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THE ATTOUCH-WETS TOPOLOGY AND A CHARACTERISATION OF NORMABLE LINEAR SPACES

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Let X and Y be metric spaces and $C(X, Y)$ be the space of all continuous functions from X to Y. If X is a locally connected space, the compact-open topology on $C(X, Y)$ is weaker than the Attouch-Wets topology on $C(X, Y)$. The result is applied on the space of continuous linear functions. Let X be a locally convex topological linear space metrisable with an invariant metric and X^* be a continuous dual. X is normable if and only if the strong topology on X^* and the Attouch-Wets topology coincide.

1. INTRODUCTION

Convex analysts all agree on the most appropriate convergence notion for sequences of closed convex sets in finite dimensions: classical Kuratowski convergence of sets [**13**]. Convergence in this sense is stable with respect to duality: if $\{A_n\}$ is a sequence of closed convex sets in R^n Kuratowski convergent to A, we have the convergence of the polar sequence $\{A_n^0\}$ to A^0 [**17**].

Attempts to obtain a suitable infinite dimensional generalisation of this convergence notion have focused on the notion of Mosco convergence [**14, 15, 7**]. Unfortunately, this convergence does not work well without reflexivity. It appears now that the correct generalisation is the topology τ of uniform convergence of distance functionals on bounded sets. This topology is stable with respect to duality without reflexivity or even completeness [**6**]. It seems particularly well-suited to problems involving estimation, approximation, and optimisation [**1, 2, 3, 4**].

It is the purpose of this article to show the connections between standard convergence notions for functions and convergence of their graphs with respect to the Attouch-Wets topology.

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2. Preliminaries

(X, d) will denote a metrisable space X with a compatible metric d. The open d-ball with centre $x_0 \in X$ and radius $\varepsilon > 0$ will be denoted by $S_d[x_0, \varepsilon]$, and the ε-parallel body $\bigcup_{a \in A} S_d[a, \varepsilon]$ for a subset A of X will be denoted by $S_d[A, \varepsilon]$.

Let $CL(X)$ be a family of nonempty closed subsets of a metric space (X, d). If $A \in CL(X)$, the distance functional $d(\cdot, A) : X \to [0, \infty)$ is described by the familiar formula $d(x, A) = \inf\{d(x, a) : a \in A\}$.

We shall denote by $\tau_{aw}(d)$ the topology on $CL(X)$ of uniform convergence of distance functionals on bounded subsets of X corresponding to a fixed metric d on X (the Attouch-Wets topology). The topology $\tau_{aw}(d)$ is most naturally presented as a uniform topology, determined by the uniformity Ω_d on $CL(X)$ with the countable base of entourages $\{V_d[x_0, n] : n \in Z^+\}$, where for each n

$$V_d[x_0, n] = \{(A, B) \colon \sup_{d(x, x_0) < n} |d(x, A) - d(x, B)| < 1/n\}.$$

The point x_0 is a fixed but arbitrary point of X, and the uniformity is independent of its choice.

We will consider a uniformity Σ_d on $CL(X)$ which has a countable base consisting of all sets of the form

$$U_d[x_0, n] = \{(A, B) \colon A \cap S_d[x_0, n] \subset S_d[B, 1/n] \text{ and } B \cap S_d[x_0, n] \subset S_d[A, 1/n]\}$$

where again x_0 is a fixed but arbitrary point of X and $n \in Z^+$. This uniformity also determines $\tau_{aw}(d)$ [8].

Now let (X, d) and (Y, ρ) be metric spaces and let $d \times \rho$ denote the box metric on $X \times Y$. If $f \colon X \to Y$ is a function, denote $G(f) = \{(x, f(x)) \colon x \in X\}$ the graph of f. Denote $C(X, Y)$ the family of all continuous functions from X to Y. We can identify the members of $C(X, Y)$ with their graphs and consider $C(X, Y)$ as a subspace of $CL(X \times Y)$ with the induced $\tau_{aw}(d \times \rho)$ topology.

It is easy to see from the definition of the uniformity $\Sigma_{d \times \rho}$ on $CL(X \times Y)$ that the $\tau_{aw}(d \times \rho)$ topology on $C(X, Y)$ is weaker than the Hausdorff metric topology [11].

In general $\tau_{aw}(d \times \rho)$ convergence of a sequence $\{f_n\}$ from $C(X, Y)$ to $f \in C(X, Y)$ does not imply pointwise convergence.

