3

Fractional Sobolev Spaces

3.1 Definitions

Let Ω be a non-empty open subset of $\mathbb{R}^n$, suppose that $p \in [1, \infty)$ and let $s > 0$, $s \notin \mathbb{N}$. The so-called fractional Sobolev space $W_p^s(\Omega)$ arose in an attempt to fill the gaps between $L_p(\Omega)$, $W_p^1(\Omega)$, $W_p^2(\Omega)$, ...; it was introduced independently and more or less simultaneously by Aronszajn [13], Gagliardo [86] and Slobodeckij [161]. We begin by dealing with the case in which $s \in (0, 1)$. Then

$$W_p^s(\Omega) := \left\{ u \in L_p(\Omega) \colon\ (x, y) \longmapsto \frac{|u(x) - u(y)|}{|x - y|^{\frac{n}{p}+s}} \in L_p(\Omega \times \Omega) \right\};$$

endowed with the norm

$$\left\| u | W_p^s(\Omega) \right\|_{s,p,\Omega} := \left(\int_\Omega |u(x)|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p},$$

it is a Banach space. Associated with this norm is the *Gagliardo seminorm*

$$[u]_{s,p,\Omega} := \left(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}.$$

If $1 < p < \infty$, the space $W_p^s(\Omega)$ is reflexive. To establish this, define

$$T \colon W_p^s(\Omega) \to L_p(\Omega) \times L_p(\Omega \times \Omega) := E$$

by $Tu = (u, U)$, where

$$U(x, y) = \frac{u(x) - u(y)}{|x - y|^{\frac{n}{p}+s}}.$$

When E is furnished with the norm

$$\|(u, v)\|_E := \left(\|u\|_{p,\Omega}^p + \|v\|_{p,\Omega\times\Omega}^p \right)^{1/p},$$

it is reflexive; since T is a linear isometry, $T\left(W_p^s(\Omega)\right)$ is a closed subspace of E and is therefore also reflexive. The reflexivity of $W_p^s(\Omega)$ follows; in fact, the same argument establishes uniform convexity and uniform smoothness of this space.

Higher-order fractional spaces are introduced in a natural way: suppose that $s = k + \sigma$, where $k \in \mathbb{N}_0$ and $\sigma \in (0, 1)$. Then

$$W_p^s(\Omega) := \left\{u \in W_p^k(\Omega) : D^\alpha u \in W_p^\sigma(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| = k\right\};$$

equipped with the norm

$$\left\|u|W_p^s(\Omega)\right\| := \left(\left\|u|W_p^k(\Omega)\right\|^p + \sum_{|\alpha|=k} \left\|D^\alpha u|W_p^\sigma(\Omega)\right\|^p\right)^{1/p},$$

it becomes a Banach space; it is reflexive if $1 < p < \infty$.

Another attempt to fill the gaps between the classical spaces is provided, when $\Omega = \mathbb{R}^n$, by the spaces $H_p^s(\mathbb{R}^n)$ defined via the Fourier transform by (2.1.6) above. As we have seen,

$$H_p^k(\mathbb{R}^n) = W_p^k(\mathbb{R}^n) \text{ if } 1 < p < \infty \text{ and } k \in \mathbb{N}_0,$$

but (see [95], p. 82) if $s > 0$, $s \notin \mathbb{N}$ and $1 < p < \infty$, then

$$H_p^s(\mathbb{R}^n) = W_p^s(\mathbb{R}^n) \text{ if and only if } p = 2.$$

Moreover, $H_p^s(\mathbb{R}^n)$ and $W_p^s(\mathbb{R}^n)$ are isomorphic if and only if $p = 2$: see [95], pp. 84–85. These facts suggest that the most natural extension of the classical Sobolev spaces involving an arbitrary smoothness parameter s is not $W_p^s(\mathbb{R}^n)$ but $H_p^s(\mathbb{R}^n)$. However, the more explicit definition of the norm on $W_p^s(\mathbb{R}^n)$ has advantages, notably in connection with the description of trace spaces associated with the restrictions to hyperplanes in $\mathbb{R}^n$ of functions belonging to Sobolev spaces, and also in the determination of optimal constants in various inequalities.

3.2 Basic Properties

We begin with an illustration of the effect of change of the smoothness parameter s.

Proposition 3.1 *Let Ω be an open subset of $\mathbb{R}^n$ and suppose that $p \in [1, \infty)$.*

(i) If $0 < s_1 \le s_2 < 1$, then $W_p^{s_2}(\Omega) \hookrightarrow W_p^{s_1}(\Omega)$.

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(ii) If Ω has a bounded Lipschitz boundary or $\Omega = \mathbb{R}^n$, then $W_p^1(\Omega) \hookrightarrow W_p^s(\Omega)$ for all $s \in (0, 1)$.

Proof To deal with (i), let $u \in W_p^{s_2}(\Omega)$. Then

$$\int_\Omega \int_{\Omega \cap \{|x-y| \geq 1\}} \frac{|u(x)|^p}{|x-y|^{n+s_1 p}}\, dx\, dy \leq \int_\Omega \left(\int_{|z| \geq 1} \frac{dz}{|z|^{n+s_1 p}} \right) |u(x)|^p\, dx$$
$$= \|u\|_{p,\Omega}^p\, \omega_{n-1}/(s_1 p).$$

Hence

$$\int_\Omega \int_{\Omega \cap \{|x-y| \geq 1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+s_1 p}}\, dx\, dy$$
$$\leq 2^{p-1} \int_\Omega \int_{\Omega \cap \{|x-y| \geq 1\}} \frac{|u(x)|^p + |u(y)|^p}{|x-y|^{n+s_1 p}}\, dx\, dy$$
$$\leq 2^p\, \|u\|_{p,\Omega}^p\, \omega_{n-1}/(s_1 p). \tag{3.2.1}$$

Moreover,

$$\int_\Omega \int_{\Omega \cap \{|x-y| < 1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+s_1 p}}\, dx\, dy \leq \int_\Omega \int_{\Omega \cap \{|x-y| < 1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+s_2 p}}\, dx\, dy,$$

and so

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+s_1 p}}\, dx\, dy \leq 2^p\, \|u\|_{p,\Omega}^p\, \omega_{n-1}/(s_1 p) + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+s_2 p}}\, dx\, dy,$$

from which (i) follows.

As for (ii), given $u \in W_p^1(\Omega)$, in view of the assumptions on $\partial\Omega$ there is an extension $\widetilde{u} \in W_p^1(\mathbb{R}^n)$ of u such that $\|\widetilde{u}\|_{1,p,\mathbb{R}^n} \leq C\, \|u\|_{1,p,\Omega}$ for some constant C independent of u (see [64], V.4). This extension is, of course, not needed if $\Omega = \mathbb{R}^n$). With $z = y - x$ and B representing the unit ball centred at 0 we have (see (2.2.3))

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$$
\begin{aligned}
\int_{\Omega}\int_{\Omega\cap\{|x-y|<1\}} & \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy \\
&\le \int_{\Omega}\int_{B} \frac{|u(x)-u(z+x)|^p}{|z|^{n+sp}}\,dz\,dx \\
&= \int_{\Omega}\int_{B} \frac{|u(x)-u(z+x)|^p}{|z|^p}\cdot\frac{1}{|z|^{n+(s-1)p}}\,dz\,dx \\
&\le \int_{\Omega}\int_{B}\left(\int_0^1 \frac{|\nabla u(x+tz)|}{|z|^{\frac{n}{p}+s-1}}dt\right)^p\,dz\,dx \\
&\le \int_{\mathbb{R}^n}\int_{B}\int_0^1 \frac{|\nabla \widetilde{u}(x+tz)|^p}{|z|^{n+p(s-1)}}\,dt\,dz\,dx \\
&\le \int_{B}\int_0^1 \frac{\|\nabla\widetilde{u}\|_{p,\mathbb{R}^n}^p}{|z|^{n+p(s-1)}}\,dt\,dz,
\end{aligned}
$$

which is bounded above by a constant multiple of $\|u\|_{1,p,\Omega}^p$. Together with (3.2.1) this gives the result. □

Remark 3.2 The necessity of some condition on $\partial\Omega$ for (ii) to hold is illustrated by an example in Section 9 of [142], where it is shown that given any $s \in (0,1)$, there exist $p \in (1/s, \infty)$ and an open set $\Omega \subset \mathbb{R}^2$, with boundary of cusp type, such that $W_p^1(\Omega) \subsetneq W_p^s(\Omega)$. More precisely, let

$$\Omega = (\mathbb{R}^2 \setminus \mathcal{C}) \cap B(0,1), \text{ where } \mathcal{C} = \{(x_1,x_2)\colon x_1 \le 0, |x_2| \le |x_1|^{\kappa}\},$$

where $\kappa > (p+1)/(p-1)$; describe points $x = (x_1,x_2) \in \mathbb{R}^2 \setminus \mathcal{C}$ by polar co-ordinates $\rho(x) \in (0,\infty)$ and $\theta(x) \in (-\pi,\pi)$, and define $u(x) = \rho(x)\theta(x)$. Computations then show that $u \in W_p^1(\Omega) \setminus W_p^s(\Omega)$.

Corollary 3.3 *Let Ω be an open subset of $\mathbb{R}^n$ with bounded Lipschitz boundary, suppose that $p \in [1,\infty)$ and let $s_1, s_2 > 1$, with $s_2 \ge s_1$. Then $W_p^{s_2}(\Omega) \hookrightarrow W_p^{s_1}(\Omega)$.*

Proof The result is clear if $s_1, s_2 \in \mathbb{N}$. Suppose that $s_i = k_i + \sigma_i$, with $k_i \in \mathbb{N}$ and $\sigma_i \in (0,1)$ $(i = 1,2)$. If $k_1 = k_2$, then the claim follows from Proposition 3.1 (i). If $k_2 \ge k_1 + 1$, then use of (i) and (ii) of Proposition 3.1 gives

$$W_p^{s_2}(\Omega) \hookrightarrow W_p^{k_2}(\Omega) \hookrightarrow W_p^{k_1+1}(\Omega) \hookrightarrow W_p^{k_1+\sigma_1}(\Omega).$$

The remaining case, in which exactly one of k_1, k_2 is an integer, is handled in a similar fashion. □

Another example of an embedding of a Sobolev space with no condition imposed on the boundary can be obtained by using the following fractional version of Theorem 2.7.

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Theorem 3.4 *Let Ω be an open subset of $\mathbb{R}^n$, suppose that $s \in (0,1)$ and let $p \in (1,\infty)$; assume that Y, Z are Banach function spaces over Ω (endowed with Lebesgue n-measure) such that $W_p^s(\Omega) \hookrightarrow Y$ and $Y \overset{*}{\hookrightarrow} Z$. Then $W_p^s(\Omega) \hookrightarrow\hookrightarrow Z$.*

Proof This is essentially the same as that given by Slavíková in the non-fractional case, but for the reader's convenience we give some details. Let $\{B_k\}$ be a sequence of balls contained in Ω that covers Ω and let $\{g_k\}$ be a bounded sequence in $W_p^s(\Omega)$. We claim that for each $m \in \mathbb{N}$, there is a subsequence $\left\{g_k^m\right\}_{k=1}^{\infty}$ of $\left\{g_k^{m-1}\right\}_{k=1}^{\infty}$ (with $g_k^0 = g_k$) that converges a.e. on B_m. To establish this, suppose that the sequence $\left\{g_k^{m-1}\right\}_{k=1}^{\infty}$ is known for some fixed m. Then $\left\{g_k^{m-1}\right\}_{k=1}^{\infty}$ is bounded in $W_p^s(B_m)$, and so, since $W_p^s(B_m)$ is compactly embedded in $L_p(B_m)$ (because B_m is bounded and has smooth boundary and so the fractional space can be identified with a Besov space for which this compact embedding is known), there is a subsequence of $\left\{g_k^{m-1}\right\}_{k=1}^{\infty}$ that converges in $L_p(B_m)$. There is therefore another subsequence, which we denote by $\left\{g_k^m\right\}_{k=1}^{\infty}$, that converges a.e. on B_m, and the inductive step is complete. The diagonal sequence $\left\{g_m^m\right\}_{m=1}^{\infty}$ converges a.e. on Ω to some function, g say: since $W_p^s(\Omega) \hookrightarrow Y$ it follows that $\left\{g_m^m\right\}_{m=1}^{\infty}$ is bounded in Y: by the Fatou lemma for Banach function spaces (see, for example, [146], Lemma 6.1.12),

$$\|g\|_Y \leq \liminf_{m\to\infty} \left\|g_m^m\right\|_Y < \infty.$$

Hence $g \in Y$; by property (iii) of Section 1.3.3 (characterising almost compactness) we have $\left\|g_m^m - g\right\|_Z \to 0$. Hence $W_p^s(\Omega) \hookrightarrow\hookrightarrow Z$. □

The example we have in mind arises when we take Ω to be bounded, $Y = L_p(\Omega)$ and $Z = L_q(\Omega)$, where $q \in [1,p)$. Since $L_p(\Omega) \overset{*}{\hookrightarrow} L_q(\Omega)$ (see Section 1.3.3), the theorem implies that $W_p^s(\Omega) \hookrightarrow\hookrightarrow L_q(\Omega)$. This is a fractional extension of [64], Theorem V. 4.16, for bounded Ω and $k = 1$. Note that no condition on $\partial\Omega$ is required.

Let $\overset{0}{W}{}_p^s(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W_p^s(\Omega)$. In general this is a proper closed subspace of $W_p^s(\Omega)$, but

$$\overset{0}{W}{}_p^s(\mathbb{R}^n) = W_p^s(\mathbb{R}^n) \text{ for all } s > 0. \tag{3.2.2}$$

For a proof we refer to [2], Theorem 7.38.

