\frac{k-2}{2}})\Big) \\ &= \langle c_k^\#, c_k^\#\rangle L^*(f_k^\#/K', \chi, k/2)\end{aligned}$$

as reformulated in [2, Proposition 3.3] yields the result, in light of the normalization (4.6). ∎

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Theorem 5.18 *Assume p, split in K' and let $\mathfrak{p} \mid p$ be a prime of K' above p. Then*

$$L_p(f/K',\chi,k_0) = \left(1-\chi(\mathfrak{p})p^{\frac{k_0-2}{2}}a_p(k_0)^{-1}\right)\left(1-\chi^{-1}(\mathfrak{p})p^{\frac{k_0-2}{2}}a_p(k_0)^{-1}\right) \cdot \langle c_{k_0}, c_{k_0}\rangle L^*(f_{k_0}/K',\chi,k_0/2)$$

and for all $k \in \mathbb{Z}^{\geq 2, \neq k_0} \cap U$,

$$L_p(f/K',\chi,k) = \lambda_B^2(k)\left(a_p(k) + p^{k-2}a_p(k)^{-1} - p^{\frac{k-2}{2}}\chi(\mathfrak{p}) - p^{\frac{k-2}{2}}\chi^{-1}(\mathfrak{p})\right)^2 \cdot L^*(f_k^{\#}/K',\chi,k/2).$$

Proof Again the proof of [2, Theorem 3.12] works in this setting (even with more general characters). We do not recall which are the main ingredients, since the computation is more involved than Theorem 5.17. As explained in [2, Proposition 3.4], the appearance of the factor $\langle c_{k_0}, c_{k_0}\rangle$ at $k = k_0$ is due to the fact that no normalization condition was imposed on the modular form c_{k_0}, so that in the Hatcher–Hui Xue's formula (5.3) this factor needs to be considered. ■

We now specialize the above theorem to a genus character χ of the imaginary quadratic field K', say attached to the pair of Dirichlet characters (χ_1, χ_2). Note that, since p is split, $\chi(\mathfrak{p}) = \chi_i(p)$ does not depend on i. Furthermore, since $\chi(\mathfrak{p})^2 = 1$, the Euler factor appearing in Theorem 5.18 can be rewritten, and one deduces the following corollary.

Corollary 5.19 *Assume p split in K', let $\mathfrak{p} \mid p$ be a prime of K' above p, and let χ be the genus character attached to the pair of Dirichlet characters (χ_1, χ_2). Then*

$$L_p(f/K',\chi,k_0) = \left(1-\chi_i(p)p^{\frac{k_0-2}{2}}a_p(k_0)^{-1}\right)^2 \langle c_{k_0}, c_{k_0}\rangle L^*(f_{k_0}/K',\chi,k/2)$$

and for all $k \in \mathbb{Z}^{\geq 2, \neq k_0} \cap U$,

$$L_p(f/K',\chi,k) = \lambda_B(k)^2 a_p(k)^2\left(1-\chi_i(p)p^{\frac{k-2}{2}}a_p(k)^{-1}\right)^4 L^*(f_k^{\#}/K',\chi,k/2).$$

Definition 5.20 Let $\eta\colon \mathbb{Z}^{\geq 2} \to \mathbb{C}_p$ be the function

$$\eta(k) := \begin{cases} \frac{\lambda_B(k)^2 a_p(k)^2}{\lambda^+(k)\lambda^-(k)} D_{K'}^{\frac{k-2}{2}} i^{k-2} & \text{for } k \neq k_0, \\ \langle c_{k_0}, c_{k_0}\rangle D_{K'}^{\frac{k_0-2}{2}} i^{k_0-2} & \text{for } k = k_0. \end{cases}$$

Theorem 5.21 *The function $\eta(k)$ uniquely extends to an analytic function such that $\eta(\kappa) \neq 0$ on U (up to shrinking it). Moreover, for every genus character χ, say attached to the couple of Dirichlet characters (χ_1, χ_2),*

$$L_p(f/K',\chi,\kappa) = \eta(\kappa) L_p(f,\chi_1,\kappa,\kappa/2) L_p(f,\chi_2,\kappa,\kappa/2)$$

on U.

Proof The proof is the same as that of [2, Corollary 5.3], after noticing that the main result of [22] extends to our higher weight setting. ■

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5.3.2 Derivatives of p-adic L-functions Attached to Imaginary Quadratic Fields

Assume until the end of this section that p is inert in K'. Let $X = X_{N^+,pN^-}$ be the Shimura curve attached to the indefinite quaternion algebra ramified at the primes pN^-. As explained in [1, Section 1.5], the Shimura curve X is endowed with Hecke operators T_l for $l \nmid N$ as well as Atkin–Lehner involutions W_l^+ for $l \mid N^+$ and Atkin–Lehner involutions W_l^- for $l \mid pN^-$. The Atkin–Lehner involution W_p^- will be of particular interest for us. Write $X_{W_p^-}$ to denote the twist of the Shimura curve X by the cocycle in $H^1(G_{\mathbb{Q}_{p^2}\mathbb{Q}_p}, \mathrm{Aut}(X))$, which maps the non trivial element $\mathrm{Frob}_p \in G_{\mathbb{Q}_{p^2}\mathbb{Q}_p}$ to W_p^-. Recall the group $\widetilde{\Gamma}'$ defined by (4.5) and denote by Γ' the subgroup of norm one elements. By the Cerednik–Drinfeld Theorem, $X_{W_p^-}$ admits a rigid analytic uniformization over $\mathbb{Q}_p$:

$$\Gamma' \backslash \mathcal{H}_p = X^{an}_{W_p^-},$$

where $X^{an}_{W_p^-}$ is the analytification of $X_{W_p^-}$. We will make an abuse of notation by writing $X = X^{an}$ or $X_{W_p^-} = X^{an}_{W_p^-}$.

The optimal embeddings of a conductor N^+ admit a particular simple description. More precisely, fix an embedding $\sigma_p \colon H = H_{\mathcal{O}} \hookrightarrow \mathbb{Q}_{p^2}$ (this is possible, since p is inert in K' and hence it splits completely in H). The p-adic uniformization allow us to view $\Gamma' \backslash \mathcal{E}mb(\mathcal{O}, R)$ as a subset of $X(\mathbb{Q}_{p^2})$:

$$\Gamma' \backslash \mathcal{E}mb(\mathcal{O}, R) \hookrightarrow \Gamma' \backslash \mathcal{E}mb(\mathbb{Q}_{p^2}, B_p) = \Gamma' \backslash \mathcal{E}mb(\mathbb{Q}_{p^2}, \mathbb{M}_2(\mathbb{Q}_p)) = X(\mathbb{Q}_{p^2}).$$

In this way we shall identify (the class of) Ψ with its image in $X(\mathbb{Q}_{p^2})$.

Remark 5.22 In view of the above twist that enters in the rigid analytic parametrization, the optimal embedding $\overline{\Psi}$ corresponds in $X(\mathbb{Q}_{p^2})$ to the optimal embedding $W_p^- \,\mathrm{Frob}_p\, \Psi$, regarded like a point of X.

Recall the rigid analytic modular form f^{rig} that was attached to the modular form f by means of the Jacquet–Langlands correspondence. It satisfies the following relation with respect to the action of the Atkin-Lehner involution W_p^- (see [1, Theorem 1.2]):

$$f^{\mathrm{rig}} \mid W_p^- = -w_p f^{\mathrm{rig}}, \tag{5.4}$$

where w_p is the sign of the Atkin–Lehner involution W_p acting on f.