Example 1: Let $X = [0, 1] \setminus \{1, 1/2, \dots 1/n, \dots\}$ with the usual metric u. Let $f \colon X \to R$ be the zero function and let $f_n = n\chi_{[0, 1/n] \cap X}$. Evidently we have $\tau_{aw}(u \times u)$-convergence of $\{f_n\}$ to f. But pointwise convergence of $\{f_n\}$ to f at 0 fails.

But the following trivial Proposition is true:

Proposition 1. *Let (X, d) and (Y, ρ) be metric spaces, and let $f, f_1, f_2, \dots$ be elements of $C(X, Y)$. If for each $x \in X$, $\{f_n(x): n \in Z^+\}$ is a bounded set in (Y, ρ), then $\tau_{aw}(d \times \rho)$ convergence of $\{f_n\}$ to f implies the pointwise convergence of $\{f_n\}$ to f.*

Proof: Let $x \in X$, $\varepsilon > 0$. We fix (x_0, y_0) in $X \times Y$ to serve as centre for $d \times \rho$-balls in $X \times Y$. There is $N \in Z^+$ such that

$$\big(\{x\} \times \{f_n(x): n \in Z^+\}\big) \cup S_{d\times\rho}[(x, f(x)), \varepsilon] \subset S_{d\times\rho}[(x_0, y_0), N].$$

The continuity of f at x implies that there is $0 < \delta < \varepsilon$ such that $\rho(f(x), f(z)) < \varepsilon/2$ whenever $d(x, z) < \delta$.

Put $\eta = \min\{\varepsilon/2, \delta\}$. Let $M \in Z^+$ be such that $M > N$ and $M > 1/\eta$. There is $M_0 \in Z^+$ such that for each $n \geqslant M_0$ we have

$$(G(f_n), G(f)) \in U_{d\times\rho}[(x_0, y_0), M].$$

It is easy to verify that $\rho(f_n(x), f(x)) < \varepsilon$ for each $n \geqslant M_0$.

∎

3. Main results

Theorem 1. *Let (X, d) be a locally connected metric space and (Y, ρ) be a metric space. Then the topology of uniform convergence on compact subsets of X on $C(X, Y)$ is weaker than the $\tau_{aw}(d \times \rho)$ topology on $C(X, Y)$.*

The proof of Theorem 1 is based on the following lemma.

Lemma 1. *Let (X, d) and (Y, ρ) be metric spaces. Let $f, f_1, f_2, \dots \in C(X, Y)$. If for each $x \in X$ there is a neighbourhood O_x of x and there is $N \in Z^+$ such that $\{f_n: n \geqslant N\}$ is uniformly bounded on O_x, then the convergence of $\{f_n\}$ to f in the $\tau_{aw}(d \times \rho)$ topology ensures uniform convergence of $\{f_n\}$ to f on compact subsets of X.*

Proof of Lemma 1: We fix (x_0, y_0) in $X \times Y$ to serve as centre for $d \times \rho$-balls in $X \times Y$. By $U_{d\times\rho}[(x_0, y_0), k](G(f))$ we mean the $\tau_{aw}(d \times \rho)$ neighbourhood of f in $CL(X \times Y)$.

Let K be a compact set in X and $\varepsilon > 0$. We show that there is $k \in Z^+$ such that for each $n \geqslant k$ and for each $x \in K$ we have $\rho(f_n(x), f(x)) < \varepsilon$.

By assumption there is $N \in Z^+$ and a bounded open set G in X such that $G \supset K$ and $\{f_n : n \geqslant N\}$ is uniformly bounded on G. Since f is uniformly continuous on K there is $\delta > 0$ such that whenever $z \in K$ and $x \in X$ and $d(x, z) < \delta$ then $\rho(f(z), f(x)) < \varepsilon/2$.

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Let $M \in Z^+$ be such that $\cup\{G \times f_n(G): n \geqslant N\} \subset S_{d\times\rho}[(x_0, y_0), M]$. Pick an integer L such that $L > \max\{M, 2/\varepsilon, 1/\delta\}$. There is $N_1 \in Z^+$, $N_1 \geqslant N$ such that

(1) $$G(f_n) \in U_{d\times\rho}[(x_0, y_0), L](G(f)) \quad \text{for each } n \geqslant N_1.$$

Let $x \in K$ and $n \geqslant N_1$. (1) guarantees the existence $u \in X$ such that $d \times \rho[(x, f_n(x)), (u, f(u))] < 1/L$, that is, $d(x, u) < 1/L < \delta$ and $\rho(f_n(x), f(u)) < \varepsilon/2$, that is, $\rho(f_n(x), f(x)) < \varepsilon$. □