It is often convenient to work, not with $\overset{0}{W}{}_p^s(\Omega)$ as defined above, but with the space $\overset{0}{X}{}_p^s(\Omega)$, where

$$\overset{0}{X}{}_p^s(\Omega) := \text{ completion of } C_0^\infty(\Omega) \text{ with respect to the norm } [\cdot]_{s,p,\mathbb{R}^n} + \|\cdot\|_{p,\Omega}.$$

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Plainly $\overset{0}{X}{}^s_p(\Omega) \subset \overset{0}{W}{}^s_p(\Omega)$. Note that if $u \in C_0^\infty(\Omega)$, then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy = \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dx\,dy + 2\int_\Omega\int_{\mathbb{R}^n\setminus\Omega} \frac{|u(x)|^p}{|x-y|^{n+sp}}\,dx\,dy,$$

that is,

$$[u]^p_{s,p,\mathbb{R}^n} = [u]^p_{s,p,\Omega} + 2\int_\Omega\int_{\mathbb{R}^n\setminus\Omega} \frac{|u(x)|^p}{|x-y|^{n+sp}}\,dx\,dy;$$

the last term need not be zero, and might even be infinite even though supp $u \subset \Omega$; below we show that this cannot happen if the boundary of Ω is smooth enough. First we give an inequality of Friedrichs type: we say that, given $s \in (0,1)$ and $p \in [1,\infty)$, the open set $\Omega \subset \mathbb{R}^n$ *supports the* (s,p)*-Friedrichs inequality* if there is a positive constant c such that for all $u \in C_0^\infty(\Omega)$,

$$\|u\|^p_{p,\Omega} \le c\,[u]^p_{s,p,\mathbb{R}^n}\,.$$

The next Proposition (given in [28]) shows that every bounded open set Ω has this property.

Proposition 3.5 *Let $p \in [1,\infty)$, $s \in (0,1)$ and suppose that Ω is a bounded open subset of $\mathbb{R}^n$. Then for every $u \in C_0^\infty(\Omega)$,*

$$\|u\|^p_{p,\Omega} \le C\,[u]^p_{s,p,\mathbb{R}^n}\,,$$

where

$$C = C(n,s,p,\Omega) = \min\left\{\frac{\operatorname{diam}\,(\Omega\cup B)^{n+sp}}{|B|} : B \subset \mathbb{R}^n\setminus\Omega \textit{ is a ball}\right\}.$$

Proof Let $u \in C_0^\infty(\Omega)$ and let B_R be a ball of radius R contained in $\mathbb{R}^n\setminus\Omega$. For all $x \in \Omega$ and $y \in B_R$,

$$|u(x)|^p = \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,|x-y|^{n+sp}\,,$$

which gives

$$|B_R|\,|u(x)|^p \le \left(\sup_{z\in\Omega,\,y\in B_R} |z-y|^{n+sp}\right)\int_{B_R} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\,dy.$$

The result follows. □

From this it is clear that when Ω is bounded the space $\overset{0}{X}{}^s_p(\Omega)$ can be equivalently defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $[\cdot]_{s,p,\mathbb{R}^n}$. The case in which Ω is contained between parallel hyperplanes, and may be unbounded, is discussed in [25], Remark 1.6; see also [42] and the references contained in that paper.

The following Proposition (see, for example, [28], Proposition B.1) implies that for $\overset{0}{X}{}^s_p(\Omega)$ to coincide with $\overset{0}{W}{}^s_p(\Omega)$ it is sufficient that $ps \neq 1$ and that Ω should be bounded and have a Lipschitz boundary.

Proposition 3.6 *Let $s \in (0, 1)$ and $p \in (1, \infty)$ be such that $ps \neq 1$; suppose Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. Then there is a positive constant $C = C(n, p, s, \Omega)$ such that for all $u \in C_0^\infty(\Omega)$,*

$$[u]_{s,p,\mathbb{R}^n} + \|u\|_{p,\Omega} \leq C \left([u]_{s,p,\Omega} + \|u\|_{p,\Omega}\right).$$

Proof Let $u \in C_0^\infty(\Omega)$ and put $\delta(x) = \inf_{y \in \mathbb{R}^n \setminus \Omega} |x - y|$ $(x \in \Omega)$. Then

$$\mathbb{R}^n \setminus \Omega \subset \mathbb{R}^n \setminus B\left(x, \delta(x)\right),$$

and so

$$\begin{aligned}\int_\Omega \int_{\mathbb{R}^n \setminus \Omega} \frac{|u(x)|^p}{|x - y|^{n+sp}}\, dx\, dy &\leq \int_\Omega \int_{\mathbb{R}^n \setminus B(x,\delta(x))} \frac{|u(x)|^p}{|x - y|^{n+sp}}\, dy\, dx \\ &= \int_\Omega |u(x)|^p \left(n\omega_n \int_{\delta(x)}^\infty r^{-1-sp} dr\right) dx \\ &= \frac{n\omega_n}{sp} \int_\Omega \frac{|u(x)|^p}{\delta(x)^{sp}}\, dx.\end{aligned}$$

Suppose $ps > 1$. Then the fractional Hardy inequality (see [55], Theorem 1)

$$\int_\Omega \frac{|u(x)|^p}{\delta(x)^{sp}}\, dx \leq c\, [u]^p_{s,p,\Omega}$$

gives the result. When $ps < 1$ we use the inequality (see [55] and [44])

$$\int_\Omega \frac{|u(x)|^p}{\delta(x)^{sp}}\, dx \leq c \left([u]^p_{s,p,\Omega} + \|u\|^p_{p,\Omega}\right).$$

□

In the borderline case $ps = 1$ it turns out that $\overset{0}{X}{}^s_p(\Omega) \neq \overset{0}{W}{}^s_p(\Omega)$: for details see [29], Remark 2.1, where it is shown that $\chi_\Omega \in \overset{0}{W}{}^s_p(\Omega) \setminus \overset{0}{X}{}^s_p(\Omega)$.

Consideration of nonlocal Dirichlet boundary conditions outside an open set $\Omega \subset \mathbb{R}^n$ makes it convenient to consider the homogeneous Sobolev–Slobodeckij space $\overset{0}{\mathcal{D}}{}^s_p(\Omega)$ defined by

$$\overset{0}{\mathcal{D}}{}^s_p(\Omega) = \text{ completion of } C_0^\infty(\Omega) \text{ with respect to the norm } [\cdot]_{s,p,\mathbb{R}^n}.$$

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The embedding $i \colon \overset{0}{\mathcal{D}}{}^s_p(\Omega) \to \overset{0}{\mathcal{D}}{}^s_p(\mathbb{R}^n)$ which associates to each $u \in \overset{0}{\mathcal{D}}{}^s_p(\Omega)$ its extension by 0 to all of $\mathbb{R}^n$ is well defined and continuous. If Ω supports the (s, p)-Friedrichs inequality, $\overset{0}{\mathcal{D}}{}^s_p(\Omega)$ is a space of functions continuously embedded in $L_p(\Omega)$; it then coincides with the closure in $W^s_p(\mathbb{R}^n)$ of $C_0^\infty(\Omega)$, namely $\overset{0}{X}{}^s_p(\Omega)$; and in view of the density of $C_0^\infty(\Omega)$ in $\overset{0}{\mathcal{D}}{}^s_p(\Omega)$ and Theorem 1.4.2.2 of [90] we see that if Ω supports the (s, p)-Friedrichs inequality, then

$$\overset{0}{\mathcal{D}}{}^s_p(\Omega) = \left\{ u \in W^s_p(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \backslash \Omega \right\}.$$

For the reader's convenience we now summarise in the next proposition the definitions and relationships between the various fractional spaces that have been introduced.

Proposition 3.7 *Let Ω be an open subset of $\mathbb{R}^n$, let $s \in (0, 1)$, suppose that $p \in (1, \infty)$ and let*

(i) $\overset{0}{W}{}^s_p(\Omega) :=$ *the closure of $C_0^\infty(\Omega)$ in $W^s_p(\Omega)$; equivalently, it is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{p,\Omega} + [\cdot]_{s,p,\Omega}$;*

(ii) $\overset{0}{X}{}^s_p(\Omega) :=$ *completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{p,\Omega} + [\cdot]_{s,p,\mathbb{R}^n}$;*

(iii) $\overset{0}{\mathcal{D}}{}^s_p(\Omega) :=$ *completion of $C_0^\infty(\Omega)$ with respect to the norm $[\cdot]_{s,p,\mathbb{R}^n}$.*

Then $\overset{0}{X}{}^s_p(\Omega) \subset \overset{0}{W}{}^s_p(\Omega)$; if Ω supports the (s, p)-Friedrichs inequality (and so, in particular, if Ω is bounded),

$$\overset{0}{X}{}^s_p(\Omega) = \overset{0}{\mathcal{D}}{}^s_p(\Omega) = \left\{ u \in W^s_p(\mathbb{R}^n) : u = 0 \textit{ a.e. in } \mathbb{R}^n \backslash \Omega \right\}.$$

If Ω is bounded and has Lipschitz boundary, then if $sp \neq 1$,

$$\overset{0}{X}{}^s_p(\Omega) = \overset{0}{W}{}^s_p(\Omega);$$

while if $sp = 1$,

$$\overset{0}{X}{}^s_p(\Omega) \neq \overset{0}{W}{}^s_p(\Omega).$$

The uniform convexity and uniform smoothness of these spaces when $1 < p < \infty$ may be established much as in the case of $W^s_p(\Omega)$.

We next establish Hölder-continuity of the functions in fractional Sobolev spaces under certain conditions, just as for the classical spaces. The argument follows that given in Proposition 2.9 of [28].

Proposition 3.8 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, let $p \in (1, \infty)$, $s \in (0, 1)$ and suppose that $sp > n$. Then $\overset{0}{X}{}^s_p(\Omega) \hookrightarrow C^\alpha(\mathbb{R}^n)$, where $\alpha = s - n/p$: for all $u \in \overset{0}{X}{}^s_p(\Omega)$,*

$$|u(x) - u(y)| \leq C(n, s, p) \left\| u|\overset{0}{X}{}^s_p(\Omega) \right\| |x - y|^\alpha \quad (x, y \in \mathbb{R}^n) \tag{3.2.3}$$

and

$$\|u\|_{\infty,\mathbb{R}^n} \leq C(n, s, p) \left\| u|\overset{0}{X}{}^s_p(\Omega) \right\| (\operatorname{diam} \Omega)^\alpha . \tag{3.2.4}$$

Proof Let $u \in \overset{0}{X}{}^s_p(\Omega)$; by Proposition 3.7 we may regard u as an element of $\overset{0}{\mathcal{D}}{}^s_p(\Omega)$. Let $x_0 \in \mathbb{R}^n$ and $\delta > 0$; denote the mean value of u in $B(x_0, \delta)$ by $\overline{u}_{x_0,\delta}$. Then

$$\int_{B(x_0,\delta)} \left|u(x) - \overline{u}_{x_0,\delta}\right|^p dx \leq \frac{1}{|B(x_0, \delta)|} \int_{B(x_0,\delta)} \int_{B(x_0,\delta)} |u(x) - u(y)|^p \, dx\, dy.$$

Since $|x - y| \leq 2\delta$ for all $x, y \in B(x_0, \delta)$,

$$\int_{B(x_0,\delta)} \left|u(x) - \overline{u}_{x_0,\delta}\right|^p dx \leq C\delta^{sp} [u]^p_{s,p,\mathbb{R}^n} ,$$

and so

$$|B(x_0, \delta)|^{-sp/n} \int_{B(x_0,\delta)} \left|u(x) - \overline{u}_{x_0,\delta}\right|^p dx \leq C' [u]^p_{s,p,\mathbb{R}^n} .$$

Hence u belongs to the Campanato space $\mathcal{L}^{p,sp}(\mathbb{R}^n)$, which is isomorphic to $C^\alpha(\mathbb{R}^n)$ with $\alpha = s - n/p$: see Section 1.3.2. From this, (3.2.3) follows. To obtain (3.2.4) take $y \in \mathbb{R}^n \backslash \operatorname{supp} u$. □

A counterpart of the (s, p)-Friedrichs inequality, valid for all elements of $W^s_p(\Omega)$, is the (s, p)-*Poincaré inequality*. If $|\Omega| < \infty$, we say that Ω supports this inequality if there exists $K > 0$ such that for all $u \in W^s_p(\Omega)$,

$$\inf_{c \in \Phi} \|u - c\|_{p,\Omega} \leq K [u]_{s,p,\Omega} ,$$

where Φ denotes the underlying field of scalars. Put

$$u_\Omega = |\Omega|^{-1} \int_\Omega u(x)\, dx$$

and observe that for all $c \in \Phi$,

$$\|u - u_\Omega\|_{p,\Omega} = \|u - c - (u - c)_\Omega\|_{p,\Omega} \leq 2 \|u - c\|_{p,\Omega} ,$$

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from which it follows that the Poincaré inequality may be equivalently written as

$$\|u - u_\Omega\|_{p,\Omega} \le K_1\, [u]_{s,p,\Omega}\,.$$

When Ω is bounded the following result holds.

Theorem 3.9 *Let $p \in [1, \infty)$, $s \in (0, 1)$ and suppose that Ω is bounded. Then for all $u \in W^s_p(\Omega)$,*

$$\|u - u_\Omega\|^p_{p,\Omega} \le \frac{(\text{diam }\Omega)^{n+sp}}{|\Omega|} \int_\Omega \int_\Omega \frac{|u(y) - u(x)|^p}{|y - x|^{n+sp}}\, dy\, dx,$$

where $u_\Omega = |\Omega|^{-1} \int_\Omega u(x)\, dx$.

Proof By Jensen's inequality (see, for example, [97], p. 202),

$$\begin{aligned}
\|u - u_\Omega\|^p_{p,\Omega} &= \int_\Omega \left| |\Omega|^{-1} \int_\Omega (u(y) - u(x))\, dx \right|^p dy \\
&\le |\Omega|^{-1} \int_\Omega \int_\Omega |u(y) - u(x)|^p\, dx\, dy \\
&\le \frac{(\text{diam }\Omega)^{n+sp}}{|\Omega|} \cdot \int_\Omega \int_\Omega \frac{|u(y) - u(x)|^p}{|y - x|^{n+sp}}\, dy\, dx.
\end{aligned}$$

□

Remark 3.10

(i) This result underlines the difference between fractional Sobolev spaces and their classical counterparts: the (s, p)-Poincaré inequality in $W^s_p(\Omega)$ holds for all (bounded) Ω, while the classical Poincaré inequality in $W^1_p(\Omega)$ does not hold in all bounded sets Ω: see [64], Theorem V.4.21(iii). The validity of the fractional case does exhibit some domain dependence if the stronger inequality

$$\inf_{c\in\mathbb{R}} \|u - c\|^p_{p,\Omega} \le C \int_\Omega \int_{\Omega\cap B(x,\tau dist(x,\partial\Omega))} \frac{|u(y) - u(x)|^p}{|y - x|^{n+sp}}\, dy\, dx$$

(with $\tau \in (0, 1)$) is considered. In [52] it is shown that this may fail in certain $\beta-$John domains (see [52] and [99] for the definition of these domains), while in [99] the inequality is established for $1-$John domains. The paper [52] also discusses the influence of weights. Moreover, the double integral on the right-hand side of the displayed formula is comparable to the full Gagliardo seminorm under suitable conditions on Ω, such as being Lipschitz [55] or uniform [149].