Let $\mathcal{M}_n$ be the Chow motive (over $\mathbb{Q}$) of weight k_0 modular forms constructed in [15, Appendix]. As explained in [15, Appendix] one can attach to an optimal embedding Ψ an element $y_\Psi^{(n)} \in CH^{m+1}(\mathcal{M}_{n,H})$, the Chow group of codimension $m+1$ cycles of $\mathcal{M}_n$ base changed to $H := H_{\mathcal{O}}$. The p-adic realization $V(m+1) := H_p(\mathcal{M}_{n,\overline{\mathbb{Q}}}, \mathbb{Q}_p(m+1))$ of the motive $\mathcal{M}_n$ affords representations attached to cusp forms that are new at pN^-. Consider the p-adic Abel–Jacobi map

$$cl_0^{m+1} \colon CH^{m+1}(\mathcal{M}_{n,H}) \to \mathrm{Ext}^1_{G_H}\big(\mathbb{Q}_p, V(m+1)\big).$$

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After a base change from H to $F_p \supset \mathbb{Q}_{p^2}$, the p-adic Abel–Jacobi map can be identified with

$$\Phi^{AJ}\colon CH^{m+1}(\mathcal{M}_{n,F_p}) \to \operatorname{Ext}^1_{MF}\big(F_p, \mathbb{D}(m+1)\big) = \frac{\mathbb{D}_{F_p}}{F^{m+1}\mathbb{D}_{F_p}},$$

where we write $\mathbb{D} := \mathbb{D}_{st}(V)$. Here $\mathbb{D}_{st}$ is the Fontaine functor attaching to a Galois representation of G_{F_p} a filtered Frobenius module over F_p. As recalled in the introduction, the above ext group is explicitly computed in [15, (49)], and the p-adic étale Abel–Jacobi map can be can be interpreted as

$$\log \Phi^{AJ}\colon CH^{m+1}(\mathcal{M}_{n,F_p}) \to M_k(\Gamma', F_p)^{\vee},$$

where $M_k(\Gamma', F_p)^{\vee}$ denotes the F_p-dual space.

Note that the Frobenius Frob_p introduced in Remark 5.22 also acts on the Chow group $CH^{m+1}(\mathcal{M}_{n,\mathbb{Q}_{p^2}})$.

Lemma 5.23 $y^{(n)}_{\operatorname{Frob}_p \Psi} = (-1)^m \operatorname{Frob}_p y^{(n)}_{\Psi}$.

Proof Let W be the group generated by the Atkin–Lehner involutions $W_l^{\pm}$ for $l \mid N$. The proof is easily reduced to the case $m = 1$, *i.e.,* weight $k_0 = 4$. In this case the Heegner cycles $y^{(2)}_{\Psi}$ are defined by fixing $y^{(2)}_{\Psi_0}$ for some Ψ_0 and then exploiting the simply transitive action of $G_{H/K'} \times W$ on the optimal embeddings in order to make these cycles compatible with the action of this group. Indeed, the elements $y^{(2)}_{\Psi}$ are only canonical up to sign. More precisely, they correspond to $z^{(2)}_{\Psi} \in \operatorname{End}_{\mathcal{R}}(A_{\Psi}) = \mathcal{O}$ that are only defined up to sign. Since $A_{\Psi} = E^2_{\Psi}$, where E_{Ψ} is an elliptic curve such that $\operatorname{End}(E_{\Psi}) = \operatorname{End}_{\mathcal{R}}(A_{\Psi})$, we can reduce to consider elliptic curves. In this case we can fix an isomorphism $[-]_{\Psi}\colon \mathcal{O} \simeq \operatorname{End}(E_{\Psi})$ with the property that for every $\sigma \in \operatorname{Aut}(\mathbb{C})$ and $\alpha \in \mathcal{O}$, we have $\sigma[\alpha]_{\Psi} = [\alpha^{\sigma}]_{\sigma\Psi}$ and define the element $z^{(2)}_{\Psi}$ by making it correspond to the choice of a fixed root $\sqrt{D_{K'}} \in K'$, the other choice $-\sqrt{D_{K'}}$ giving rise to the element $-z^{(2)}_{\Psi}$. With this choice the elements $z^{(2)}_{\Psi}$ are compatible for the action of the group $G_{H/K'} \times W$. Furthermore, since Frob_p is induced by the complex conjugation τ (because p is inert) we find that $\tau[\sqrt{D_{K'}}]_{\Psi} = [-\sqrt{D_{K'}}]_{\tau\Psi}$, which gives $\operatorname{Frob}_p z^{(2)}_{\Psi} = -z^{(n)}_{\operatorname{Frob}_p \Psi}$. ■

Theorem 5.24 *Let $\Psi \in \mathcal{E}mb^{+}(\mathcal{O}, \mathcal{R})$. Then*

$$\frac{d}{d\kappa}\big[\mathcal{L}_p(f/K', \Psi, \kappa)\big]_{\kappa=k_0} = \frac{1}{2}\Big(\log \Phi^{AJ}(y^{(n)}_{\Psi})(f^{\mathrm{rig}}) - w_p \log \Phi^{AJ}(\operatorname{Frob}_p y^{(n)}_{\Psi})(f^{\mathrm{rig}})\Big).$$

Proof By the main result of [24]:

$$\frac{d}{d\kappa}\big[\mathcal{L}_p(f/K', \Psi, \kappa)\big]_{\kappa=k_0} = \frac{1}{2}\Big(\log \Phi^{AJ}(y^{(n)}_{\Psi})(f^{\mathrm{rig}}) + (-1)^m \log \Phi^{AJ}(y^{(n)}_{\overline{\Psi}})(f^{\mathrm{rig}})\Big).$$

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By Remark 5.22 and Lemma 5.23,

$$y^{(n)}_{\Psi} = y^{(n)}_{W_p^- \operatorname{Frob}_p \Psi} = W_p^- y^{(n)}_{\operatorname{Frob}_p \Psi} = (-1)^m W_p^- \operatorname{Frob}_p y^{(n)}_{\Psi}.$$

Now the claim follows from (5.4). ■

Whenever F is a field, let us write $MW_f(F)$ to denote the image of the Chow group over F in the group $\operatorname{Ext}^1_{G_F}(\mathbb{Q}_p, V_{[f]}(m+1))$, *i.e.*, the image obtained by $cl^{m+1}_{0,f} := e_{[f]} \circ cl^{m+1}_0$. By the theory of complex multiplication,

$$y^{\chi} := \sum_{\sigma \in G_{H_{\mathcal{O}}/K'}} \chi^{-1}(\sigma) y^{(n)}_{\sigma\Psi} \in CH^{m+1}(\mathcal{M}_{n,H_\chi})^{\chi},$$

where H_χ/K' is the subextension of H/K' that corresponds to the kernel of χ. Hence $cl^{m+1}_{0,f}(y^\chi) \in MW_f(H_\chi)^\chi$.

Corollary 5.25 *Let $\chi\colon G_{H_{\mathcal{O}}/K'} \to \mathbb{C}^\times$ be a character. Then*

$$\begin{aligned}\frac{d^2}{d\kappa^2}\big[L_p(f/K',\chi,\kappa)\big]_{\kappa=k_0} &= \frac{1}{2}\big(\log \Phi^{AJ}(y^\chi)(f^{\mathrm{rig}}) - w_p \log \Phi^{AJ}(\operatorname{Frob}_p y^\chi)(f^{\mathrm{rig}})\big)\\ &\quad\cdot \big(\log \Phi^{AJ}(y^{\chi^{-1}})(f^{\mathrm{rig}}) - w_p \log \Phi^{AJ}(\operatorname{Frob}_p y^{\chi^{-1}})(f^{\mathrm{rig}})\big).\end{aligned}$$

Proof This is a consequence of Theorem 5.24, in light of Proposition 5.16. ■

For the remainder of this section let us focus on a genus character χ attached to the couple (χ_1,χ_2). We note that the signs of the twisted L-functions $L(f,\chi_i,s)$ are given by (see [25, Theorem 3.66]):

$$(-1)^{\frac{k_0}{2}} w_N \chi_i(-N). \tag{5.5}$$

Furthermore, since the number of the inert primes pN^- dividing N is even and $\varepsilon_{K'} = \chi_1\chi_2$, where $\varepsilon_{K'}$ is the Dirichlet character attached to the imaginary quadratic extension $K'/\mathbb{Q}$,

$$\chi_1(-N)\chi_2(-N) = \varepsilon_{K'}(-1) = -1.$$

Hence the signs of the twisted complex L-functions $L(f,\chi_i,s)$ are opposite to each other. The genus character χ cuts out a biquadratic extension of $\mathbb{Q}$. Write $\mathbb{Q}_{\chi_i}$ to denote the quadratic extension that corresponds to the Dirichlet character χ_i.