PROOF OF THEOREM 1: Denote the topology of uniform convergence on compact subsets of X on $C(X, Y)$ by $\mathcal{U}$. Suppose that $\mathcal{U}$ is not contained in $\tau_{aw}(d \times \rho)$. There is $U \in \mathcal{U}$ such that $U \notin \tau_{aw}(d \times \rho)$. Thus there is $f \in U$ with the following property

(*) for every $\tau_{aw}(d \times \rho)$ neighbourhood V of , $V \not\subset U$.

Here (*) ensures the existence of a sequence $\{f_n\}$, $f_n \in C(X, Y)$ $n = 1, 2, \ldots$ which converges in the $\tau_{aw}(d \times \rho)$ topology to f, but fails to converge in $\mathcal{U}$ to f.

We show that $\{f_n\}$ uniformly converges to f on compact subsets of X and that will be a contradiction. By Lemma 1 it is sufficient to prove that for each $x \in X$ there is a neighbourhood O_x of x and there is $N \in Z^+$ such that $\{f_n : n \geqslant N\}$ is uniformly bounded on O_x.

We fix (x_0, y_0) in $X \times Y$ to serve as centre for $d \times \rho$ -balls in $X \times Y$. Let $x \in X$. There is δ $(0 < \delta < 1)$ such that

(2) $$\text{for each } z \in S_d[x, \delta] \text{ we have } \rho(f(x), f(z)) < 1/2.$$

Let O_x be a connected neighbourhood of x such that $O_x \subset S_d[x, \delta/2]$ and let δ_0 be such that $S_d[x, \delta_0] \subset O_x$. We claim that there is $N \in Z^+$ such that for each $n \geqslant N$, $f_n(O_x) \subset S_\rho[f(x), 2]$.

Suppose that for each $n \in Z^+$ there is $m \geqslant n$ such that

(3) $$f_m(O_x) \not\subset S_\rho[f(x), 2].$$

There is $M \in Z^+$ such that $S_{d\times\rho}[(x, f(x)), 3] \subset S_{d\times\rho}[(x_0, y_0), M]$. Let $n_0 \in Z^+$ be such that $n_0 > \max\{1/\delta_0, M\}$. The $\tau_{aw}(d \times \rho)$ convergence of $\{f_n\}$ to f implies that there is $N_1 \in Z^+$ such that for each $n \geqslant N_1$,

(4) $$(G(f_n), G(f)) \in U_{d\times\rho}[(x_0, y_0), n_0].$$

Put $A = \{y \in Y : \rho(f(x), y) = 2\}$. By (3) there is $m \geqslant N_1$ and there is $u \in O_x$ such that $f_m(u) \notin S_\rho[f(x), 2]$ and by (4) there is x_m such that

$d \times \rho((x, f(x)), (x_m, f_m(x_m))) < 1/n_0$, that is, $d(x, x_m) < 1/n_0 < \delta_0$; thus $x_m \in O_x$. The connectedness of O_x and continuity of f_m imply that there is $v \in O_x$ such that $f_m(v) \in A$.

(4) guarantees the existence of z such that $d \times \rho((z, f(z))(v, f_m(v))) < 1/n_0$, that is, $d(z, v) < \delta_0$ and thus $d(z, x) < \delta$; but $2 = \rho(f(x), f_m(v)) \leqslant \rho(f(x), f(z)) + \rho(f(z), f_m(v))$, that is, $2 - (1/n_0) \leqslant \rho(f(x), f(z))$, and that is a contradiction to (2). $\square$

Example 1 shows that the assumption local connectedness of X in Theorem 1 is essential.

Theorem 1 has a very useful consequence.

Corollary 1. *Let (X, d) be a locally connected metric space and (Y, ρ) be a metric space. Let $f, f_1, f_2, \ldots \in C(X, Y)$. If $\{f_n\}$ converges to f in the $\tau_{aw}(d \times \rho)$ topology, then $\{f_n\}$ is a pointwise equicontinuous sequence. ($\{f_n : n \in Z^+\}$ is pointwise equicontinuous at x if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\rho(f_n(x), f_n(z)) < \varepsilon$ whenever $d(x, z) < \delta$ and $n \in Z^+$.)*

Let (X, d) be a metric space. A metric space (X, d) is called *b*-compact if every bounded subset of X has compact closure [**12**].