Definitions of plumpness different from that given above and used by Zhou may be found in the literature. Sets that are plump in his sense are often called *lower Ahlfors regular*.

(ii) Note that when Ω is a cube Q, the theorem gives

$$\left\| u - u_Q \right\|_{p,Q} \leq n^{(n+sp)/(2p)} \left| Q \right|^{s/n} [u]_{s,p,\Omega} .$$

The constant in this inequality is not sharp: see Theorem 1 of [23].

When $s \in \mathbb{N}$ and $p \in [1, \infty)$, elements of $W_p^s(\Omega)$ may be extended to be functions in $W_p^s(\mathbb{R}^n)$ if the domain Ω has some regularity properties: we have already used such a result in proving Proposition 3.1 (ii). Here we describe work by Zhou [173] that characterises those Ω for which corresponding extension results hold for arbitrary $s \in (0, 1)$.

Definition 3.11 Let Ω be a domain in $\mathbb{R}^n$ $(n \geq 2)$ and suppose that $p \in [1, \infty)$ and $s > 0$. The set Ω is called a W_p^s extension domain if given any $u \in W_p^s(\Omega)$ there exists $\widetilde{u} \in W_p^s(\mathbb{R}^n)$ such that $\widetilde{u}|_\Omega = u$ and

$$\left\| \widetilde{u} | W_p^s(\mathbb{R}^n) \right\| \leq C \left\| u | W_p^s(\Omega) \right\| ,$$

where C is a constant that depends on n, p, s, Ω but not on u. It is said to be plump if there is a constant $c > 0$ such that for all $x \in \Omega$ and all $r \in (0, 1]$,

$$|B(x, r) \cap \Omega| \geq cr^n.$$

When $s \in \mathbb{N}$ every Ω with minimally smooth boundary (see [64], V.4) is a W_p^s extension domain; in particular, every bounded Ω with boundary of class $C^{0,1}$ is an extension domain. Zhou's result implies the following.

Theorem 3.12 *Let Ω be a domain in $\mathbb{R}^n$ $(n \geq 2)$. Then Ω is a W_p^s extension domain for all $s \in (0, 1)$ and all $p \in [1, \infty)$ if and only if it is plump.*

Remark 3.13

1. When $s > 0$, $s \notin \mathbb{N}$ and $p \in [1, \infty)$, the space $W_p^s(\mathbb{R}^n)$ defined above coincides with the Besov space $B_{p,p}^s(\mathbb{R}^n)$, which itself coincides with the Lizorkin–Triebel space $F_{p,p}^s(\mathbb{R}^n)$: see [169], 2.3–2.5 and [95], 3.6. This means that all the embedding theorems known for $B_{p,p}^s(\mathbb{R}^n)$ (see, for example, [70], 2.3.3) are also available for $W_p^s(\mathbb{R}^n)$. When Ω is an open subset of $\mathbb{R}^n$ the spaces $B_{p,p}^s(\Omega)$ (see, for example, the Proposition in [169], 3.28) may be defined intrinsically or by restriction of elements of $B_{p,p}^s(\mathbb{R}^n)$: if $\partial\Omega$ is smooth enough the resulting space is the same whichever procedure is adopted, and so $W_p^s(\Omega)$ may be identified with $B_{p,p}^s(\Omega)$ (see, for example, [169]). Thus embedding results known for $B_{p,p}^s(\Omega)$ (see, for example, [70], 2.5) hold for $W_p^s(\Omega)$. Hence Proposition 3.1 (ii) could have been deduced from the known embeddings for Besov spaces: we chose to give an independent proof so as to acclimatise the reader to the techniques to be used later on. To illustrate what may be obtained from results known for Besov spaces, we give the following:

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1. Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^∞ boundary, let $s_1, s_2 \in (0,\infty)\backslash\mathbb{N}$ and $p_1, p_2 \in (1,\infty)$. Then if $s_1 - s_2 - n\max(1/p_1 - 1/p_2, 0) \geq 0$,
$$W_{p_1}^{s_1}(\Omega) \hookrightarrow W_{p_2}^{s_2}(\Omega);$$
the embedding is compact if $s_1 - s_2 - n\max(1/p_1 - 1/p_2, 0) > 0$. Similar statements hold for the spaces $\overset{0}{W}{}_{p_i}^{s_i}(\Omega)$; and if $s_i p_i \neq 1$ $(i = 1, 2)$, each $\overset{0}{W}{}_{p_i}^{s_i}(\Omega)$ can be replaced by $\overset{0}{X}{}_{p_i}^{s_i}(\Omega)$ (see Proposition 3.7).
2. Given $u \in \mathcal{S}(\mathbb{R}^n)$, its restriction to the hyperplane
$$\Gamma := \left\{x = \left(x', x_n\right) \in \mathbb{R}^n : x_n = 0\right\},$$
that is, the function $v \in \mathcal{S}\left(\mathbb{R}^{n-1}\right)$ given by $v\left(x'\right) = u\left(x', 0\right)$ $\left(x' \in \mathbb{R}^{n-1}\right)$, is called the trace of u on Γ. It turns out that, given any $s > 1/2$, the map $u \longmapsto v$ can be extended to a continuous, surjective map $tr_\Gamma : W_2^s(\mathbb{R}^n) \to W_2^{s-1/2}\left(\mathbb{R}^{n-1}\right)$, the trace map. For a proof of this assertion, its extension to traces on the boundary of bounded, smoothly bounded subsets of $\mathbb{R}^n$ and in the context of more general function spaces, we refer to [95], Chapter 4 and [142], Section 3.
3. If $p \in [1,\infty)$ and $s \in (0,1)$, then the real space $W_p^s(\mathbb{R}^n)$ has the truncation property: if u belongs to $W_p^s(\mathbb{R}^n)$ then so does $u_+ := \max(u, 0)$ and
$$\left\|u_+|W_p^s(\mathbb{R}^n)\right\| \leq c\left\|u|W_p^s(\mathbb{R}^n)\right\|,$$
where c is a constant independent of u. For this, and many more general results, we refer to [171], Section 25. □

We have seen earlier that when Ω is bounded and $1 \leq q < p < \infty$, the space $W_p^s(\Omega)$ is compactly embedded in $L_q(\Omega)$, no conditions on $\partial\Omega$ being necessary. When $q = p$ this embedding is not always compact. On the positive side, when Ω has smooth enough boundary, $W_p^s(\Omega)$ can be identified with a Besov space and the compactness of $I : W_p^s(\Omega) \to L_p(\Omega)$ follows from known properties of these spaces. The same holds when Ω is an extension domain: see Theorem 7.1 of [142], which also contains an example showing that some condition on $\partial\Omega$ is needed for I to be compact. To investigate the position a little more closely we introduce the notions of entropy and approximation numbers, and the measure of noncompactness of a map, following the line of argument for the non-fractional case given in [64], V.5. where background material may be found.

Given a bounded linear map T between Banach spaces X and Y, for each $k \in \mathbb{N}$ the kth *entropy number* $e_k(T)$ of T is defined by
$$e_k(T) = \inf\left\{\varepsilon > 0 : T(B_X) \text{ can be covered by } 2^{k-1} \text{ balls in } Y \text{ with radius } \varepsilon\right\},$$

where B_X is the closed unit ball in X. Since T is compact if and only if $\lim_{k\to\infty} e_k(T) = 0$, this limit is called the *measure of noncompactness* of T; we denote it by $\beta(T)$. Evidently $0 \le \beta(T) \le \|T\|$; if $\beta(T) = \|T\|$ we say that T is *maximally noncompact*: for examples of such maps and further details see [69] and the references given in that paper. In fact, [69] shows that when $p \in (1,\infty)$ and Ω is an infinite strip, the natural embedding $I_p\colon W_p^1(\Omega) \to L_p(\Omega)$ is maximally noncompact in the sense that $\beta\left(I_p\right) = \left\|I_p\right\|$; from [68] it turns out that I_p is not strictly singular. For each $k \in \mathbb{N}$ the kth *approximation number* $a_k(T)$ of T is defined by

$$a_k(T) = \inf\{\|T - F\| : F \in B(X,Y),\ \operatorname{rank} F < k\};$$

$\lim_{k\to\infty} a_k(T)$ is denoted by $\alpha(T)$. If $\alpha(T) = 0$ the map T is compact; the converse is true if Y has the approximation property. By Proposition II.2.7 of [64],

$$\beta(T) \le \alpha(T).$$

We now consider the embedding map $I\colon W_p^s(\Omega) \to L_p(\Omega)$, where $|\Omega| < \infty$, $s \in (0,1)$ and $p \in (1,\infty)$. First suppose that Ω is bounded. Given $\varepsilon > 0$, write $\Omega_\varepsilon = \{x \in \Omega\colon d(x,\partial\Omega) < \varepsilon\}$; by Theorem V.4.20 of [64] there is a domain U_ε with analytic boundary such that $\Omega\backslash\Omega_\varepsilon \subset U_\varepsilon \subset \overline{U_\varepsilon} \subset \Omega$; put $\mathcal{U} = \cup_{\varepsilon>0} U_\varepsilon$. For each $\varepsilon > 0$, the natural embedding $I_\varepsilon\colon W_p^s(\Omega) \to L_p(U_\varepsilon)$ is compact since it can be represented as the composition of the maps

$$W_p^s(\Omega) \hookrightarrow W_p^s(U_\varepsilon) \hookrightarrow\hookrightarrow L_p(U_\varepsilon).$$

Let $F(\Omega)$ (resp. $F(U_\varepsilon)$) stand for the family of all bounded linear maps from $W_p^s(\Omega)$ to $L_p(\Omega)$ (resp. $L_p(U_\varepsilon)$) that have finite rank. As $L_p(U_\varepsilon)$ has the approximation property, it follows from Theorem 1.2.25 of [61] that there exists $P \in F(U_\varepsilon)$ such that for all $f \in W_p^s(\Omega)$,

$$\|f - Pf\|_{p,U_\varepsilon} \le \varepsilon \|f\|_{s,p,\Omega}.$$

Since $L_p(\Omega)$ has the approximation property,

$$\begin{aligned}\alpha(I) &= \operatorname{dist}\left(I, K\left(W_p^s(\Omega), L_p(\Omega)\right)\right)\\ &= \inf\left\{\|I - K\| : K \in K\left(W_p^s(\Omega), L_p(\Omega)\right)\right\}.\end{aligned}$$

In fact, $\beta(I) = \alpha(I)$. To prove this we first establish a lemma.

Lemma 3.14 *Let $P \in F(\Omega)$. Then given $\varepsilon > 0$, there exists $R \in F(\Omega)$ and a domain $\Omega' \subset\subset \Omega$ such that $\|P - R\| < \varepsilon$ and $R\left(W_p^s(\Omega)\right) \subset C_0^\infty\left(\Omega'\right)$. If Ω_0 ($\subset \Omega$) is open, $\varepsilon > 0$ and $P \in F(\Omega)$ are given, then there exists $R \in F(\Omega)$ such that $\|(P - R)f\|_{p,\Omega_0} \le \varepsilon \|f\|_{s,p,\Omega}$ for all $f \in W_p^s(\Omega)$ and $R\left(W_p^s(\Omega)\right) \subset C_0^\infty(\Omega_0)$.*

Proof There are linearly independent functions $u_1, ..., u_N$, with each $\|u_i\|_{p,\Omega} = 1$, such that for each $f \in W_p^s(\Omega)$, $Pf = \sum_{i=1}^N c_i(f)u_i$. Since all norms on the finite-dimensional range of P are equivalent, there exists K such that

$$\sum_{i=1}^{N} |c_i(f)| \leq K \|Pf\|_{p,\Omega} \leq K \|P\| \|f\|_{s,p,\Omega} .$$

For each i let $\phi_i \in C_0^\infty(\Omega)$ be such that $\|u_i - \phi_i\|_{p,\Omega} < \varepsilon/(K\|P\|)$ and set $Rf = \sum_{i=1}^N c_i(f)\phi_i$. Then $R \in F(\Omega)$ and

$$\|Pf - Rf\|_{p,\Omega} \leq \sum_{i=1}^{N} |c_i(f)| \|u_i - \phi_i\|_{p,\Omega} \leq \varepsilon \|f\|_{s,p,\Omega} ;$$

since $\operatorname{supp} Rf \subset \cup_{i=1}^N \operatorname{supp} \phi_i \subset\subset \Omega$, the first part of the lemma follows. The proof of the second part is similar, noting that $f \longmapsto \chi_{\Omega_0} Pf \in F(\Omega_0)$ and choosing $\phi_i \in C_0^\infty(\Omega_0)$. □

Given any $\varepsilon > 0$, let

$$\Gamma(\varepsilon) := \sup\left\{\|u\|_{p,\Omega_\varepsilon}^p : \|u\|_{s,p,\Omega} = 1\right\},$$

and

$$\Gamma(0) := \lim_{\varepsilon\to 0} \Gamma(\varepsilon).$$

Theorem 3.15 *When Ω is bounded,* $\Gamma(0) = \beta(I)^p = \alpha(I)^p$.

Proof This is exactly the same as that of Theorem V.5.7 of [64]. It is in this proof that the material involving $\mathcal{U} = \cup_{\varepsilon>0} U_\varepsilon$ is used. □

The next theorem shows how the universal validity of the fractional Poincaré inequality (in bounded sets Ω) affects the measure of noncompactness $\beta(I)$.

Theorem 3.16 *Suppose that Ω is bounded. Then* $\beta(I) = \alpha(I) < 1$.