Whenever V is a $\mathbb{Q}_p[G_{H_\chi/\mathbb{Q}}]$-module, let us write $V^\pm$ to denote the subspace on which the complex conjugation τ acts as $\pm$, so that $V = V^+ \oplus V^-$. Since $\operatorname{Ind}^{G_\mathbb{Q}}_{G_{K'}}(\chi) = \chi_1 \oplus \chi_2$, we also have $V^\chi = V^{\chi_1} \oplus V^{\chi_2}$, where the left-hand side is viewed as a $G_{H_\chi/K'}$-module and the right-hand side as a $G_{H_\chi/\mathbb{Q}}$-module. Since $\chi_2(-1) = -\chi_1(-1)$, we may order (χ_1,χ_2) in such a way that $\mathbb{Q}_{\chi_1}/\mathbb{Q}$ is a real field. Then we have $V^{\chi_1} \subset V^{\chi,+}$ and $V^{\chi_2} \subset V^{\chi,-}$, so that $V^{\chi_1} = V^{\chi,+}$ and $V^{\chi_1} = V^{\chi,-}$.

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This remark applied to $V = CH^{m+1}(\mathcal{M}_{n,H_\chi})$ and $V = \mathrm{Ext}^1_{G_{H_\chi}}(\mathbb{Q}_p, V_{[f]})$ implies that $CH^{m+1}(\mathcal{M}_{n,H_\chi})^\chi$ and $MW_f(H_\chi)^\chi$ both have a direct sum decomposition with

$$\begin{aligned} CH^{m+1}(\mathcal{M}_{n,H_\chi})^{\chi,+} &= CH^{m+1}(\mathcal{M}_{n,\mathbb{Q}_{\chi_1}})^{\chi_1}, \\ CH^{m+1}(\mathcal{M}_{n,H_\chi})^{\chi,-} &= CH^{m+1}(\mathcal{M}_{n,\mathbb{Q}_{\chi_2}})^{\chi_2}, \\ MW_f(H_\chi)^{\chi,+} &= MW_f(\mathbb{Q}_{\chi_1})^{\chi_1}, \\ MW_f(H_\chi)^{\chi,-} &= MW_f(\mathbb{Q}_{\chi_2})^{\chi_2}. \end{aligned}$$

Whenever Ψ is an oriented optimal embedding of level N^+, viewed as an element of $X(H)$, $\tau\Psi$ is an optimal embedding whose orientations at the primes dividing N^+ have been reversed. Define

$$W_N := \prod_{l|N^+} W_l^+ \prod_{l|pN^-} W_l^-.$$

Since $\#\{l : l \mid pN^-\}$ is even, it follows from the analogue of (5.4) at the primes dividing pN^-, that $W_N f^{\mathrm{rig}} = w_N f^{\mathrm{rig}}$, where w_N is the sign of the Atkin–Lehner involution acting on f. Since $W_N\Psi$ reverses all the orientations too, we have

$$\tau\Psi = W_N\delta\Psi, \text{ for some } \delta \in G_{H/K'}. \tag{5.6}$$

It is easily checked using Lemma 5.23 and (5.6) that $cl_{0,f}^{m+1}(y^\chi) \in MW_f(H_\chi)^{\chi,\pm}$ for a suitable choice of a sign. Let H be the Hilbert ring class field.

Corollary 5.26 *Let $\chi\colon G_{H/K} \to \mathbb{C}^\times$ be a genus character. If*

$$cl_{0,f}^{m+1}(y^\chi) \in MW_f(\mathbb{Q}_{\chi_i})^{\chi_i},$$

then

$$\frac{d^2}{d\kappa^2}\big[L_p(f/K', \chi, \kappa)\big]_{\kappa=k_0} = \frac{1}{2}\big(1 + a_p p^{-\frac{k_0-2}{2}}\chi_i(p)\big)^2 \log\Phi^{AJ}(y^\chi)(f^{\mathrm{rig}})^2.$$

Proof Since $cl_{0,f}^{m+1}(y^\chi) \in MW_f(\mathbb{Q}_{\chi_i})^{\chi_i}$, we have

$$cl_{0,f}^{m+1}(\mathrm{Frob}_p\, y^\chi) = \mathrm{Frob}_p\, cl_{0,f}^{m+1}(y^\chi) = \chi_i(p) cl_{0,f}^{m+1}(y^\chi).$$

The p-adic Abel–Jacobi map $\Phi^{AJ}(-)(f^{\mathrm{rig}})$ factors through $cl_{0,f}^{m+1}$ by definition, so that we find

$$\log\Phi^{AJ}(\mathrm{Frob}_p\, y^\chi)(f^{\mathrm{rig}}) = \chi_i(p)\log\Phi^{AJ}(y^\chi)(f^{\mathrm{rig}}).$$

The claim follows from Corollary 5.25, since we have $\chi = \chi^{-1}$ and $-w_p = a_p p^{-\frac{k_0-2}{2}}$. ■

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Corollary 5.27 *Let $\chi\colon G_{H/K} \to \mathbb{C}^\times$ be a genus character. If*

$$cl_{0,f}^{m+1}(y^\chi) \in MW_f(\mathbb{Q}_{\chi_i})^{\chi_i},$$

then

$$\frac{d^2}{d\kappa^2}\big[L_p(f/K',\chi,\kappa)\big]_{\kappa=k_0} = \begin{cases} 2\log\Phi^{AJ}(y^\chi)(f^{\mathrm{rig}})^2 & \text{if } \chi_i(p) = a_p p^{-\frac{k_0-2}{2}} = -w_p,\\ 0 & \text{if } \chi_i(p) = -a_p p^{-\frac{k_0-2}{2}} = w_p. \end{cases}$$

We will also need the following deep result of Kato.

Lemma 5.28 *If $\chi_i(-N) = (-1)^{\frac{k_0-2}{2}} w_N$ and $L(f,\chi_j,k_0/2) \neq 0$ with $i \neq j$ or if $\chi_i(-N) = (-1)^{\frac{k_0-2}{2}} w_N$ and $\chi_i(pN^-) = 1$, then $cl_{0,f}^{m+1}(y^\chi) \in MW_f(\mathbb{Q}_{\chi_i})^{\chi_i}$.*

Proof We only prove the first statement, which is the one we need in the subsequent section. If $\chi_i(-N) = (-1)^{k_0-2/2} w_N$, the sign of $L(f,\chi_i,s)$ is negative and we assume that $L(f,\chi_j,k_0/2) \neq 0$. Then $MW_f(\mathbb{Q}_{\chi_j})^{\chi_j} = 0$ by [16, Theorem 14.2 (2)]. ■

Remark 5.29 Suppose that $0 \neq cl_{0,f}^{m+1}(y_{\mathfrak{p}}^\chi) \in MW_{f,\mathfrak{p}}(\mathbb{Q}_{\chi_i})^{\chi_i}$. As an application of Kolyvagin methods developed in [17, 18], one can show that $K_{f,\mathfrak{p}} cl_{0,f}^{m+1}(y_{\mathfrak{p}}^\chi) = MW_{f,\mathfrak{p}}(\mathbb{Q}_{\chi_i})^{\chi_i}$.

6 Proof of the Main Results

Recall our factorization $N = pN^+N^- = pM$ into factors prime each other, where N^- is squarefree and divisible by an odd number of prime factors. In the following theorem we will assume the existence of a prime $q \parallel M$ and the consideration of a factorization with $q \mid N^-$ will be implicit in order to apply the results of the previous section. Recall the harmonic cocycle $c^{\mathrm{har}} = c_f^{\mathrm{har}}$ that was associated with f in Subsection 4.2. We may assume that $c^{\mathrm{har}} \in \mathcal{C}_{\mathrm{har}}(\mathcal{E}, \mathbf{V}_n(K_f))^{\Gamma'}$, so that $\langle c_{k_0}, c_{k_0}\rangle \in K_f^\times$.

