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A SUBSEMILATTICE OF THE LATTICE OF VARIETIES OF LATTICE ORDERED GROUPS

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1. Introduction. Each variety of lattice ordered groups $\mathscr{V}$ determines a variety of groups, namely the variety of groups generated by the groups in $\mathscr{V}$. In this paper a completely new and different correspondence between varieties of groups and varieties of lattice ordered groups is developed. It is known that the variety of representable lattice ordered groups is defined by the law $z^+ \wedge u^{-1}z^-u = 1$. Here we consider the varieties defined by laws of this form where u is restricted to lie in some fully invariant subgroup of the free group F_X on a countable set X. All the varieties considered contain the variety of representable l-groups and therefore the free group with appropriate ordering.

The fundamental theorem (in Section 2) establishes that for any lattice ordered group G (or variety of lattice ordered groups $\mathscr{V}$) the set of $u \in F_X$ for which $l(u) = 1$ is a law in G (respectively $\mathscr{V}$) is a fully invariant subgroup $\mathscr{F}(G)$ (respectively, $\mathscr{F}(\mathscr{V})$) of F_X. On the other hand the class of lattice ordered groups $\mathscr{R}(U)$ satisfying the laws $l(u) = 1$ for $u \in U$, a fully invariant subgroup of F_X, is clearly a variety. The mappings $\mathscr{V} \to \mathscr{F}(\mathscr{V})$ and $U \to \mathscr{R}(U)$ are studied in detail in Sections 3 and 4. In particular, $\mathscr{F}\mathscr{R}$ is shown to be the identity mapping and $\mathscr{R}\mathscr{F}$ a closure operator. The mapping $\mathscr{R}$ is then used to construct a meet semilattice isomorphism of the lattice of varieties of groups into the lattice of varieties of lattice ordered groups. The structural properties of elements of $\mathscr{R}(U)$ are discussed in Section 5, where it is shown that they have properties reminiscent of representable lattice ordered groups. For instance, minimal prime subgroups are "U-normal".

The reader is referred to [**2**] and [**3**] for background information, terminology and notation for lattice ordered groups (henceforth called l-groups). For background information on varieties of l-groups the reader is referred to [**5**].

We will denote by $\mathscr{L}$ the lattice of varieties of l-groups. For any l-group G (family of l-groups $\{G_i\}$) we denote by $\mathscr{V}(G)$ (respectively, $\mathscr{V}\{G_i\}$) the variety of l-groups generated by G (respectively, $\{G_i\}$).

Let $X' = X \cup \{z\}$, where $z \notin X$, be a countable alphabet which will be used to express laws. Let $F = F_X$ denote the free group on the set X, which will be considered as a subset of F.

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Let φ be a mapping of X' into a group G. Then the restriction of φ to X extends to a unique homomorphism of F into G. In this way φ determines a unique mapping of $\{z\} \cup F$ into G which is a homomorphism on F and the restriction of which to X' is just φ. We will denote this extension by φ also.

For any variety of l-groups $\mathscr{V}$, we define

$$\mathscr{F}\mathscr{V} = \{u \in F : z^+ \wedge u^{-1}z^-u = 1, \text{ is a law in } \mathscr{V}\},$$

and, for any l-group G, we define

$$\mathscr{F}G = \{u \in F : z^+ \wedge u^{-1}z^-u = 1, \text{ is a law in } G\}.$$

Clearly $\mathscr{F}G = \mathscr{F}\mathscr{V}(G)$ and $\mathscr{F}\mathscr{V} = \cap \{\mathscr{F}G : G \in \mathscr{V}\}$. We will sometimes write

$$l(u) = z^+ \wedge u^{-1}z^-u.$$

2. Fundamental theorem. The following observation is fundamental to this paper.

THEOREM 2.1. *For any l-group G (or variety of l-groups $\mathscr{V}$), $\mathscr{F}G$ (respectively, $\mathscr{F}\mathscr{V}$) is a fully invariant subgroup of F.*