Proof The map $f \longmapsto f_\Omega$ belongs to $F(\Omega)$, and so, by Lemma 3.14, given $\varepsilon > 0$, there exist $R \in F(\Omega)$ and $\delta > 0$ such that for all $f \in W_p^s(\Omega)$,

$$\|f_\Omega - Rf\|_{p,\Omega}^p \leq 2^{1-p}\varepsilon \|f\|_{s,p,\Omega}^p$$

and $\operatorname{supp} Rf \subset \Omega \setminus \Omega^\delta := \{x \in \Omega : d(x, \partial\Omega) > \delta\}$. Hence by the Poincaré inequality,

$$\begin{aligned}\|f - Rf\|_{p,\Omega}^p &\leq 2^{p-1}K^p [f]_{s,p,\Omega}^p + \varepsilon \|f\|_{s,p,\Omega}^p \\ &\leq (K_1 + \varepsilon) [f]_{s,p,\Omega}^p + \varepsilon \|f\|_{p,\Omega}^p ,\end{aligned}$$

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where $K_1 = 2^{p-1}K$. Thus

$$\begin{aligned}\|f\|^p_{p,\Omega\backslash\Omega^\delta} &\leq (K_1+\varepsilon)\,[f]^p_{s,p,\Omega} + \varepsilon\,\|f\|^p_{p,\Omega} \\ &= (K_1+\varepsilon)\left(\|f\|^p_{s,p,\Omega} - \|f\|^p_{p,\Omega}\right) + \varepsilon\,\|f\|^p_{p,\Omega} \\ &\leq (K_1+\varepsilon)\,\|f\|^p_{s,p,\Omega} - K_1\,\|f\|^p_{p,\Omega^\delta},\end{aligned}$$

and so

$$\|f\|^p_{p,\Omega\backslash\Omega^\delta} \leq \left[(K_1+\varepsilon)/(K_1+1)\right]\|f\|^p_{s,p,\Omega}.$$

Thus by Theorem 3.15, $\alpha(I) < 1$. □

This result shows that $\beta(I)^p \leq 1 - 1/(K_1+1)$, where

$$K_1 = 2^p\,(\text{diam }\Omega)^{n+sp}/|\Omega| = 2^pL(\Omega)\,(\text{diam }\Omega)^{sp},$$

where

$$L(\Omega) = (\text{diam }\Omega)^n/|\Omega| \geq 1.$$

Some idea of the dependence of $\beta(I)$ on s may be obtained from this. For example, it can be shown that

$$\beta(I)^p \leq 1 - \frac{1}{2\max\{1, 2^pL(\Omega)\,(\text{diam }\Omega)^{sp}\}}.$$

In particular, if $(\text{diam }\Omega)^{sp} > 2^{-p}/L(\Omega)$, then

$$\beta(I)^p \leq 1 - \frac{(\text{diam }\Omega)^{-sp}}{2^{p+1}L(\Omega)}.$$

Thus if Ω is the unit cube $(0,1)^n$, so that $|\Omega| = 1$ and $\text{diam }\Omega = \sqrt{n}$,

$$\beta(I)^p \leq 1 - 2^{-p-1}n^{-(sp+n)/2}.$$

Now suppose that the open set Ω is merely required to have finite measure. We aim to obtain results analogous to those given in [64], V.5.3 for the classical Sobolev case that involve the map $f \mapsto \nabla f$ of $W^1_p(\Omega)$ to $\left[L_p(\Omega)\right]^n$. Define $T\colon W^s_p(\Omega) \to L_p(\Omega\times\Omega)$ by

$$(Tf)(x,y) = \frac{f(x)-f(y)}{|x-y|^{\frac{n}{p}+s}}\,((x,y)\in\Omega\times\Omega).$$

The reduced minimum modulus $\gamma(T)$ of T is defined by

$$\gamma(T) = \inf\left\{\|Tf\|_{p,\Omega\times\Omega}/\text{dist }(f,\ker T) : f\in W^s_p(\Omega)\backslash\{0\}\right\}.$$

Note that $\ker T$ can be identified with Φ, the underlying field of scalars, and

$$\text{dist }(f,\ker T) = \inf_{\Phi}\|f-c\|_{s,p,\Omega}.$$

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We recall that the (s, p)-Poincaré inequality asserts that there exists $K > 0$ such that for all $f \in W_p^s(\Omega)$,

$$\inf_{c\in\mathbb{C}} \|f - c\|_{p,\Omega} \leq K \left[f\right]_{s,p,\Omega},$$

and that if Ω is bounded, this form of the Poincaré inequality holds with no additional assumption on Ω.

Assume that the (s, p)-Poincaré inequality holds. Then for all $f \in W_p^s(\Omega)\backslash\{0\}$ and all $c \in \Phi$,

$$\begin{aligned}\|Tf\|_{p,\Omega\times\Omega} / \text{dist } (f, \ker T) &\geq \left[f\right]_{s,p,\Omega} / \|f - c\|_{s,p,\Omega} \\ &= \frac{\left[f\right]_{s,p,\Omega}}{\left[\|f - c\|_{p,\Omega}^p + \left[f\right]_{s,p,\Omega}^p\right]^{1/p}} \\ &\geq \frac{\left[f\right]_{s,p,\Omega}}{\left[K^p \left[f\right]_{s,p,\Omega}^p + \left[f\right]_{s,p,\Omega}^p\right]^{1/p}} \\ &= (1 + K^p)^{-1/p}.\end{aligned}$$

Thus $\gamma(T) > 0$ and so T has closed range, by Theorem 1.3.4 of [64]. Conversely, if $\gamma(T) > 0$, then for all $f \in W_p^s(\Omega)$,

$$\inf_{c\in\mathbb{C}} \|f - c\|_{s,p,\Omega} \leq \gamma(T)^{-1} \|Tf\|_{p,\Omega\times\Omega},$$

so that the Poincaré inequality holds and T has closed range.

To summarise the position:

(i) if $|\Omega| < \infty$, then the Poincaré inequality holds if and only if T has closed range;
and

(ii) if Ω is bounded, then the Poincaré inequality holds, T has closed range and for the embedding $I\colon\ W_p^s(\Omega) \to L_p(\Omega)$ we have

$$\alpha(I) = \beta(I) < 1;$$

more precisely (see Theorem 3.16),

$$\alpha^p(I) = \beta^p(I) \leq 1 - 1/(C + 1),$$

where

$$C = 2^p\, (\text{diam } \Omega)^{n+sp} / |\Omega|.$$

A similar analysis may be carried out for the Friedrichs inequality and its connection with the embedding of $\overset{0}{X}{}_p^s(\Omega)$ in $L_p(\Omega)$. □

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Next we give a fractional analogue of Proposition 2.6 in the form given by Lemma A.1 of [28], dealing with the behaviour of functions under translation.

Proposition 3.17 *Let $p \in [1, \infty)$ and $s \in (0, 1)$. Then there is a constant $C = C(n, p)$ such that for every $u \in C_0^\infty(\mathbb{R}^n)$,*

$$\sup_{|h|>0} \int_{\mathbb{R}^n} \frac{|u(x+h) - u(x)|^p}{|h|^{sp}} dx \le C(1-s)\,[u]_{s,p,\mathbb{R}^n}^p .$$

Proof Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be non-negative, with supp $\rho \subset B(0,1)\backslash B(0,1/2)$ and $\int_{\mathbb{R}^n} \rho dx = 1$. Given $h \in \mathbb{R}^n \backslash \{0\}$, put

$$\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon) \ (x \in \mathbb{R}^n, 0 < \varepsilon < |h|).$$

Using suitable changes of variable we see that for all $x \in \mathbb{R}^n$,

$$\begin{aligned} |u(x+h) - u(x)| &= \left| \int_{\mathbb{R}^n} (u(x+h) - u(x))\, \rho_\varepsilon(y)\, dy \right| \\ &= \left| \int_{\mathbb{R}^n} \begin{Bmatrix} \left[u(x+h) - u(x+h-y)\right] + u(x+h-y) \\ - \left[u(x) - u(x-y)\right] - u(x-y) \end{Bmatrix} \rho_\varepsilon(y)\, dy \right| \\ &\le \left| \int_{\mathbb{R}^n} u(y) \left[\rho_\varepsilon(x+h-y) - \rho_\varepsilon(x-y)\right] dy \right| \\ &\quad + \int_{\mathbb{R}^n} |u(x+h) - u(x+h-y)|\, \rho_\varepsilon(y)\, dy \\ &\quad + \int_{\mathbb{R}^n} |u(x) - u(x-y)|\, \rho_\varepsilon(y)\, dy. \end{aligned}$$

Since $\int_{\mathbb{R}^n} \nabla \rho_\varepsilon \, dx = 0$ it follows that

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} u(y) \left[\rho_\varepsilon(x+h-y) - \rho_\varepsilon(x-y)\right] dy \right| \\ &\qquad = \left| \int_0^1 \int_{\mathbb{R}^n} u(y)\, \langle \nabla \rho_\varepsilon(x - y + sh), h \rangle \, dy\, ds \right| \end{aligned}$$

may be written as

$$\begin{aligned} &\left| \int_0^1 \int_{\mathbb{R}^n} \left[u(y) - u(x+sh)\right] \langle \nabla \rho_\varepsilon(x-y+sh), h\rangle \, dy\, ds \right| \\ &\le \|\nabla\rho\|_\infty\, |h|\, \varepsilon^{-n-1} \int_0^1 \int_{B(x+sh,\varepsilon)\backslash B(x+sh,\varepsilon/2)\backslash} |u(y) - u(x+sh)|\, dy\, ds \\ &= \|\nabla\rho\|_\infty\, |h|\, \varepsilon^{-n-1} \int_0^1 \int_{B(0,\varepsilon)\backslash B(0,\varepsilon/2)} |u(x+z+sh) - u(x+sh)|\, dz\, ds. \end{aligned}$$

Use of Jensen's inequality (see, for example, [126], Theorem 2.2) together with the translation invariance of the L_p norm now shows that

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$$\int_{\mathbb{R}^n} |u(x+h) - u(x)|^p \, dx$$

is bounded above by

$$C\,|h|^p\,\varepsilon^{-n-p} \int_{B(0,\varepsilon)\setminus B(0,\varepsilon/2)} \int_{\mathbb{R}^n} |u(x+z) - u(x)|^p \, dx\, dz$$

$$+ C\,\|\rho\|_\infty\,\varepsilon^{-n} \int_{B(0,\varepsilon)\setminus B(0,\varepsilon/2)} \int_{\mathbb{R}^n} |u(x+z) - u(x)|^p \, dx\, dz.$$

As $\varepsilon < |h|$ we obtain

$$\int_{\mathbb{R}^n} |u(x+h) - u(x)|^p \, dx$$

$$\leq C_1\,|h|^p\,\varepsilon^{-n-p} \int_{B(0,\varepsilon)\setminus B(0,\varepsilon/2)} \int_{\mathbb{R}^n} |u(x+z) - u(x)|^p \, dx\, dz.$$

Hence

$$\int_{\mathbb{R}^n} \frac{|u(x+h) - u(x)|^p}{|h|^{sp}} \, dx$$

$$\leq C_1\,|h|^{p(1-s)}\,\varepsilon^{-n-p} \int_{B(0,\varepsilon)\setminus B(0,\varepsilon/2)} \int_{\mathbb{R}^n} |u(x+z) - u(x)|^p \; dx\, dz.$$

Multiply both sides by $\varepsilon^{p(1-s)-1}$, integrate with respect to ε from 0 to $|h|$ and simplify: we find that

$$\frac{1}{p(1-s)} \int_{\mathbb{R}^n} \frac{|u(x+h) - u(x)|^p}{|h|^{sp}} \, dx$$

is bounded above by

$$C_1 \int_0^{|h|} \varepsilon^{-n-ps-1} \int_{B(0,\varepsilon)} \int_{\mathbb{R}^n} |u(x+z) - u(x)|^p \, dx\, dz\, d\varepsilon.$$

Next put

$$G(\varepsilon) = \int_{B(0,\varepsilon)} \int_{\mathbb{R}^n} |u(x+z) - u(x)|^p \, dx\, dz,\ 0 < \varepsilon < |h|\,.$$

Observe that $G(0) = 0$ and G is increasing. For small positive τ,

$$\int_0^{|h|} \frac{(G(t) - \tau)_+}{t^{n+ps+1}} \, dt = \frac{-1}{n+ps} \cdot \frac{(G(|h|) - \tau)_+}{|h|^{n+ps}} + \frac{1}{n+ps} \int_{\{G(t)>\tau\}} \frac{G'(t)}{t^{n+ps}} \, dt$$

$$\leq \frac{1}{n+ps} \int_0^{|h|} \frac{G'(t)}{t^{n+ps}} \, dt.$$

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Letting $\tau \to 0$ we see that

$$(n+ps)\int_0^{|h|} t^{-n-ps-1}G(t)\,dt \le \int_0^{|h|} t^{-n-ps}G'(t)\,dt$$

which equals

$$\int_0^{|h|} t^{-n-ps}\int_{\partial B(0,t)}\int_{\mathbb{R}^n} |u(x+z)-u(x)|^p\,dx\,d\sigma(z)\,dt$$
$$= \int_{B(0,|h|)}\int_{\mathbb{R}^n} \frac{|u(x+z)-u(x)|^p}{|z|^{n+ps}}dx\,dz \le [u]^p_{s,p,\mathbb{R}^n}\,,$$

from which the result follows. □

We now turn to the limiting behaviour of the fractional spaces: more precisely, to their connection with the classical Sobolev spaces as the parameter s approaches 1 or 0. To do this we introduce a family $(\rho_\varepsilon)_{\varepsilon>0}$ of non-negative functions, each belonging to $L_{1,loc}(0,\infty)$, such that

$$\int_0^\infty \rho_\varepsilon(r)r^{n-1}dr = 1\ (\varepsilon > 0), \tag{3.2.5}$$

and

$$\lim_{\varepsilon\to 0}\int_\delta^\infty \rho_\varepsilon(r)r^{n-1}dr = 0 \text{ for all } \delta > 0. \tag{3.2.6}$$

Note that for such a family,

$$\lim_{\varepsilon\to 0}\int_0^R \rho_\varepsilon(r)r^{p+n-1}dr = 0\ (0 < R < \infty,\ 1 \le p < \infty). \tag{3.2.7}$$

It is plainly enough to establish this for $R \in (1,\infty)$. Then for all $\delta \in (0,1)$,

$$\int_0^R \rho_\varepsilon(r)r^{p+n-1}dr = \int_0^\delta \rho_\varepsilon(r)r^{p+n-1}dr + \int_\delta^R \rho_\varepsilon(r)r^{p+n-1}dr$$
$$\le \delta^p + R^p\int_\delta^R \rho_\varepsilon(r)r^{n-1}dr,$$

so that

$$\limsup_{\varepsilon\to 0}\int_0^R \rho_\varepsilon(r)r^{p+n-1}dr \le \delta^p.$$

Since this is true for all $\delta \in (0,1)$, the claim follows.