Theorem 6.1 *Suppose there exists $q \parallel M$. Let ω be a quadratic Dirichlet character of conductor prime to N such that*

$$\omega(-N) = (-1)^{\frac{k_0-2}{2}} w_N \quad \text{and} \quad \omega(p) = a_p p^{-\frac{k_0-2}{2}} = -w_p.$$

Then:

(i) *the p-adic L-function $L_p(f,\omega,\kappa,\kappa/2)$ vanishes to order*

$$\mathrm{ord}_{\kappa=k_0} L_p(f,\omega,\kappa,\kappa/2) \geq 2;$$

(ii) *there exists $y^\omega \in CH^{m+1}(\mathcal{M}_{n,\mathbb{Q}_\omega})^\omega$ and $t_f \in K_f^\times$ such that*

$$\frac{d^2}{d\kappa^2}\big[L_p(f/K,\omega,\kappa,\kappa/2)\big]_{\kappa=k_0} = t_f \cdot \log\Phi^{AJ}(y^\omega)(f^{\mathrm{rig}})^2;$$

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(iii) *if* $cl^{m+1}_{0,f}(y^{\omega}_{\mathfrak{p}}) \neq 0$, *then* $MW_{f,\mathfrak{p}}(\mathbb{Q}_{\omega})^{\omega} = K_{f,\mathfrak{p}} cl^{m+1}_{0,f}(y^{\omega}_{\mathfrak{p}})$*;*

(iv) *we have*

$$t_f \equiv L^*(f, \psi, k_0/2) \text{ in } K_f^{\times}/K_f^{\times 2}$$

for any quadratic Dirichlet character ψ *such that* $\psi(l) = \omega(l)$ *for every* $l \mid M := N/p$, $\psi(p) = -\omega(p)$ *and* $L(f, \psi, 1) \neq 0$.

Proof Set $\omega = \chi_1$ and choose an auxiliary quadratic Dirichlet character χ_2 of conductor prime to the conductor of χ_1 such that:

(i) $\chi_2(l) = \chi_1(l)$ for all $l \mid N^+$;
(ii) $\chi_2(l) = -\chi_1(l)$ for all $l \mid pN^-$ and $\chi_2(-1) = -\chi_1(-1)$;
(iii) $L(f, \chi_2, k_0/2) \neq 0$.

This is possible, since the main result of [22] generalizes to higher weight modular forms. The Dirichlet character $\varepsilon_{K'} := \chi_1\chi_2$ cuts out an imaginary quadratic extension $K'/\mathbb{Q}$, and there is a genus character χ attached to the pair (χ_1, χ_2). Furthermore, note that the sign of $L(f, \chi_1, s)$ is -1 in light of the assumption $\chi_1(-N) = (-1)^{\frac{k_0-2}{2}} w_N$ (by (5.5), while the sign of $L(f, \chi_2, s)$ is 1. Note that, thanks to (iii) and Lemma 5.28, we can apply Corollary 5.27 with $\chi_i = \chi_1$.

By Theorem 5.21,

$$L_p(f/K', \chi, \kappa) = \eta(\kappa) L_p(f, \chi_1, \kappa, \kappa/2) L_p(f, \chi_2, \kappa, \kappa/2). \tag{6.1}$$

The factor $\eta(\kappa)L_p(f, \chi_2, \kappa, \kappa/2)$ does not vanish at the critical point $\kappa = k_0$, since $\eta(\kappa) \neq 0$ on U, and we have

$$\begin{aligned} L_p(f, \chi_2, k_0, k_0/2) &= \left(1 - \chi_2(p) p^{\frac{k_0-2}{2}} a_p^{-1}\right) L^*(f, \chi_2, k_0/2) \\ &= 2L^*(f, \chi_2, k_0/2) \neq 0. \end{aligned} \tag{6.2}$$

Indeed the first equality follows by Corollary 5.4; the second one follows by the assumption $\chi_1(p) = p^{-k_0-2/2} a_p$, together with (ii) assuring us that $\chi_2(p) = -\chi_1(p)$, and the non-vanishing is a consequence of (iii).

On the other hand, the factor $L_p(f, \chi_1, \kappa, \kappa/2)$ vanishes at the critical point $\kappa = k_0$, again by Corollary 5.4 and the assumption $\chi_1(p) = p^{-k_0-2/2} a_p$, or thanks to the fact that $L^*(f_k^{\#}, \chi_1, k_0/2) = 0$ by the above considerations on the complex L-functions. Hence,

$$L_p(f, \chi_1, k_0, k_0/2) = 0. \tag{6.3}$$

This preliminary discussion has the effect of avoiding appealing to [2, Remark 1.13], since we have not exploited the Mazur–Kitagawa p-adic L-function as a two variable function.

(i) A formal computation using (6.1) and (6.3) yields

$$\frac{d}{d\kappa}\left[L_p(f/K', \chi, \kappa)\right]_{\kappa=k_0} = \frac{d}{d\kappa}\left[L_p(f, \chi_1, \kappa, \kappa/2)\right]_{\kappa=k_0} \eta(k_0) L_p(f, \chi_2, k_0, k_0/2).$$

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Note that $\varepsilon_K(p) = -1$ so that we are in the inert case, and the left-hand side vanishes by Proposition 5.16. Now (6.2) implies that

$$\frac{d}{d\kappa}\big[L_p(f,\chi_1,\kappa,\kappa/2)\big]_{\kappa=k_0} = 0, \tag{6.4}$$

so that the claim (i) follows. Note that that the same sign considerations of [2, Theorem 5.4] apply in order to deduce the order two vanishing along the line (k_0, s) of $L_p(f,\chi_1,\kappa,s)$, and hence the order two vanishing of the two variable p-adic L-function.

(ii) A formal computation using (6.1), (6.3), and (6.4) yields

$$\frac{d^2}{d\kappa^2}\big[L_p(f/K',\chi,\kappa)\big]_{\kappa=k_0} = \frac{d^2}{d\kappa^2}[L_p(f,\chi_1,\kappa,\kappa/2)]_{\kappa=k_0}\eta(k_0)L_p(f,\chi_2,k_0,k_0/2). \tag{6.5}$$

By (6.2) and Corollary 5.27 we can write

$$\frac{d^2}{d\kappa^2}\big[L_p(f,\chi_1,\kappa,\kappa/2)\big]_{\kappa=k_0} = \frac{1}{2}\langle c_{k_0},c_{k_0}\rangle^{-1}D_K^{\frac{2-k_0}{2}}i^{2-k_0}\cdot L^*(f,\chi_2,k_0/2)^{-1}\log\Phi^{AJ}(y^\chi)(f^{\mathrm{rig}})^2,$$

where $y^\chi \in CH^{m+1}(\mathcal{M}_{n,\mathbb{Q}_\omega})^\omega$. Now claim (ii) follows, since we have, writing $D_{K'} = -D \in \mathbb{Z}$ with $D > 0$ and $k_0 = 2h$,

$$D_{K'}^{\frac{2-k_0}{2}}i^{2-k_0} = (-1)^{2-k_0}D^{1-h} \in \mathbb{Z}.$$

(iii) This is a consequence of Remark 5.29.