Proof. Let $u, v \in \mathscr{F}G$. Then $l(u) = 1$ is a law in G. Hence, for any mapping $\varphi : X' \to G$ with $z\varphi = g, u\varphi = h, v\varphi = k$,

$$g^+ \wedge h^{-1}g^-h = 1.$$

Let $f = g^+(h^{-1}g^-h)^{-1}$. Then $f^+ = g^+$ and $f^- = h^{-1}g^-h$. Since $l(v)$ is a law in G, we have

$$f^+ \wedge k^{-1}f^-k = 1$$

or

$$g^+ \wedge (hk)^{-1}g^-(hk) = 1.$$

That is,

$$(z\varphi)^+ \wedge ((uv)\varphi)^{-1}(z\varphi)^-(uv)\varphi = 1.$$

Therefore, $l(uv) = 1$ is a law in G and $uv \in \mathscr{F}_G$.

Furthermore, for any $g \in G$,

$$(g^{-1})^+ \wedge u^{-1}(g^{-1})^-u = 1 \quad \text{or} \quad g^- \wedge u^{-1}g^+u = 1.$$

Therefore,

$$(u^{-1})^{-1}g^-(u^{-1}) \wedge g^+ = 1$$

and $u^{-1} \in \mathscr{F}G$. Thus $\mathscr{F}G$ is a subgroup of F.

To see that $\mathscr{F}G$ is fully invariant, let α be any endomorphism of F. Let $\varphi : X' \to G$ be any mapping. Define $\theta : X' \to G$ by $z\theta = z\varphi$, $x\theta = (x\alpha)\varphi$ $(x \in X)$. Let $u \in \mathscr{F}G$. Then $l(u)$ is a law in G. Therefore,

$$(z\theta)^+ \wedge (u\theta)^{-1}(z\theta)^-(u\theta) = 1,$$

that is,

$$(z\varphi)^+ \wedge ((u\alpha)\varphi)^{-1}(z\varphi)^-((u\alpha)\varphi) = 1.$$

Since φ was chosen arbitrarily, it follows that $l(u\alpha) = 1$ is a law in G and that $u\alpha \in \mathscr{F}G$. Hence $\mathscr{F}G$ is fully invariant. The result for $\mathscr{F}\mathscr{V}$ then follows easily.

3. Group varieties to l-group varieties. Theorem 2.1 associates a fully invariant subgroup of F with each variety of l-groups. Conversely, each full invariant subgroup of F may be used to define a variety of l-groups as follows. We denote by $\mathscr{I}F$ the lattice of fully invariant subgroups of F.

Definition 3.1. Let $U \in \mathscr{I}F$ be a fully invariant subgroup of F. Then $\mathscr{R}U$ will denote the variety of l-groups satisfying the laws $l(u) = 1$, for all $u \in U$. The two limiting cases are well known. If $U = \{1\}$, the trivial subgroup of F, then $\mathscr{R}U = \mathscr{L}$, the variety of all l-groups while $\mathscr{R}F$ is simply the variety $\mathscr{R}$ of all representable l-groups. We will call an l-group G *quasi-representable* if $G \in \mathscr{R}U$ for some $U \in \mathscr{I}F$.

We will now develop some basic properties of the two mappings $\mathscr{F} : \mathscr{V} \to F\mathscr{V}$ and $\mathscr{R} : U \to \mathscr{R}U$. To do so we will need to have certain examples at our disposal.

We recall the construction of the wreath product of two l-permutation groups. Let (H, Θ) and (G, Ω) be l-permutation groups and let $\Lambda = \Theta \times \Omega$ be ordered lexicographically from the right. Then the wreath product $(W, \Lambda) = (H, \Theta)$ Wr (G, Ω) of (H, Θ) and (G, Ω) is the l-permutation group of order preserving permutations of Λ of the form (A, a) where $A : \Omega \to H$ and $(\theta, \omega)\ (A, a) = (\theta A(\omega), \omega a)$. The resulting group consists of all ordered pairs (A, a) where $A : \Omega \to H$ and $a \in G$ with the group operation

$$(A, a)(B, b) = (AB^a, ab)$$

where $B^a(\alpha) = B(\alpha a)$.

Since the l-group W so constructed is independent of Θ, we will simply write $W = H$ Wr (G, Ω). For basic properties of the wreath product and its role in the study of $\mathscr{L}$ see [**4**] and [**5**].