Lemma 3.18 *Let $p \in (1,\infty)$, suppose that $f \in W_p^1(\mathbb{R}^n)$ and let $\rho \in L_1(\mathbb{R}^n)$, $\rho \ge 0$. Then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^p}\rho(x-y)\,dx\,dy \le \|\rho\|_1\int_{\mathbb{R}^n} |\nabla f|^p\,dx. \tag{3.2.8}$$

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Proof By Proposition 2.6,

$$\left(\int_{\mathbb{R}^n} |f(x+h) - f(x)|^p \, dx \right)^{1/p} \leq |h| \, \| |\nabla f| \|_p$$

for all $h \in \mathbb{R}^n$. Hence

$$\begin{aligned}\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho\,(x-y)\; dx\, dy &= \int_{\mathbb{R}^n} \frac{\rho(h)}{|h|^p} \int_{\mathbb{R}^n} |f(x+h) - f(x)|^p \, dx\, dh \\ &\leq \|\rho\|_1 \int_{\mathbb{R}^n} |\nabla f|^p \, dx.\end{aligned}$$

□

Theorem 3.19 *Suppose that* $f \in L_p(\mathbb{R}^n)$ $(1 < p < \infty)$. *Then, with* ρ_ε *defined as above,*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon\,(|x-y|)\; dx\, dy = K(p,n) \int_{\mathbb{R}^n} |\nabla f|^p \, dx, \quad (3.2.9)$$

where

$$K(p,n) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n+p}{2}\right)},$$

with the understanding that $\int_{\mathbb{R}^n} |\nabla f|^p \, dx = \infty$ *iff* $f \notin W_p^1(\mathbb{R}^n)$.

Proof Let

$$F_\varepsilon(x,y) = \frac{|f(x) - f(y)|}{|x-y|} \rho_\varepsilon^{1/p}\,(|x-y|)\; (x, y \in \mathbb{R}^n, x \neq y, \varepsilon > 0),$$

and suppose first that $f \in W_p^1(\mathbb{R}^n)$. We have to show that

$$\lim_{\varepsilon \to 0} \|F_\varepsilon\|_{p, \mathbb{R}^n \times \mathbb{R}^n}^p = K \, \| |\nabla f| \|_p^p \quad (3.2.10)$$

with $K = K(p,n)$. By Lemma 3.18, for all $\varepsilon > 0$ and all $g \in W_p^1(\mathbb{R}^n)$,

$$\left| \|F_\varepsilon\|_p - \|G_\varepsilon\|_p \right| \leq \|F_\varepsilon - G_\varepsilon\|_p \leq \omega_{n-1} \, \| |\nabla (f-g)| \|_p$$

where G_ε is defined in the same way as F_ε. It is therefore enough to prove (3.2.10) when f belongs to any dense subset of $W_p^1(\mathbb{R}^n)$. Thus we assume that $f \in C_0^2(\mathbb{R}^n)$, with supp f contained in a bounded open set $\Omega \subset \mathbb{R}^n$; for each fixed $x \in \Omega$ let $R = \operatorname{dist}(x, \partial\Omega)$; note that

$$\frac{|f(x) - f(y)|}{|x-y|} = \left| (\nabla f)(x) \cdot \frac{x-y}{|x-y|} \right| + O(|x-y|).$$

Then

$$\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon\,(|x-y|)\, dy = \int_{B(x,R)} + \int_{\mathbb{R}^n \setminus B(x,R)} := I_1(x) + I_2(x).$$

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In view of (3.2.6),

$$I_2(x) \leq \frac{|f(x)|^p}{R^p} \int_{\mathbb{R}^n \backslash B(x,R)} \rho_\varepsilon \left(|x-y|\right) dy \to 0 \text{ as } \varepsilon \to 0.$$

Moreover,

$$\begin{aligned} I_1(x) &= \int_0^R \rho_\varepsilon(r) \int_{|y-x|=r} \left(\left| (\nabla f)(x) \cdot \frac{x-y}{|x-y|} \right|^p + O\left(|x-y|^p\right) \right) d\sigma\, dr \\ &= \int_0^R \rho_\varepsilon(r) \int_{|\omega|=r} \left(\left| (\nabla f)(x) \cdot \frac{\omega}{|\omega|} \right|^p + O\left(r^p\right) \right) d\sigma\, dr \\ &= |(\nabla f)(x)|^p \left(\int_{S^{n-1}} |\omega \cdot \mathbf{e}|^p \, d\sigma \right) \int_0^R r^{n-1} \rho_\varepsilon(r)\, dr \\ &\quad + O\left(\int_0^R r^{n+p-1} \rho_\varepsilon(r)\, dr \right) \end{aligned}$$

where $\mathbf{e}$ is a unit vector in $\mathbb{R}^n$: by (3.3.7) of [15],

$$\int_{S^{n-1}} |\omega \cdot \mathbf{e}|^p \, d\sigma = K(p,n).$$

Hence

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon \left(|x-y|\right) dy = K(p,n) \left|(\nabla f)(x)\right|^p \text{ for all } x \in \Omega. \tag{3.2.11}$$

Since $f \in C_0^2(\mathbb{R}^n)$, there exists M such that $|f(x) - f(y)| \leq M|x-y|$ for all $x, y \in \Omega$: thus

$$\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon \left(|x-y|\right) dy \leq M^p \omega_{n-1} \; (x \in \Omega, k \in \mathbb{N}).$$

Together with (3.2.11) and the dominated convergence theorem this establishes (3.2.10) for all $f \in C_0^2(\mathbb{R}^n)$ and hence for all $f \in W_p^1(\mathbb{R}^n)$.

To complete the proof it will be enough to show that if $f \in L_p(\mathbb{R}^n)$ and

$$A_p := \liminf_{\varepsilon \to 0} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_\varepsilon \left(|x-y|\right) \, dx\, dy \right)^{1/p} < \infty,$$

then $f \in W_p^1(\mathbb{R}^n)$. To do this we use the following result:

Let ρ be a non-negative, radial function belonging to $L_1(\mathbb{R}^n)$, $g \in L_1(\mathbb{R}^n)$, $\phi \in C_0^2(\mathbb{R}^n)$ and $\mathbf{e} \in \mathbb{R}^n$, $|\mathbf{e}| = 1$. Then

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} g(x)\, dx \int_{(y-x)\cdot \mathbf{e} \geq 0} \frac{\phi(y) - \phi(x)}{|y-x|} \rho\left(y-x\right) dy \right| \\ &\qquad \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|g(x) - g(y)|}{|x-y|} |\phi(y)| \, \rho\left(x-y\right) \, dx\, dy. \end{aligned} \tag{3.2.12}$$

To prove this, let $\delta > 0$ and define

$$\rho^{(\delta)}(y) = \begin{cases} 0, & |y| < \delta, \\ \rho(y), & |y| > \delta. \end{cases}$$

It is enough to prove the result when ρ is replaced by $\rho^{(\delta)}$ and then allow $\delta \to 0$. Let

$$\begin{aligned} I &:= \int_{\mathbb{R}^n} g(x)\,dx \int_{(y-x)\cdot\mathbf{e}\geq 0} \frac{\phi(y)-\phi(x)}{|y-x|}\rho^{(\delta)}(y-x)\,dy \\ &= \int\int_{(y-x)\cdot\mathbf{e}\geq 0} g(x)\phi(y)\frac{\rho^{(\delta)}(y-x)}{|y-x|}\,dx\,dy \\ &\quad - \int\int_{(y-x)\cdot\mathbf{e}\geq 0} g(x)\phi(x)\frac{\rho^{(\delta)}(y-x)}{|y-x|}\,dx\,dy \\ &:= I_1 - I_{2,} \end{aligned}$$

the separation being justified since the integrands in I_1 and I_2 belong to $L_1(\mathbb{R}^n \times \mathbb{R}^n)$. Interchange of x and y in I_2 and use of the radial property of $\rho^{(\delta)}$ shows that I_2 equals

$$\int\int_{(x-y)\cdot\mathbf{e}\geq 0} g(y)\phi(y)\frac{\rho^{(\delta)}(x-y)}{|x-y|}\,dx\,dy = \int\int_{(y-x)\cdot\mathbf{e}\geq 0} g(y)\phi(y)\frac{\rho^{(\delta)}(x-y)}{|x-y|}\,dx\,dy,$$

and so

$$\begin{aligned} I &= \int\int_{(y-x)\cdot\mathbf{e}\geq 0} \phi(y)\frac{g(x)-g(y)}{|y-x|}\rho^{(\delta)}(y-x)\,dx\,dy \\ &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|g(x)-g(y)|}{|x-y|}|\phi(y)|\,\rho^{(\delta)}(x-y)\,dx\,dy; \end{aligned}$$

thus (3.2.12) holds, as required. Note that the assumption that $g \in L_1(\mathbb{R}^n)$ can be relaxed to $g \in L_{p,loc}(\mathbb{R}^n)$ as this weaker requirement is sufficient to ensure the existence of the various integrals because ϕ has compact support.

Finally, let $\phi \in C_0^\infty(\mathbb{R}^n)$, suppose that $e \in \mathbb{R}^n$, $|e| = 1$, and observe that arguments similar to those used when proving (3.2.11) show that for all $x \in \mathbb{R}^n$,

$$\int_{(y-x)\cdot\mathbf{e}\geq 0} \frac{\phi(y)-\phi(x)}{|y-x|}\rho_\varepsilon(|y-x|)\,dy \to K\nabla\phi(x)\cdot e \tag{3.2.13}$$

as $\varepsilon \to 0$, where

$$K = \frac{1}{2}\int_{\omega\in S^{n-1}} |\omega\cdot e|\,d\sigma := \frac{1}{2}K_{1,n}. \tag{3.2.14}$$

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We may apply (3.2.12) to $f \in L_p(\mathbb{R}^n)$ (see the comment in the proof of this result): together with Hölder's inequality and (3.2.5) this gives

$$\begin{aligned} J_\varepsilon &:= \left| \int_{\mathbb{R}^n} f(x)\,dx \int_{(y-x)\cdot e \geq 0} \frac{\phi(y)-\phi(x)}{|y-x|} \rho_\varepsilon(|y-x|)\,dy \right| \\ &\leq \int_{\mathbb{R}^n} dx \int_{\operatorname{supp}\phi} \frac{|f(x)-f(y)|}{|x-y|} |\phi(y)|\,\rho_\varepsilon(|y-x|)\,dy \\ &\leq \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{|x-y|} |\phi(y)|\,\rho_\varepsilon(|y-x|)\,dy \\ &\leq \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_\varepsilon(|y-x|)\,dx\,dy \right)^{1/p} \|\phi\|_{p'} . \end{aligned}$$

Letting $\varepsilon \to 0$ we have

$$K \left| \int_{\mathbb{R}^n} f(x)\,(\nabla\phi(x)\cdot e)\,dx \right| \leq A_p \|\phi\|_{p'} .$$

The choice of e as the co-ordinate unit vector e_i $(i = 1, ..., n)$ shows that

$$\left| \int_{\mathbb{R}^n} f D_i\phi\,dx \right| \leq A_p \|\phi\|_{p'} /K,$$

so that $f \in W_p^1(\mathbb{R}^n)$. □

Corollary 3.20 *Let $p \in (1,\infty)$. Then there is a constant K, depending only on p and n, such that for all $f \in L_p(\mathbb{R}^n)$ (and with the same understanding as in Theorem 3.19),*

$$\lim_{s\to 1-} (1-s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy = \frac{K(p,n)}{p} \int_{\mathbb{R}^n} |\nabla f(x)|^p\,dx. \quad (3.2.15)$$

Proof We apply the last theorem with the particular choice of ρ_ε given by

$$\rho_\varepsilon(r) = \left\{ \begin{array}{cc} \frac{\varepsilon}{r^{n-\varepsilon}}, & 0 < r < 1, \\ 0, & r > 1 \end{array} \right\}.$$

This shows that

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\mathbb{R}^n} \int_{|x-y|<1} \frac{|f(x)-f(y)|^p}{|x-y|^{n+p-\varepsilon}}\,dx\,dy = K(p,n) \int_{\mathbb{R}^n} |\nabla f(x)|^p\,dx.$$

Since

$$\int_{\mathbb{R}^n} \int_{|x-y|\geq 1} \frac{|f(x)-f(y)|^p}{|x-y|^{n+p-\varepsilon}}\,dx\,dy \leq C \|f\|_p^p$$

(see, for example, (3.2.1)), it follows that

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+p-\varepsilon}}\,dx\,dy = K(p,n) \int_{\mathbb{R}^n} |\nabla f(x)|^p\,dx,$$

which gives the corollary. □

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Remark 3.21 The proof of Theorem 3.19 given here is based on [23], which also deals with the case in which the functions are defined on a bounded domain Ω in $\mathbb{R}^n$ with smooth boundary, rather than on the whole of $\mathbb{R}^n$. Other approaches are given in [24] and [33]. Note that Corollary 3.20 immediately implies that if

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy = o\left(\frac{1}{1-s}\right) \text{ as } s\to 1-, \tag{3.2.16}$$

then f is a constant function. The same holds when $\mathbb{R}^n$ is replaced by a smoothly bounded, connected open subset Ω of $\mathbb{R}^n$. In [33] Brezis shows that for any connected open subset Ω of $\mathbb{R}^n$, the condition

$$\int_{\Omega}\int_{\Omega}\frac{|f(x)-f(y)|^p}{|x-y|^{n+p}}\,dx\,dy < \infty \tag{3.2.17}$$

is sufficient to ensure that f is constant. An elegant, simple proof of a result of this type is given in [158], pp. 214–215. A consequence is that the inequality

$$\left\|\frac{f(x)-f(y)}{|x-y|^{\frac{n}{p}+1}}\right\|_{L_p(\Omega\times\Omega)} \le C(n)^{1/p}\|\nabla f\|_{L_p(\Omega)}\forall f\in C_0^\infty(\Omega)$$

does not hold. In [36] it is shown that there is a valid inequality

$$\left\|\frac{f(x)-f(y)}{|x-y|^{\frac{n}{p}+1}}\right\|_{L_{p,\infty}(\mathbb{R}^n\times\mathbb{R}^n)} \le C(n)^{1/p}\|\nabla f\|_{L_p(\mathbb{R}^n)}, \tag{3.2.18}$$

where $L_{p,\infty}$ is the Marcinkiewicz (or weak L_p) space. A natural question addressed in [35] is whether an improvement of (3.2.18) is possible in the Lorentz scale $L_{p,q}$, where $1\le p<\infty$ and $1\le q\le\infty$. For $q\in(p,\infty)$ the answer is proved to be negative and there is the generalisation of (3.2.17)

$$\left\|\frac{f(x)-f(y)}{|x-y|^{\frac{n}{p}+1}}\right\|_{L_{p,q}(\mathbb{R}^n\times\mathbb{R}^n)} \le C(n,p,q)\|\nabla f\|_{L_p(\mathbb{R}^n)} < \infty \Rightarrow f \text{ is a constant.} \tag{3.2.19}$$

The case when $s\to 0+$ was settled in [134], where it was shown that for all $f\in\cup_{0<s<1}W_p^s(\mathbb{R}^n)$,

$$\lim_{s\to 0+} s\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy = C'(n,p)\int_{\mathbb{R}^n}|f(x)|^p\,dx.$$

3.3 An Approach via Interpolation Theory

Here we indicate how the limiting behaviour of the Gagliardo seminorm mentioned in Corollary 3.20 and Remark 3.21 may be explained as a consequence of interpolation theory. The approach we follow owes much to [107], [109] and [136]. To begin with, we recall some basic ideas.