(iv) Let $\psi := \chi_1'$ be any Dirichlet character satisfying the conditions of 4. and consider the Dirichlet character $\varepsilon_{K''} := \chi_1'\chi_2$. It cuts out an imaginary quadratic field $K''/\mathbb{Q}$. There is a genus character χ' attached to the couple (χ_1',χ_2), but now p is split in K''. In particular $\chi_i(p) = \chi'(\mathfrak{p})$ for any $\mathfrak{p} \mid p$ and Corollary 5.19 yields, in light of the fact that $\chi_2(p) = -p^{-\frac{k_0-2}{2}}a_p$,

$$\begin{aligned} L_p(f/K'',\chi',k_0) &= \big(1 - p^{\frac{k_0-2}{2}}a_p^{-1}\chi_2(p)\big)^2\langle c_{k_0},c_{k_0}\rangle L^*(f_{k_0}/K'',\chi',k_0/2)\\ &= 4\langle c_{k_0},c_{k_0}\rangle L^*(f_{k_0}/K'',\chi',k_0/2).\end{aligned}$$

By (6.1) relative to (χ_1',χ_2) together with (6.2) relative to χ_1',

$$2\langle c_{k_0},c_{k_0}\rangle L^*(f_{k_0}/K'',\chi',k_0/2) = L^*(f,\chi_1',k_0/2)\eta(k_0)L_p(f,\chi_2,k_0,k_0/2).$$

Besides, thanks to (6.5), $t_f/2 = \eta(k_0)^{-1}L_p(f,\chi_2,k_0,k_0/2)^{-1}$, so that

$$4\langle c_{k_0},c_{k_0}\rangle L^*(f_{k_0}/K'',\chi',k_0/2) \equiv t_fL^*(f,\chi_1',k_0/2) \mod K_f^{\times 2}.$$

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But the left-hand side is a square by the Hatcher–Hui Xue formula applied to the newform c_{k_0} of level pN^+, that can be reformulated in a similar way as it is done in [2, Proposition 3.4] when $k_0 = 2$, thus getting a formula having the same shape as (5.3) but involving optimal embeddings of level pN^+. ■

We now turn to the case where $K/\mathbb{Q}$ is a real quadratic field and χ is a genus character attached to (χ_1, χ_2). Recall that, by Remark 5.10, $\chi_1(-M) = \chi_2(-M)$. Assume again that $c^{\mathrm{har}} \in \mathcal{C}_{\mathrm{har}}(\mathcal{E}, \mathbf{V}_n(K_f))^{\Gamma'}$.

Theorem 6.2 *Suppose $N = pM$, that there exists $q \parallel M$, and that*

$$\chi_i(-M) = (-1)^{\frac{k_0}{2}} w_M.$$

Then:

(i) *there exist $y^\chi \in CH^{m+1}(\mathcal{M}_{n,H_\chi})^\chi$ and $s_f \in K_f^\times$ such that*

$$\log \Phi^{AJ}(j^\chi)(I_f) = s_f \cdot \log \Phi^{AJ}(y^\chi)(f^{\mathrm{rig}});$$

(ii) $\chi_2(-N) = -\chi_1(-N)$ *and* $cl_{0,f}^{m+1}(y^\chi) \in MW_{f,\mathfrak{p}}(\mathbb{Q}_{\chi_i})^{\chi_i}$, *where* $\chi_i(-N) = (-1)^{\frac{k_0-2}{2}} w_N$ *and, if we assume* $cl_{0,f}^{m+1}(y_{\mathfrak{p}}^\chi) \neq 0$,

$$MW_{f,\mathfrak{p}}(H_\chi)^\chi = MW_{f,\mathfrak{p}}(\mathbb{Q}_{\chi_i})^{\chi_i} = K_{f,\mathfrak{p}} cl_{0,f}^{m+1}(y_{\chi,\mathfrak{p}}).$$

Proof Consider the functional equation given by Theorem 5.9:

$$L_p(f/K, \chi, \kappa) = D_K^{\frac{\kappa-2}{2}} L_p(f, \chi_1, \kappa, \kappa/2) L_p(f, \chi_2, \kappa, \kappa/2). \tag{6.6}$$

In light of the assumption $\chi_i(-M) = (-1)^{\frac{k_0}{2}} w_M$, we find that

$$\chi_i(-N) = (-1)^{\frac{k_0}{2}} w_M \chi_i(p) \tag{6.7}$$

Furthermore, since p is inert in K, we have $\chi_1(p) = -\chi_2(p)$. In particular, one of the two Dirichlet characters, say χ_1, will be such that $\chi_1(p) = -w_p$; then (6.7) tells us that

$$\chi_1(-N) = (-1)^{\frac{k_0-2}{2}} w_N. \tag{6.8}$$

It follows that the Dirichlet character $\chi_1 = \omega$ satisfies the assumption of Theorem 6.1. By (i) we know that the order of vanishing of the p-adic L-function $L_p(f, \chi_1, \kappa, \kappa/2)$ is at least 2. A formal computation using this information and (6.6)

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gives

(6.9)
$$\frac{d^2}{d\kappa^2}\left[L_p(f/K,\chi,\kappa)\right]_{\kappa=k_0}=\frac{d^2}{d\kappa^2}[L_p(f,\chi_1,\kappa,\kappa/2)]_{\kappa=k_0}D_K^{\frac{k_0-2}{2}}L_p(f,\chi_2,k_0,k_0/2).$$

By Corollary 5.14 (again use $\chi_i(-M)=(-1)^{\frac{k_0}{2}}w_M$), Theorem 6.1(ii), and (6.9),

$$\log\Phi^{AJ}(j^\chi)(I_f)^2=t_f/2\cdot\log\Phi^{AJ}(y^\omega)(f^{\mathrm{rig}})^2L_p(f,\chi_2,k_0,k_0/2). \tag{6.10}$$

Again note that $\chi_2(p)=-\chi_1(p)$, so that (6.7) and (6.8) tell us that $\chi_2(p)=w_p=-p^{\frac{k_0-2}{2}}a_p^{-1}$. Hence thanks to (6.2) we can rewrite (6.10) as

$$\log\Phi^{AJ}(j^\chi)(I_f)^2=t_f\cdot\log\Phi^{AJ}(y^\omega)(f^{\mathrm{rig}})^2L^*(f,\chi_2,k_0/2).$$

If $L^*(f,\chi_2,k_0/2)=0$, we deduce that $\log\Phi^{AJ}(j^\chi)(I_f)=0$ and the first part of the theorem is trivially true by setting $y_\chi=0$. Hence suppose $L^*(f,\chi_2,k_0/2)\neq 0$. In this case, note that χ_2 satisfies the assumption that was made on ψ in Theorem 6.1(iv). Hence we know that $t_fL^*(f,\chi_2,k_0/2)$ is a square in $K_f^\times$. The first claim follows by setting $y^\chi=y^\omega$ and extracting the square roots.

For the second statement we have already proved that $\chi_2(-N)=-\chi_1(-N)$, we are assuming that $\chi_1(-N)=(-1)^{k_0-2/2}w_N$, and we know that y^ω belongs to $MW_f(\mathbb{Q}_{\chi_1})^{\chi_1}$, so that y^χ belongs to $MW_f(\mathbb{Q}_{\chi_1})^{\chi_1}$ by construction. Since $\mathrm{Ind}_{G_K}^{G_\mathbb{Q}}(\chi)=\chi_1\oplus\chi_2$, we can write

$$MW_f(H_\chi)^\chi=MW_f(\mathbb{Q}_{\chi_1})^{\chi_1}\oplus MW_f(\mathbb{Q}_{\chi_2})^{\chi_2}.$$

When $cl_{0,f}^{m+1}(y_\mathfrak{p}^\chi)\neq 0$ we are in the case $L^*(f,\chi_2,k_0/2)\neq 0$ and then $y^\chi=y^\omega$. It follows from Theorem 6.1(iii) that we have $MW_{f,\mathfrak{p}}(\mathbb{Q}_{\chi_1})^{\chi_1}=K_{f,\mathfrak{p}}cl_{0,f}^{m+1}(y_\mathfrak{p}^\chi)$, and, since $L^*(f,\chi_2,k_0/2)\neq 0$, [16, Theorem 14.2 (2)] implies $MW_{f,\mathfrak{p}}(\mathbb{Q}_{\chi_2})^{\chi_2}=0$. ■

Remark 6.3 Let $\sigma(f)$ be the companion form of f obtained by applying the automorphism σ to the Fourier coefficients of f. If we choose $c_{\sigma(f)}^{\mathrm{har}}:=\sigma(c_f^{\mathrm{har}})$, the quantities s_f appearing in the statement of Theorem 6.2 satisfy the relation $\sigma(s_f)=s_{\sigma(f)}$. It follows that there is $s\in K_f\otimes_\mathbb{Q}F_p$ inducing $s_{\sigma(f)}$ on the $\sigma(f)$-component. Recall that $F_p/\mathbb{Q}_p$ denotes an extension such that $K_{[f]}\subset F_p$, where $K_{[f]}$ is the field generated by the Fourier coefficients of f and its companion forms.