If G is a totally ordered group, then (G, G) will denote G as a group of order preserving permutations of the totally ordered set G where the action is by right translations.

For the purpose of constructing the examples below, let F be endowed with some fixed total order.

Let U be a fully invariant subgroup of F. Then we define a subset L_U of Z Wr (F, F) as follows:

$$L_U = \{(A, a) : A(g) = A(h), \quad \text{if } Ug = Uh\}.$$

LEMMA 3.2. *L_U is an l-subgroup of Z Wr (F, F).*

Proof. Let $(A_i, a_i) \in L_U$ $(i = 1, 2)$. Then

$$(A_1, a_1)(A_2, a_2) = (A_1 A_2^{a_1}, a_1 a_2).$$

Let $Ug = Uh$. Then $Uga_1 = Uha_1$ and so

$$(A_1 A_2^{a_1})(g) = A_1(g)A_2(ga_1) = A_1(h)A_2(ha_1) = (A_1 A_2^{a_1})(h).$$

Thus L_U is closed under products. For any $(A, a) \in L_U$,

$$(A, a)^{-1} = ((A^b)^{-1}, a^{-1})$$

where $b = a^{-1}$ and so

$$(A^b)^{-1}(g) = A(ga^{-1})^{-1} = A(ha^{-1})^{-1} = (A^b)^{-1}(h).$$

Therefore L_U is closed under inverses and is a subgroup. It can readily be verified that, for any $(A, a) \in L_U$,

$$(A, a) \vee 1 = \begin{cases} (A, a) & \text{if } a > 1 \\ 1 & \text{if } a < 1 \\ (A \vee I, 1) & \text{if } a = 1 \end{cases}$$

where 1 denotes the identity of L_U or F, as appropriate, and I denotes the constant map of F onto $1 \in \mathscr{F}$. Therefore L_U is an l-subgroup, as required.

The l-groups L_U relate to the varieties $\mathscr{R}\,U$ through the following result.

THEOREM 3.3. *Let U be a fully invariant subgroup of F. Then*
(1) L_U satisfies the laws $l(u) = 1$ $(u \in U)$;
(2) L_U does not satisfy the law $l(v) = 1$, if $v \notin U$.

Proof. (1) Let $u \in U$ and let $\varphi : X' \to L_U$ and let $w = z\varphi = (B, b)$. If $b \neq 1$ then, since F is totally ordered, either $b > 1$ or $b < 1$ so that either $w > 1$ or $w < 1$. This means that either $w^- = 1$ or $w^+ = 1$ and therefore

$$w^+ \wedge (u\varphi)^{-1} w^- (u\varphi) = 1.$$

Now suppose that $w = (B, 1)$ and let $x = u\varphi = (A, a)$. Let $\pi : L_U \to F$ be the mapping that projects onto the second component: $(C, c) \to c$. Then $\varphi\pi$ (restricted to F) is an endomorphism of F. Since U is fully

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invariant, it follows that $a \in U$ and so also a^{-1}. We have, with $f = a^{-1}$,

$$x^{-1}w^{-}x = ((A^{f})^{-1}, a^{-1})(B^{-}, 1)(A, a)$$
$$= ((A^{f})^{-1}(B^{-})^{f}A^{f}, 1) = ((B^{-})^{f}, 1).$$

Since $a^{-1} \in U$, for any y,

$$(B^{-})^{f}(y) = B^{-}(ya^{-1}) = B^{-}((ya^{-1}y^{-1})y) = B^{-}(y),$$

since $ya^{-1}y^{-1} \in U$, by the normality of U, and since $w \in L_U$. Thus the support of $(B^{-})^{f}$ equals the support of B^{-}. Hence

$$w^{+} \wedge x^{-1}w^{-}x = (B^{+}, 1) \wedge ((B^{-})^{f}, 1) = (B^{+} \wedge (B^{-})^{f}, 1) = (I, 1).$$

Therefore $l(u) = 1$ is a law in L_U.