A pair (A_0, A_1) of Banach spaces with norms $\|\cdot|A_0\|$, $\|\cdot|A_1\|$ is said to be *compatible* if there is a Hausdorff topological vector space A in which both A_0 and A_1 are continuously embedded. This implies that the sum $A_0 + A_1$ and the intersection $A_0 \cap A_1$ are well defined; endowed with the norms defined by

$$\|a|A_0 + A_1\| = \inf\{\|a_0|A_0\| + \|a_1|A_1\| : a = a_0 + a_1, a_i \in A_i \ (i = 0, 1)\}$$

and

$$\|a|A_0 \cap A_1\| = \max\{\|a|A_0\|, \|a|A_1\|\},$$

respectively, they are Banach spaces. Given such a compatible pair, $t \in (0, \infty)$ and $a \in A_0 + A_1$, the *K-functional* $K(t, a; A_0, A_1)$ is defined to be

$$\inf\{\|a_0|A_0\| + t\,\|a_1|A_1\| : a = a_0 + a_1, a_i \in A_i \ (i = 0, 1)\}. \tag{3.3.1}$$

Note that for all $t \in (0, \infty)$ and all $a \in A_0 + A_1$,

$$K(t, a; A_0, A_1) = tK(1/t, a; A_1, A_0). \tag{3.3.2}$$

Moreover (see, for example [21], Proposition 5.1.2), on $(0, \infty)$ the map $t \mapsto K(t, a; A_0, A_1)$ is increasing and $t \mapsto t^{-1}K(t, a; A_0, A_1)$ is decreasing. Thus

$$\lim_{t\to 0+} t^{-1}K(t, a; A_0, A_1) \text{ and } \lim_{t\to\infty} K(t, a; A_0, A_1)$$

exist (and may be infinite).

Let (A_0, A_1) be a compatible pair of Banach spaces; suppose that $s \in (0, 1)$ and $q \in [1, \infty)$. Then

$$(A_0, A_1)_{s,q} := \left\{a \in A_0 + A_1 : \ \left\|a|\,(A_0, A_1)_{s,q}\right\| < \infty\right\},$$

where

$$\left\|a|\,(A_0, A_1)_{s,q}\right\| := \left(\int_0^\infty \left(t^{-s}K(t, a; A_0, A_1)\right)^q \frac{dt}{t}\right)^{1/q}.$$

The space $(A_0, A_1)_{s,\infty}$ is defined analogously, with

$$\left\|a|\,(A_0, A_1)_{s,\infty}\right\| := \sup_{0<t<\infty} t^{-s}K(t, a; A_0, A_1).$$

These are Banach spaces when equipped with the norms $\left\|\cdot|\,(A_0, A_1)_{s,q}\right\|$ and are said to be of *real interpolation type*. If no ambiguity is likely we shall simply write $K(t, a)$, $\|\cdot\|_{s,q}$, etc. We summarise some of the important properties of these interpolation spaces in the following theorem, giving indications of proofs for the convenience of the reader.

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Theorem 3.22 *Let (A_0, A_1), (B_0, B_1) be compatible pairs of Banach spaces, let $s \in (0, 1)$ and suppose that $q \in [1, \infty]$. Then:*

(i) $(A_0, A_1)_{s,q} = (A_1, A_0)_{1-s,q}$.

(ii) There is a constant $c = c(s, q)$ such that for all $a \in (A_0, A_1)_{s,q}$ and all $t \in (0, \infty)$,

$$K(t, a) \leq ct^s \|a\|_{s,q} .$$

(iii) If $q \leq p \leq \infty$, then

$$(A_0, A_1)_{s,1} \hookrightarrow (A_0, A_1)_{s,q} \hookrightarrow (A_0, A_1)_{s,p} \hookrightarrow (A_0, A_1)_{s,\infty} .$$

(iv) $A_0 = (A_0, A_0)_{s,q}$ and for all $a \in A_0$,

$$\|a_0|A_0\| = (s(1-s)q)^{1/q} \left\| a| (A_0, A_0)_{s,q} \right\| .$$

(v)

$$A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{s,q} \hookrightarrow A_0 + A_1 .$$

(vi) Let $T\colon A_0 + A_1 \to B_0 + B_1$ be linear and such that $T|_{A_0} \in B(A_0, B_0)$ and $T|_{A_1} \in B(A_1, B_1)$. Then

$$\left\| T|B\left((A_0, A_1)_{s,q}, (B_0, B_1)_{s,q}\right) \right\| \leq \|T|B(A_0, B_0)\|^{1-s} \|T|B(A_1, B_1)\|^s .$$

(vii) There exists $c = c(s, q)$ such that for all $a \in A_0 \cap A_1$,

$$\left\| a| (A_0, A_1)_{s,q} \right\| \leq c \|a|A_0\|^{1-s} \|a|A_1\|^s .$$

Proof (i) This follows immediately from (3.3.2).

(ii) If $q < \infty$, then since $t \longmapsto K(t, a)$ is obviously monotonic increasing,

$$t^{-s} K(t, a) = (sq)^{1/q} K(t, a) \left(\int_t^\infty u^{-sq} \frac{du}{u} \right)^{1/q} \leq (sq)^{1/q} \|a\|_{s,q} .$$

When $q = \infty$ the result is trivial.

(iii) It is enough to deal with the case in which $q < p < \infty$. By (ii), for all $a \in (A_0, A_1)_{s,q}$,

$$\begin{aligned} \|a\|_{s,p} &\leq \left(\int_0^\infty \left(t^{-s} K(t, a)\right)^q \frac{dt}{t} \right)^{1/p} \left(\sup_t \, t^{-s} K(t, a) \right)^{1-q/p} \\ &\leq c \|a\|_{s,q} , \end{aligned}$$

and (iii) follows.

(iv) Let $a \in A_0$. It is easy to see that

$$K(t, a; A_0, A_0) = \begin{cases} t \|a|A_0\| , & 0 \leq t \leq 1, \\ \|a|A_0\| , & 1 < t < \infty. \end{cases}$$

From this (iv) is immediate.

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(v) This is a direct consequence of (ii) and the fact that

$$K(t,a) \leq \min\{1,t\} \|a|A_0 \cap A_1\|, \ a \in A_0 \cap A_1.$$

(vi) If $T|_{A_0} \neq 0$,

$$\begin{aligned} K(t, Ta; B_0, B_1) &\leq \inf_{a=a_0+a_1} \left(\|Ta_0\|_{B_0} + t\,\|Ta_1\|_{B_1} \right) \\ &\leq \|T|B(A_0,B_0)\|\, K\left(\frac{\|T|B(A_1,B_1)\|}{\|T|B(A_0,B_0)\|} t, a; A_0, A_1 \right). \end{aligned}$$

The transformation $\tau = t\,\|T|B(A_1,B_1)\| / \|T|B(A_0,B_0)\|$ now leads to the result. If $T|_{A_0} = 0$, replacement of $\|T|B(A_0,B_0)\|$ in the above argument by an arbitrarily small $\varepsilon > 0$ followed by passage of ε to 0 completes the argument.

(vii) See [170], p. 27. □

As pointed out in Remark 3.13, when $s > 0, s \notin \mathbb{N}, p \in [1,\infty)$ and Ω is either the whole of $\mathbb{R}^n$ or a bounded open subset of $\mathbb{R}^n$ with smooth boundary, the space $W_p^s(\Omega)$ coincides with the Besov space $B_{p,p}^s(\Omega)$. The behaviour of the fractional Sobolev spaces under real interpolation can thus be deduced from that of Besov spaces, namely that if $p_0, p_1 \in [1,\infty), \theta \in (0,1)$ and $s_0, s_1 \in (0,\infty)$, while $s = (1-\theta)s_0 + \theta s_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, then (see [169], 3.3.6)

$$\left(B_{p_0,p_0}^{s_0}(\Omega), B_{p_1,p_1}^{s_1}(\Omega) \right)_{\theta,p} = B_{p,p}^s(\Omega).$$

A compatible pair (A_0, A_1) is said to be *normal* if

$$\lim_{t\to 0+} t^{-1} K(t,f;A_0,A_1) = \|f|A_1\| \ \text{ for all } f \in A_1, \tag{3.3.3}$$

and

$$\lim_{t\to\infty} K(t,f;A_0,A_1) = \|f|A_0\| \ \text{ for all } f \in A_0. \tag{3.3.4}$$

It is called *quasi-normal* if

$$\lim_{t\to 0+} t^{-1} K(t,\cdot;A_0,A_1) \ \text{ is equivalent to the norm } \ \|\cdot|A_1\| \ \text{ on } A_1,$$

and

$$\lim_{t\to\infty} K(t,\cdot;A_0,A_1) \ \text{ is equivalent to the norm } \ \|\cdot|A_0\| \ \text{ on } A_0.$$

These notions are key in what follows.

Theorem 3.23 *Let* (A_0, A_1) *be normal. Then:*

(i) If $1 \le q < \infty$ *and* $f \in A_0 \cap A_1$,

$$\lim_{s \to 1-} (qs(1-s))^{1/q} \|f\|_{s,q} = \|f|A_1\|$$

and

$$\lim_{s \to 0+} (qs(1-s))^{1/q} \|f\|_{s,q} = \|f|A_0\| .$$

(ii) If $1 \le q < \infty$ *and* $f \in A_0 \cap \cup_{s \in (0,1)} (A_0, A_1)_{s,q}$, *then*

$$\lim_{s \to 0+} (qs(1-s))^{1/q} \|f\|_{s,q} = \|f\|_{A_0} .$$

Proof (i) First consider the case when $s \to 1-$. Given $\varepsilon > 0$, by (3.3.3) there exists $\delta > 0$ such that

$$\left| \left(\frac{K(t, f; A_0, A_1)}{t} \right)^q - \|f|A_1\|^q \right| < \varepsilon \text{ if } 0 < t < \delta. \tag{3.3.5}$$

Note that

$$\left| qs(1-s) \|f\|_{s,q}^q - \|f|A_1\|^q \right| = \left| qs(1-s) \int_0^\infty \left(t^{-s} K(t,f) \right)^q \frac{dt}{t} - \|f|A_1\|^q \right| ;$$

we write $\|f|A_1\|^q$ in the form

$$\|f|A_1\|^q \, q(1-s) \delta^{-(1-s)q} \int_0^\delta t^{(1-s)q} \frac{dt}{t}.$$

With $\int_0^\infty = \int_0^\delta + \int_\delta^\infty$ we thus obtain

$$\left| qs(1-s) \|f\|_{s,q}^q - \|f|A_1\|^q \right| \le I_1 + I_2 + I_3,$$

where

$$I_1 = qs(1-s) \int_0^\delta \left| t^{(1-s)q} \left(\left(\frac{K(t,f)}{t} \right)^q - \|f|A_1\|^q \right) \right| \frac{dt}{t},$$

$$I_2 = qs(1-s) \int_0^\delta \left| t^{(1-s)q} \left(\delta^{-(1-s)q} \frac{\|f|A_1\|^q}{s} - \|f|A_1\|^q \right) \right| \frac{dt}{t}$$

and

$$I_3 = qs(1-s) \int_\delta^\infty t^{(1-s)q} \left(\frac{K(t,f)}{t} \right)^q \frac{dt}{t}.$$

Use of (3.3.5) shows that

$$I_1 \le \varepsilon \delta^{(1-s)q} s, \tag{3.3.6}$$

while plainly

$$I_2 = s \left| \left(s^{-1} - \delta^{-(1-s)q} \right) \delta^{(1-s)q} \right| \|f|A_1\|^q = \|f|A_1\|^q \, s \left| s^{-1} \delta^{(1-s)q} - 1 \right| . \tag{3.3.7}$$

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Since $K(t,f) \leq \|f|A_0\|$ we see that

$$I_3 \leq \delta^{-sq}(1-s)\,\|f|A_0\|^q\,. \tag{3.3.8}$$

From these estimates it follows that if $1-s$ is small enough, then

$$\left|qs(1-s)\,\|f\|_{s,q}^q - \|f|A_1\|^q\right| < \varepsilon,$$

and the first part of (i) is established.

For the second part, we know from Theorem 3.22 (i) that $(A_1, A_0)_{1-s,q} = (A_0, A_1)_{s,q}$ and $\left\|f|\,(A_1, A_0)_{1-s,q}\right\| = \left\|f|\,(A_0, A_1)_{s,q}\right\|$, from which the rest of (i) follows.