Lemma 2. *Let (X, d) and (Y, ρ) be metric spaces. Let (X, d) be a b-compact space. Then the $\tau_{aw}(d \times \rho)$ topology on $C(X, Y)$ is weaker than uniform topology on compact subsets of X on $C(X, Y)$.*

Proof: For an easy proof see [**8 Theorem 4.1 (a)**]. $\square$

Proposition 2. *Let (X, d) be a metric space. The following are equivalent:*

(i) *(X, d) is b-compact;*
(ii) *for every metric space (Y, ρ) the $\tau_{aw}(d \times \rho)$ topology on $C(X, Y)$ is weaker than the topology of uniform convergence on compact subsets of X on $C(X, Y)$.*

Proof: (i) $\Rightarrow$ (ii) is clear from Lemma 2.

(ii) $\Rightarrow$ (i) Suppose there is a bounded set A in X such that the closure $\overline{A}$ of A is not compact. There is a sequence $\{x_n\}$ in $\overline{A}$ which has no cluster point in X. Choose ε_n for any $n \in Z^+$ such that $0 < \varepsilon_n < 1/n$ and such that the family $\{S_d[x_n, \varepsilon_n] : n \in Z^+\}$ is pairwise disjoint. Define g_n from $C(X, R)$ by

$$g_n(x) = \begin{cases} 1 - (d(x_n, x)/\varepsilon_n) & \text{if } x \in S_d[x_n, \varepsilon_n] \\ 0 & \text{for other } x. \end{cases}$$

The sequence $\{g_n\}$ is convergent to the function $g\colon X \to R$ identically equal to 0 in the topology of uniform convergence on compact subsets of X. (If K is a compact

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set in X, then K meets only finitely many sets $S_d[x_n, \varepsilon_n]$.) It is easy to see that the sequence $\{g_n\}$ fails to converge to g in the $\tau_{aw}(d \times |\cdot|)$ topology. ∎

THEOREM 2. *Let (X, d) be a locally connected b-compact metric space. Let (Y, ρ) be a metric space. Then the $\tau_{aw}(d \times \rho)$ topology on $C(X, Y)$ and the topology of uniform convergence on compact subsets of X on $C(X, Y)$ coincide.*

COROLLARY 2. *Let (Y, ρ) be a metric space. Let u_n be the usual metric on R^n $(n = 1, 2, \ldots)$. Then the $\tau_{aw}(u_n \times \rho)$ topology on $C(R^n, Y)$ and the topology of uniform convergence on compact subsets of R^n on $C(R^n, Y)$ coincide.*

COROLLARY 3. *Let (X, d) be a locally connected compact metric space. Let (Y, ρ) be a metric space. Then the $\tau_{aw}(d \times \rho)$ topology on $C(X, Y)$ and the Hausdorff metric topology on $C(X, Y)$ coincide.*

PROOF: For an easy proof see [**9**] and Theorem 2. ∎

Further relations between standard convergence notions and the Attouch-Wets convergence may be obtained from [**9**].

4. APPLICATIONS

In the subsequent part of our paper we give some applications of previous results to the space of continuous linear functions.

In the sequel, X will be a locally convex topological linear space metrisable with an invariant metric d, with origin θ. X^* will be a space of all continuous linear functions from X to R. In the sequel, the product $X \times R$ will be understood to be equipped with the box metric, denoted by $d \times |\cdot|$.

Let $\mathcal{S}$ denote the strong topology on X^*, that is if $U_{\varepsilon, A} = \{f \in X^* : |f(x)| < \varepsilon \text{ for every } x \in A\}$, then $\mathcal{S}$ is generated by the family $\{U_{\varepsilon, A}\}$, where ε runs over all positive reals and A runs over all linearly bounded sets in X. (A is a linearly bounded set if for every neighbourhood V of θ there is $n > 0$ such that $\chi V \supset A$ for every χ: $|\chi| > n$.)

THEOREM 3. *Let X be a locally convex topological linear space metrisable with an invariant metric d. Then the strong topology $\mathcal{S}$ on X^* is weaker than the $\tau_{aw}(d \times |\cdot|)$ topology on X^*.*

PROOF: It is sufficient to prove that $\tau_{aw}(d \times |\cdot|)$ convergence of a sequence $\{f_n\}$ from X^* to $f \in X^*$ implies $\mathcal{S}$ convergence of $\{f_n\}$ to f.