(2) Now let $v \in F\backslash U$. Let $A : F \to \mathbf{Z}$ be defined by

$$A(y) = \begin{cases} 1 & \text{if } y \in U \\ -1 & \text{if } y \in Uv^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Then $w = (A, 1) \in L_U$. Define $\varphi : X' \to L_U$ by $z\varphi = w$, $x\varphi = (I, x)$ for $x \in X$. Then $v\varphi = (I, v)$, and, with $u = v^{-1}$

$$(A^{-})^{u}(1) = A^{-}(v^{-1}) = 1 = A^{+}(1).$$

Thus

$$A^{+} \wedge (A^{-})^{u} \neq I,$$

and so

$$(z\varphi)^{+} \wedge (v\varphi)^{-1}(z\varphi)^{-}(v\varphi) \neq 1.$$

Therefore $l(v) = 1$ is not a law in L_U.

4. Basic properties of the correspondences. The next theorem summarizes the basic properties of the mappings $\mathscr{F} : \mathscr{L} \to \mathscr{I}(F)$, $\mathscr{R} : \mathscr{I}(F) \to \mathscr{L}$ of the lattice of varieties of l-groups to the lattice of varieties of fully invariant subgroups of F and conversely.

THEOREM 4.1. *Let* $\mathscr{U}, \mathscr{V}, \mathscr{U}_\alpha$ $(\alpha \in A) \in \mathscr{L}$ *and* $U, V, U_\alpha (\alpha \in A) \in \mathscr{I}(F)$.

(1) $\mathscr{R}$ *is one-to-one.*

(2) $\mathscr{U} \subseteq \mathscr{V} \Rightarrow \mathscr{F}(\mathscr{V}) \subseteq \mathscr{F}(\mathscr{U})$.

(3) $U \subseteq V \Leftrightarrow \mathscr{R}(V) \subseteq \mathscr{R}(U)$.

(4) $\mathscr{U} \subseteq \mathscr{R}\mathscr{F}(\mathscr{U})$.

(5) $U = \mathscr{F}\mathscr{R}(U)$.

(6) $\cap \mathscr{F}(\mathscr{U}_\alpha) = \mathscr{F}(\vee \mathscr{U}_\alpha)$.

(7) $\cap \mathscr{R}(U_\alpha) = \mathscr{R}(\vee U_\alpha)$.

(8) $\vee \mathscr{F}(\mathscr{U}_\alpha) \subseteq \mathscr{F}(\cap \mathscr{U}_\alpha)$.

(9) $\vee \mathscr{R}(U_\alpha) \subseteq \mathscr{R}(\cap U_\alpha)$.

Proof. (1) Let U, V be distinct elements of $\mathscr{I}(F)$. Without loss of generality, let there be an element $v \in V\backslash U$. It then follows from Theorem 3.3 that $L_U \in \mathscr{R}(U)\backslash\mathscr{R}(V)$. Hence $\mathscr{R}$ is one-to-one. The direct part of (3), (2) and (4) are immediate from the definitions of $\mathscr{F}$ and $\mathscr{R}$.

(5) Again it is immediate from the definitions that $U \subseteq \mathscr{F}\mathscr{R}(U)$. Let $v \in F\backslash U$. By Theorem 3.3, $L_U \in \mathscr{R}(U)$ and L_U does not satisfy $l(v) = 1$. Hence $v \notin \mathscr{F}(\mathscr{R}(U))$ and so we must have equality.

(6) By (2), $\mathscr{F}(\vee \mathscr{U}_\alpha) \subseteq \cap \mathscr{F}(\mathscr{U}_\alpha)$. Now, for each α, $\mathscr{F}(\mathscr{U}_\alpha) \supseteq \cap \mathscr{F}(\mathscr{U}_\alpha)$ and so, by (3) and (4),

$$\mathscr{U}_\alpha \subseteq \mathscr{R}\mathscr{F}(\mathscr{U}_\alpha) \subseteq \mathscr{R}(\cap \mathscr{F}(\mathscr{U}_\alpha)).$$

Hence

$$\vee \mathscr{U}_\alpha \subseteq \mathscr{R}(\cap \mathscr{F}(\mathscr{U}_\alpha)).$$

Applying $\mathscr{F}$ to both sides, we obtain

$$\mathscr{F}(\vee \mathscr{U}_\alpha) \supseteq \mathscr{F}\mathscr{R}(\cap \mathscr{F}(\mathscr{U}_\alpha))$$

which, by (5), reduces to $\mathscr{F}(\vee \mathscr{U}_\alpha) \supseteq \cap \mathscr{F}(\mathscr{U}_\alpha)$. Therefore, we have equality.