It remains to prove (ii). Let $f \in A_0 \cap (A_0, A_1)_{s,q}$. Using the hypothesis of normality we see that, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|K\,(t,f;A_0,A_1)^q - \|f|A_0\|^q\right| < \varepsilon \text{ when } t > \delta. \tag{3.3.9}$$

Moreover

$$\begin{aligned}
&\left|s(1-s)q\,\|f\|_{s,q}^q - \|f|A_0\|^q\right| \\
&\quad = \left|s(1-s)q\left\{\int_0^\infty \left(t^{-s}K\left(t,f;A_{0,}A_1\right)\right)^q \frac{dt}{t}\right\} - \|f|A_0\|^q\right| \\
&\quad \leq J_1 + J_2 + J_3,
\end{aligned}$$

where

$$J_1 = s(1-s)q\int_0^\delta \left(t^{-s}K\,(t,f;A_0,A_1)\right)^q \frac{dt}{t},$$

$$J_2 = s(1-s)q\int_\delta^\infty t^{-sq}\left|K\,(t,f;A_0,A_1)^q - \|f|A_0\|^q\right| \frac{dt}{t}$$

and

$$J_3 = \|f|A_0\|^q\,((1-s)\,\delta^{sq} - 1).$$

Suppose that $s < s_0$. Then

$$\begin{aligned}
J_1 &= s(1-s)q\delta^{-sq}\int_0^\delta \left((t/\delta)^{-s}\,K\,(t,f;A_0,A_1)\right)^q \frac{dt}{t} \\
&\leq s(1-s)q\delta^{-sq}\int_0^\delta \left((t/\delta)^{-s_0}\,K\,(t,f;A_0,A_1)\right)^q \frac{dt}{t} \\
&\leq s(1-s)q\delta^{q(s_0-s)}\int_0^\delta \left(t^{-s_0}K\,(t,f;A_0,A_1)\right)^q \frac{dt}{t} \\
&\leq s(1-s)q\delta^{q(s_0-s)}\,\|f\|_{s_0,q}^q\,,
\end{aligned}$$

and so $\lim_{s\to 0} J_1 = 0$. Using (3.3.9) we see that $J_2 < \varepsilon$ for small enough s; that $\lim_{s\to 0} J_3 = 0$ is clear. □

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Remark 3.24 If the requirement of normality is weakened to quasi-normality, minor changes to the proof show that a modified form of the theorem still holds. More precisely, (i) and (ii) hold with $\|f|A_0\|$ and $\|f|A_1\|$ replaced by $\lim_{t\to\infty} K(t,f;A_0,A_1)$ and $\lim_{t\to 0+} t^{-1}K(t,f;A_0,A_1)$, respectively.

We can now apply this abstract result to fractional Sobolev spaces. Let $p \in [1,\infty)$ and denote by $W^1_{p,0}(\mathbb{R}^n)$ the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm given by

$$\left\|f|W^1_{p,0}(\mathbb{R}^n)\right\| := \||\nabla f|\|_{p,\mathbb{R}^n}.$$

With the understanding that functions in $W^1_{p,0}(\mathbb{R}^n)$ that differ by a constant are identified, the pair $\left(L_p(\mathbb{R}^n), W^1_{p,0}(\mathbb{R}^n)\right)$ of Banach spaces is compatible. The K-functional for this pair satisfies

$$K\left(t,f;L_p(\mathbb{R}^n), W^1_{p,0}(\mathbb{R}^n)\right) \approx w_p(f,t),\ t>0, \tag{3.3.10}$$

where

$$w_p(f,t) := \sup_{|h|\le t} \|\Delta_h f\|_{p,\mathbb{R}^n},\ \Delta_h f(x) := f(x+h)-f(x). \tag{3.3.11}$$

For this we refer to [21], p. 341; the constants of equivalence are independent of f and t. Denoting by $X_{s,p}$ the interpolation space $\left(L_p(\mathbb{R}^n), W^1_{p,0}(\mathbb{R}^n)\right)_{s,p}$, it follows that

$$\left\|f|X_{s,p}\right\| \approx \left(\int_0^\infty \left(t^{-s}w_p(f,t)\right)^p \frac{dt}{t}\right)^{1/p}. \tag{3.3.12}$$

Lemma 3.25 *Let $s\in(0,1)$, $p\in[1,\infty)$ and denote by $W^s_{p,0}(\mathbb{R}^n)$ the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm given by*

$$\left\|f|W^s_{p,0}(\mathbb{R}^n)\right\| := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right)^{1/p} = [f]_{s,p,\mathbb{R}^n}.$$

Then

$$X_{s,p} = \left(L_p(\mathbb{R}^n), W^1_{p,0}(\mathbb{R}^n)\right)_{s,p} = W^s_{p,0}(\mathbb{R}^n) \tag{3.3.13}$$

and

$$\left\|f|W^s_{p,0}(\mathbb{R}^n)\right\| \approx (n+sp)^{1/p}\left\|f|X_{s,p}\right\|. \tag{3.3.14}$$

Proof Of course $\left\|\cdot|W^s_{p,0}(\mathbb{R}^n)\right\|$ is just the Gagliardo seminorm, which is a norm in this context. Also $W^s_{p,0}(\mathbb{R}^n)$ coincides with the space $\overset{0}{\mathcal{D}}{}^s_p(\mathbb{R}^n)$ men-

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tioned in Proposition 3.7; the present notation is used here as it is more suggestive in this context. We use the fact that

$$w_p(f,t) \approx \left(t^{-n} \int_{|h|\le t} \| \Delta_h f \|^p_{p,\mathbb{R}^n} \, dh \right)^{1/p}. \tag{3.3.15}$$

Assuming this for the moment, Fubini's theorem shows that

$$\int_0^\infty \left(t^{-s} w_p(f,t) \right)^p \frac{dt}{t} \approx (n+sp)^{-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} \, dx \, dy,$$

from which the result follows.

To establish (3.3.15), observe that the estimate of $w_p(f,t)$ from below is obvious. For the upper bound write, for $p \in (1,\infty)$ and $q \in (0,\infty]$,

$$\tau_q(f,t) = \left\{ t^{-n} \int_{|h|\le t} \| \Delta_h f \|^q_{p,\mathbb{R}^n} \, dh \right\}^{1/q} \text{ if } 0 < q < \infty, \;\; \tau_\infty(f,t) = w_p(f,t).$$

Plainly

$$\tau_q(f,t) \le |B(0,1)|^{1/q-1/r} \, \tau_r(f,t) \text{ if } q < r < \infty.$$

It is thus enough to show that there is a constant $c = c(q)$ such that

$$\tau_\infty(f,t) \le c(q) \tau_q(f,t), \; 0 < q < 1.$$

To do this, let $|h|, |\xi| \le t$ and note that

$$\| \Delta_h f \|_p \le \left\| \Delta_{\xi-h} f \right\|_p + \left\| \Delta_\xi f \right\|_p \le 2 \left\| \Delta_{(\xi-h)/2} f \right\|_p + \left\| \Delta_\xi f \right\|_p.$$

Since $|(\xi-h)/2| \le t$,

$$\| \Delta_h f \|_p^q \le 2^q \left\| \Delta_{(\xi-h)/2} f \right\|_p^q + \left\| \Delta_\xi f \right\|_p^q$$

and

$$\int_{|\xi|\le t} \| \Delta_h f \|_p^q \, d\xi \le (2^q+1) \int_{|\xi|\le t} \left\| \Delta_\xi f \right\|_p^q \, d\xi.$$

Hence there is a constant $C = C(q)$ such that

$$\| \Delta_h f \|_p \le C \tau_q(f,t), \; 0 < q < 1, \; |h| \le t,$$

and the proof is complete. □

Lemma 3.26 *The pair* $\left(L_p(\mathbb{R}^n), W^1_{p,0}(\mathbb{R}^n) \right)$ $(p \in [1,\infty))$ *is quasi-normal.*

Proof Let $f \in W^1_{p,0}(\mathbb{R}^n)$. Then for each $h \in \mathbb{R}^n$ and almost every $x \in \mathbb{R}^n$,

$$f(x+h) - f(x) = \int_0^1 \nabla f(x+th) \cdot h \, dt$$

(see Proposition 2.6). Then

$$\left(\int_{\mathbb{R}^n} |f(x+h)-f(x)|^p \, dx\right)^{1/p} \leq |h| \, \|\nabla f\|_p \, ,$$

so that (see (3.3.11))

$$w_p(f,\delta) \leq \delta \, \|\nabla f\|_p \, . \tag{3.3.16}$$

Given $\varepsilon > 0$, there exists $f_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ such that $\|\nabla(f-f_\varepsilon)\|_p < \varepsilon$. Moreover,

$$\mu_\varepsilon(\delta) := \sup_{|h|=\delta} \|f_\varepsilon(\cdot+h) - f_\varepsilon(\cdot) - \nabla f_\varepsilon(\cdot)\cdot h\|_p / \delta \to 0 \text{ as } \delta \to 0,$$

and hence there exists $\delta_\varepsilon > 0$ such that $\mu_\varepsilon(\delta) < \varepsilon$ if $0 < \delta < \delta_\varepsilon$. By (3.3.16),

$$w_p(f_\varepsilon,\delta) \leq w_p(f,\delta) + \varepsilon\delta.$$

Thus for all $\delta \in (0,\delta_\varepsilon)$,

$$\begin{aligned}\|\nabla f\|_p &\leq \|\nabla f_\varepsilon\|_p + \varepsilon \leq \mu_\varepsilon(\delta) + \delta^{-1} w_p(f_\varepsilon,\delta) + \varepsilon \\ &\leq \delta^{-1} w_p(f,\delta) + 3\varepsilon.\end{aligned}$$

Hence

$$\lim_{\delta\to 0+} \frac{w_p(f,\delta)}{\delta} = \|\nabla f\|_p \, . \tag{3.3.17}$$

Moreover, $t \longmapsto w_p(f,t)$ is monotonic increasing and $w_p(f,t) \leq 2\,\|f\|_p$; hence $\lim_{t\to\infty} w_p(f,t)$ exists and is bounded above by $2\,\|f\|_p$. To obtain a lower bound, let $f \in C_0^\infty(\mathbb{R}^n)$, and suppose $\operatorname{supp} f \subset \{x \in \mathbb{R}^n \colon |x| < M\}$, so that $f(x+h) = 0$ if $|h| \geq 2M$ and $|x| < M$. Hence

$$w_p(f,t) \geq \|f\|_p \text{ if } t \geq 2M,$$

and so

$$\|f\|_p \leq \lim_{t\to\infty} w_p(f,t) \leq 2\,\|f\|_p \, .$$

If $f \in L_p(\mathbb{R}^n)$, then let $(f_k)_{k\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ be such that $\lim_{k\to\infty} \|f-f_k\|_p = 0$ and observe that

$$\begin{aligned}\|f_k\|_p &\leq \lim_{t\to\infty} w_p(f_k,t) \leq \lim_{t\to\infty} w_p(f_k-f,t) + \lim_{t\to\infty} w_p(f,t) \\ &\leq 2\,\|f-f_k\|_p + \lim_{t\to\infty} w_p(f,t),\end{aligned}$$

which shows that

$$\|f\|_p \leq \lim_{t\to\infty} w_p(f,t),$$

and completes the proof. □

Application of this result to the pair $\left(L_p(\mathbb{R}^n), W^1_{p,0}(\mathbb{R}^n)\right)$ is thus possible. Let $f \in W^1_{p,0}(\mathbb{R}^n)$. Then

$$\lim_{s\to 1-} s^{-1/p}p^{-1/p}(1-s)^{1/p}\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right\}^{1/p}$$

coincides with

$$\begin{aligned}\lim_{s\to 1-} & s^{-1/p}p^{-1/p}(1-s)^{1/p}\left\|f|W^s_{p,0}(\mathbb{R}^n)\right\| \\ &\approx \lim_{s\to 1-}(n+sp)^{1/p}s^{-1/p}p^{-1/p}\left\|f|\left(L_p(\mathbb{R}^n), W^1_{p,0}(\mathbb{R}^n)\right)_{s,p}\right\| \\ &\approx (n+p)^{1/p}p^{-1/p}\left\|f|W^1_{p,0}(\mathbb{R}^n)\right\| \\ &= (n+p)^{1/p}p^{-1/p}\left\|\nabla f\right\|_p,\end{aligned}$$

in line with the Bourgain, Brezis, Mironescu result of [24] insofar as the rate of blow up is concerned, but without the exact constant that they obtain.

In the direction of the Maz'ya–Shaposhnikova theorem of [134] we observe that if $f \in \cup_{s\in(0,1)} W^s_{p,0}(\mathbb{R}^n)$, then

$$\begin{aligned}\lim_{s\to 0+} s\left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right\} &= \lim_{s\to 0+} s\left\|f|W^s_{p,0}(\mathbb{R}^n)\right\|^p \\ &\approx p^{-1}n\lim_{s\to 0+}\left\|f|\left(L_p(\mathbb{R}^n), W^1_{p,0}(\mathbb{R}^n)\right)_{s,p}\right\|^p \\ &= p^{-1}n\left\|f\right\|_p^p.\end{aligned}$$

The paper [31] contains much interesting information about the interpolation approach to the fractional spaces.

3.4 Connections with the Laplacian

Definition 3.27 Given any $s \in (0,1)$, the corresponding fractional power of the Laplacian is the map $(-\Delta)^s : \mathcal{S} \to L_2(\mathbb{R}^n)$ given by

$$(-\Delta)^s u = F^{-1}\left(|\xi|^{2s}F(u)\right). \tag{3.4.1}$$

Since the Fourier transform F maps the Schwartz space $\mathcal{S}$ onto itself, it is easy to see, using the dominated convergence theorem, that

$$\lim_{s\to 0+}(-\Delta)^s u(x) = u(x) \text{ and } \lim_{s\to 1-}(-\Delta)^s u(x) = -\Delta u(x) \ \ (u \in \mathcal{S}, x \in \mathbb{R}^n).$$

Note that since $\mathcal{S}$ is dense in every space $H^t(\mathbb{R}^n)$ with $t > 0$, we may and shall suppose that this definition holds for all $u \in H^t(\mathbb{R}^n)$. We first establish

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a connection between this fractional power and the corresponding Gagliardo seminorm $[u|H^s(\mathbb{R}^n)] = \left[u|W_2^s(\mathbb{R}^n)\right] := [u]_{s,2,\mathbb{R}^n}$ (see Section 3.1).