Since X is a locally connected metric space by Corollary 1 a family $\mathcal{L} = \{f_n : n = 1, 2, \ldots\} \cup \{f\}$ is pointwise equicontinuous. From the pointwise equicontinuity of $\mathcal{L}$ at θ we have the following assertion: there is δ_0 $(0 < \delta_0 < 1)$ such that for every $z \in S_d[\theta, \delta_0]$, $|f(z)| \leqslant 1$ and $|f_n(z)| \leqslant 1$ for every $n \in Z^+$.

Put $U = S_d[\theta, \delta_0]$. We show that $\{f_n\}$ converges uniformly to f on U. Let $\varepsilon > 0$. The uniform continuity of f implies that there is $\delta < \delta_0$ such that $|f(x) - f(y)| < \varepsilon/2$ whenever $d(x, y) < \delta$.

Put $\eta = \min\{\varepsilon/2, \delta\}$. Let $k \in Z^+$ be such that $k > 1/\eta$. The $\tau_{aw}(d \times |\cdot|)$ convergence of $\{f_n\}$ to f implies that there is $N_0 \in Z^+$ such that for every $n \geqslant N_0$ we have $(G(f_n), G(f)) \in U_{d\times|\cdot|}[(\theta, 0), k]$. We show that for every $n \geqslant N_0$ and every $x \in U$ we have $|f_n(x) - f(x)| < \varepsilon$.

Let $n \geqslant N_0$ and $x \in U$. There is $z \in X$ such that

$$d \times |\cdot|((x, f_n(x)), (z, f(z))) < 1/k,$$

that is $|f_n(x) - f(z)| < 1/k < \varepsilon/2$ and $d(x, z) < 1/k < \delta$. Then $|f(z) - f(x)| < \varepsilon/2$ and thus $|f_n(x) - f(x)| < \varepsilon$.

From this observation the $\mathcal{S}$ convergence of $\{f_n\}$ to f is obvious. ∎

The following theorem gives a characterisation of normed linear spaces in the class of locally convex topological linear spaces metrisable with an invariant metric.

Theorem 4. *Let X be a locally convex topological linear space metrisable with an invariant metric d. X is normable if and only if the strong topology $\mathcal{S}$ on X^* and the $\tau_{aw}(d \times |\cdot|)$ topology on X^* coincide.*

Proof: If X is a normed linear space, then by Theorem 4.3 in [5] the strong topology $\mathcal{S}$ on X^* and the $\tau_{aw}(d \times |\cdot|)$ topology on X^* coincide.

Suppose X is not normable. There is $k \in Z^+$ such that $S_d[\theta, k]$ is not linearly bounded. Put $A = S_d[\theta, k]$. There is an absolutely convex neighbourhood U of θ with the following property: for each $n \in Z^+$ there is $k_n > n$ such that $k_n U \not\supset A$. Thus there is a sequence $\{x_n\}$ such that $x_n \in A$ for each $n \in Z^+$ and $x_n \notin k_n U$ for each $n \in Z^+$. The absolute convexity of U implies that $x_n/n \notin U$ for each $n \in Z^+$. Without loss of generality we can also suppose that x_n/n is not contained in the closure $\overline{U}$ of U for each $n \in Z^+$.

From the fundamental theorem of functional analysis [16] for each $n \in Z^+$ there is a continuous linear function $f_n \colon X \to R$ such that $f_n(x_n/n) > 1$ and $|f_n(x)| \leqslant 1$ for each $x \in U$. For every $n \in Z^+$ put $h_n = f_n/n$. It is easy to verify that the sequence $\{h_n\}$ converges to the zero function h in the strong topology $\mathcal{S}$. We show that the sequence $\{h_n\}$ fails to converge to h in $\tau_{aw}(d \times |\cdot|)$ topology. There is δ $(0 < \delta < 1)$ such that for every $x, y \in X$ with $d(x, y) < \delta$ we have $|h_n(x) - h_n(y)| < 1/2$ for every $n \in Z^+$. Put $\eta = \min\{\delta, 1/2, 1/k\}$. Let $M \in Z^+$ be such that $1/M < \eta$. Then for every $n \in Z^+$ we have $(G(h_n), G(h)) \notin U_{d\times|\cdot|}[(\theta, 0), M]$. (Let $n \in Z^+$. The choice of M and $h_n(x_n) > 1$ implies that $(x_n, 0) = (x_n, h(x_n)) \notin S_{d\times|\cdot|}[G(h_n), 1/M]$.) ∎

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