For the converse implication in (3), now suppose that $\mathscr{R}(V) \subseteq \mathscr{R}(U)$. Then, by (2), $\mathscr{F}\mathscr{R}(U) \subseteq \mathscr{F}\mathscr{R}(V)$ which, by (5), reduces to $U \subseteq V$, as required.

(7) By (3), we have that $\mathscr{R}(\vee U_\alpha) \subseteq \cap \mathscr{R}(U_\alpha)$. On the other hand, for each α, $\cap \mathscr{R}(U_\alpha) \subseteq \mathscr{R}(U_\alpha)$ implies that

$$U_\alpha = \mathscr{F}\mathscr{R}(U_\alpha) \subseteq \mathscr{F}(\cap \mathscr{R}(U_\alpha)).$$

Hence,

$$\vee U_\alpha \subseteq \mathscr{F}(\cap \mathscr{R}(U_\alpha))$$

and

$$\mathscr{R}(\vee U_\alpha) \supseteq \mathscr{R}\mathscr{F}(\cap \mathscr{R}(U_\alpha)) \supseteq \cap \mathscr{R}(U_\alpha).$$

Therefore equality holds.

(8) From (2), $\mathscr{F}(\mathscr{U}_\alpha) \subseteq \mathscr{F}(\cap \mathscr{U}_\alpha)$, for each α. Therefore (8) follows.

(9) From (3), $\mathscr{R}(U_\alpha) \subseteq \mathscr{R}(\cap U_\alpha)$, for each α. Therefore (9) follows.

It was first proved by Kopitov and Medvedev [**7**] that there are uncountably many varieties of l-groups. In fact, the varieties described in [**7**] are all varieties of representable l-groups. Feil, private communication, has described another much simpler uncountable family of varieties of representable l-groups. Since the mapping $\mathscr{R}$ is one-to-one and it is known that $\mathscr{I}(F)$ is uncountable [**10**], it follows that $\mathscr{R}\mathscr{I}(F)$ constitutes a new uncountable family of varieties of l-groups. More specifically, from Theorem 4.1 (1) and (7) we have the following corollary.

COROLLARY 4.2 *$\mathscr{R}\mathscr{I}(F)$ is an uncountable $\cap$-complete lower subsemilattice of $\mathscr{L}$ with $\mathscr{R}$, the variety of all representable l-groups, as its smallest element and $\mathscr{L}$ as its largest member.*

The following application of Theorem 4.1, to answer an open question regarding the breadth of $\mathscr{L}$, was drawn to the author's attention by A. Glass. It has been shown by S. Adyan [**1**] that there exist $2^{\aleph_0}$ pairwise incomparable varieties of groups and consequently $2^{\aleph_0}$ pairwise incomparable elements of $\mathscr{I}(F)$. By Theorem 4.1 (3), $\mathscr{R}$ maps such elements onto incomparable elements in $\mathscr{L}$. Thus

COROLLARY 4.3. (A. Glass). *There exist $2^{\aleph_0}$ pairwise incomparable varieties of lattice ordered groups.*

In terms of the lattice of varieties of groups we have the following. Let $\theta : \mathscr{G} \rightarrow \mathscr{I}(F)$ be the usual anti-isomorphism of the lattice $\mathscr{G}$ of group varieties onto the lattice $\mathscr{I}(F)$ of fully invariant subgroups of F.

COROLLARY 4.4. *The composition of θ and $\mathscr{R}$ is a one-to-one complete lower semilattice homomorphism of the lattice $\mathscr{G}$ of group varieties into the lattice $\mathscr{L}$ of varieties of l-groups.*

Notation. For each positive integer n, let A_n denote the fully invariant subgroup of F corresponding to the variety of abelian groups of exponent n. If $(m, n) = 1$, that is, if m and n are relatively prime, then $A_mA_n = F$. Hence from Theorem 4.1 (7) we have the following.

COROLLARY 4.5. *If m and n are relatively prime positive integers, then $\mathscr{R}A_m \cap \mathscr{R}A_n = \mathscr{R}$.*

Finally, the following corollary is a simple consequence of parts (2)-(5) of Theorem 4.1.