Let us now prove the main result Theorem 1.1. Let $V_{[f]}$ be the p-adic representation attached to the new modular form f, with associated filtered Frobenius module $\mathbb{D}_{[f]}$. Note that $MW_f(H_\chi)^\chi$ is naturally a K_f-vector space, since the Hecke correspondences act on the rational Chow groups through the idempotent $e_{[f]}$ corresponding to the f-isotypic component. Let the assumptions be as in Theorem 6.2. Fix an isomorphism of monodromy modules $\mathbf{D}_{[f]}\simeq\mathbb{D}_{[f]}$ over $\mathbb{Q}_p$ as granted by Theorem 4.11.

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The identification $\varphi\colon \mathbf{D}_{[f]} \simeq \mathbb{D}_{[f]}$ in $MF_{\mathbb{Q}_p}(\phi, N)$ allows us to identify the tangent spaces

$$\begin{aligned}(6.11)\qquad \alpha : \mathbf{MS}_{[f]}^{c,\vee,w_\infty}(F_p) &= \frac{\mathbf{D}_{[f],F_p}}{F^{m+1}\mathbf{D}_{[f],F_p}} \overset{\varphi}{\simeq} \frac{\mathbb{D}_{[f],F_p}}{F^{m+1}\mathbb{D}_{[f],F_p}} \\ &\overset{\exp}{\simeq} H^1_{st}(K, V_{[f]}(m+1)) = \mathrm{Ext}^1_{MF}\big(F_p, \mathbb{D}_{[f]}(m+1)\big) \\ &= \frac{\mathbb{D}_{[f],F_p}}{F^{m+1}\mathbb{D}_{[f],F_p}} = (F^{m+1}\mathbb{D}_{[f],F_p})^\vee \\ &= e_{[f]}M_k(X, F_p)^\vee = e_{[f]}M_k(\Gamma', F_p)^\vee.\end{aligned}$$

The above identifications hold over any complete field extension $F_p/\mathbb{Q}_p$, with the only possible exception of the last identification, that holds assuming $F_p \supset \mathbb{Q}_{p^2}$.

The first identification is the morphism

$$f^0\colon \frac{\mathbf{D}_{[f],F_p}}{F^{m+1}\mathbf{D}_{[f],F_p}} \to \mathbf{MS}_{[f]}^{c,\vee,w_\infty}(F_p)$$

that was considered in Section 3. The last five identifications are given by

$$IS\colon H^1_{st}\big(K, V_{[f]}(m+1)\big) \to e_{[f]}M_k(\Gamma', F_p)^\vee.$$

We have the following commutative diagram

$$(6.12)\qquad \begin{array}{ccc} \frac{\mathbf{D}_{[f],F_p}}{F^{m+1}\mathbf{D}_{[f],F_p}} & \xrightarrow{\exp\circ\varphi} & H^1_{st}\big(K, V_{[f]}(m+1)\big) \\ \Big\downarrow{\scriptstyle f^0} & & \Big\downarrow{\scriptstyle IS} \\ \mathbf{MS}_{[f]}^{c,\vee,w_\infty}(F_p) & \xrightarrow{\alpha} & e_{[f]}M_k(\Gamma', F_p)^\vee. \end{array}$$

It will be convenient to give an explicit description of the monodromy module $\mathbb{D}$ by means of Teitelbaum's p-adic integration theory (as developed, for example, in [26]). More explicitly let Γ' be the arithmetic group defined in Subsection 5.3.2. As is well known, there is an analogue of Proposition 2.8 in this definite setting: the morphism

$$R\colon \mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p)) \to \mathcal{C}_{\mathrm{har}}(\mathcal{E}, \mathbf{V}_n)$$

that was considered in Subsection 2.2 induces an isomorphism $\mathcal{D}_n^0(\mathbb{P}^1(\mathbb{Q}_p))^{\Gamma'} = \mathcal{C}_{\mathrm{har}}(\mathcal{E}, \mathbf{V}_n)^{\Gamma'} =: \mathbf{C}_{\mathrm{har}}(F_p)$ (over any complete local field $F_p/\mathbb{Q}_p$). Define $\mathbf{D}^T$ over $\mathbb{Q}_p$ as follows:

$$\mathbf{D}^T := \mathbf{C}_{\mathrm{har}}(\mathbb{Q}_p)^\vee \oplus \mathbf{C}_{\mathrm{har}}(\mathbb{Q}_p)^\vee,$$

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with filtration, monodromy operator, and Frobenius formally defined exactly as in Subsection 2.3, Teitelbaum's $\mathcal{L}$-invariant replacing Orton's $\mathcal{L}$-invariant. Similarly as in Section 3, there is an identification obtained by means of $f(x, y) = -x - \mathcal{L}y$:

$$f\colon \frac{\mathbf{D}^T_{F_p}}{F^{m+1}\mathbf{D}^T_{F_p}} \overset{\simeq}{\to} \mathbf{C}_{\mathrm{har}}(F_p)^\vee.$$

By [15] Teitelbaum's $\mathcal{L}$-invariant equals the $\mathcal{L}$-invariant of the monodromy module $\mathbb{D}$ and there is an identification $\mathbf{D}^T \simeq \mathbb{D}$ in $MF_{\mathbb{Q}_p}(\phi, N)$. As it follows from the proof of [23, Lemma 4.4], in order to give an explicit identification $\mathbf{D}^T \simeq \mathbb{D}$, we can simply identify the m-isotypic components as Hecke modules. Furthermore, we can identify $\mathbf{D}^T \simeq \mathbb{D}$ in $MF_{\mathbb{Q}_{p^2}}(\phi, N)$, since we have

$$\mathrm{Hom}_{MF_{\mathbb{Q}_p}(\phi,N)}(D_1, D_2) = \mathrm{Hom}_{MF_{\mathbb{Q}_{p^2}}(\phi,N)}(D_{1,\mathbb{Q}_{p^2}}, D_{2,\mathbb{Q}_{p^2}}),$$

whenever $D_i \in MF_{\mathbb{Q}_p}(\phi, N)$ (see the proof of [23, Lemma 4.4]). As in [15], let $\mathcal{V}_n$ be the coherent sheaf on the Shimura curve X over $\mathbb{Q}_{p^2}$ associated with the representation $\mathbf{V}_n$, so that $\mathbb{D}_{\mathbb{Q}_{p^2}} = H^1(X, \mathcal{V}_n)$. As it follows from [15], the m-isotypic component of $H^1(X, \mathcal{V}_n)$ is $H^1(X, \mathcal{V}_n)^m = \iota(H^1(\Gamma', \mathbf{V}_n(\mathbb{Q}_{p^2})))$, where ι is the injection [15, (76)]. Let

$$\langle -, - \rangle_{\Gamma'}\colon \mathbf{C}_{\mathrm{har}}(\mathbb{Q}_{p^2}) \otimes H^1(\Gamma', \mathbf{V}_n(\mathbb{Q}_{p^2})) \to \mathbb{Q}_{p^2}$$

be the perfect pairing [15, (75)]. It induces an isomorphism

$$H^1\big(\Gamma', \mathbf{V}_n(\mathbb{Q}_{p^2})\big) \overset{\simeq}{\to} \mathbf{C}_{\mathrm{har}}(\mathbb{Q}_{p^2})^\vee \tag{6.13}$$

that we use to identify the m-isotypic components. Let $I\colon M_k(\Gamma', \mathbb{Q}_{p^2}) \overset{\simeq}{\to} \mathbf{C}_{\mathrm{har}}(\mathbb{Q}_{p^2})$ be the residue map, thus inducing a map $I^\vee\colon \mathbf{C}_{\mathrm{har}}(\mathbb{Q}_{p^2})^\vee \to M_k(\Gamma', \mathbb{Q}_{p^2})^\vee$.