Proposition 3.28 *Let* $s \in (0,1)$. *Then for all* $u \in H^s(\mathbb{R}^n)$,

$$\left[u|H^s(\mathbb{R}^n)\right]^2 = 2C(n,s)^{-1} \left\|(-\Delta)^{s/2} u\right\|_2^2, \tag{3.4.2}$$

where

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1-\cos\zeta_1}{|\zeta|^{n+2s}} d\zeta\right)^{-1} = 2^{2s}\pi^{-n/2}\frac{\Gamma\left(\frac{n}{2}+s\right)}{|\Gamma(-s)|}. \tag{3.4.3}$$

Proof Using a change of variable and the Plancherel formula we see that

$$\begin{aligned}
\left[u|H^s(\mathbb{R}^n)\right]^2 &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dx\right)dy \qquad (3.4.4)\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\frac{|u(z+y)-u(y)|^2}{|z|^{n+2s}}dy\right)dz\\
&= \int_{\mathbb{R}^n}\left\|\frac{u(z+\cdot)-u(\cdot)}{|z|^{n/2+s}}\right\|_{2,\mathbb{R}^n}^2 dz\\
&= \int_{\mathbb{R}^n}\left\|F\left(\frac{u(z+\cdot)-u(\cdot)}{\left|z^{n/2+s}\right|}\right)\right\|_{2,\mathbb{R}^n}^2 dz\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\left|e^{i\xi\cdot z}-1\right|^2}{|z|^{n+2s}}|Fu(\xi)|^2\,dz\,d\xi\\
&= 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(1-\cos(\xi\cdot z))}{|z|^{n+2s}}|Fu(\xi)|^2\,dz\,d\xi. \qquad (3.4.5)
\end{aligned}$$

We claim that the function $G\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$G(\xi) = \int_{\mathbb{R}^n}\frac{(1-\cos(\xi\cdot z))}{|z|^{n+2s}}dz$$

is given by

$$G(\xi) = C(n,s)^{-1}|\xi|^{2s}.$$

To see this, observe that G is rotationally invariant: $G(\xi) = G(|\xi|\, e_1)$, where $e_1(1,0,...,0) \in \mathbb{R}^n$. This is obvious when $n=1$. When $n \geq 2$, let R be the rotation for which $R(|\xi|\, e_1) = \xi$, denote its transpose by R^T, put $y = R^T z$ and note that

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$$G(\xi) = \int_{\mathbb{R}^n} \frac{1 - \cos\left((R(|\xi|\, e_1)) \cdot z\right)}{|z|^{n+2s}}\, dz$$
$$= \int_{\mathbb{R}^n} \frac{1 - \cos\left((|\xi|\, e_1) \cdot (R^T z)\right)}{|z|^{n+2s}}\, dz$$
$$= \int_{\mathbb{R}^n} \frac{1 - \cos(|\xi|\, e_1 \cdot y)}{|y|^{n+2s}}\, dy = G(|\xi|\, e_1),$$

as claimed. Hence, with $\zeta = |\xi|\, y$, we have

$$G(\xi) = G(|\xi|\, e_1) = \int_{\mathbb{R}^n} \frac{1 - \cos(|\xi|\, y_1)}{|y|^{n+2s}}\, dy$$
$$= \frac{1}{|\xi|^n} \int_{\mathbb{R}^n} \frac{1 - \cos \zeta_1}{|\zeta/|\xi||^{n+2s}} d\zeta = C(n, s)^{-1} |\xi|^{2s},$$

as desired. Together with (3.4.5) this shows that

$$\left[u|H^s(\mathbb{R}^n)\right]^2 = 2C(n, s)^{-1} \left\| |\xi|^s Fu \right\|_{2,\mathbb{R}^n}^2 = 2C(n, s)^{-1} \left\| (-\Delta)^{s/2} u \right\|_2^2.$$

The formula for $C(n, s)$ given in (3.4.3) follows from the fact that

$$C(n, s)^{-1} = \int_0^\infty \left(\left|S^{n-1}\right| - (2\pi)^{n/2} r^{-(n-2)/2} J_{(n-2)/2}(r) \right) r^{-2s-1} dr,$$

where $J_{(n-2)/2}$ is the Bessel function of the first kind of order $(n-2)/2$, and the identity

$$\int_0^\infty r^{-z} \left(J_{(n-2)/2}(r) - 2^{-(n-2)/2} \Gamma(n/2^{-1} r^{(n-2)/2} \right) dr = 2^{-z} \frac{\Gamma((n-2z)/4)}{\Gamma((n+2z)/4)}$$

for $n/2 < z < (n+4)/2$ given in (2.20) of [172]; see also [78], proof of Lemma 1. □

Remark 3.29 From (3.4.3) it is plain that the constant $C(n, s)$ has the following behaviour:

$$\lim_{s \to 0+} s^{-1} C(n, s) = 2/\omega_n, \quad \lim_{s \to 1-} (1-s)^{-1} C(n, s) = 4n/\omega_n.$$

Proposition 3.30 *Let $s \in (0, 1)$. Then for all $u \in \mathcal{S}$,*

$$(-\Delta)^s u(x) = -\frac{1}{2} C(n, s) \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}}\, dy \; (x \in \mathbb{R}^n). \tag{3.4.6}$$

Proof Denote the right-hand side of (3.4.6) by $\mathcal{L}u(x)$. Use of Taylor's theorem shows that for each fixed $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n \setminus \{0\}$,

$$\frac{|u(x+y)+u(x-y)-2u(x)|}{|y|^{n+2s}} \le C\, |y|^{2-n-2s} \sup\{|D^\alpha u(z)| : |\alpha| = 2,\ z \in B(x, 1)\},$$

so that as $u \in \mathcal{S}$, $\mathcal{L}u(x)$ exists. Moreover, the condition $u \in \mathcal{S}$ also implies that for each fixed $y \in \mathbb{R}^n \backslash \{0\}$ the map $x \longmapsto |u(x+y) + u(x-y) - 2u(x)|\, |y|^{-n-2s}$ belongs to $L_1(\mathbb{R}^n)$. Since

$$\begin{aligned}
&-\frac{1}{2}C(n,s)\int_{\mathbb{R}^n}\frac{F\left(u(\cdot+y)+u(\cdot-y)-2u(\cdot)\right)(\xi)}{|y|^{n+2s}}\,dy \\
&\qquad = -\frac{1}{2}C(n,s)\int_{\mathbb{R}^n}\frac{e^{i\xi\cdot y}+e^{-i\xi\cdot y}-2}{|y|^{n+2s}}\,(Fu)(\xi)\,dy \\
&\qquad = C(n,s)\,(Fu)(\xi)\int_{\mathbb{R}^n}\frac{1-\cos(\xi\cdot y)}{|y|^{n+2s}}\,dy = |\xi|^{2s}\,(Fu)(\xi),
\end{aligned}$$

the final step following from (3.4.3), we conclude from the Fubini–Tonelli theorem that $\mathcal{L}u(x) = F^{-1}\left(|\xi|^{2s}(Fu)(\xi)\right)(x) = (-\Delta)^s(x)$, as required. □

Using this result we can now give yet another form of the fractional Laplacian. Let $u \in \mathcal{S}$ and note that

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\backslash B(x,\varepsilon)}\frac{u(y)-u(x)}{|x-y|^{n+2s}}\,dy &:= pv\int_{\mathbb{R}^n}\frac{u(y)-u(x)}{|x-y|^{n+2s}}\,dy \\
= pv\int_{\mathbb{R}^n}\frac{u(x+z)-u(x)}{|z|^{n+2s}}\,dz &= pv\int_{\mathbb{R}^n}\frac{u(x-z)-u(x)}{|z|^{n+2s}}\,dz,
\end{aligned}$$

so that

$$\begin{aligned}
2pv\int_{\mathbb{R}^n}\frac{u(x+z)-u(x)}{|z|^{n+2s}}\,dz &= pv\int_{\mathbb{R}^n}\frac{u(x+z)-u(x)}{|z|^{n+2s}}dz + pv\int_{\mathbb{R}^n}\frac{u(x-z)-u(x)}{|z|^{n+2s}}\,dz \\
&= pv\int_{\mathbb{R}^n}\frac{u(x+z)+u(x-z)-2u(x)}{|z|^{n+2s}}\,dz.
\end{aligned}$$

As observed above,

$$\left|\frac{u(x+z)+u(x-z)-2u(x)}{|z|^{n+2s}}\right| \le |z|^{2-n-2s}\sup_{|\alpha|=2,\,y\in\mathbb{R}^n}|D^\alpha u(y)|$$

and since the right-hand side is integrable near 0, the *pv* can be removed and we have

$$\begin{aligned}
pv\int_{\mathbb{R}^n}\frac{u(y)-u(x)}{|x-y|^{n+2s}}\,dy &= \frac{1}{2}\int_{\mathbb{R}^n}\frac{u(x+z)+u(x-z)-2u(x)}{|z|^{n+2s}}\,dz \\
&= -C(n,s)^{-1}\mathcal{L}u(x),
\end{aligned}$$

giving

$$(-\Delta)^s(x) = C(n,s)pv\int_{\mathbb{R}^n}\frac{u(y)-u(x)}{|x-y|^{n+2s}}\,dy. \tag{3.4.7}$$

Companion to the developments just outlined involving fractional powers of the Laplacian there is the rapidly evolving theory of the fractional p-Laplacian.

To introduce this, suppose that Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary and let $p \in (1, \infty)$. The p-Laplacian Δ_p can be defined by its action on smooth enough functions u:

$$\Delta_p u := \sum_{j=1}^{n} D_j \left(|\nabla u|^{p-2} D_j u \right).$$

It arises naturally on seeking to minimise the Rayleigh quotient

$$R(u) := \int_\Omega |\nabla u|^p \, dx \Big/ \int_\Omega |u|^p \, dx$$

among all functions $u \in C_0^\infty(\Omega)\backslash\{0\}$. We refer to [65], Chapter 9 for some details of the basic theory.

To obtain a fractional version of this suppose $s \in (0, 1)$ and let

$$X(\Omega) := \left\{ u \in W_p^s(\mathbb{R}^n) : u = 0 \text{ a.e. in } \mathbb{R}^n \backslash \Omega \right\};$$

this is endowed with the norm $[\cdot]_{s,p,\mathbb{R}^n}$ and then becomes a uniformly convex, uniformly smooth Banach space that coincides with the space $\overset{0}{\mathcal{D}}{}_p^s(\Omega)$ defined earlier. The nonlinear map $A \colon X(\Omega) \to X(\Omega)^*$ defined by

$$\langle Au, v \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2} (u(x) - u(y)) (v(x) - v(y))}{|x - y|^{n+sp}} \, dx \, dy$$

for all $u, v \in X(\Omega)$ is a duality map (see Section 1.2):

$$\langle Au, u \rangle = \|u|X(\Omega)\|^p, \ |\langle Au, v \rangle| \le \|u|X(\Omega)\|^{p-1} \|v|X(\Omega)\|.$$

This map A is called the s fractional p-Laplacian and is denoted by $(-\Delta)_p^s$; it is the gradient of the functional J defined by

$$J(u) = \frac{1}{p} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy = \frac{1}{p} [u]_{s,p,\mathbb{R}^n}^p. \tag{3.4.8}$$

To see how this operator is connected with a boundary-value problem, let

$$S = \{u \in X(\Omega) \colon I(u) = 1\}, \text{ where } I(u) \colon = \|u\|_{p,\Omega}^p.$$

Then $I, J \in C^1(X(\Omega))$ and $\left\langle J'(u), v \right\rangle = \langle Au, v \rangle$ for all $u, v \in X(\Omega)$. Let $\widetilde{J}$ be the restriction of J to S. We claim that $\lambda > 0$ is a critical value of $\widetilde{J}$ if and only if it is an eigenvalue of the (weak) problem

$$(-\Delta)_p^s u = \lambda |u|^{p-2} u \text{ in } \Omega,$$
$$u = 0 \text{ in } \mathbb{R}^n \backslash \Omega.$$

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Here by a weak solution of this problem is meant a function $u \in X(\Omega)$ such that for all $v \in X(\Omega)$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+sp}}\,dx\,dy = \lambda \int_{\Omega} |u|^{p-2}\,uv\,dx.$$

For suppose that $u \in S$ and $\mu \in \mathbb{R}$ are such that $J(u) = \lambda$ and $J'(u) - \mu I'(u) = 0$ in $X(\Omega)^*$.Then for all $v \in X(\Omega)$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+sp}}\,dx\,dy = \mu \int_{\Omega} |u|^{p-2}\,uv\,dx;$$

taking $v = u$, we see that $\lambda = \mu$ and $u \neq 0$ is a weak solution of the problem. Conversely, if λ is an eigenvalue of the weak problem, then there is a corresponding weak eigenfunction $u \in X(\Omega)$ with $I(u) = 1$. By Proposition 3.54 of [144], $u \in S$ is a critical point of $\widetilde{J}$ at level λ.

We next show that A has a property (introduced by Browder [37]) that will prove to be useful in our discussion of the spectrum. Given a Banach space Y, a map $T\colon Y \to Y^*$ is said to be *of type* $(S)_+$ if, whenever (u_k) is a sequence in Y such that $u_k \rightharpoonup u \in Y$ and

$$\limsup_{k\to\infty} \langle Tu_k - Tu, u_k - u\rangle \leq 0,$$

then $u_k \to u$ in Y. Following [151] we show that the map A discussed above has this property.

Lemma 3.31 *The map* $A\colon X(\Omega) \to X(\Omega)^*$ *is of type* $(S)_+$.

Proof Suppose that $u_k \rightharpoonup u$ in $X(\Omega)$ and

$$\limsup_{k\to\infty} \langle Au_k - Au, u_k - u\rangle \leq 0.$$

Since

$$\langle Au_k - Au, u_k - u\rangle \geq \|u_k|X(\Omega)\|^{p-1}\left(\|u_k|X(\Omega)\| - \|u|X(\Omega)\|\right) - \langle Au, u_k - u\rangle\,,$$

we see that

$$\lim_{k\to\infty} \langle Au_k - Au, u_k - u\rangle = 0.$$

Now we use Lemma 1.11. If $p \geq 2$, this shows that

$$\|u_k - u|X(\Omega)\|^p \leq C_p\,\langle Au_k - Au, u_k - u\rangle \to 0 \text{ as } k \to \infty,$$

while when $1 < p < 2$, $\|u_k - u|X(\Omega)\|^p$ is bounded above by

$$\begin{aligned}
&C_p^{p/2}\,|\langle Au_k - Au, u_k - u\rangle|^{p/2}\,(\|u_k|X(\Omega)\|^p + \|u|X(\Omega)\|^p)^{(2-p)/2} \\
&\quad \le C_p^{p/2}\,|\langle Au_k - Au, u_k - u\rangle|^{p/2}\left(\|u_k|X(\Omega)\|^{p(2-p)/2} + \|u|X(\Omega)\|^{p(2-p)/2}\right) \\
&\quad \le C\,|\langle Au_k - Au, u_k - u\rangle|^{p/2} \to 0 \text{ as } k \to \infty.
\end{aligned}$$

The lemma follows. □

For much information about the s fractional p-Laplacian and associated boundary-value problems we refer to [28], [29], [102] and [127].

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