COROLLARY 4.6. *$\mathscr{R}\mathscr{F}$ is a closure operator.*

5. Characterizations of quasi-representable l-groups. Recall that an l-group G is *representable* if and only if there exists a family of prime subgroups $\{P_\lambda\}$ such that each P_λ is normal in G and $\cap P_\lambda = \{1\}$. Suppose now that M is a fully invariant subgroup of F and that $M(G)$ denotes the fully invariant subgroup of G corresponding to M. The subgroup $M(G)$ can also be described as the smallest normal subgroup of G such that the quotient group lies in the variety of groups defined by M or as the union of all the images of M under all homomorphisms of F into G. Note that $M(G)$ is a fully invariant subgroup of G but not necessarily an l-subgroup.

Definition 5.1. Let M be a fully invariant subgroup of F and H be a subgroup of an l-group G. Then H is said to be *M-invariant* or $M(G)$-

invariant if and only if $M(G)$ is contained in the normalizer of H in G. The l-group G is said to be *M-representable* if there exists a family of prime subgroups $\{P_\lambda\}$ such that each P_λ is $M(G)$-invariant and $\bigcap P_\lambda = \{1\}$. A prime subgroup P of G is said to be a *representing* prime subgroup if

$$\bigcap \{x^{-1}Px : x \in G\} = \{1\}.$$

This is equivalent to saying that the Holland representation [**6**] of G on the cosets of P is faithful.

If, in the following theorem, $M = F$ then the listed conditions reduce to the well-known equivalent characterizations of representability (see [**3**]). Since the proofs are entirely analogous, they are not included.

THEOREM 5.2. *Let G be an l-group and M a fully invariant subgroup of F. Then the following statements are equivalent.*

(1) *G is M-representable.*

(2) *G is a subdirect product of l-groups with M-invariant representing prime subgroups.*

(3) *$a \in G$, $u \in M(G)$ and $a \wedge u^{-1}a\,u = 1$ implies that $a = 1$.*

(4) *$a \wedge b = 1$ and $u \in M(G)$ implies that $a \wedge u^{-1}bu = 1$.*

(5) *$G \in \mathscr{R}M$.*

(6) *Every polar in G is $M(G)$-invariant.*

(7) *For any prime subgroup P of G, $J = \bigcap \{g^{-1}Pg : g \in M(G)\}$ is a prime subgroup.*

(8) *Every minimal prime subgroup of G is $M(G)$-invariant.*

(9) *If N is a prime subgroup of G and $g \in M(G)$ then N and $g^{-1}Ng$ are comparable.*

(10) *If $a \in G$, $a \neq 1$ and K is a convex l-subgroup of G which is maximal with respect to* (i) *$a \notin K$ and* (ii) *K is $M(G)$-invariant, then K is prime.*

6. Further remarks on $\mathscr{R}\mathscr{I}(F)$. We will first show that the containment in Theorem 4.1 (9) can be proper. For each $n \in N$, let

$$G_n = \{(H, h) \in \mathbf{Z} \text{ Wr } (\mathbf{Z}, \mathbf{Z}) : H(i) = H(j) \text{ if } i \equiv j \text{ and mod } n\}.$$

Then G_n is an l-subgroup of $\mathbf{Z}$ Wr $(\mathbf{Z}, \mathbf{Z})$, introduced by Martinez [**8**]. Scrimger [**11**] considered the varieties $\mathscr{S}_n$, now called *Scrimger* varieties, generated by the l-groups G_n. Each G_n is subdirectly irreducible.

For each integer n, let A_n be the fully invariant subgroup of F corresponding to the variety of abelian groups of exponent n. The verification of the following lemma is a straightforward exercise.

LEMMA 6.1. (i) $G_n \in \mathscr{R}(A_n)$ (ii) $\mathscr{S}_n \subseteq \mathscr{R}(A_n)$.

We will also need the following result due to Martinez [**9**].