Lemma 6.4 *The isomorphism* (6.13) *induces an identification $\psi : \mathbf{D}^T \simeq \mathbb{D}$ making the following diagram commutative:*

$$\begin{tikzcd}
\frac{\mathbf{D}^T_{\mathbb{Q}_{p^2}}}{F^{m+1}\mathbf{D}^T_{\mathbb{Q}_{p^2}}} \arrow[r, equal, "f"] \arrow[d, equal, "\psi"'] & \mathbf{C}_{\mathrm{har}}(\mathbb{Q}_{p^2})^\vee \arrow[d, equal, "I^\vee"] \\
\frac{\mathbb{D}_{\mathbb{Q}_{p^2}}}{F^{m+1}\mathbb{D}_{\mathbb{Q}_{p^2}}} \arrow[r, equal] & M_k(\Gamma', \mathbb{Q}_{p^2})^\vee.
\end{tikzcd}$$

Here the lower horizontal arrow is the composition of the last three identifications in the definition of α.

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Proof The morphism f maps the class $\mathbf{d} = [x, y] \in \mathbf{D}^T_{\mathbb{Q}_{p^2}}/F^{m+1}$ to the coordinate $f(\mathbf{d})$ in $\mathbf{C}_{\mathrm{har}}(\mathbb{Q}_{p^2})^\vee$ of the opposite of the unique element $-(f(\mathbf{d}), 0)$ in $\ker N = \mathbf{D}^{T,m}$ representing $\mathbf{d}$. Let $\psi(\mathbf{d}) \in \mathbb{D}_{\mathbb{Q}_{p^2}}/F^{m+1}$ be the corresponding element and denote by d the unique element of $\ker N = \mathbb{D}^{T,m}_{\mathbb{Q}_{p^2}}$ representing $\psi(\mathbf{d})$. By unicity we have $\psi((f(\mathbf{d}), 0)) = -d$. Let

$$P\colon H^1(X, \mathcal{V}_n) \to H^1\big(\Gamma', \mathbf{V}_n(\mathbb{Q}_{p^2})\big)$$

be the left inverse of ι as defined in [15, (15)]. If we write $x \in \mathbb{D}_{\mathbb{Q}_{p^2}} = H^1(X, \mathcal{V}_n)$ as $x = x^m + x^{m+1}$ according to its slope decomposition, we have $x^m = \iota(P(x))$ (see [15]). In particular we have $d = d_m = \iota(P(d))$. Since ψ is induced by (6.13), we deduce, from the equality $\psi((f(\mathbf{d}), 0)) = -d$, that $f(\mathbf{d}) = -\langle -, P(d)\rangle_{\Gamma'}$. Besides, the identification $\mathbb{D}_{\mathbb{Q}_{p^2}}/F^{m+1} = (F^{m+1}\mathbb{D}_{\mathbb{Q}_{p^2}})^\vee$ arises from Serre duality induced by cup product and the canonical identification $F^{m+1}\mathbb{D}_{\mathbb{Q}_{p^2}} = M_k(\Gamma', \mathbb{Q}_{p^2})$ (see [15, Proposition 6.1]). Let d be as above, so that $d \in \ker N = \ker I$ and d corresponds to $\langle -, d\rangle_X \in M_k(\Gamma', \mathbb{Q}_{p^2})^\vee$, where $\langle -, -\rangle_X$ is the cup product. The reciprocity law [15, Theorem 10.3] implies

$$\langle -, d\rangle_X = -\langle I(-), P(d)\rangle_X = I^\vee(f(\mathbf{d})),$$

which is the claim. ∎

Lemma 6.5 *We may choose $\varphi\colon \mathbf{D}_{[f]} \simeq \mathbb{D}_{[f]}$ in such a way that $\alpha^\vee(ev_{\sigma(f)^{\mathrm{rig}}}) = ev_{I_{\sigma(f)}}$, and Remark* 6.3 *holds.*

Proof Let us write $\mathbf{D}^T_{[f]}$ (resp. $\mathbf{C}_{har,[f]}$) to denote the space obtained by taking the f-isotypic component by means of the idempotent $e_{[f]}$. According to Lemma 6.4 there is a commutative diagram:

$$\begin{tikzcd}
 & \frac{\mathbf{D}^T_{[f]}}{F^{m+1}\mathbf{D}^T_{[f]}} \arrow[r, equals, "f"] \arrow[d, "\psi"'] & \mathbf{C}_{har,[f]}(F_p)^\vee \arrow[d, "I^\vee"] \\
\mathbf{MS}^{c,\vee,w_\infty}_{[f]}(F_p) \arrow[r, equals] & \frac{\mathbb{D}_{F_p}}{F^{m+1}\mathbb{D}_{F_p}} \arrow[r, equals] & M_k(\Gamma', F_p)^\vee.
\end{tikzcd}$$

Here the lower row comes from (6.11).

The above identifications holds even with $F_p = \mathbb{Q}_p$, the only possible exception being the last one appearing in the lower row. Denote by β the arrow from $\mathbf{MS}^{c,\vee,w_\infty}_{[f]}(\mathbb{Q}_p)$ to $\mathbf{C}_{har,[f]}(\mathbb{Q}_p)^\vee$, so that we have

$$\beta^\vee\colon \mathbf{C}_{har,[f]}(\mathbb{Q}_p)^{\vee\vee} \to \mathbf{MS}^{c,\vee\vee,w_\infty}_{[f]}(\mathbb{Q}_p).$$

Then $\mathbf{MS}^{c,w_\infty}_{[f]}(\mathbb{Q}_p)$ and $\mathbf{C}_{har,[f]}(\mathbb{Q}_p)$ are naturally endowed with the $\mathbb{Q}$-structures $\mathbf{MS}^{c,w_\infty}_{[f]}(\mathbb{Q})$ and, respectively, $\mathbf{C}_{har,[f]}(\mathbb{Q})$, and they are both rank one K_f-modules.

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Fix an isomorphism $b\colon \mathbf{C}_{har,[f]}(\mathbb{Q}) \simeq \mathbf{MS}^{c,w_\infty}_{[f]}(\mathbb{Q})$ of K_f-modules, thus inducing an isomorphism b of $K_f \otimes L$-modules $\mathbf{C}_{har,[f]}(L) \simeq \mathbf{MS}^{c,w_\infty}_{[f]}(L)$ over any field extension. Once we fix $I_f = I_f^{w_\infty} \in \mathbf{MS}^{c,w_\infty}_{[f]}(K_f)$, we may choose $I_{\sigma(f)} := \sigma(I_f) \in \mathbf{MS}^{c,w_\infty}_{[f]}(K_{\sigma(f)})$, the quantity $\Omega^{w_\infty}_{\sigma(f)}$ appearing in Proposition 2.1 being well defined only up to multiplication by an element in $K^\times_{\sigma(f)}$. Setting $c^{\mathrm{har}}_{\sigma(f)} := b^{-1}(I_{\sigma(f)}) \in \mathbf{C}_{har,[f]}(K_{\sigma(f)})$, the relation $c^{\mathrm{har}}_{\sigma(f)} = \sigma(c^{\mathrm{har}}_f)$ in Remark 6.3 is satisfied and Theorem 6.2 is in force. By biduality we find the morphism

$$b^{\vee\vee}\colon \mathbf{C}_{har,[f]}(\mathbb{Q}_p)^{\vee\vee} \to \mathbf{MS}^{c,\vee\vee,w_\infty}_{[f]}(\mathbb{Q}_p)$$