LEMMA 6.2. *Let $\mathscr{V}_1, \mathscr{V}_2 \in \mathscr{L}$. Then $G \in \mathscr{V}_1 \vee \mathscr{V}_2$ if and only if G is a subdirect product of l-groups in $\mathscr{V}_1$ and $\mathscr{V}_2$.*

Let m and n be relatively prime integers greater than 1. Then $G_{mn} \in \mathscr{R}(A_{mn})$ and $A_{mn} = A_m \cap A_n$. Suppose that $G_{mn} \in \mathscr{R}(A_m) \vee \mathscr{R}(A_n)$. Then by Lemma 6.2, G_{mn} is a subdirect product of l-groups in $\mathscr{R}(A_m)$ and $\mathscr{R}(A_n)$. However, G_{mn} is subdirectly irreducible and so we must have $G_{mn} \in \mathscr{R}(A_m)$ or $G_{mn} \in \mathscr{R}(A_n)$. It is easily verified that this is not the case and therefore we have a contradiction. Hence $G_{mn} \notin \mathscr{R}(A_m) \vee \mathscr{R}(A_n)$. Therefore

$$\mathscr{R}(A_m) \vee \mathscr{R}(A_n) \subsetneqq \mathscr{R}(A_m \cap A_n) = \mathscr{R}(A_{mn})$$

and we see that it is possible to have proper containment in Theorem 4.1 (9).

With regard to the placement of the quasi-representable varieties in the lattice of varieties we have seen (Theorem 4.1) that the smallest quasi-representable variety is $\mathscr{R}F$ which is simply the variety of representable l-groups. Thus all quasi-representable varieties contain $\mathscr{R}$. In Lemma 6.1 we have observed that $\mathscr{S}_n \subseteq \mathscr{R}A_n$. Now let B_n denote the fully invariant subgroup of F_X defining the Burnside variety of exponent n.

The variety $\mathscr{L}_n$ of l-groups defined by the law $x^n y^n = y^n x^n$ was introduced by Martinez [**8**] and also studied by Scrimger [**11**] and Smith [**12**].

LEMMA 6.3. *For any integer* n, $\mathscr{L}_n \subseteq \mathscr{R}B_n$.

The containment in Lemma 6.3 is clearly proper since $\mathscr{R} \subset \mathscr{R}B_n$ but $\mathscr{R} \not\subset \mathscr{L}_n$.

The smallest quasi-representable varieties that properly contain $\mathscr{R}$ are clearly those of the form $\mathscr{R}A_p$, where p is a prime. If p and q are distinct primes, then by Corollary 4.4, $\mathscr{R}A_p \cap \mathscr{R}A_q = \mathscr{R}$. It is therefore natural to consider whether these varieties actually cover $\mathscr{R}$. By Lemma 6.1, $\mathscr{S}_n \subset \mathscr{R}A_n$. Thus

$$\mathscr{R} \vee \mathscr{S}_n \subseteq \mathscr{R}A_n$$

for all integers $n > 1$. Now $L_{A_n} \in \mathscr{R}A_n$. Moreover L_{A_n} is subdirectly irreducible. Hence, if $L_{A_n} \in \mathscr{R} \vee \mathscr{S}_n$ then it must lie either in $\mathscr{R}$ or $\mathscr{S}_n$. Since neither is the case it follows that $L_{A_n} \notin \mathscr{R} \vee \mathscr{S}_n$. Therefore we have the following proposition.

PROPOSITION 6.4. *For any integer* $n > 1$,

$$\mathscr{R} \vee \mathscr{S}_n \subsetneqq \mathscr{R}A_n.$$

For any group G let

$$G^{(0)} = G, G^{(1)} = [G, G], \ldots, G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$$

be the derived series for G. Let $\mathscr{A}$ denote the variety of all abelian l-groups and let $\mathscr{A}^n$ the product of $\mathscr{A}$ with itself under the Mal'cev product as discussed by Glass et al. [**5**]. Then $\mathscr{A}^n$ is solvable $(\mathscr{A}^n)^{(n)} = 1$ and so $\mathscr{A}^n \subset \mathscr{R}F^{(n)}$. It is established in [**5**] that $\vee \mathscr{A}^n = \mathscr{N}$, the variety of normal valued l-groups. Hence we have.

PROPOSITION 6.5. $\vee\{\mathscr{R}U : U \in \mathscr{I}F,\ U \neq \{1\}\} = \mathscr{N}$.

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