such that $b^{\vee\vee}(ev_{c^{\mathrm{har}}_{\sigma(f)}}) = ev_{I_{\sigma(f)}}$ (after extending the scalars to $F_p \supset K_{[f]}$). Since $\mathbf{MS}^{c,\vee\vee,w_\infty}_{[f]}(\mathbb{Q}_p) \simeq K_f \otimes \mathbb{Q}_p$, there exists $t \in (K_f \otimes \mathbb{Q}_p)^\times$ such that $b^{\vee\vee} = t \circ \beta^\vee$. By [23, Lemma 4.4] $\mathrm{End}_{MF_{\mathbb{Q}_p}(\phi,N)}(\mathbb{D}_{[f]}) = K_f \otimes \mathbb{Q}_p$. Replacing φ by $t\circ\varphi$, the morphism $\beta^\vee$ turns into $t \circ \beta^\vee = b^{\vee\vee}$, because the above morphisms are Hecke equivariant. Hence we may assume that $\beta^\vee(ev_{c^{\mathrm{har}}_{\sigma(f)}}) = ev_{I_{\sigma(f)}}$. Recall that the rigid analytic modular form $\sigma(f)^{\mathrm{rig}}$ was obtained as $I(\sigma(f)^{\mathrm{rig}}) = c^{\mathrm{har}}_{\sigma(f)}$, so that $I^{\vee\vee}(ev_{\sigma(f)^{\mathrm{rig}}}) = ev_{c^{\mathrm{har}}_{\sigma(f)}}$. We have $\alpha = I^\vee \circ \beta$, hence $\alpha^\vee = \beta^\vee \circ I^{\vee\vee}$ satisfies $\alpha^\vee(ev_{\sigma(f)^{\mathrm{rig}}}) = \beta^\vee(ev_{c^{\mathrm{har}}_{\sigma(f)}}) = ev_{I_{\sigma(f)}}$. ■

By Lemma 6.5 we have $\alpha^\vee(ev_{\sigma(f)^{\mathrm{rig}}}) = ev_{I_{\sigma(f)}}$. Hence, by Remark 6.3 (which is in force in light of Lemma 6.5), Theorem 6.2 implies

$$\alpha\big(\log \Phi^{AJ}_{[f]}(j^\chi)\big) = \log \Phi^{AJ}_{[f]}(sy^\chi) = IS\big(cl^{m+1}_{0,f}(sy^\chi)\big). \tag{6.14}$$

Then we have

$$\exp\Big(\varphi\big(\Phi^{AJ}_{[f]}(j^\chi)\big)\Big) = cl^{m+1}_{0,f}(sy^\chi)$$

if and only if we have

$$\mathrm{IS}\Big(\exp\big(\varphi(\Phi^{AJ}_{[f]}(j^\chi))\big)\Big) = IS\big(cl^{m+1}_{0,f}(sy^\chi)\big).$$

This is true since the left-hand side is $\alpha(f^0(\Phi^{AJ}_{[f]}(j^\chi))) = \alpha(\log \Phi^{AJ}_{[f]}(j^\chi))$, thanks to the commutativity of (6.12). The claim follows from (6.14).

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References

[1] M. Bertolini and H. Darmon, *Heegner points on Mumford-Tate curves.* Invent. Math. **126**(1996), no. 3, 413–456. http://dx.doi.org/10.1007/s002220050105
[2] ———, *Hida families and rational points on elliptic curves.* Invent. Math. **168**(2007), no. 2, 371–431. http://dx.doi.org/10.1007/s00222-007-0035-4
[3] ———, *The rationality of Stark-Heegner points over genus fields of real quadratic fields.* Ann. of Math. **170**(2009), no. 1, 343–370. http://dx.doi.org/10.4007/annals.2009.170.343

https://doi.org/10.4153/CJM-2011-076-8 Published online by Cambridge University Press

[4] M. Bertolini, H. Darmon, and A. Iovita, *Families of automorphic forms on definite quaternion algebras and Teitelbaum's conjecture.* Astérisque, to appear.
[5] M. Bertolini, H. Darmon, A. Iovita, and M. Spiess, *Teitelbaum's exceptional zero conjecture in the anticyclotomic setting.* Amer. J. Math. **124**(2002), no. 2, 411–449. http://dx.doi.org/10.1353/ajm.2002.0009
[6] M. Bertolini, H. Darmon, and P. Green, *Periods and points attached to quadratic algebras.* In: Heegner points and Rankin L-series, Math. Sci. Res. Inst. Publ., 49, Cambridge University Press, Cambridge, MA, 2004, pp. 323–367.
[7] G. Chenevier, *Une correspondance de Jaquet-Langlands p-adique.* Duke Math. J. **126**(2005), no. 3, 161–194. http://dx.doi.org/10.1215/S0012-7094-04-12615-6
[8] R. F. Coleman, *p-adic Banach spaces and families of modular forms.* Invent. Math. **127**(1997), no. 3, 417–479. http://dx.doi.org/10.1007/s002220050127
[9] P. Colmez, *Invariants $\mathcal{L}$ et dérivées de valeurs propres de Frobenius.* Astérisque **331**(2010), 13–28.
[10] H. Darmon, *Integration on $\mathcal{H}_p \times \mathcal{H}$ and arithmetic applications.* Ann. of Math. **154**(2001), no. 3, 589–639. http://dx.doi.org/10.2307/3062142
[11] S. Dasgupta, *Stark-Heegner points on modular Jacobians.* Ann. Scient. École Norm. Sup. (4) **38**(2005), no. 3, 427–469.
[12] S. Dasgupta and J. Teitelbaum, *The p-adic upper half plane.* In: p-adic geometry. Univ. Lecture Ser., 45, American Mathematical Society, Providence, RI, 2008, pp. 65–121.
[13] M. Greenberg, *Stark-Heegner points and the cohomology of quaternionic Shimura varieties.* Duke Math. J. **147**(2009), no. 3, 541–575. http://dx.doi.org/10.1215/00127094-2009-017
[14] H. Hida, *Elementary theory of L-functions and Eisenstein series.* London Mathematical Society Student Texts, 26, Cambridge University Press, Cambridge, 1993.
[15] A. Iovita and M. Spieß, *Derivatives of p-adic L-functions, Heegner cycles and monodromy modules attached to modular forms.* Invent. Math. **154**(2003), no. 2, 333–384. http://dx.doi.org/10.1007/s00222-003-0306-7
[16] K. Kato, *p-adic Hodge theory and values of zeta functions of modular forms. Cohomologies p-adiques et applications arithmétiques. III.* Astérisque **295**(2004), ix, 117–290.
[17] J. Nekovář, *Kolyvagin's method for Chow groups of Kuga-Sato varieties.* Invent. Math. **107**(1992), no. 1, 99–125. http://dx.doi.org/10.1007/BF01231883
[18] ———, *The Euler system method for CM-points on Shimura curves.* In: L-functions and Galois representations, London Math. Soc. Lecture Note Ser., 320, Cambridge University Press, Cambridge, 2007, pp. 471–547.
[19] L. Orton, *An elementary proof of a weak exceptional zero conjecture.* Canad. J. Math. **56**(2004), no. 2, 373–405. http://dx.doi.org/10.4153/CJM-2004-018-4
[20] A. Popa, *Central values of Rankin L-series over real quadratic fields.* Compos. Math. **142**(2006), no. 4, 811–866. http://dx.doi.org/10.1112/S0010437X06002259
[21] B. Mazur, J. Tate, and J. Teitelbaum, *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer.* Invent. Math. **84**(1986), no. 1, 1–48. (1986). http://dx.doi.org/10.1007/BF01388731
[22] M. Ram Murty and K. V. Murty, *Non-vanishing of L-functions and applications.* Prog. Math., 157. Birkhäuser, Basel, 1997.
[23] V. Rotger and M. A. Seveso, *$\mathcal{L}$-invariants and Darmon cycles attached to modular forms.* http://sites.google.com/site/sevesomarco/publications.
[24] M. A. Seveso, *Heegner cycles and derivatives of two variable p-adic L-functions.* http://sites.google.com/site/sevesomarco/publications.
[25] G. Shimura, *Introduction to the arithmetic theory of automorphic functions.* Kanô Memorial Lectures, 1, Publications of the Mathematical Society of Japan, 11, Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971.
[26] J. Teitelbaum, *Values of p-adic L-functions and a p-adic Poisson kernel.* Invent. Math. **101**(1990), no. 2, 395–410. http://dx.doi.org/10.1007/BF01231508

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